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Implementing the Kustin-Miller complex construction

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ABSTRACT. The Kustin-Miller complex construction, due to A. Kustin and M. Miller, can be applied to a pair of resolutions of Gorenstein rings with certain properties to obtain a new Gorenstein ring and a resolution of it. It gives a tool to construct and analyze Gorenstein rings of high codimension. We describe the Kustin-Miller complex, its implementation in the Macaulay2 package *KustinMiller*, and explain how it can be applied to explicit examples.

1. INTRODUCTION. Many important rings in commutative algebra and algebraic geometry turn out to be Gorenstein rings, i.e. commutative rings such that the localization at each prime ideal is a Noetherian local ring R with finite injective dimension as an R-module. Examples are canonical rings of regular algebraic surfaces of general type, anticanonical rings of Fano varieties, and Stanley-Reisner rings of triangulations of spheres. Gorenstein rings with an embedding codimension at most 2 are known to be complete intersections, and those with embedding codimension 3 are described by the theorem of Buchsbaum-Eisenbud [BE, Theorem 2.1] as Pfaffians of a skew-symmetric matrix. Structure theorems in higher codimension are lacking. One goal of unprojection theory, which was introduced by A. Kustin, M. Miller and M. Reid and developed further by the second author (see [KM, R, PR, P]), is to act as a substitute for a structure theorem in codimension at least 4 by providing a construction that (under certain conditions) relates a more complicated Gorenstein ring to two simpler ones. The geometric motivation is to provide inverses of certain projections in birational geometry. The process can be considered as a version of Castelnuovo blow-down.

Examples of applications range from the construction of Campedelli surfaces [NP1] to results on the structure of Stanley-Reisner rings [BP2]. For an outline of more applications see [R], the introduction of [BP1], and §3 below.

We describe the Kustin-Miller complex construction [KM], which is the key tool to obtain resolutions of unprojection rings, and discuss our implementation in the *Macaulay2* [M2] package *KustinMiller*. We illustrate the construction with examples and applications.

2. IMPLEMENTATION OF THE KUSTIN-MILLER COMPLEX CONSTRUCTION. We will consider the following setup. Let R be a positively graded polynomial ring over a field $\Bbbk$ and let $I,J \subset R$ be homogeneous ideals of R such that R/I and R/J are Gorenstein, $I \subset J$, and $\dim R/J = \dim R/I - 1$. By [BH, Proposition 3.6.11], there exist $k_1, k_2 \in \mathbb{Z}$ such that $\omega_{R/I} = R/I(k_1)$ and $\omega_{R/J} = R/J(k_2)$. Assume that $k_1 > k_2$ so that the unprojection ring defined below is also positively graded.

Definition ([PR, Definition 1.2]). Let $\varphi \in \operatorname{Hom}_{R/I}(J/I, R/I)$ be a homomorphism of degree $k_1 - k_2$ such that $\operatorname{Hom}_{R/I}(J/I, R/I)$ is generated as an R/I-module by φ and the inclusion morphism i. Let

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KustinMiller version 1.4.

T be a variable of degree k_1-k_2. We call the graded algebra $(R/I)[T]/(Tu-\varphi(u) \mid u\in J/I)$ the ***Kustin-Miller unprojection ring*** of the pair $I\subset J$ defined by φ.

The Kustin-Miller unprojection ring is naturally isomorphic to $R[T]/U$, where $U\subset R[T]$ is the inverse image of the ideal $(Tu-\varphi(u)\mid u\in J/I)$ of $(R/I)[T]$ under the natural map $R[T]\to(R/I)[T]$. In the following, we will consider $R[T]/U$ as the Kustin-Miller unprojection ring.

Proposition ([KM], [PR, Theorem 1.5])**.** *The R-algebra $R[T]/U$ is Gorenstein and independent of the choice of φ (up to isomorphism).*

Following [KM], we now describe the construction of a graded free resolution of $R[T]/U$ from those of R/I and R/J. We will refer to this as the ***Kustin-Miller complex construction***. Denote by $g=\dim R-\dim R/J$ the codimension of the ideal J of R, and suppose $g\geq 4$ (the special cases $g=2,3$ may be treated in a similar way). Let $C_J\colon \frac{R}{J}\longleftarrow A_0\xleftarrow{a_1}A_1\xleftarrow{a_2}\cdots\xleftarrow{a_{g-1}}A_{g-1}\xleftarrow{a_g}A_g\longleftarrow 0$ and $C_I\colon\frac{R}{I}\longleftarrow B_0\xleftarrow{b_1}B_1\xleftarrow{b_2}\cdots\xleftarrow{b_{g-1}}B_{g-1}\longleftarrow 0$ be minimal graded free resolutions (self-dual by the Gorenstein property [E, Corollary 21.16]) of R/J and R/I as R-modules with $A_0=B_0=R$, $A_g=R(-k_2-\eta)$ and $B_{g-1}=R(-k_1-\eta)$, where η is the sum of the degrees of the variables of R.

Consider the complex

$$C_U\colon\frac{R[T]}{U}\longleftarrow F_0\xleftarrow{f_1}F_1\xleftarrow{f_2}\cdots\xleftarrow{f_{g-1}}F_{g-1}\xleftarrow{f_g}\leftarrow F_g\longleftarrow 0$$

with $F_0=B_0'$, $F_1=B_1'\oplus A_1'(k_2-k_1)$, $F_i=B_i'\oplus A_i'(k_2-k_1)\oplus B_{i-1}'(k_2-k_1)$ for all $2\leq i\leq g-2$, $F_{g-1}=A_{g-1}'(k_2-k_1)\oplus B_{g-2}'(k_2-k_1)$, and $F_g=B_{g-1}'(k_2-k_1)$; for an R-module M, we simply write $M':=M\otimes_R R[T]$. By specifying chain maps $\alpha\colon C_I\to C_J$, $\beta\colon C_J\to C_I[-1]$, and a homotopy map (not necessarily a chain map) $h\colon C_I\to C_I$ (with $\alpha_i\colon B_i\to A_i$ of degree 0 and $\beta_i\colon A_i\to B_{i-1}$ and $h_i\colon B_i\to B_i$ of degree k_1-k_2 for all i) we will define the differentials as

$$f_1=\begin{bmatrix} b_1 & \beta_1+T\cdot a_1\end{bmatrix},\qquad f_2=\begin{bmatrix} b_2 & \beta_2 & h_1+T\cdot \mathrm{I}_1\\ 0 & -a_2 & -\alpha_1\end{bmatrix},$$

$$f_i=\begin{bmatrix} b_i & \beta_i & h_{i-1}+(-1)^iT\cdot\mathrm{I}_{i-1}\\ 0 & -a_i & -\alpha_{i-1}\\ 0 & 0 & b_{i-1}\end{bmatrix}\qquad \text{for } 3\leq i\leq g-2,$$

$$f_{g-1}=\begin{bmatrix}\beta_{g-1} & h_{g-2}+(-1)^{g-1}T\cdot\mathrm{I}_{g-2}\\ -a_{g-1} & -\alpha_{g-2}\\ 0 & b_{g-2}\end{bmatrix},\qquad f_g=\begin{bmatrix}-\alpha_{g-1}+(-1)^g\frac{1}{\beta_g(1)}T\cdot a_g\\ b_{g-1}\end{bmatrix},$$

where I_t denotes the $\mathrm{rank}(B_t)\times\mathrm{rank}(B_t)$ identity matrix. We now discuss the construction of α, β, and h. Fix R-module bases $e_1,\ldots,e_{t_1}$ of A_1 and $\hat e_1,\ldots,\hat e_{t_1}$ of A_{g-1}. Write $\sum_{i=1}^{t_1}\hat c_i\cdot\hat e_i:=a_g(1_R)$ and $c_i\cdot 1_R:=a_1(e_i)$ for $1\leq i\leq t_1$ where by the Gorenstein property $c_i,\hat c_i\in J$ for all i. Denote by $\ell_i,\hat\ell_i\in R$ lifts of $\varphi(c_i),\varphi(\hat c_i)\in R/I$ respectively. For an R-module A, we write $A^*=\mathrm{Hom}_R(A,R)$ and for an R-basis $f_1,\ldots,f_t$ of A we denote by $f_1^*,\ldots,f_t^*$ the dual basis of A^*. Now consider the R-homomorphism $A_{g-1}^*\to R=B_{g-1}^*$ defined by $\hat e_i^*\mapsto\hat\ell_i\cdot 1_R$ which (by self-duality of C_I and C_J) extends to a chain map $C_J^*\to C_I^*$. Denote by $\widetilde\alpha\colon C_I\to C_J$ its dual. The map $\widetilde\alpha_0\colon B_0=R\to R=A_0$ is multiplication by an invertible element of R, cf. [P, §3]. Set $\alpha=\widetilde\alpha/\widetilde\alpha_0(1_R)$. We obtain $\beta\colon C_J\to C_I[-1]$ by extending the map $\beta_1 \colon A_1 \to R = B_0$ given by $e_i \mapsto -\ell_i \cdot 1_R$. Finally, by [KM, p.308], there is a homotopy $h \colon C_I \to C_I$ with $h_0 = h_{g-1} = 0$ and $\beta_i \alpha_i = h_{i-1} b_i + b_i h_i$ for $1 \le i \le g$.

Theorem ([KM])**.** *The complex C_U is a graded free resolution of $R[T]/U$ as an $R[T]$-module.*

It is important to remark that C_U is not necessarily minimal, although in many examples coming from algebraic geometry it is.

We now describe how to compute C_U, as implemented in our *Macaulay2* package *KustinMiller*. First note, that we can determine φ via the *Macaulay2* commands `Hom(sub(J,R/I),(R/I)^1)` and `homomorphism`. Furthermore one can extend homomorphisms to chain maps by the command `extend`.

Algorithm (Kustin-Miller complex)**.**

Input: Resolutions C_I and C_J, denoted as above, for homogeneous ideals $I \subset J$ in a polynomial ring R with R/I and R/J Gorenstein, and $\dim R/J = \dim R/I - 1$.

Output: The Kustin-Miller complex C_U associated to I and J.

Compute φ as in the Definition and lift $A_1 \twoheadrightarrow J \twoheadrightarrow J/I \xrightarrow{\varphi} R/I$ to $\varphi' \colon A_1 \to B_0$.
Compute the dual C_J^* of C_J and express the first differential as the product of a_1 with a square matrix $Q \colon A_{g-1}^* \to A_1$.
Extend $\varphi' \circ Q \colon A_{g-1}^* \to B_0 \cong B_{g-1}^*$ to a chain map $\alpha^* \colon C_J^* \to C_I^*$ and dualize to obtain $\widetilde{\alpha} \colon C_I \to C_J$. Dividing all differentials of $\widetilde{\alpha}$ by the inverse of the entry of $\widetilde{\alpha}_0$ yields $\alpha \colon C_I \to C_J$.
Extend φ' to a map $C_J \to C_I[-1]$ and multiply the differentials by -1 to obtain $\beta \colon C_J \to C_I[-1]$.
Set $h_0 := 0_R$. For $i = 1$ to $g-1$ do
    Set $h_i' := \beta_i \alpha_i - h_{i-1} b_i$.
    Using the `extend` command obtain h_i in the following diagram
end for
Return the differentials f_i according to the formulas given above.

$$\begin{array}{ccc} B_i & \xrightarrow{h_i'} & B_i \\ \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle b_i} \\ B_i & \xrightarrow{h_i} & B_i \end{array}$$

3. APPLICATIONS. We comment on some applications of the Kustin-Miller complex construction involving the authors. For more examples, see the documentation for the *KustinMiller* package.

Cyclic polytopes. For a polynomial ring $R = \Bbbk[x_1, \dots, x_n]$, let $I_d(R)$ denote the Stanley-Reisner ideal of the boundary complex of the cyclic polytope of dimension d with n-vertices. As shown in [BP2, Theorem 6.1], the Kustin-Miller complex construction yields a recursion for a minimal resolution of $I_d(R)$. Specifically for d even, apply Algorithm 2 with $T = x_n$ to minimal resolutions C_I and C_J of $I = I_d(\Bbbk[x_1, \dots, x_{n-1}])$ and $J = I_{d-2}(\Bbbk[z, x_2, \dots, x_{n-2}])$ considered as ideals in $\Bbbk[z, x_1, \dots, x_{n-1}]$ and quotient by the ideal (z). For d odd, one can proceed in a similar way.

Stellar subdivisions. Suppose C is a Gorenstein* simplicial complex on the variables of $\Bbbk[x_1, \dots, x_n]$ and F is a face of C. Let C_F be obtained by the stellar subdivision of C with respect to F, introducing the new variable x_{n+1}. Denote by I the image of the Stanley-Reisner ideal of C in $\Bbbk[z, x_1, \dots, x_n]$ and by $J = (z) + I : (\prod_{i \in F} x_i)$ the ideal corresponding to the link of F. Apply the Algorithm to minimal resolutions of I and J with $T = x_{n+1}$ and quotient by the ideal (z). By [BP1, §5.1], this yields a resolution of the Stanley-Reisner ring of C_F.

Constructions in algebraic geometry. In the paper [NP1], a series of Kustin-Miller unprojections were used to give the first examples of Campedelli algebraic surfaces of general type with algebraic fundamental group $\mathbb{Z}/6$. A similar technique produced in [NP2] seven families of Calabi-Yau 3-folds of high codimension. In both cases, the Kustin-Miller complex construction was used to control the numerical invariants of the new varieties.

Example. Using the package we discuss the *Tom* example given in [P, §5] passing from a codimension 3 to a codimension 4 ideal. Over the polynomial ring

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage "KustinMiller";

i2 : R = QQ[x_1..x_4,z_1..z_4];
```

consider the skew-symmetric matrix

```
i3 : b2 = matrix{{0,x_1,x_2,x_3,x_4}, {-x_1,0,0,z_1,z_2}, {-x_2,0,0,z_3,z_4},
       {-x_3,-z_1,-z_3,0,0},{-x_4,-z_2,-z_4,0,0}}

o3 = | 0    x_1  x_2  x_3 x_4 |
     | -x_1 0    0    z_1 z_2 |
     | -x_2 0    0    z_3 z_4 |
     | -x_3 -z_1 -z_3 0   0   |
     | -x_4 -z_2 -z_4 0   0   |

              5       5
o3 : Matrix R  <--- R
```

The Buchsbaum-Eisenbud complex

```
i4 : betti(cI = resBE b2)

            0 1 2 3
o4 = total: 1 5 5 1
         0: 1 . . .
         1: . 5 5 .
         2: . . . 1

o4 : BettiTally
```

resolves the ideal $I = (b_1) \subset R$ generated by the 4×4-Pfaffians

```
i5 : b1 = cI.dd_1

o5 = | z_2z_3-z_1z_4 -x_4z_3+x_3z_4 x_4z_1-x_3z_2 x_2z_2-x_1z_4 -x_2z_1+x_1z_3 |

              1       5
o5 : Matrix R  <--- R
```

of the skew-symmetric matrix b_2. Consider the unprojection locus J with Koszul resolution

```
i6 : J = ideal(z_1..z_4);

o6 : Ideal of R

i7 : betti(cJ = res J)

            0 1 2 3 4
o7 = total: 1 4 6 4 1
         0: 1 4 6 4 1

o7 : BettiTally
```

Applying the Algorithm, we obtain the Kustin-Miller resolution of the unprojection ideal $U \subset R[T]$, in this case the ideal of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$,

```
i8 : betti(cU = kustinMillerComplex(cI,cJ,QQ[T]))

            0 1  2 3 4
o8 = total: 1 9 16 9 1
         0: 1 .  . . .
         1: . 9 16 9 .
         2: . .  . . 1

o8 : BettiTally
```

with generators

```
i9 : S = ring cU;

i10 : f1 = cU.dd_1

o10 = | z_2z_3-z_1z_4 -x_4z_3+x_3z_4 x_4z_1-x_3z_2 x_2z_2-x_1z_4 -x_2z_1+x_1z_3
      ------------------------------------------------------------------------
      -x_1x_3+Tz_1 -x_1x_4+Tz_2 -x_2x_3+Tz_3 -x_2x_4+Tz_4 |

              1       9
o10 : Matrix S  <--- S
```

and syzygy matrix

```
i11 : f2 = cU.dd_2

o11 = {2} | 0    x_1  x_2  x_3 x_4 0    0    0    0    0    0    T    0    0    0    0    |
      {2} | -x_1 0    0    z_1 z_2 0    0    -x_1 0    0    x_2  0    T    0    0    0    |
      {2} | -x_2 0    0    z_3 z_4 -x_1 0    0    -x_2 0    0    0    0    T    0    0    |
      {2} | -x_3 -z_1 -z_3 0   0   0    0    -x_3 -x_3 -x_4 0    -x_3 0    0    T    0    |
      {2} | -x_4 -z_2 -z_4 0   0   0    x_3  0    0    0    0    0    0    0    0    T    |
      {2} | 0    0    0    0   0   z_2  z_3  0    z_4  0    0    z_4  0    -x_4 0    x_2  |
      {2} | 0    0    0    0   0   -z_1 0    z_3  0    z_4  0    0    0    x_3  -x_2 0    |
      {2} | 0    0    0    0   0   0    -z_1 -z_2 0    0    z_4  -z_2 x_4  0    0    -x_1 |
      {2} | 0    0    0    0   0   0    0    0    -z_1 -z_2 -z_3 0    -x_3 0    x_1  0    |

              9       16
o11 : Matrix S  <--- S
```

The code computing this example and various others related to the applications mentioned above can be found in the documentation of the package *KustinMiller*.

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