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Intersection numbers on $\overline{\mathscr{M}}_{g,n}$

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ABSTRACT. We introduce the package *HodgeIntegrals*, which calculates top intersection numbers among tautological classes on $\overline{\mathscr{M}}_{g,n}$. As an application, we show that the tautological ring of the moduli space $\mathscr{M}_{3,0}^{\lambda_2}$ of genus three curves whose dual graph has at most one loop is not Gorenstein.

Let $\overline{\mathscr{M}}_{g,n}$ denote the moduli space of stable curves of genus g with n marked points. The tautological rings $R^*(\overline{\mathscr{M}}_{g,n})$ are defined to be the smallest system of $\mathbb{Q}$-subalgebras of the Chow rings $A^*(\overline{\mathscr{M}}_{g,n})$ that is closed under the natural forgetful morphisms $\pi_{n+1}\colon \overline{\mathscr{M}}_{g,n+1} \to \overline{\mathscr{M}}_{g,n}$ and the gluing morphisms $\iota_{\text{irr}}\colon \overline{\mathscr{M}}_{g-1,n+2} \to \overline{\mathscr{M}}_{g,n}$, $\iota_{g_1,S}\colon \overline{\mathscr{M}}_{g_1,|S|+1} \times \overline{\mathscr{M}}_{g_2,|S^c|+1} \to \overline{\mathscr{M}}_{g_1+g_2,n}$; here S denotes a subset of $\{1,\ldots,n\}$ and S^c its complement. Tautological rings contain fundamental classes of boundary strata, Mumford-Morita κ classes, cotangent ψ classes, and the Chern classes of the Hodge bundle $\lambda_i := c_i(\mathbb{E})$. For definitions and properties of these tautological classes, see [M, AC].

Around 1997, Faber [F2] implemented the program *KaLa5* in Maple, which calculates top intersection numbers among κ, λ and ψ classes. The *Macaulay2* package *HodgeIntegrals* is modeled after Faber's program, though the algorithm presented here is different. The main advantage of *HodgeIntegrals* over *KaLa5* is that it is entirely recursive. By contrast, *KaLa5* uses look-up tables, which limits the calculations to $\dim \overline{\mathscr{M}}_{g,n} \leq 20$. What limits *HodgeIntegrals* is, as with all recursions, the need for memory. In practice, integrals involving κ and ψ classes are computed quickly up to $\dim \overline{\mathscr{M}}_{g,n} \leq 40$. Integrals involving λ classes are considerably slower. Here are some examples:

```
i1 : loadPackage "HodgeIntegrals";

i2 : R = hodgeRing(15,0);

i3 : time integral(15,0,kappa_42)
     -- used 37.9246 seconds

                        1
o3 = ---------------------------------
     660188928419744764258399813632000

o3 : R

i4 : time integral(4,0,lambda_1^9)
     -- used 15.4209 seconds

         1
o4 = ------
     113400

o4 : R

i5 : time integral(5,0,lambda_1^12)
     -- used 79.4923 seconds
```

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HodgeIntegrals version 1.2.

```
        31
o5 = ------
     680400

o5 : R
```

The Gorenstein conjectures describe the structure of the tautological rings of $\overline{\mathscr{M}}_{g,n}$ and the related moduli spaces of curves with compact Jacobians, $\mathscr{M}^{\mathrm{ct}}_{g,n}$, and curves with rational tails, $\mathscr{M}^{\mathrm{rt}}_{g,n}$, where the tautological rings of $\mathscr{M}^{\mathrm{ct}}_{g,n}$ and $\mathscr{M}^{\mathrm{rt}}_{g,n}$ are defined by restriction. For definitions and precise statements, see [F1, P]. According to these conjectures, multiplication followed by integration over a homology class of $3g-3+n$, respectively $2g-3+n$ and $g-2+n$, gives a perfect pairing on the ring $R^*(\overline{\mathscr{M}}_{,g,n})$, respectively $R^*(\mathscr{M}^{\mathrm{ct}}_{g,n})$ and $R^*(\mathscr{M}^{\mathrm{rt}}_{g,n})$.

Let $\mathscr{M}^{\lambda_2}_{3,0}$ denote the moduli space of genus three curves whose dual graph has at most one loop; equivalently, this is the locus of curves in $\overline{\mathscr{M}}_{3,0}$ where the sum of the geometric genera of the components is at least 2. We use the package *HodgeIntegrals* to show that the tautological ring of $\mathscr{M}^{\lambda_2}_{3,0}$, which is defined by restriction, does not have perfect pairing.

1. Integrals among ψ, κ, and λ classes. Top intersection numbers among ψ classes are determined with the Theorem 1.1 of [LX]:

$$
\begin{aligned}
(2g+n-1)(2g+n-2)\int_{\overline{\mathscr{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n} = \\
&\frac{2d_1+3}{12}\int_{\overline{\mathscr{M}}_{g-1,n+4}} \psi_1^{d_1+1}\psi_2^{d_2}\cdots\psi_n^{d_n} - \frac{2g+n-1}{6}\int_{\overline{\mathscr{M}}_{g-1,n+3}} \psi_1^{d_1}\cdots\psi_n^{d_n} \\
&+\sum_{I\sqcup J=\{2,\ldots,n\}} (2d_1+3)\int_{\overline{\mathscr{M}}_{g',|I|+3}} \psi_1^{d_1+1}\prod_{i\in I}\psi_i^{d_i}\int_{\overline{\mathscr{M}}_{g-g',|J|+2}}\prod_{j\in J}\psi_i^{d_i} \\
&-\sum_{I\sqcup J=\{2,\ldots,n\}} (2g-n-1)\int_{\overline{\mathscr{M}}_{g',|I|+2}} \psi_1^{d_1}\prod_{i\in I}\psi_i^{d_i}\int_{\overline{\mathscr{M}}_{g-g',|J|+2}}\prod_{j\in J}\psi_j^{d_j}.
\end{aligned}
$$

This reduces an integral in ψ classes to a sum of four terms involving integrals on strictly lower genera. Our base cases come from the well-known formula

$$
\int_{\overline{\mathscr{M}}_{0,n}} \psi_1^{d_1}\cdots\psi_n^{d_n} = \binom{n-3}{d_1,\ldots,d_n}
$$

which follows directly from the equation $1=\int_{\overline{\mathscr{M}}_{0,4}} 1$, along with the string and dilaton equations.

Integrals involving both ψ and κ classes can be reduced to the case above using the pullback formulas $\pi^*_{n+1}\kappa_b=\kappa_b-\psi^b_{n+1}$ and $\pi^*_{n+1}\psi_i=\psi_i-D_{i,n+1}$. The term $D_{i,n+1}$ is the class of the boundary divisor that is the closure of the locus of curves consisting of a rational component with two marked points p_i and p_{n+1} attached to a genus g curve carrying the remaining marked points. It is immediate from this definition that the product $\psi_{n+1}D_{i,n+1}$ vanishes for all i. These allow us to compute

$$\int_{\overline{\mathscr{M}}_{g,n}} \psi_1^{\alpha_1}\cdots\psi_n^{\alpha_n}\kappa_{b_1}\cdots\kappa_{b_m} = \int_{\overline{\mathscr{M}}_{g,n}} \kappa_{b_1}\prod_{i=1}^n \psi_i^{\alpha_i}\prod_{j=2}^m \kappa_{b_j} = \int_{\overline{\mathscr{M}}_{g,n}} (\pi_{n+1*}\psi_{n+1}^{a+1})\prod_{i=1}^n \psi_i^{\alpha_i}\prod_{j=2}^m \kappa_{b_j}$$
$$= \int_{\overline{\mathscr{M}}_{g,n+1}} (\psi_{n+1}^{a+1})\pi_{n+1}^*\left(\prod_{i=1}^n \psi_i^{\alpha_i}\prod_{j=2}^m \kappa_{b_j}\right)$$
$$= \int_{\overline{\mathscr{M}}_{g,n+1}} \psi_{n+1}^{a+1}\prod_{i=1}^n(\psi_i^{\alpha_i} - D_{i,n+1})\prod_{j=2}^m(\kappa_{b_j} - \psi_{n+1}^{b_j+1})$$
$$= \int_{\overline{\mathscr{M}}_{g,n+1}} \psi_{n+1}^{a+1}\prod_{i=1}^n \psi_i^{\alpha_i}\prod_{j=2}^m(\kappa_{b_j} - \psi_{n+1}^{b_j+1}).$$

The last expression can be expanded to a sum of integrals which contain one fewer κ class in their integrands at the cost of introducing a marked point. Repeated iteration of this equation allows us to eliminate κ classes entirely.

Integrals involving λ classes are more complicated. The first step is to express λ classes in terms of the Chern character of $\mathbb{E}$ using the formula $1+\lambda_1 t+\cdots+\lambda_g t^g = \exp\left(\sum_{i=1}^g (2i-2)!\,\mathrm{ch}_{2i-1}t^{2i-1}\right)$. Applying the Grothendieck-Riemann-Roch formula to the universal family $\pi\colon \overline{\mathscr{M}}_{g,1}\to\overline{\mathscr{M}}_{g,0}$ and pulling this back to $\overline{\mathscr{M}}_{g,n}$ gives us an expression [M, eq. 5.2] for the Chern character of $\mathbb{E}$ in terms of κ, ψ, and boundary classes:

$$\mathrm{ch}_a = \frac{B_{a+1}}{(a+1)!}\left(\kappa_a - \sum_{i=1}^n \psi_i^a + \frac{1}{2}\sum_{i=0}^{a-1}(-1)^i\sum_{\iota_\beta}(\iota_\beta)_*\,\psi_\star^i\psi_\bullet^{a-1-i}\right).$$

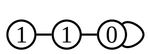
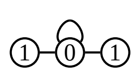
Here B_i denotes the i-th Bernoulli number, and ι_β ranges over all possible gluing morphisms.

$R^*(\mathscr{M}_{3,0}^{\lambda_2})$ IS NOT GORENSTEIN. The tautological ring $R^*(\mathscr{M}_{3,0}^{\lambda_2})$ is one-dimensional in degree 4 and vanishes in higher degree [CY, Proposition 1], thus we have an intersection pairing

$$R^i(\mathscr{M}_{3,0}^{\lambda_2})\times R^{4-i}(\mathscr{M}_{3,0}^{\lambda_2})\to R^4(\mathscr{M}_{3,0}^{\lambda_2})\cong\mathbb{Q}. \tag{1}$$

The Chern class λ_2 does not vanish on the generator of $R^4(\mathscr{M}_{3,0}^{\lambda_2})$ and serves as an evaluation class for the pairing. We use *HodgeIntegrals* to show that this pairing is degenerate.

The dual graph Γ_C of a curve C is a graph which encodes the topological type of C. Vertices of Γ are labelled with a genus g_i and correspond to irreducible components of genus g_i, and edges between labelled vertices correspond to nodes between the corresponding components. Let Γ_1 and Γ_2 denote the two graphs:

Γ_1:  Γ_2: 

Define X_1 and X_2 to be the associated boundary strata, that is, the closure of the locus of curves whose dual graphs are Γ_1 and Γ_2. It is straightforward to check that $(X_1 - X_2)\lambda_2 = 0$. We now show that X_1 and X_2 are not linearly equivalent in $\mathscr{M}_{3,0}^{\lambda_2}$. Since $R^1(\overline{\mathscr{M}}_{3,0}\setminus\mathscr{M}_{3,0}^{\lambda_2}) = A^1(\overline{\mathscr{M}}_{3,0}\setminus\mathscr{M}_{3,0}^{\lambda_2})$, the tautological restriction sequence

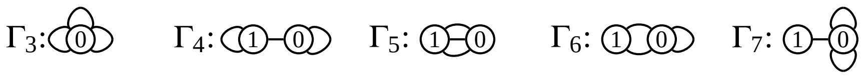
$$R^1(\overline{\mathscr{M}}_{3,0}\setminus\mathscr{M}_{3,0}^{\lambda_2})\longrightarrow R^3(\overline{\mathscr{M}}_{3,0})\longrightarrow R^3(\mathscr{M}_{3,0}^{\lambda_2})\longrightarrow 0, \tag{2}$$

is exact, where the first map is inclusion and the second is restriction. The shift in degrees is due to the fact that any curve in $\overline{\mathscr{M}}_{,3,0} \setminus \mathscr{M}_{3,0}^{\lambda_2}$ necessarily has at least two nodes. Note that in most cases, the sequence (2) is not known exact in the middle.

The exactness of (2) implies that, to show that X_1 and X_2 are not linearly equivalent, we need to check if their extensions to $\overline{\mathscr{M}}_{3,0}$ satisfy

$$X_1 - X_2 \in R^1(\overline{\mathscr{M}}_{3,0} \setminus \mathscr{M}_{3,0}^{\lambda_2}). \tag{3}$$

The generators of $R^3(\overline{\mathscr{M}}_{3,0} \setminus \mathscr{M}_{3,0}^{\lambda_2})$ are the boundary strata X_i associated to the the graphs:



The intersection pairing of $X_1, \ldots, X_7$ against the five tautological classes κ_3, $\kappa_1\kappa_2$, κ_1^3, $\kappa_2\lambda_2$, and $\kappa_1^2\lambda_1$ is computed.

```
i6 : R = hodgeRing(3,0);

i7 : List * List := (A, B) -> apply(A, B, (x, y) -> x * y);

i8 : tempFactors = (FactorList, n) -> (
          if #FactorList === 0 then return {splice{n : 1}} else (
               tempList := tempFactors(drop(FactorList, 1), n);
               a := first FactorList;
               newList := new List;
               for i from 1 to n do (
                    aList = splice{i - 1 : 1, a, n - i : 1};
                    newList = append(newList, apply(tempList, x -> aList * x)));
               return flatten newList));
```

The function `tempFactors` returns a list of how the factors of a monomial $\lambda_1\kappa_{a_1}\cdots\kappa_{a_k}$ can be distributed among components of a boundary stratum.

```
i9 : gnList = {{(1,1), (1,2), (0,3)}, {(1,1), (1,1), (0,4)}, {(0,6)}, {(1,3), (0,3)},
               {(1,3), (0,3)}, {(1,2), (0,4)}, {(1,1), (0,5)}};

i10 : klpList = {{kappa_3}, {kappa_1, kappa_2}, {kappa_1, kappa_1, kappa_1},
            {kappa_2, lambda_1}, {kappa_1, kappa_1, lambda_1}};

i11 : M = matrix table(klpList, gnList, (x,y) -> (sum(tempFactors(x,#y),
                z-> product(#y, i -> integral(y#i#0, y#i#1, z#i)))));

               5       7
o11 : Matrix R  <--- R

i12 : kernel M

o12 = image | 0  -2304 |
            | 0  -1152 |
            | 0  -1    |
            | -1 24    |
            | 1  0     |
            | 0  96    |
            | 0  72    |

                              7
o12 : R-module, submodule of R
```

The kernel of M is incompatible with (3), since the first two coordinates of the two vectors above correspond to X_1 and X_2 and there is no vector in the kernel of the form whose first and second coordinates are respectively 1 and -1. Thus $X_1 - X_2$ is nonzero in $R^3(\mathscr{M}_{3,0}^{\lambda_2})$, and the pairing (1) is not perfect.

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