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DEFORMATION OF HETEROGENEOUS MICROSTRETCH ELASTIC BARS

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The material points of microstretch continua undergo a uniform microdilatation and a rigid microrotation. This paper is concerned with the deformation of a bar composed by two different microstretch elastic materials. The intended applications of the solution are to bone implants and various compound cylinders. The bar is reinforced by a longitudinal rod and is subjected to extension, bending, torsion and flexure. First, the problem of bending and extension is investigated. The solution involves the solving of three plane strain problems. Then, we study the problem of torsion and flexure. The results are used to investigate the extension of a right circular cylinder reinforced by a circular rod.

1. Introduction

A microstretch continuum is a material with microstructure in which the microelements can stretch and contract independently of their translations and rotations. The theory of microstretch continua was introduced in [Eringen 1999] as a generalization of the Cosserat theory. When the microdilatation is zero, the microstretch continuum reduces to the Cosserat model. The applications of the theory of microstretch continua are to composite materials, porous solids, bones and various materials with inner structure. The Cosserat elastic solid was used as model for carbon nanotubes and composite materials [Lakes 2001; Chandraseker et al. 2009; Ha et al. 2016] and for bones [Lakes 1982; Fatemi et al. 2002]. Lakes [1982] presented some experimental observations on the mechanical behavior of bones and remarked that "Human bone, a natural fibrous composite, displays size effects in torsion and bending which are consistent with Cosserat elasticity rather than classical elasticity". The cancellous bone is considered as a porous body [Kohles and Roberts 2002] so the linear theory of microstretch elastic solids is adequate to describe the mechanical behavior of bones. Various papers have been devoted to the study of bone implants and anisotropic cylinders [Hanumantharaju and Shivanand 2009; Thielen et al. 2009; Taliercio and Veber 2016]. The bone and the implant form a body which can be modeled as a continuum composed of different materials.

We study the deformation of a heterogeneous bar which is made of two materials, welded together along the surface of separation. The deformation of homogeneous microstretch elastic cylinders has been investigated in [Ieşan and Nappa 1995; Ieşan 2008; 2019a; 2019b]. We assume that the bar is composed of two homogenous and isotropic microstretch elastic materials and is subjected to extension, bending, torsion and flexure.

The paper is structured as follows. First, we present the basic equations of isotropic microstretch elastic solids and formulate the problem of a reinforced rod. Then, we define the plane strain problem associated to a heterogeneous body. In the following section we present the solution of the problem of extension and bending. It is shown that this problem reduces to the study of some two-dimensional

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problems in which the external data depend only on the constitutive coefficients. Then, we study the problem of torsion and flexure. The flexure of the cylinder produces a torsion of the bar about its axis. Finally, we use the method to investigate the extension of a circular cylinder reinforced by a circular rod.

2. Preliminaries

We present the basic equations of the microstretch elastic solids and the formulation of the problem. We consider a continuum which in undeformed state occupies the regular domain B of Euclidean three-dimensional space and is bounded by the surface ∂B. Throughout this paper a fixed system of rectangular cartesian axes Ox_k, $(k = 1, 2, 3)$, is used. We shall employ standard indicial notations: Greek subscripts take on the values 1 and 2 whereas Latin subscripts (unless otherwise specified) are understood to range over $(1, 2, 3)$, and summation over repeated subscripts is implied. We denote by n_j the components of the outward unit normal of ∂B and introduce the notation $f_{,k} = \partial f/\partial x_k$. We study the deformation of isotropic solids in the linear theory of microstretch elastic continua. Let u_k be the displacement vector, let φ_k be the microrotation vector, and let ψ be the microstretch function. We denote by ε_{ijk} the alternating symbol. The strain measures are defined by

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k, \quad \kappa_{ij} = \varphi_{j,i}, \quad \gamma_i = \psi_{,i}. \tag{2-1}$$

Let t_{ij} be the stress tensor, m_{ij} be the couple stress tensor, σ_i be the microstretch stress vector, and ζ be the microstress function. The constitutive equations are

$$\begin{aligned} &t_{ij} = \lambda e_{rr}\delta_{ij} + (\mu+\kappa)\, e_{ij} + \mu e_{ji} + \lambda_0\psi\delta_{ij}, \quad && m_{ij} = \alpha\kappa_{rr}\delta_{ij} + \beta\kappa_{ji} + \gamma\kappa_{ij} + b_0\varepsilon_{kji}\gamma_k, \\ &\sigma_i = a_0\gamma_i + b_0\varepsilon_{irs}\kappa_{sr}, && \zeta = \lambda_0 e_{rr} + \lambda_1\psi, \end{aligned} \tag{2-2}$$

where λ, μ, κ, λ_0, λ_1, α, β, γ, b_0 and a_0 are constitutive coefficients. In the linear elasticity the Cosserat model is characterized by the coefficients λ, μ, κ, α, β, γ. The tractions acting at a regular point of ∂B are defined by

$$t_i = t_{ji}n_j, \quad m_i = m_{ji}n_j, \quad \sigma = \sigma_i n_i. \tag{2-3}$$

In the absence of body loads, the equilibrium equations are

$$t_{ji,j} = 0, \quad m_{ji,j} + \varepsilon_{irs}t_{rs} = 0, \quad \sigma_{j,j} - \zeta = 0. \tag{2-4}$$

In what follows we assume that the domain B is a right cylinder of length h with the cross-section Σ and the lateral boundary Π. The rectangular coordinate frame is chosen such that the x_3-axis is parallel to the generator of B. We denote by Σ_1 and Σ_2 the terminal cross-sections and assume that these are located at $x_3 = 0$ and $x_3 = h$, respectively. Let L be the boundary of Σ_1. We assume that the cylinder is free of lateral loading. We have

$$t_{\alpha i}n_\alpha = 0, \quad m_{\alpha i}n_\alpha = 0, \quad \sigma_\alpha n_\alpha = 0 \quad \text{on } \Pi. \tag{2-5}$$

We assume that the load of the cylinder is distributed over its ends in a way which fulfills the equilibrium conditions of the body. We use Saint-Venant's formulation in which the pointwise assignment of the terminal tractions is replaced by prescribing the corresponding resultant force and resultant moment. Let the loading applied on Σ_1 be equivalent to the resultant force $\boldsymbol{R} = (R_1, R_2, R_3)$ and the resultant moment $\boldsymbol{M} = (M_1, M_2, M_3)$. The cross-section Σ_2 is subjected to tractions that satisfy the conditions of the equilibrium of the cylinder. The conditions on the end Σ_1 can be expressed as

$$\int_{\Sigma_1} t_{3i}\, da = -R_i, \quad \int_{\Sigma_1} (x_\alpha t_{33} - \varepsilon_{3\alpha\beta} m_{3\beta})\, da = \varepsilon_{\alpha\beta 3} M_\beta, \quad \int_{\Sigma_1} (\varepsilon_{3\alpha\beta} x_\alpha t_{3\beta} + m_{33})\, da = -M_3. \tag{2-6}$$

The formulation of Saint-Venant leads to the four basic problems of extension ($R_\alpha = 0,\ M_j = 0$), bending ($R_j = 0,\ M_3 = 0$), torsion ($R_j = 0,\ M_\alpha = 0$) and flexure ($R_3 = 0,\ M_j = 0$).

Let Γ be a closed curve contained in Σ_1, which is the boundary of a regular domain Ω_2 contained in Σ_1. We suppose that Γ and L have no common points and denote by Ω_1 the domain bounded by Γ and L. Let B_ρ be the cylinder:

$$B_\rho = \{(x_1, x_2, x_3) : (x_1, x_2) \in \Omega_\rho, \quad 0 < x_3 < h\}, \quad \rho = 1, 2.$$

We suppose that B_ρ is occupied by an elastic material with the constitutive coefficients $\lambda^{(\rho)}$, $\mu^{(\rho)}$, $\kappa^{(\rho)}$, $\lambda_0^{(\rho)}$, $\lambda_1^{(\rho)}$, $\alpha^{(\rho)}$, $\beta^{(\rho)}$, $\gamma^{(\rho)}$, $b_0^{(\rho)}$ and $a_0^{(\rho)}$. We assume that the elastic potential corresponding to the material that occupies the cylinder B_ρ is a positive definite quadratic form in the independent constitutive variables.

Let S be the surface of separation of the two materials. We assume that the cylinder B is composed of two different materials which are welded together along S (Figure 1) and that there is no separation of material along S in the course of deformation. The functions u_j, φ_j, ψ, t_j, m_j and σ must be continuous in passing from one medium to another, so that we have

$$\begin{aligned} &[u_j]_1 = [u_j]_2, && [\varphi_j]_1 = [\varphi_j]_2, && [\psi]_1 = [\psi]_2, \\ &[t_{\alpha j}]_1 n_\alpha^0 = [t_{\alpha j}]_2 n_\alpha^0, && [m_{\alpha j}]_1 n_\alpha^0 = [m_{\alpha j}]_2 n_\alpha^0, && [\sigma_\alpha]_1 n_\alpha^0 = [\sigma_\alpha]_2 n_\alpha^0 \quad \text{on } S, \end{aligned} \tag{2-7}$$

where $(n_1^0, n_2^0, 0)$ are the components of the unit normal to S, and the expressions in brackets are calculated for the domains B_1 and B_2, respectively.

Saint-Venant's problem consists in the determination of the functions u_j, φ_j and ψ which satisfy the equations (2-1), (2-2) and (2-4) on B_1 and B_2, the conditions (2-5) on the lateral surface, the conditions (2-6) on Σ_1 and the conditions (2-7) on S. The constants R_j and M_j, and the constitutive coefficients are prescribed. In what follows we use the method established in [Ieşan 1976a; 1976b; 1976c] to study the deformation of inhomogeneous cylinders. This method has been extended to study generalized models in [Lyons et al. 2002; Ieşan and Scalia 2009; Ieşan 2008; Bîrsan and Altenbach 2011; Bîrsan et al. 2012].

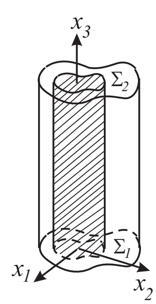

Figure 1. A heterogeneous rod.

3. Plane strain

We introduce the plane strain problem associated to a cylinder composed by two materials. We assume now that the cylinder is subjected to a body force $f_j^{(\rho)}$, a body couple $g_j^{(\rho)}$, and to the microstretch body force $l^{(\rho)}$, on B_ρ, $(\rho = 1, 2)$. Moreover, we suppose that the lateral surface of the cylinder is subjected to surface tractions $\tilde{t}_j$, to surface moments $\widetilde{m}_j$ and to microstretch traction $\tilde{\sigma}$. We assume that all external data are independent of the axial coordinate and that $f_3^{(\rho)} = 0$, $g_\alpha^{(\rho)} = 0$, $\tilde{t}_3 = 0$, $\widetilde{m}_\alpha = 0$.

We say that the cylinder B is in a state of plane strain, parallel to the x_1Ox_2-plane if the functions u_j, φ_j and ψ have the properties

$$u_\alpha = u_\alpha(x_1, x_2), \quad u_3 = 0, \quad \varphi_\alpha = 0, \quad \varphi_3 = \varphi_3(x_1, x_2), \quad \psi = \psi(x_1, x_2) \quad \text{on } B. \tag{3-1}$$

It follows from (2-1), (2-2) and (3-1) that e_{ij}, κ_{ij}, γ_i, t_{ij}, m_{ij}, σ_i and ζ are all independent of the axial coordinate. The nonzero strain measures are

$$e_{\alpha\beta} = u_{\beta,\alpha} + \varepsilon_{\beta\alpha 3}\,\varphi_3, \quad \kappa_{\alpha 3} = \varphi_{3,\alpha}, \quad \gamma_\alpha = \psi_{,\alpha}. \tag{3-2}$$

The constitutive equations (2-2) imply that, in the case of plane strain, the nonzero components of the stress tensor, couple stress tensor and microstretch stress vector are $t_{j\beta}$, $m_{\alpha 3}$, $m_{3\alpha}$ and σ_α. Further,

$$\begin{gathered} t_{a\beta} = \lambda^{(\rho)} e_{\nu\nu}\,\delta_{\alpha\beta} + (\mu^{(\rho)} + \kappa^{(\rho)})\, e_{\alpha\beta} + \mu^{(\rho)} e_{\beta\alpha} + \lambda_0^{(\rho)}\psi\,\delta_{\alpha\beta}, \\ m_{\alpha 3} = \gamma^{(\rho)}\kappa_{\alpha 3} + b_0^{(\rho)}\varepsilon_{3\alpha\beta}\,\psi_{,\beta}, \quad \sigma_\alpha = a_0^{(\rho)}\gamma_\alpha + b_0^{(\rho)}\varepsilon_{3\beta\alpha}\,\kappa_{\beta 3}, \quad \zeta = \lambda_0^{(\rho)} e_{\nu\nu} + \lambda_1^{(\rho)}\psi. \end{gathered} \tag{3-3}$$

The equations of equilibrium in the case of plane strain can be expressed as

$$t_{\beta\alpha,\beta} + f_\alpha^{(\rho)} = 0, \quad m_{\alpha 3,\alpha} + \varepsilon_{3\alpha\beta}\, t_{\alpha\beta} + g_3^{(\rho)} = 0, \quad \sigma_{\alpha,\alpha} - \zeta + l^{(\rho)} = 0 \quad \text{on } \Omega_\rho. \tag{3-4}$$

The conditions (2-7) reduce to

$$\begin{gathered} [u_\alpha]_1 = [u_\alpha]_2, \qquad [\varphi_3]_1 = [\varphi_3]_2, \qquad [\psi]_1 = [\psi]_2, \\ [t_{\beta\alpha}]_1 n_\beta^0 = [t_{\beta\alpha}]_2 n_\beta^0, \quad [m_{\alpha 3}]_1 n_\alpha^0 = [m_{\alpha 3}]_2 n_\alpha^0, \quad [\sigma_\alpha]_1 n_\alpha^0 = [\sigma_\alpha]_2 n_\alpha^0 \quad \text{on } \Gamma. \end{gathered} \tag{3-5}$$

The conditions on the boundary Π become

$$[t_{\beta\alpha}]_1 n_\beta = \tilde{t}_\alpha, \quad [m_{\alpha 3}]_1 n_\alpha = \widetilde{m}_3, \quad [\sigma_\alpha]_1 n_\alpha = \tilde{\sigma} \quad \text{on } L. \tag{3-6}$$

The plane strain problem consists in finding the functions u_α, φ_3 and ψ which satisfy the equations (3-2)–(3-4) on Ω_ρ, the conditions (3-5) on Γ and the conditions (3-6) on L. The functions $f_\alpha^{(\rho)}$, $g_3^{(\rho)}$, $l^{(\rho)}$, $\tilde{t}_\alpha$, $\widetilde{m}_3$ and $\tilde{\sigma}$ are given C^∞-fields. The necessary and sufficient conditons for the existence of a solution to the plane strain problem are [Fichera 1973]

$$\int_L \tilde{t}_\alpha\, ds + \sum_{\rho=1}^{2} \int_{\Omega_\rho} f_i^{(\rho)}\, da = 0, \quad \int_L (\varepsilon_{3\alpha\beta}\, x_\alpha\, \tilde{t}_\beta + \widetilde{m}_3)\, ds + \sum_{\rho=1}^{2} \int_{\Omega_\rho} (\varepsilon_{3\alpha\beta}\, x_\alpha f_\beta^{(\rho)} + g_3^{(\rho)})\, da = 0. \tag{3-7}$$

If the conditions (3-5) are replaced by

$$\begin{gathered} [u_\alpha]_1 = [u_\alpha]_2, \qquad [\varphi_3]_1 = [\varphi_3]_2, \qquad [\psi]_1 = [\psi]_2, \\ [t_{\beta\alpha}]_1 n_\beta^0 = [t_{\beta\alpha}]_2 n_\beta^0 + p_\alpha, \quad [m_{\alpha 3}]_1 n_\alpha^0 = [m_{\alpha 3}]_2 n_\alpha^0 + q, \quad [\sigma_\alpha]_1 n_\alpha^0 = [\sigma_\alpha]_2 n_\alpha^0 + s, \end{gathered} \tag{3-8}$$

where p_α, q and s are prescribed functions, then the necessary and sufficient conditions for the existence of the solution are

$$\begin{gathered}\int_L \tilde{t}_\alpha\, ds + \sum_{\rho=1}^{2} \int_{\Omega_\rho} f_\alpha^{(\rho)}\, ds + \int_\Gamma p_\alpha\, ds = 0,\\ \int_L (\varepsilon_{3\alpha\beta}\, x_\alpha\, \tilde{t}_\beta + \widetilde{m}_3)\, ds + \sum_{\rho=1}^{2} \int_{\Omega_\rho} (\varepsilon_{3\alpha\beta}\, x_\alpha f_\beta^{(\rho)} + g_3^{(\rho)})\, da + \int_\Gamma (\varepsilon_{3\alpha\beta}\, x_\alpha\, p_\beta + q)\, ds = 0.\end{gathered} \tag{3-9}$$

If $\widetilde{m}_3 = 0$ and $g_3^{(\rho)} = 0$, then the conditions (3-9) reduce to those used in classical elasticity to study the deformation of heterogeneous cylinders [Muskhelishvili 1953]. By using (3-2) and (3-3) we can express the equilibrium equations (3-4) in terms of functions u_α, φ_3 and ψ. We obtain the equations of equilibrium:

$$\begin{aligned}&(\mu^{(\rho)} + \kappa^{(\rho)})\, \Delta u_\alpha + (\lambda^{(\rho)} + \mu^{(\rho)})\, u_{\nu,\nu\alpha} + \kappa^{(\rho)} \varepsilon_{3\alpha\beta}\, \varphi_{3,\beta} + \lambda_0^{(\rho)} \psi_{,\alpha} + f_\alpha^{(\rho)} = 0,\\ &\gamma^{(\rho)} \Delta\varphi_3 + \kappa^{(\rho)} \varepsilon_{3\alpha\beta}\, u_{\beta,\alpha} - 2\kappa^{(\rho)} \varphi_3 + g_3^{(\rho)} = 0, \quad a_0^{(\rho)} \Delta\psi - \lambda_0^{(\rho)} u_{\nu,\nu} - \lambda_1^{(\rho)} \psi + l^{(\rho)} = 0\end{aligned} \tag{3-10}$$

on Ω_ρ, $(\rho = 1, 2)$, where Δ is the Laplacian.

4. Extension and bending of heterogeneous cylinders

We assume that the loading applied on the end located at $x_3 = 0$ is statically equivalent to the force $(0, 0, R_3)$ and the moment $(M_1, M_2, 0)$. The conditions on Σ_1 become

$$\int_{\Sigma_1} t_{3\alpha}\, da = 0, \tag{4-1}$$

$$\int_{\Sigma_1} t_{33}\, da = -R_3, \tag{4-2}$$

$$\int_{\Sigma_1} (x_\alpha\, t_{33} - \varepsilon_{3\alpha\beta}\, m_{3\beta})\, da = \varepsilon_{\alpha\beta 3}\, M_\beta, \tag{4-3}$$

$$\int_{\Sigma_1} (\varepsilon_{3\alpha\beta}\, x_\alpha\, t_{3\beta} + m_{33})\, da = 0. \tag{4-4}$$

The problem of extension and bending consists in finding the functions u_j, φ_j and ψ which satisfy the equations (2-1), (2-2) and (2-4) on B_ρ, the conditions (2-5) on Π, the conditions (2-7) on S, and the conditions (4-1)–(4-4) for $x_3 = 0$. Following [Ieşan 2019b] we try to solve this problem assuming that

$$\begin{gathered}u_\alpha = -\frac{1}{2} c_\alpha x_3^2 + \sum_{k=1}^{3} c_k\, u_\alpha^{(k)}, \quad u_3 = (c_1 x_1 + c_2 x_2 + c_3)\, x_3,\\ \varphi_\alpha = \varepsilon_{3\alpha\beta}\, c_\beta\, x_3, \quad \varphi_3 = \sum_{k=1}^{3} c_k\, \varphi_3^{(k)}, \quad \psi = \sum_{k=1}^{3} c_k \psi^{(k)},\end{gathered} \tag{4-5}$$

where $u_\alpha^{(k)}$, $\varphi_3^{(k)}$ and $\psi^{(k)}$ are unknown functions which are independent of x_3, and c_k are unknown constants. In what follows we shall prove that the functions $u_\alpha^{(k)}$, $\varphi_3^{(k)}$ and $\psi^{(k)}$ satisfy a plane strain problem $P^{(k)}$, $(k=1,2,3)$, associated to the composed cylinder B.

Let us denote by $e_{\alpha\beta}^{(k)}$, $\kappa_{\alpha 3}^{(k)}$ and $\gamma^{(k)}$ the strain measures (3-2) associated to the functions $u_\alpha^{(k)}$, $\varphi_3^{(k)}$ and $\psi^{(k)}$, $(k=1,2,3)$. Thus, we have

$$e_{\alpha\beta}^{(k)}=u_{\beta,\alpha}^{(k)}+\varepsilon_{\beta\alpha 3}\,\varphi_3^{(k)},\quad \kappa_{\alpha 3}^{(k)}=\varphi_{3,\alpha}^{(k)},\quad \gamma_\alpha^{(k)}=\psi_{,\alpha}^{(k)}. \tag{4-6}$$

We introduce the notations

$$\begin{gathered} t_{\alpha\beta}^{(k)}=\lambda^{(\rho)}e_{\nu\nu}^{(k)}\delta_{\alpha\beta}+(\mu^{(\rho)}+\kappa^{(\rho)})\,e_{\alpha\beta}^{(k)}+\mu^{(\rho)}e_{\beta\alpha}^{(k)}+\lambda_0^{(\rho)}\psi\,\delta_{\alpha\beta},\\ m_{\alpha 3}^{(k)}=\gamma^{(\rho)}\kappa_{\alpha 3}^{(k)}+b_0^{(\rho)}\varepsilon_{3\alpha\beta}\,\gamma_\beta^{(k)},\quad \sigma_\alpha^{(k)}=a_0^{(\rho)}\gamma_\alpha^{(k)}+b_0^{(\rho)}\varepsilon_{3\beta\alpha}\,\kappa_{\beta 3},\quad \zeta^{(k)}=\lambda_0^{(\rho)}e_{\nu\nu}^{(k)}+\lambda_1^{(\rho)}\psi^{(k)},\\ t_{33}^{(k)}=\lambda^{(\rho)}e_{\nu\nu}^{(k)}+\lambda_0^{(\rho)}\psi^{(k)},\quad m_{3\alpha}^{(k)}=\beta^{(\rho)}\kappa_{\alpha 3}^{(k)}+b_0^{(\rho)}\varepsilon_{3\alpha\nu}\,\gamma_\nu^{(k)}\quad \text{on } \Omega_\rho. \end{gathered} \tag{4-7}$$

In view of (2-1), (4-5)–(4-7), the constitutive equations (2-2) imply

$$\begin{gathered} t_{\alpha\beta}=\lambda^{(k)}(c_1x_1+c_2x_2+c_3)\,\delta_{\alpha\beta}+\sum_{k=1}^{3}c_k\,t_{\alpha\beta}^{(k)},\quad t_{\alpha 3}=t_{3\alpha}=0,\\ t_{33}=(\lambda^{(\rho)}+2\mu^{(\rho)}+\kappa^{(\rho)})(c_1x_1+c_2x_2+c_3)+\sum_{k=1}^{3}c_k\,t_{33}^{(k)},\\ m_{\alpha 3}=m_{33}=0,\quad m_{\alpha 3}=\beta^{(\rho)}\varepsilon_{3\alpha\beta}\,c_\beta+\sum_{k=1}^{3}c_k\,m_{\alpha 3}^{(k)},\quad m_{3\alpha}=\gamma^{(\rho)}\varepsilon_{3\alpha\nu}\,c_\nu+\sum_{k=1}^{3}c_k\,m_{3\alpha}^{(k)},\\ \sigma_\alpha=-b_0^{(\rho)}c_\alpha+\sum_{k=1}^{3}c_k\,\sigma_\alpha^{(k)},\quad \sigma_3=0,\quad \zeta=\lambda_0^{(\rho)}(c_1x_1+c_2x_2+c_3)+\sum_{k=1}^{3}c_k\,\zeta^{(k)}\quad \text{on } B_\rho. \end{gathered} \tag{4-8}$$

Let us substitute (4-8) into the equations of equilibrium (2-4). We require that the resulting equations be satisfied for any constant c_k. Thus, we find that the functions $t_{\alpha\beta}^{(k)}$, $m_{\alpha 3}^{(k)}$, $\sigma_\alpha^{(k)}$ and $\zeta^{(k)}$, $(k=1,2,3)$, satisfy

$$\begin{gathered} t_{\beta\alpha,\beta}^{(\nu)}+\lambda^{(\rho)}\delta_{\alpha\nu}=0,\quad t_{\beta\alpha,\beta}^{(3)}=0,\quad m_{\beta\alpha,\beta}^{(k)}+\varepsilon_{3\rho\nu}\,t_{\rho\nu}^{(k)}=0,\\ \sigma_{\alpha,\alpha}^{(\nu)}-\zeta^{(\nu)}-\lambda_0^{(\rho)}x_\nu=0,\quad \sigma_{\alpha,\alpha}^{(3)}-\zeta^{(3)}-\lambda_0^{(\rho)}=0,\quad \nu=1,2\quad \text{on } \Omega_\rho. \end{gathered} \tag{4-9}$$

The equilibrium equations are satisfied if (4-9) holds. It follows from (2-5) and (4-8) that the conditions on the lateral surface are satisfied for any constants c_1, c_2 and c_3 if we have

$$[t_{\beta\alpha}^{(k)}]_1\,n_\beta=\tilde{t}_\alpha^{(k)},\quad [m_{\alpha 3}^{(k)}]_1\,n_\alpha=\widetilde{m}_3^{(k)},\quad [\sigma_\alpha^{(k)}]_1\,n_\alpha=\tilde{\sigma}^{(k)}\quad \text{on } L, \tag{4-10}$$

where we have used the notations

$$\tilde{t}_\beta^{(\alpha)}=-\lambda^{(1)}x_\alpha n_\beta,\ \ \tilde{t}_\beta^{(3)}=-\lambda^{(1)}n_\beta,\ \ \widetilde{m}_3^{(\alpha)}=\beta^{(1)}\varepsilon_{3\alpha\nu}\,n_\nu,\ \ \widetilde{m}_3^{(3)}=0,\ \ \tilde{\sigma}^{(\alpha)}=b_0^{(1)}n_\alpha,\ \ \tilde{\sigma}^{(3)}=0. \tag{4-11}$$

Similarly, we conclude the conditions (2-7) on the surface S are satisfied if the unknown functions satisfy

$$\begin{gathered}[u_\alpha^{(k)}]_1=[u_\alpha^{(k)}]_2,\quad [\varphi_3^{(k)}]_1=[\varphi_3^{(k)}]_2,\quad [\psi^{(k)}]_1=[\psi^{(k)}]_2,\quad [t_{\beta\alpha}^{(k)}]_1 n_\beta^0=[t_{\beta\alpha}^{(k)}]_2 n_p+p_\alpha^{(k)},\\ [m_{\alpha 3}^{(k)}]\, n_\alpha^0=[m_{\alpha 3}^{(k)}]_2\, n_\alpha^0+q^{(k)},\quad [\sigma_\alpha^{(k)}]_1 n_\alpha^0=[\sigma_\alpha^{(k)}]_2\, n_\alpha^0+s^{(k)}\quad \text{on } \Gamma.\end{gathered} \tag{4-12}$$

Here, $p_\alpha^{(k)}$, $q^{(k)}$ and $s^{(k)}$ are defined by

$$\begin{aligned} &p_\alpha^{(\beta)}=(\lambda^{(2)}-\lambda^{(1)})\, x_\beta\, n_\alpha^0, && p_\alpha^{(3)}=(\lambda^{(2)}-\lambda^{(1)})\, n_\alpha^0,\\ &q^{(\eta)}=(\beta^{(1)}-\beta^{(2)})\, \varepsilon_{3\eta\alpha}\, n_\alpha^0,\quad q^{(3)}=0, && s^{(\beta)}=(b_0^{(1)}-b_0^{(2)})\, n_\beta^0,\quad s^{(3)}=0.\end{aligned} \tag{4-13}$$

Thus, the functions $u_\alpha^{(k)}$, $\varphi_3^{(k)}$ and $\psi^{(k)}$ are the solutions of the plane strain problems $P^{(k)}$ which are characterized by the geometrical equations (4-6), the constitutive equations (4-7) and the equilibrium equations (4-9) on Ω_ρ, and the boundary conditions (4-10) and (4-11). The functions $t_{33}^{(k)}$ and $m_{3\alpha}^{(k)}$ can be found after the determination of $u_\alpha^{(k)}$, $\varphi_3^{(k)}$ and $\psi^{(k)}$ from the problems $P^{(k)}$, $(k=1,2,3)$. It is easy to see that the necessary and sufficient conditions (3-9) for the existence of the solution are satisfied for each problem $P^{(k)}$, $(k=1,2,3)$. We note that the external loading in the problems $P^{(k)}$ depend only the constitutive coefficients and the cross-section of the cylinder.

Let us prove that the constants c_1, c_2 and c_3 can be determined from the conditions on the ends. First, we note that the conditions (4-1) and (4-4) are satisfied on the basis of equations (4-8).

The conditions (4-2) and (4-3) reduce to the following system for the constants c_1, c_2 and c_3:

$$A_{\alpha j}\, c_j=\varepsilon_{3\alpha\beta} M_\beta,\quad A_{3j}\, c_j=-R_3. \tag{4-14}$$

Here, we have used the notations

$$\begin{aligned} A_{\alpha\beta}&=\sum_{\rho=1}^{2}\int_{\Omega_\rho}\{(\lambda^{(\rho)}+2\mu^{(\rho)}+\kappa^{(\rho)})\, x_\alpha x_\beta+x_\alpha t_{33}^{(\beta)}+\varepsilon_{3\nu\alpha}\, m_{3\nu}^{(\beta)}+\gamma^{(\rho)}\delta_{\alpha\beta}\}\, da,\\ A_{\alpha 3}&=\sum_{\rho=1}^{2}\int_{\Omega_\rho}\{(\lambda^{(\rho)}+2\mu^{(\rho)}+\kappa^{(\rho)})\, x_\alpha+x_\alpha t_{33}^{(3)}+\varepsilon_{3\nu\alpha}\, m_{3\nu}^{(3)}\}\, da,\\ A_{3\alpha}&=\sum_{\rho=1}^{2}\int_{\Omega_\rho}[(\lambda^{(\rho)}+2\mu^{(\rho)}+\kappa^{(\rho)})\, x_\alpha+t_{33}^{(\alpha)}]\, da,\\ A_{33}&=\sum_{\rho=1}^{2}\int_{\Omega_\rho}(\lambda^{(\rho)}+2\mu^{(\rho)}+\kappa^{(\rho)}+t_{33}^{(3)})\, da.\end{aligned} \tag{4-15}$$

The constants A_{ij} can be found if we know the functions $u_\alpha^{(k)}$, $\varphi_3^{(k)}$ and $\psi^{(k)}$, $(k=1,2,3)$. By using the methods of the classical elasticity [Muskhelishvili 1953; Ieşan 2008] we can prove that the positive definiteness of the elastic potential and the reciprocal theorem imply

$$\det(A_{ij})>0,\quad A_{ij}=A_{ji}. \tag{4-16}$$

The constants c_j are determined by the system (4-14). Thus, the solution of the problem is given by (4-5).

5. Torsion and flexure

The problem of torsion and flexure consists in finding the displacement vector, the microrotation vector and the microstretch function, which satisfy the equations (2-1), (2-2) and (2-4), the conditions (2-5) on the lateral surface, the conditions (2-7) on the surface separation, and the conditions (2-6) on Σ_1, when $R_3 = 0$ and $M_\alpha = 0$. We seek the solution of the problem in the form

$$u_\alpha = -\frac{1}{6}a_\alpha x_3^3 + x_3 \sum_{k=1}^{3} a_k u_\alpha^{(k)} + d x_3 \varepsilon_{\beta\alpha 3} x_\beta, \quad u_3 = \frac{1}{2}(a_1 x_1 + a_2 x_2 + a_3) x_3^2 + F_3,$$
$$\varphi_\alpha = \frac{1}{2} x_3^2 \varepsilon_{3\alpha\beta} a_\beta + F_\beta, \quad \varphi_3 = d x_3 + x_3 \sum_{k=1}^{3} a_k \varphi_3^{(k)}, \quad \psi = x_3 \sum_{k=1}^{3} a_k \psi^{(k)}, \tag{5-1}$$

where a_k and d are unknown constants, $u_\alpha^{(k)}$, $\varphi_3^{(k)}$ and $\psi^{(k)}$ are the solution to the problem $P^{(k)}$, and F_j are unknown functions of x_1 and x_2. We denote by $\omega = (F_1, F_2, F_3)$ the ordered triplet of functions F_1, F_2 and F_3, and introduce the notations

$$T_{\alpha 3}^{(\rho)}\omega = (\mu^{(\rho)} + \kappa^{(\rho)}) F_{3,\alpha} + \kappa^{(\rho)} \varepsilon_{\alpha\beta 3} F_\beta, \qquad T_{3\alpha}^{(\rho)}\omega = \mu^{(\rho)} F_{3,\alpha} + \kappa^{(\rho)} \varepsilon_{3\beta\alpha} F_\beta,$$
$$M_{\alpha\beta}^{(\rho)}\omega = \alpha^{(\rho)} F_{\eta,\eta} \delta_{\alpha\beta} + \beta^{(\rho)} F_{\alpha,\beta} + \gamma^{(\rho)} F_{\beta,\alpha}, \quad N_\alpha^{(\rho)}\omega = n_\beta M_{\beta\alpha}^{(\rho)}\omega, \quad N_3^{(\rho)}\omega = n_\alpha T_{\alpha 3}^{(\rho)}\omega. \tag{5-2}$$

and

$$L_\nu^{(\rho)}\omega = \alpha^{(\rho)} F_{\eta,\eta\nu} + \beta^{(\rho)} F_{\lambda,\nu\lambda} + \gamma^{(\rho)} F_{\nu,\lambda\lambda} + \varepsilon_{\nu\eta 3} \kappa^{(\rho)} F_{3,\eta} - 2\kappa^{(\rho)} F_\nu,$$
$$L_3^{(\rho)}\omega = (\mu^{(\rho)} + \kappa^{(\rho)}) F_{3,\alpha\alpha} + \varepsilon_{\alpha\beta 3} \kappa^{(\rho)} F_{\beta,\alpha} \quad \text{on } \Omega_\rho. \tag{5-3}$$

From (2-1), (5-1) and the constitutive equations we obtain

$$t_{\alpha\beta} = \lambda^{(\rho)}(a_1 x_1 + a_2 x_2 + a_3) x_3 \delta_{\alpha\beta} + x_3 \sum_{k=1}^{3} a_k t_{\alpha\beta}^{(k)},$$
$$t_{33} = (\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})(a_1 x_1 + a_2 x_2 + a_3) x_3 + x_3 \sum_{k=1}^{3} a_k (\lambda^{(\rho)} e_{\nu\nu}^{(\rho)} + \lambda_0^{(\rho)} \psi^{(k)}),$$
$$t_{\alpha 3} = T_{\alpha 3}^{(\rho)}\omega + d\mu^{(\rho)} \varepsilon_{3\beta\alpha} x_\beta + \mu^{(\rho)} \sum_{k=1}^{3} a_k u_\alpha^{(k)},$$
$$t_{3\alpha} = T_{3\alpha}^{(\rho)}\omega + d \varepsilon_{3\beta\alpha} (\mu^{(\rho)} + \kappa^{(\rho)}) x_\beta + (\mu^{(\rho)} + \kappa^{(\rho)}) \sum_{k=1}^{3} a_k u_\alpha^{(k)}, \tag{5-4a}$$
$$m_{\nu\eta} = M_{\nu\eta}^{(\rho)}\omega + \alpha^{(\rho)} d \delta_{\nu\eta} + \sum_{k=1}^{3} a_k (\alpha^{(\rho)} \delta_{\nu\eta} \varphi_3^{(k)} + \varepsilon_{3\eta\nu} b_0^{(\rho)} \psi^{(k)}),$$
$$m_{33} = (\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)}) \left(d + \sum_{k=1}^{3} a_k \varphi_3^{(k)} \right) + \alpha^{(\rho)} F_{\eta,\eta},$$
$$m_{\alpha 3} = \left(\beta^{(\rho)} \varepsilon_{\alpha\nu 3} a_\nu + \sum_{k=1}^{3} a_k m_{\alpha 3}^{(k)} \right) x_3, \quad m_{3\alpha} = \left(\gamma^{(\rho)} \varepsilon_{\alpha\beta 3} a_\beta + \sum_{k=1}^{3} a_k m_{3\alpha}^{(k)} \right) x_3,$$

$$\sigma_\alpha = \left(\sum_{k=1}^{3} a_k \sigma_\alpha^{(\rho)} - b_0^{(\rho)} a_\alpha\right) x_3, \quad \sigma_3 = b_0^{(\rho)} \varepsilon_{3\alpha\beta} F_{\alpha,\beta} + a_0^{(\rho)} \sum_{k=1}^{3} a_k \psi^{(k)},$$
$$\zeta = \left(\lambda_0^{(\rho)}(a_1x_1 + a_2x_2 + a_3) + \sum_{k=1}^{3} a_k \zeta^{(k)}\right) x_3 \quad \text{on } B_\rho. \tag{5-4b}$$

By using (4-9), (5-3), (5-4a) and (5-4b) we see that the equilibrium equations (2-4) reduce to

$$L_j^{(\rho)} \omega = S_j^{(\rho)} \quad \text{on } \Omega_\rho, \tag{5-5}$$

where

$$S_\nu^{(\rho)} = d\kappa^{(\rho)} x_\nu + \gamma^{(\rho)} \varepsilon_{3\beta\nu} a_\beta - \sum_{k=1}^{3} a_k [m_{3\nu}^{(k)} + \alpha^{(\rho)} \varphi_{3,\nu}^{(k)} - \varepsilon_{\nu\beta 3} \kappa^{(\rho)} u_\beta^{(k)}],$$
$$S_3^{(\rho)} = -(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})(a_1x_1 + a_2x_2 + a_3) - \sum_{k=1}^{3} a_k [(\lambda^{(\rho)} + \mu^{(\rho)}) e_{\alpha\alpha}^{(k)} + \lambda_0^{(\rho)} \psi^{(k)}]. \tag{5-6}$$

In view of (4-10), (4-11), (5-2), (5-4a) and (5-4b), the boundary conditions (2-5) can be written as

$$N_j^{(1)} \omega = h_j \quad \text{on } L, \tag{5-7}$$

where

$$h_\alpha = -\alpha^{(1)} d n_\alpha - n_\nu \sum_{k=1}^{3} a_k (\alpha^{(1)} \delta_{\nu\alpha} \varphi_3^{(k)} + \varepsilon_{3\alpha\nu} b_0^{(1)} \psi^{(k)}),$$
$$h_3 = d\mu^{(1)} \varepsilon_{3\alpha\beta} x_\beta n_\alpha - n_\alpha \mu^{(1)} \sum_{k=1}^{3} a_k u_\alpha^{(k)}. \tag{5-8}$$

The conditions on the surface of separation S reduce to

$$[F_j]_1 = [F_j]_2, \quad (N_j^{(1)} \omega)(n^0) = (N_j^{(2)} \omega)(n^0) + \varepsilon_j \ \text{ on } \Gamma. \tag{5-9}$$

Here, $(N_j^{(\rho)} \omega)(n^0)$ denotes the operator $N_j^{(\rho)} \omega$ for $n_\alpha = n_\alpha^0$ and ε_j are defined by

$$\varepsilon_\nu = (\alpha^{(2)} - \alpha^{(1)}) n_\nu^0, \quad \varepsilon_3 = (\mu^{(1)} - \mu^{(2)}) \left(d\varepsilon_{3\alpha\beta} x_\beta - \sum_{k=1}^{3} a_k u_\alpha^{(k)} \right) n_\alpha^0. \tag{5-10}$$

It is known [Fichera 1973] that the necessary and sufficient condition for the existence of a solution to the problem (5-5), (5-7) and (5-9) is

$$\int_L h_3 \, ds + \int_\Gamma \varepsilon_3 \, ds = \sum_{\rho=1}^{2} \int_{\Omega_\rho} S_3^{(\rho)} \, da. \tag{5-11}$$

In view of (5-6), (5-8) and (5-10), the condition (5-11) reduces to

$$A_{3j} a_j = 0, \tag{5-12}$$

where A_{3j} are defined in (4-15). By using the equations of equilibrium (2-4) we get

$$t_{3\alpha} = t_{\alpha 3} + \varepsilon_{3\beta\alpha}\, m_{j\beta,j} = t_{\alpha 3} + x_\alpha\, t_{k3,k} + \varepsilon_{3\beta\alpha}\, m_{j\beta,j} = (x_\alpha\, t_{\nu 3} + \varepsilon_{3\beta\alpha}\, m_{\nu\beta})_{,\nu} + x_\alpha\, t_{33,3} + \varepsilon_{3\beta\alpha}\, m_{3\beta,3}.$$

Thus, in view of (2-5) and (2-7), we obtain

$$\int_{\Sigma_1} t_{3\alpha}\, da = \int_{\Sigma_1} (x_\alpha\, t_{33,3} + \varepsilon_{3\beta\alpha}\, m_{3\beta,3})\, da. \tag{5-13}$$

With the help of (5-4a), (5-13) and (4-15), the first two conditions from (2-6) become

$$A_{\alpha j}\, a_j = -R_\alpha. \tag{5-14}$$

On the basis of (4-16), the equations (5-12) and (5-14) determine the constants a_k. Let us determine now the constant d. We introduce the notation $V = (G_1, G_2, G_3)$, where G_j satisfy the following boundary value problem

$$\begin{gathered} L_\alpha^{(\rho)} V = \kappa^{(\rho)} x_\alpha, \quad L_3^{(\rho)} V = 0 \quad \text{on } \Omega_\rho, \qquad [V_j]_1 = [V_j]_2, \quad (N_\alpha^{(1)} V)(n^0) = (N_\alpha^{(2)} V)(n^0), \\ (N_3^{(1)} V)(n^0) = (N_3^{(2)} V)(n^0) + (\mu^{(1)} - \mu^{(2)})\, \varepsilon_{3\alpha\beta}\, x_\beta\, n_\alpha^0, \quad \text{on } \Gamma, \\ N_\nu^{(1)} V = -\alpha^{(1)} n_\nu, \quad N_3^{(1)} V = \mu^{(1)} \varepsilon_{3\alpha\beta}\, x_\beta\, n_\alpha \quad \text{on } L. \end{gathered} \tag{5-15}$$

If we define the functions F_j^0 by

$$F_j^0 = F_j - d G_j, \tag{5-16}$$

and denote $\omega^0 = (F_1^0, F_2^0, F_3^0)$, then ω^0 satisfies

$$L_j^{(\rho)} \omega^0 = \gamma^{(\rho)} \varepsilon_{3\beta\nu}\, a_\beta - \sum_{k=1}^{3} a_k [m_{3\nu}^{(k)} + \alpha^{(\rho)} \varphi_{3,\nu}^{(k)} - \varepsilon_{\nu\beta 3}\, \kappa^{(\rho)} u_\beta^{(k)}], \quad L_3^{(\rho)} \omega^0 = S_3^{(\rho)} \quad \text{on } \Omega_\rho, \tag{5-17}$$

and the conditions

$$\begin{gathered} [F_j^0]_1 = [F_j^0]_2, \quad (N_\nu^{(1)} \omega^0)(n^0) = (N_\nu^{(2)} \omega^0)(n^0) + \varepsilon_\nu, \\ (N_3^{(1)} \omega^0)(n^0) = (N_3^{(2)} \omega^0)(n^0) - (\mu^{(1)} - \mu^{(2)})\, n_\alpha^0 \sum_{k=1}^{3} a_k\, u_\alpha^{(k)} \quad \text{on } \Gamma, \\ N_\alpha^{(1)} \omega^0 = -n_\nu \sum_{k=1}^{3} a_k (\alpha^{(1)} \delta_{\nu\alpha}\, \varphi_3^{(k)} + \varepsilon_{3\alpha\nu}\, b_0^{(1)} \psi^{(k)}), \quad N_3^{(1)} \omega^0 = -\mu^{(1)} n_\alpha \sum_{k=1}^{3} a_k\, u_\alpha^{(k)} \quad \text{on } L. \end{gathered} \tag{5-18}$$

Clearly, the conditions for the existence of the functions G_j and F_j^0 are satisfied. The functions $t_{3\alpha}$ and m_{33} can be expressed as

$$\begin{aligned} t_{3\alpha} &= T_{3\alpha}^{(\rho)} \omega^0 + d\,[T_{3\alpha}^{(\rho)} V + \varepsilon_{3\beta\alpha} (\mu^{(\rho)} + \kappa^{(\rho)})\, x_\beta] + (\mu^{(\rho)} + \kappa^{(\rho)}) \sum_{k=1}^{3} a_k\, u_\alpha^{(k)}, \\ m_{33} &= (\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)}) \sum_{k=1}^{3} a_k \varphi_3^{(k)} + \alpha^{(\rho)} F_{\eta,\eta}^0 + d(\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)} + \alpha^{(\rho)} G_{\eta,\eta}). \end{aligned} \tag{5-19}$$

In view of (5-19), the last condition from (2-6) determines the constant d,

$$d = -D^{-1}(M_3 + M_3^*), \tag{5-20}$$

where

$$D = \sum_{\rho=1}^{2} \int_{\Omega_\rho} [\varepsilon_{3\alpha\beta}\, x_\alpha T_{3\beta}^{(\rho)} V + (\mu^{(\rho)} + \kappa^{(\rho)}) x_\nu x_\nu + a^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)} + \alpha^{(\rho)} G_{\nu,\nu}]\, da,$$

$$M_3^* = \sum_{\rho=1}^{2} \int_{\Omega_\rho} \{\varepsilon_{3\alpha\beta}\, x_\alpha T_{3\beta}^{(\rho)} + \alpha^{(\rho)} F_{\nu,\nu}^0 + \sum_{k=1}^{3} a_k[(\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)})\, v_3^{(k)}] + \varepsilon_{3\alpha\beta}\, x_\alpha (\mu^{(\rho)} + \kappa^{(\rho)}) u_\beta^{(k)}\}\, da. \tag{5-21}$$

As in classical elasticity we can show that the torsional rigidity D is different from zero. The other conditions from (2-6) are satisfied on the basis of relations (5-4a). Thus, the solution of the problem is given by (5-1).

In the case of flexure we have $M_3 = 0$, but the constant d could be different from zero. In general, the flexure of the bar is accompanied by torsion. The torsion problem is characterized by $R_j = 0$ and $M_\alpha = 0$. In this case, from (5-12) and (5-14) we find that the constants a_k are equal to zero. It follows that the torsion does not produce a microdilatation. The solution of the flexure problem shows that the microrotation vector and the microstretch function are, in general, different from zero.

6. Extension of a cylinder reinforced by a longitudinal rod

This section is concerned with the problem of extension of a circular cylinder composed by two different microstretch elastic materials. In this case in the conditions (2-6) we have $R_\alpha = 0$ and $M_j = 0$. We assume that the domains Ω_1 and Ω_2 are defined by $\Omega_1 = \{(x_1, x_2, x_3) : r_2^2 < x_1^2 + x_2^2 < r_1^2, x_3 = 0\}$ and $\Omega_2 = \{(x_1, x_2, x_3) : 0 \le x_1^2 + x_2^2 < r_2^2, x_3 = 0\}$, where r_1 and r_2 are the radiuses of the concentric circles L and Γ, respectively. To investigate this problem we use the solution (4-5). First, we have to study the plane strain problems $P^{(k)}$, and then to calculate the constants c_j. We seek the solution of the problem $P^{(3)}$ in the form

$$u_\alpha^{(3)} = U_{,\alpha}, \quad \varphi^{(3)} = 0, \quad \psi^{(3)} = \Phi, \tag{6-1}$$

where U and Φ are unknown functions which depend only on the variable $r = (x_1^2 + x_2^2)^{1/2}$. Clearly,

$$u_{\alpha,\alpha}^{(3)} = \Delta U = \tfrac{1}{r}(rU')', \quad e_{\alpha\beta}^{(3)} = U_{,\alpha\beta} = \delta_{\alpha\beta} r^{-1} U' - x_\alpha x_\beta r^{-3} U' + x_\alpha x_\beta r^{-2} U'', \tag{6-2}$$

where $U' = dU/dr$. From (4-7) and (6-1) we find that

$$t_{\alpha\beta}^{(3)} = \lambda^{(\rho)} \delta_{\alpha\beta} \Delta U + (2\mu^{(\rho)} + \kappa^{(\rho)})\, U_{,\alpha\beta} + \lambda_0^{(\rho)} \Phi\, \delta_{\alpha\beta},$$
$$m_{\alpha 3}^{(3)} = 0, \quad \sigma_\alpha^{(3)} = a_0^{(\rho)} \Phi_{,\alpha}, \quad \zeta^{(3)} = \lambda_0^{(\rho)} \Delta U + \lambda_1^{(\rho)} \Phi \quad \text{on } \Omega_\rho. \tag{6-3}$$

In view of (3-10), the equilibrium equations (4-9) reduce to

$$\Delta U + e_{(\rho)} \Phi = A_1^{(\rho)}, \quad \Delta \Phi - \tau_{(\rho)}^2 \Phi = \frac{\lambda_0^{(\rho)}}{a_0^{(\rho)}}(1 + A_1^{(\rho)}) \quad \text{on } \Omega_\rho, \tag{6-4}$$

where $A_1^{(1)}$ and $A_1^{(2)}$ are arbitrary constants and

$$e_{(\rho)} = \lambda_0^{(\rho)}/(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)}), \quad \tau_{(\rho)}^2 = \frac{\lambda_1^{(\rho)}}{a_0^{(\rho)}} - \frac{\lambda_0^{(\rho)}}{a_0^{(\rho)}} e_{(\rho)}. \tag{6-5}$$

The positive definiteness of elastic potential imply that $\tau_{(\rho)}^2 > 0$. From the equations (6-4) we find

$$\begin{aligned} U' &= -\eta_{(\rho)}\Phi_0' + \chi_{(\rho)} r + \xi_{(\rho)} A_1^{(\rho)} r + A_2^{(\rho)} r^{-1}, \\ \Phi &= \Phi_0 - 2d_{(\rho)}(1 + A_1^{(\rho)}), \\ \Phi_0 &= C_1^{(\rho)} I_0(\tau_{(\rho)} r) + C_2^{(\rho)} K_0(\tau_{(\rho)} r) \quad \text{on } \Omega_\rho, \end{aligned} \tag{6-6}$$

where $A_2^{(\rho)}$, $C_1^{(\rho)}$ and $C_2^{(\rho)}$ are arbitrary constants, I_n and K_n are modified Bessel functions of order n, and we have used the notations

$$\eta_{(\rho)} = e_{(\rho)}/\tau_{(\rho)}^2, \quad \chi_{(\rho)} = e_{(\rho)} d_{(\rho)}, \quad d_{(\rho)} = \lambda_0^{(\rho)}(2\tau_{(\rho)}^2 a_0^{(\rho)})^{-1}, \quad \xi_{(\rho)} = \lambda_1^{(\rho)}(d_{(\rho)}/\lambda_0^{(\rho)}). \tag{6-7}$$

Since U' and Φ must be bounded for $r = 0$, we conclude that

$$A_2^{(2)} = 0, \quad C_2^{(2)} = 0. \tag{6-8}$$

With the help of (6-1) and (6-6) we obtain

$$\begin{aligned} u_\alpha^{(3)} &= x_\alpha(-\eta_{(\rho)}\Phi_0' r^{-1} + \chi_{(\rho)} + \xi_{(\rho)} A_1^{(\rho)} + A_2^{(\rho)} r^{-2}) \quad \text{on } \Omega_\rho, \\ t_{\alpha\beta}^{(3)} n_\alpha &= n_\beta\{(2\mu^{(1)} + \kappa^{(1)})\eta_{(\rho)}\Phi_0' r^{-1} - (2\mu^{(1)} + \kappa^{(1)})\chi_{(\rho)} + k_{(\rho)} A_1^{(1)} - (2\mu^{(1)} + \kappa^{(1)}) r^{-2} A_2^{(1)}\}, \\ \sigma_\alpha^{(3)} n_\alpha &= a_0^{(1)}\Phi_0' \quad \text{on } L, \end{aligned} \tag{6-9}$$

where

$$k_{(\rho)} = d_{(\rho)}[\lambda_1^{(\rho)}(2\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)}) - 2(\lambda_0^{(\rho)})^2]/\lambda_0^{(\rho)}. \tag{6-10}$$

If we impose the conditions (4-10) and (4-12) corresponding to the problem $P^{(3)}$, then we obtain a linear system of equations for the constants $A_1^{(\rho)}$, $C_1^{(\rho)}$, $(\rho = 1, 2)$, $A_2^{(1)}$ and $C_2^{(1)}$. Thus, the condition $\sigma_\alpha^{(3)} n_\alpha = 0$ on L reduces to

$$C_2^{(1)} = \nu_1 C_1^{(2)}, \tag{6-11}$$

where

$$\nu_1 = I_1(\tau_{(1)} r_1)/K_1(\tau_{(1)} r_1). \tag{6-12}$$

From (6-6) and (6-11) we get

$$\begin{aligned} \Phi_0 &= C_1^{(1)} Q(r), & \Phi_0' &= C_1^{(1)} \tau_{(1)} \Lambda(r) \quad \text{on } \Omega_1, \\ \Phi_0 &= C_1^{(2)} I_0(\tau_{(2)} r), & \Phi_0' &= C_1^{(2)} \tau_{(2)} I_1(\tau_{(2)} r) \quad \text{on } \Omega_2, \end{aligned} \tag{6-13}$$

where

$$Q(r) = I_0(\tau_{(1)} r) + \nu_1 K_0(\tau_{(1)} r), \quad \Lambda(r) = I_1(\tau_{(1)} r) - \nu_1 K_1(\tau_{(1)} r). \tag{6-14}$$

The condition imposed to the function $\sigma^{(3)}$ on Γ leads to

$$C_1^{(2)} = \nu_2 C_1^{(1)}, \tag{6-15}$$

where

$$\nu_2 = a_0^{(1)} \tau_{(1)} \Lambda(r_2) [a_0^{(2)} \tau_{(2)} I_1(\tau_{(2)} r_2)]^{-1}. \tag{6-16}$$

Let us introduce the notations

$$A_1^{(1)} = X_1, \quad A_2^{(1)} = X_2, \quad A_1^{(2)} = X_3, \quad C_1^{(1)} = X_4. \tag{6-17}$$

The remaining conditions from (4-10) and (4-12) reduce to the following system for the constants $A_\alpha^{(1)}$, $A_1^{(2)}$ and $C_1^{(1)}$,

$$\sum_{s=1}^{4} a_{rs} X_s = b_r, \quad r = 1, 2, 3, 4, \tag{6-18}$$

where

$$\begin{aligned}
&a_{11} = -2d_{(1)}, \quad a_{12} = 0, \quad a_{13} = 2d_{(2)}, \quad a_{14} = Q(r_2) - \nu_2 I_0(\tau_{(2)} r_2),\\
&a_{21} = \xi_{(1)}, \quad a_{22} = r_2^{-1}, \quad a_{23} = -\xi_{(2)},\\
&a_{24} = \eta_{(2)} \tau_{(2)} \nu_2 r_2^{-1} I_1(\tau_{(2)} r_2) - \eta_{(1)} \tau_{(1)} r_2^{-1} \Lambda(r_2),\\
&a_{31} = k_{(1)}, \quad a_{32} = -(2\mu^{(1)} + \kappa^{(1)}) r_2^{-2}, \quad a_{33} = -k_{(2)},\\
&a_{34} = (2\mu^{(1)} + \kappa^{(1)}) \eta_{(1)} \tau_{(1)} r_2^{-1} \Lambda(r_2) - (2\mu^{(2)} + \kappa^{(2)}) \eta_{(2)} \nu_2 r_2^{-1} \tau_{(2)} I_1(\tau_{(2)} r_2),\\
&a_{41} = k_{(1)}, \quad a_{42} = -(2\mu^{(1)} + \kappa^{(1)}) r_1^{-2}, \quad a_{43} = 0,\\
&a_{44} = (2\mu^{(1)} + \kappa^{(1)}) \eta_{(2)} \tau_{(1)} r_1^{-1} \Lambda(r_1),\\
&b_1 = 2(d_{(1)} - d_{(2)}), \quad b_2 = \chi_{(2)} - \chi_{(1)},\\
&b_3 = (2\mu^{(1)} + \kappa^{(1)}) \chi_{(1)} - (2\mu^{(2)} + \kappa^{(2)}) \chi_{(2)} + \lambda^{(2)} - \lambda^{(1)},\\
&b_4 = (2\mu^{(1)} + \kappa^{(1)}) \chi_{(1)} - \lambda^{(1)}.
\end{aligned} \tag{6-19}$$

We assume that the constitutive coefficients are independent. From (6-18) we can determine the constants $A_\alpha^{(1)}$, $A_1^{(2)}$ and $C_1^{(1)}$. The constants $C_2^{(1)}$ and $C_1^{(2)}$ can be calculated by (6-11) and (6-15). Thus, from (6-1), (6-6), (6-9) and (6-13) we find that the solution of the problem $P^{(3)}$ is given by

$$\begin{aligned}
&u_\alpha^{(3)} = x_\alpha \{-\eta_{(1)} C_1^{(1)} \tau_{(1)} r^{-1} \Lambda(r) + \chi_{(1)} + \xi_{(1)} A_1^{(1)} + A_2^{(1)} r^{-2}\} \quad \text{on } \Omega_1,\\
&u_\alpha^{(3)} = x_\alpha \{-\eta_{(2)} \tau_{(2)} C_1^{(2)} r^{-1} I_1(\tau_{(2)} r) + \chi_{(2)} + \xi_{(2)} A_1^{(2)}\} \quad \text{on } \Omega_2,\\
&\psi^{(3)} = \{C_1^{(1)} Q(r) - 2d_{(1)}(1 + A_1^{(1)})\} \quad \text{on } \Omega_1,\\
&\psi^{(3)} = \{C_1^{(2)} I_0(\tau_{(2)} r) - 2d_{(2)}(1 + A_1^{(2)})\} \quad \text{on } \Omega_2, \quad \varphi^{(3)} = 0.
\end{aligned} \tag{6-20}$$

From (4-7) and (6-2) we find

$$t_{33}^{(3)} = \lambda^{(\rho)} r^{-1} (r' U')' + \lambda_0^{(\rho)} [\Phi_0 - 2d_{(\rho)}(1 + A_1^{(\rho)})] \quad \text{on } \Omega_\rho. \tag{6-21}$$

In view of (6-6), (6-13), (6-21) and (4-15), we obtain

$$A_{\alpha 3} = 0, \quad A_{33} = H, \tag{6-22}$$

where

$$H = \pi(\lambda^{(2)} + 2\mu^{(2)} + \kappa^{(2)}) r_2^2 + \pi(\lambda^{(2)} + 2\mu^{(1)} + \kappa^{(1)})(r_1^2 - r_2^2) + 2\pi(h_{11}A_1^{(1)} + h_{12}A_1^{(2)} + h_{13}C_1^{(1)} + h_{14}C_1^{(2)} + h_0),$$

$$\begin{aligned} h_{11} &= (r_1^2 - r_2^2)(\lambda^{(1)} - d_{(1)}\lambda_0^{(1)}), \quad & h_{12} &= r_2^2(\lambda^{(2)} - d_{(2)}\lambda_0^{(2)}), \\ h_{13} &= G(r_1) - G(r_2), & h_{14} &= r_2(\lambda_0^{(2)}\tau_{(2)}^{-1} - \lambda^{(2)}e_{(2)})\, I_1(\tau_{(2)}r_2), \\ h_0 &= (r_1^2 - r_2^2)(\lambda^{(1)}\chi_{(1)} - d_{(1)}\lambda_0^{(1)}) - d_{(2)}\lambda_0^{(2)}r_2^2 + \lambda^{(2)}r_2^2\chi_{(2)}, \\ G(r) &= r\{\lambda_0^{(1)}\tau_{(1)}^{-1}[I_1(\tau_{(1)}r) - \nu_1 K_1(\tau_{(1)}r)] - \lambda^{(2)}e_{(1)}\Lambda(r)\}. \end{aligned} \tag{6-23}$$

With the help of (4-16) and (6-22) we obtain $A_{3\alpha} = 0$. In the case of extension we have $M_j = 0$, so that the system (4-14) has the solution

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = -R_3/H. \tag{6-24}$$

Thus, the solution of the extension problem can be expressed as

$$u_\alpha = c_3 u_\alpha^{(3)}, \quad u_3 = c_3 x_3, \quad \varphi_j = 0, \quad \psi = c_3 \psi^{(3)},$$

where $u_\alpha^{(3)}$ and $\psi^{(3)}$ are given by (6-20).

7. Conclusions

The paper is concerned with the deformation of a bar composed by two different microstretch elastic materials welded together along the surface of separation. The results established in this paper can be summarized as follows:

(a) We study the deformation of a heterogeneous bar which is subjected to extension, bending, torsion and flexure.

(b) We show that the solution of the problem of extension and bending can be reduced to the study of some two-dimensional problems.

(c) We establish the solution of the problem of torsion and flexure by a transversal force.

(d) We use the method to investigate the extension of a circular cylinder reinforced by a longitudinal rod.

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