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APPROXIMATE CONFORMAL MAPPINGS AND ELASTICITY PROBLEMS FOR NONCIRCULAR TUBES

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We present a method for analytic stress evaluation in elliptic and oval tubes based on approximate conformal mappings from annuli onto oval and elliptical doubly connected domains. The approximate conformal mapping is realized by the boundary reparametrization method. We also solve two elasticity problems for such tubes.

1. Introduction

Elastic deformations of circular tubes always attracted attention of scientists [Yella Reddy and Reid 1979; Moore 1990; Nayak and Mondal 2011]. Recently the researchers began to consider noncircular tubes with flattened, quasi-triangular, quasi-square, elliptical, rectangular and hexagon cross sections [Bazehhouri and Rezaeepazhand 2012; Baroutaji et al. 2014]. Such tubes can serve, for example, as energy absorbers in different mechanisms. Also the scientists analyzed the stresses in tubes under different load, compression and twist deformations applying both FEM and analytical methods [Zheng et al. 2015; Rizzetto et al. 2019]. Elliptic and oval tubes are manufactured and sold by different modern firms. The developing analytic technique and the computer progress allow appearance of new analytical evaluation methods of the tube characteristics.

We apply the boundary reparametrization method [Abzalilov and Shirokova 2017] for construction of the approximate conformal mapping from an annulus onto a doubly connected domain and then consider the analytical solution of some plane problems based on the analytic function theory [Muskhelishvili 1977]. Similar problem for a simply connected domain was solved in [Ivanshin and Shirokova 2016]. We apply the formulas of [Muskhelishvili 1977] for evaluation of the tubes torsion. We also consider the 3D element of the tubes with the oval and elliptic cuts and its deformation under bending.

The basic ideas of the boundary reparametrization method were presented in [Shirokova 2014] where the author constructed a method of the unit disk conformal mapping onto a simply connected domain. The boundary reparametrization method is based on application of an integral equation solution to finding the reparametrization function $t(\theta)$, $\theta \in [0, 2\pi]$. This reparametrization function transformes the representation $z(t)$ of the given domain boundary to the boundary value $z(t(\theta))$ of the function analytic in some circular domain (e.g., the unit disk, the annulus, the unit disk with concentric circular slits). We restore the analytic function in the circular domain via the Cauchy integral formula after we find the boundary values of the function.

We find the approximate solution of the integral equation reducing the integral equation to an infinite linear system and then to a truncated finite system. We reduce the Cauchy integral representation of

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the analytic function in an annulus to a Laurent series in this annulus. We find the essential coefficients of the Laurent series in order to construct the approximate analytic function in the form of a Laurent polynomial applicable to further investigations.

2. Laurent polynomial approximate conformal mapping from an annulus onto an oval cut and an elliptic cut

We approximately map the annuli onto the oval tube cross-section (Figure 1, left) and onto the elliptical tube cross-section (Figure 1, right) applying the boundary reparametrization method [Abzalilov and Shirokova 2017].

First we approximate the boundary curves – the interior curve $z = z_1(t)$, $t \in [0, 2\pi]$, and the exterior curve $z = z_2(t)$, $t \in [0, 2\pi]$ – of each of the cross-sections by the Fourier polynomials

$$z_j(t) = \sum_{k=-T_j}^{T_j} C_k^j e^{ikt}, \quad t \in [0, 2\pi], \quad j = 1, 2.$$

The boundary reparametrization method is to find the reparametrizing functions $t_j(\theta)$, $\theta \in [0, 2\pi]$, $j = 1, 2$, such that the expressions

$$\sum_{k=-T_j}^{T_j} C_k^j e^{ikt_j(\theta)} = \sum_{k=-M_j}^{M_j} D_k^j e^{ik\theta}, \quad j = 1, 2,$$

are the boundary values (at the interior circle $|\zeta| = r$ and at the exterior circle $|\zeta| = 1$) of a function analytic in the annulus $r < |\zeta| < 1$.

Consider a finite doubly-connected domain D_z bounded by the curves $z = z_j(t)$, $t \in [0, 2\pi]$, $j = 1, 2$. We consider at first the analytic in the domain D_z function $\zeta(z)$ which maps the domain D_z to an annulus $D_\zeta = \{\zeta : r < |\zeta| < 1\}$ and the analytic in D_z function $\log \frac{z}{\zeta(z)}$. Let $\theta_s(t)$, $s = 1, 2$, be the polar angle of the annulus boundary point corresponding to the boundary point $z_s(t)$ of the domain D_z. We introduce the function $q_s(t) = \arg z_s(t) - \theta_s(t)$, $s = 1, 2$. We apply the necessary and sufficient condition for $\log \frac{z}{\zeta(z)}$ to be analytic in D_z and obtain the integral Fredholm equation for the vector function $(q_1(t), q_2(t))$, $t \in [0, 2\pi]$:

$$q_s(t) = -\sum_{j=1}^{2} \frac{1}{\pi} \int_0^{2\pi} \log \frac{|z_j(\tau)|}{R_j} \left(\log |z_j(\tau) - z_s(t)|\right)'_\tau \, d\tau + \sum_{j=1}^{2} \frac{1}{\pi} \int_0^{2\pi} q_j(\tau) \left(\arg[z_j(\tau) - z_s(t)]\right)'_\tau \, d\tau, \quad s = 1, 2,$$

as in [Abzalilov and Shirokova 2017], where $R_1 = r$, $R_2 = 1$. Now the principal value singular integral can be represented as

$$\frac{1}{\pi} \int_0^{2\pi} \log \frac{|z_j(\tau)|}{R_j} \left(\log |z_j(\tau) - z_j(t)|\right)'_\tau \, d\tau \equiv \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|z_j(\tau)|}{R_j} \cot \frac{\tau - t}{2} d\tau + \frac{1}{\pi} \int_0^{2\pi} \log \frac{|z_j(\tau)|}{R_j} L_j(\tau, t) \, d\tau;$$

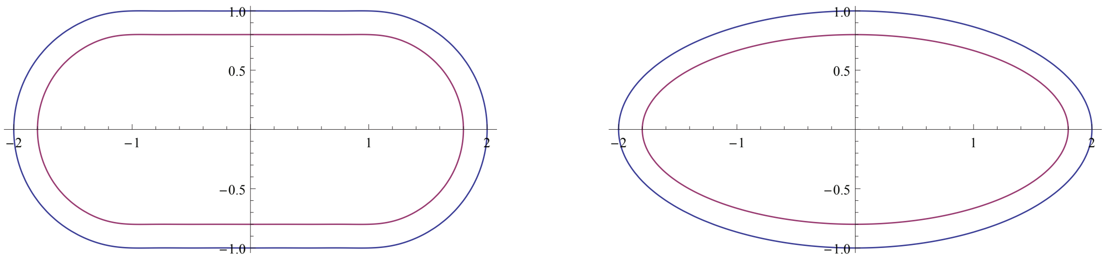


Figure 1. Shape of the cross-section of the oval (left) and elliptical (right) tubes.

here the kernel $L_j(\tau, t)$ is continuous. If the value of r were known, the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{|z_1(\tau)|}{r}\cot\frac{\tau-t}{2}d\tau$$

could be calculated via the Hilbert formula. Similar integral equation was applied to find the function $q(t)$ and the reparametrizing function $t(\theta)$ in the case of a simply connected domain [Shirokova 2014].

We do not know the value of r in the case of a doubly connected domain. So we have to differentiate both sides of the previous integral equation. After integrating the right-hand side of the resulting equation by parts, we obtain the following relations on the functions $q_s'(t)$:

$$q_s'(t) = -\sum_{j=0}^{1}\frac{1}{\pi}\int_0^{2\pi} q_j'(\tau)\left(\arg[z_j(\tau)-z_s(t)]\right)_t' d\tau + \sum_{j=0}^{1}\frac{1}{\pi}\int_0^{2\pi}[\log|z_j(\tau)|]'\left(\log|z_j(\tau)-z_s(t)|\right)_t' d\tau, \quad s=1,2.$$

We separate the singularities in the kernel $(\log|z_s(\tau)-z_s(t)|)_t'$ in the form of $\cot\frac{\tau-t}{2}$ and obtain the corresponding integrals with this principal value singular kernel exactly as it was described above.

The final integral equation can be represented as the Fredholm equation with an unknown vector function $M = (q_1'(s), q_2'(s))$ in the form of $M = AM + B$. If the unknown functions have the form

$$q_s'(t) = \sum_{j=1}^{\infty}\alpha_{j,s}\cos jt + \beta_{j,s}\sin jt, \ \ t\in[0,2\pi], \qquad s=1,2.$$

we reduce the integral equation to the solution of an infinite linear system with unknown Fourier coefficients of the functions $q_s'(t)$, $s = 1, 2$. Then we reduce the infinite linear system to a truncated linear system, a 2D generalization of the following result:

Lemma [Ivanshin and Shirokova 2016]. Let there exist the numbers $j, m > 1$ and a constant $U > 0$ so that $\left|\partial^{j+m}G(\tau,t)/\partial t^j\partial\tau^m\right| \le U$ and the function $Y(t)$ possess the bounded second derivative: $|Y''(t)| < T$. Then the approximate solution of the uniquely resolvable Fredholm integral equation of the second kind

$$X(t) = \int_0^{2\pi} G(\tau,t)X(\tau)\,d\tau + Y(t),$$

where $Y(t)$ is 2π periodic and $G(\tau, t)$ is 2π periodic with respect to both variables, can be reduced to the solution of a finite linear system with error estimated by $O(1/N^2)$. Here N is the rank of the finite linear system.

Now we obtain the monotone functions $\theta_s(t) = \arg z_s(t) - q_s(t)$, $s = 1, 2$. Note that one of the functions $q_s(t)$, $s = 1, 2$, can be restored via its derivative with an arbitrary constant summand, e.g. 0, but the other one must contain the special constant summand, because the function $\log(z(\zeta)/\zeta)/\zeta$ must be analytical in the annulus D_ζ. So the relation

$$\int_0^{2\pi} q_1(t)\theta_1'(t)\,dt = -\int_0^{2\pi} q_2(t)\theta_2'(t)\,dt$$

holds true due to Cauchy theorem. We put the expressions $\theta_s(t) = \arg z_s(t) - q_s(t)$ into the last relation and achieve the equality

$$\int_0^{2\pi} q_1(t)(\arg z_1(t))'dt + \int_0^{2\pi} q_2(t)(\arg z_2(t))'dt = 0,$$

which determines the value of the constant summand for the second function $q_s(t)$ restored via its derivative.

After the relations between t and θ are found at the both boundary components we can obtain the mapping function $z(\zeta)$. We restore this function via its boundary values $z(t(\theta))$ by the Cauchy integral formula. This Cauchy integral and its derivatives vanish at the point $\zeta = 0$. Therefore the inner radius r of the annulus D_ζ can be found via one of the formulas

$$\int_0^{2\pi} z_1(t)e^{ik\theta_0(t)}\theta_1'(t)\,dt + r^k\int_0^{2\pi} z_2(t)e^{ik\theta_1(t)}\theta_2'(t)\,dt = 0, \quad k = 1, 2, \ldots$$

or by the least-squares method.

The Laurent series coefficients of the analytic function $z(\zeta)$ mapping the annulus $r < |\zeta| < 1$ onto the domain D_z can be restored via the formulas

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} z_1(t)e^{-ik\theta_1(t)}\theta_0'(t)\,dt, \quad k = 0, 1, 2, \ldots,$$
$$c_{-k} = -\frac{r^k}{2\pi}\int_0^{2\pi} z_1(t)e^{ik\theta_1(t)}\theta_1'(t)\,dt, \quad k = 1, 2, \ldots.$$

We tested the reparametrization method in the approximate conformal mapping of the annulus given by $2-\sqrt{3} < |\zeta| < 1$ onto the doubly connected domain $D_z = \{z \mid |z| < 2, |z - 0.5| < 0.5\}$. The function

$$z = 2\frac{\zeta(2+\sqrt{3}) - 1}{\zeta - 2 - \sqrt{3}}.$$

gives the exact conformal mapping of the annulus onto the given domain. We took the mapping polynomial with 200 coefficients and compared the values of the exact mapping function and the approximate mapping function at the points of the circle $|\zeta| = 0.5$. The error was less than 0.0005.

k	a_k	b_k	k	a_k	b_k	k	a_k	b_k
−17	−0.00040	0.00001	−5	−0.00165	0.00354	7	−0.00911	0.01532
−15	0.00036	0.00002	−3	−0.04314	0.01779	9	0.00178	0.00723
−13	0.00088	0.00005	−1	0.39657	0.40846	11	0.00195	0.00368
−11	−0.00135	0.00013	1	1.52248	1.44051	13	−0.00121	0.00197
−9	−0.00112	0.00035	3	0.16082	0.12051	15	−0.00035	0.00109
−7	0.00543	0.00103	5	0.00369	0.03707	17	0.00071	0.00061

Table 1. Coefficients of Laurent polynomial for oval cut and for the elliptical cut.

We apply the described reparametrization method for the oval domain and for the elliptical domain. We find the following analytic functions. The function

$$z_o(\zeta) = \sum_{k=0}^{17} a_k \zeta^k + \sum_{k=-17}^{-1} a_k \left(\frac{\zeta}{r_o}\right)^k$$

maps approximately the annulus $r_o < |\zeta| < 1$, ($r_o = 0.87785$) onto the given oval cross-section presented in Figure 1, left. The function

$$z_e(\zeta) = \sum_{k=0}^{17} b_k \zeta^k + \sum_{k=-17}^{-1} b_k \left(\frac{\zeta}{r_e}\right)^k$$

maps approximately the annulus $r_e < |\zeta| < 1$, ($r_e = 0.87432$) onto the given elliptical cross-section presented on Figure 1, right. The coefficients a_k and b_k are presented in Table 1. The absolute values of the Laurent coefficients vanish while the absolute values of their numbers increase. Therefore we take the essential polynomial coefficients' indices only in the range $[-17, 17]$. The other coefficients do not bring significant difference to the results of calculations.

We apply these mappings to the solution of two elasticity theory problems.

3. Solution of the torsion problem for the oval tube and for the elliptical tube

We consider the boundary shear stresses on the exterior surfaces of the given tubes twisted in the plane cross-sections over the center point of the cross-section. We base the torsion problem solution on relation (13) of [Muskhelishvili 1977], Chapter 7: the value of the shear stress on the outer boundary of the orthogonal cross-section of a tube is proportional to the expression

$$S(\theta) = \frac{1}{|z'(e^{i\theta})|} \operatorname{Im}\bigl[e^{i\theta}\bigl(\varphi'(e^{i\theta}) - i\overline{z(e^{i\theta})}z'(e^{i\theta})\bigr)\bigr].$$

Here $z(\zeta)$ is the Laurent polynomial mapping the corresponding annulus onto the tube cut and $\varphi(\zeta)$ is the analytic in the annulus function with the boundary condition

$$\operatorname{Im}\varphi(re^{i\theta}) = |z(re^{i\theta})|^2/2,$$

where $r = 1$ for each tube at the exterior boundary and $r = r_o$ for the oval tube or $r = r_e$ for the elliptical tube at the interior boundary.

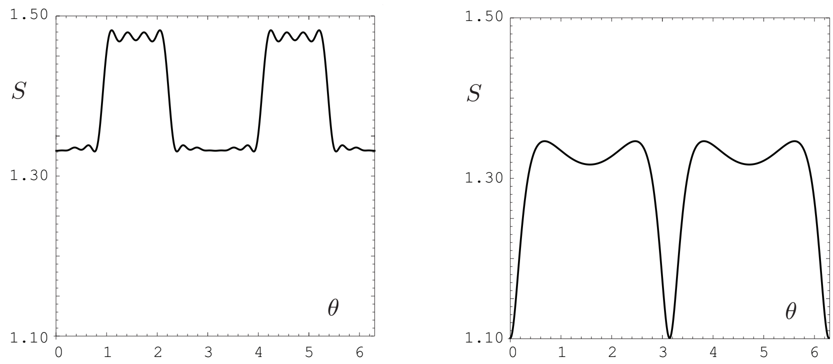


Figure 2. Stress $S(\theta)$ in the twisted oval (left) and elliptical (right) tubes.

The corresponding graphs of the function $S(\theta)$ for the oval and elliptical tubes are presented on Figure 2.

We see that the boundary shear stress of each of the given tubes twisted in the plane xOy over the cross-section center of symmetry changes from its minimal value to its maximal value in small neighbourhoods of four symmetrically located boundary points. The maximal shear stress values are larger for the oval tube. So the oval tube is more destructible then the elliptical one under twisting.

4. Spline-interpolation solution of the bending problem for the oval tube and for the elliptical tube

Consider the stresses at the exterior surfaces of the tubes in the space (x, y, h) with the cut cross-sections D parallel to the plane xOy. Let the exterior surfaces of both tubes be fixed at the level $h = 0$ and the shift in Ox direction on the exterior surfaces at the level $h = H$ equal a. Such a deformation happens when one bends the tube in Ox direction. We assume that the interior surfaces of the tubes are free from stresses. For small values of H and a we apply the linear spline-interpolation method [Ivanshin and Shirokova 2011; Shirokova 2004]. The linear spline-interpolation method of 3D elasticity problem solution for a tube is to find the stresses in this tube when the displacement coordinates at the points of a small segment $D \times [0, H]$ are assumed to be linear over the coordinate h. The problem is reduced to a set of mixed boundary value problems in an annulus.

According to the assumption the coordinates of the displacement vector take the form

$$\begin{aligned} u &= u_0(x, y) + u_1(x, y)h, \quad v = v_0(x, y) + v_1(x, y)h, \\ w &= w_0(x, y) + w_1(x, y)h, \quad (x, y) \in D, \quad h \in [0, H]. \end{aligned} \tag{1}$$

The interior surface null pressure assumption gives the relations

$$[\sigma_{k1} \cos(n, x) + \sigma_{k2} \cos(n, y) + \sigma_{k3} \cos(n, h)]_{x=x_1(s), y=y_1(s)} = 0, \quad k = 1, 2, 3,$$

for the points (x, y, h) on the interior surface of the tube segment, where σ_{kj}, $k, j = 1, 2, 3$, are tensor components, n is the unit normal to the interior surface at the corresponding point. Note that $\cos(n, h) = 0$ on the interior surface of the tube.

Due to coordinate linearity (1) on h the latter relations take the following form for the points $(x, y) = (x_1(s), y_1(s))$ of the interior boundary of D:

$$\left\{\left(\lambda\left[\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}+w_1\right]+2\mu\frac{\partial u_0}{\partial x}\right)dy_1(s)-\mu\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)dx_1(s)\right\}_{x=x_1(s),y=y_1(s)}=0, \tag{2}$$

$$\left\{\left(\lambda\left[\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}+w_1\right]+2\mu\frac{\partial v_0}{\partial y}\right)dx_1(s)-\mu\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)dy_1(s)\right\}_{x=x_1(s),y=y_1(s)}=0, \tag{3}$$

$$\left\{\left(u_1+\frac{\partial w_0}{\partial x}\right)dy_1(s)-\left(v_1+\frac{\partial w_0}{\partial y}\right)dx_1(s)\right\}_{x=x_1(s),y=y_1(s)}=0, \tag{4}$$

$$\left\{\left(\lambda\left[\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right]+2\mu\frac{\partial u_1}{\partial x}\right)dy_1(s)-\mu\left(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}\right)dx_1(s)\right\}_{x=x_1(s),y=y_1(s)}=0, \tag{5}$$

$$\left\{\left(\lambda\left[\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right]+2\mu\frac{\partial v_1}{\partial y}\right)dx_1(s)-\mu\left(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}\right)dy_1(s)\right\}_{x=x_1(s),y=y_1(s)}=0, \tag{6}$$

$$\left\{\frac{\partial w_1}{\partial x}dy_1(s)-\frac{\partial w_1}{\partial y}dx_1(s)\right\}_{x=x_1(s),y=y_1(s)}=0, \tag{7}$$

where λ and μ are Lame coefficients.

The boundary conditions on the exterior surface of the tube segment yield the following relations at the points $(x, y) = (x_2(s), y_2(s))$ of the exterior boundary of D:

$$u_0(x_2(s), y_2(s)) = 0, \quad v_0(x_2(s), y_2(s)) = 0, \quad w_0(x_2(s), y_2(s)) = 0,$$
$$u_1(x_2(s), y_2(s))H = a, \quad v_1(x_2(s), y_2(s)) = 0, \quad w_1(x_2(s), y_2(s)) = 0.$$

The equilibrium equations

$$\frac{\partial\sigma_{k1}}{\partial x}+\frac{\partial\sigma_{k2}}{\partial y}+\frac{\partial\sigma_{k3}}{\partial h}=0, \quad k=1,2,3,$$

must be met everywhere in the tube segment. Due to the displacement coordinates linearity in h the equilibrium equations take the form

$$\lambda\left\{\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial w_1}{\partial x}+\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial x\partial y}\right)h\right\}+\mu\left\{\sum_{k=0}^{1}\left(2\frac{\partial^2 u_k}{\partial x^2}+\frac{\partial^2 u_k}{\partial y^2}+\frac{\partial^2 v_k}{\partial x\partial y}\right)h^k+\frac{\partial w_1}{\partial x}\right\}=0, \tag{8}$$

$$\lambda\left\{\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_1}{\partial y}+(\frac{\partial^2 u_1}{\partial x\partial y}+\frac{\partial^2 v_1}{\partial y^2})h\right\}+\mu\left\{\sum_{k=0}^{1}\left(2\frac{\partial^2 v_k}{\partial y^2}+\frac{\partial^2 u_k}{\partial x\partial y}+\frac{\partial^2 v_k}{\partial x^2}\right)h^k+\frac{\partial w_1}{\partial y}\right\}=0, \tag{9}$$

$$\lambda\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)+\mu\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}+\left[\frac{\partial^2 w_1}{\partial x^2}+\frac{\partial^2 w_1}{\partial y^2}\right]h\right)=0. \tag{10}$$

The coefficients with h in relations (8) and (9) form the system

$$\begin{cases} \dfrac{\partial}{\partial x}\left[(\lambda+2\mu)\left(\dfrac{\partial u_1}{\partial x}+\dfrac{\partial v_1}{\partial y}\right)\right]-\dfrac{\partial}{\partial y}\left[\mu\left(\dfrac{\partial v_1}{\partial x}-\dfrac{\partial u_1}{\partial y}\right)\right]=0, \\ \dfrac{\partial}{\partial y}\left[(\lambda+2\mu)\left(\dfrac{\partial u_1}{\partial x}+\dfrac{\partial v_1}{\partial y}\right)\right]+\dfrac{\partial}{\partial x}\left[\mu\left(\dfrac{\partial v_1}{\partial x}-\dfrac{\partial u_1}{\partial y}\right)\right]=0, \end{cases}$$

which is equivalent to the equation

$$\frac{\partial}{\partial \overline{z}}\left[(\lambda+2\mu)\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)+i\mu\left(\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}\right)\right]=0,$$

where $z=x+iy$, $\overline{z}=x-iy$. So

$$(\lambda+2\mu)\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)+i\mu\left(\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}\right)=F_1(z),$$

where $F_1(z)$ is a function analytical in D. Now we express the derivative $\frac{\partial}{\partial z}(u_1+iv_1)$ in terms of F_1 and obtain

$$u_1(x,y)+iv_1(x,y)=\frac{\lambda+3\mu}{4\mu(\lambda+2\mu)}\int F_1(z)dz-\frac{\lambda+\mu}{4\mu(\lambda+2\mu)}z\overline{F_1(z)}+\overline{G_1(z)},$$

where $G_1(z)$ is a function analytical in D. Finally we have this representation of the plane displacement vector u_1+iv_1, analogous to the plane displacement vector representation of [Muskhelishvili 1977]:

$$-2\mu(u_1(x,y)+iv_1(x,y))=-\kappa f_1(z)+z\overline{f_1'(z)}+\overline{g_1(z)},$$

where

$$\frac{\lambda+\mu}{2(\lambda+2\mu)}\int F_1(z)dz\equiv f_1(z),\quad -2\mu G_1(z)\equiv g_1(z),\quad \frac{\lambda+3\mu}{\lambda+\mu}\equiv\kappa.$$

The coefficient with h in (10) yields $\dfrac{\partial^2 w_1}{\partial x^2}+\dfrac{\partial^2 w_1}{\partial y^2}=0$. So $w_1=\operatorname{Re} q_1(z)$, where $q_1(z)$ is a function analytical in D.

We have to restore the analytical in D functions $f_1(z)$, $g_1(z)$ and $q_1(z)$ using the boundary conditions (5)–(7) on the interior boundary of D and using the given displacements $u_1(x_2(s),y_2(s))H=a$, $v_1(x_2(s),y_2(s))=0$, $w_1(x_2(s),y_2(s))=0$ at the exterior boundary of D.

So we have the following boundary conditions for the functions $f_1(z)$ and $g_1(z)$, which are analytical in D:

$$\left[f_1(z)+z\overline{f_1'(z)}+\overline{g_1(z)}\right]_{z=x_1(s)+iy_1(s)}=0,\quad \left[-\kappa f_1(z)+z\overline{f_1'(z)}+\overline{g_1(z)}\right]_{z=x_2(s)+iy_2(s)}=-2\mu\frac{a}{H},\quad (11)$$

The interior boundary condition in relation (11) is the boundary condition of the first boundary value problem of the plane elasticity theory, the exterior boundary condition is the boundary condition of the second boundary value problem of the plane elasticity theory [Muskhelishvili 1977].

We have the following boundary conditions for the function $q_1(z)$, analytical in D:

$$\frac{\partial}{\partial n}\left[\operatorname{Re} q_1(z)\right]_{z=x_1(s)+iy_1(s)}=0,\quad \left[\operatorname{Re} q_1(z))\right]_{z=x_2(s)+iy_2(s)}=0. \qquad (12)$$

The interior boundary condition in (12) is the Neumann condition for a harmonic function which can be rewritten as $\frac{\partial}{\partial s}[\operatorname{Im} q_1(z))]_{z=x_1(s)+iy_1(s)} = 0$; the exterior boundary condition is the Dirichlet boundary condition for a harmonic function.

Clearly this boundary value problem has the solution $f_1(z) \equiv \dfrac{2\mu a}{(1+\kappa)H}$, $g_1(z) \equiv -\dfrac{2\mu a}{(1+\kappa)H}$, $q_1(z) \equiv 0$, $z \in D$.

After restoring the components $u_1(x, y) \equiv a/H$, $v_1(x, y) \equiv 0$ and $w_1(x, y) \equiv 0$ we reconstruct the components $u_0(x, y)$, $v_0(x, y)$ and $w_0(x, y)$. To do this we apply the interior boundary conditions (2)–(4) and the exterior boundary conditions $u_0(x_2(s), y_2(s)) = 0$, $v_0(x_2(s), y_2(s)) = 0$, $w_0(x_2(s), y_2(s)) = 0$ and introduce functions $f_0(z)$, $g_0(z)$ and $q_0(z)$, analytical in D. We have for these functions the boundary relations

$$\begin{aligned} &\left[f_0(z) + z\overline{f_0'(z)} + \overline{g_0(z)}\right]_{z=x_1(s)+iy_1(s)} = 0, \qquad &&\frac{\partial}{\partial n}[\operatorname{Re} q_0(z)]_{z=x_1(s)+iy_1(s)} = -\frac{a}{H}, \\ &\left[-\kappa f_0(z) + z\overline{f_0'(z)} + \overline{g_0(z)}\right]_{z=x_2(s)+iy_2(s)} = 0, \qquad &&[\operatorname{Re} q_0(z)]_{z=x_2(s)+iy_2(s)} = 0. \end{aligned} \tag{13}$$

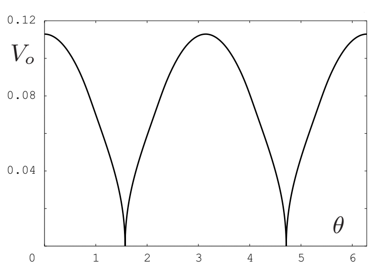
The boundary conditions in the left column of (13) yield $f_0(z) \equiv 0$, $g_0(z) \equiv 0$, so $u_0(x, y) \equiv 0$, $v_0(x, y) \equiv 0$, but the problem of the function $q_0(z)$ restoration via boundary conditions (14) is not so easy. Application of the additional mapping from an annulus to the domain D allows us to solve this boundary value problem. Let $z(\zeta)$ be the analytic function mapping the annulus $r < |\zeta| < 1$ onto D. Consider $\chi_0(\zeta) = q_0(z(\zeta))$. Now in order to restore the function $\chi_0(\zeta)$ in the annulus $r < |\zeta| < 1$ we have the boundary conditions

$$[\operatorname{Re}(\zeta\chi_0'(\zeta))]_{z=re^{i\theta}} = -\frac{a}{H}|z'(re^{i\theta})|, \quad \operatorname{Re}(\chi_0(e^{i\theta})) = 0.$$

This boundary value problem in the annulus is resolvable approximately through the relative series expansion and coefficient comparison.

We consider $z(\zeta) = z_o(\zeta)$ for the oval tube, $z(\zeta) = z_e(\zeta)$ for the elliptical tube, $r = r_o$ for the oval tube, $r = r_e$ for the elliptical tube and examine the resulting exterior boundary stresses for the oval and elliptical tubes.

We find the absolute value of the stress vector $\sqrt{\sigma_{n1}^2 + \sigma_{n2}^2}$ at the level $h = 0$ on the exterior surfaces for both tubes and for $\kappa = 2$. The formula expressing the stresses value dependence on the polar angle is $100a\,V_o(\theta)/H$ for the oval tube and $100a\,V_e(\theta)/H$ for the elliptical tube. The graphs of $V_o(\theta)$ and $V_e(\theta)$ are shown in Figure 3. The maximal absolute value of stress vector for the bent tube reaches maximum both at the bending points and at the points opposite to them, this maximal value for the elliptical tube is larger than that for the oval one but the difference is not essential.



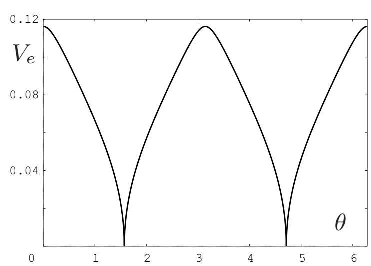


Figure 3. Stresses in the bent oval (left) and elliptical (right) tubes.

5. Conclusion

The conformal mapping method presented here is computationally efficient for twisted and bent tubes with noncircular cross-sections. It provides us with a Fourier polynomial mapping function. This approximate conformal mapping method makes it possible to apply the conformal mapping approach to many problems of elasticity.

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