Implementing the Kustin-Miller complex construction

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Abstract. The Kustin-Miller complex construction, due to A. Kustin and M. Miller, can be applied to a pair of resolutions of Gorenstein rings with certain properties to obtain a new Gorenstein ring and a resolution of it. It gives a tool to construct and analyze Gorenstein rings of high codimension. We describe the Kustin-Miller complex, its implementation in the Macaulay2 package KustinMiller, and explain how it can be applied to explicit examples.

1. Introduction. Many important rings in commutative algebra and algebraic geometry turn out to be Gorenstein rings, i.e. commutative rings such that the localization at each prime ideal is a Noetherian local ring $R$ with finite injective dimension as an $R$-module. Examples are canonical rings of regular algebraic surfaces of general type, anticanonical rings of Fano varieties, and Stanley-Reisner rings of triangulations of spheres. Gorenstein rings with an embedding codimension at most 2 are known to be complete intersections, and those with embedding codimension 3 are described by the theorem of Buchsbaum-Eisenbud [BE, Theorem 2.1] as Pfaffians of a skew-symmetric matrix. Structure theorems in higher codimension are lacking. One goal of unprojection theory, which was introduced by A. Kustin, M. Miller and M. Reid and developed further by the second author (see [KM, R, PR, P]), is to act as a substitute for a structure theorem in codimension at least 4 by providing a construction that (under certain conditions) relates a more complicated Gorenstein ring to two simpler ones. The geometric motivation is to provide inverses of certain projections in birational geometry. The process can be considered as a version of Castelnuovo blow-down.

Examples of applications range from the construction of Campedelli surfaces [NP1] to results on the structure of Stanley-Reisner rings [BP2]. For an outline of more applications see [R], the introduction of [BP1], and §3 below.

We describe the Kustin-Miller complex construction [KM], which is the key tool to obtain resolutions of unprojection rings, and discuss our implementation in the Macaulay2 [M2] package KustinMiller. We illustrate the construction with examples and applications.

2. Implementation of the Kustin-Miller complex construction. We will consider the following setup. Let $R$ be a positively graded polynomial ring over a field $k$ and let $I, J \subset R$ be homogeneous ideals of $R$ such that $R/I$ and $R/J$ are Gorenstein, $I \subset J$, and $\dim R/J = \dim R/I - 1$. By [BH, Proposition 3.6.11], there exist $k_1, k_2 \in \mathbb{Z}$ such that $\omega_{R/I} = R/I(k_1)$ and $\omega_{R/J} = R/J(k_2)$. Assume that $k_1 > k_2$ so that the unprojection ring defined below is also positively graded.

Definition ([PR, Definition 1.2]). Let $\varphi \in \text{Hom}_{R/I}(J/I, R/I)$ be a homomorphism of degree $k_1 - k_2$ such that $\text{Hom}_{R/I}(J/I, R/I)$ is generated as an $R/I$-module by $\varphi$ and the inclusion morphism $i$. Let

\textit{KustinMiller} version 1.4.
Let $T$ be a variable of degree $k_1 - k_2$. We call the graded algebra $(R/I)[T]/(Tu - \varphi(u) \mid u \in J/I)$ the **Kustin-Miller unprojection ring** of the pair $I \subset J$ defined by $\varphi$.

The Kustin-Miller unprojection ring is naturally isomorphic to $R[T]/U$, where $U \subset R[T]$ is the inverse image of the ideal $(Tu - \varphi(u) \mid u \in J/I)$ of $(R/I)[T]$ under the natural map $R[T] \to (R/I)[T]$. In the following, we will consider $R[T]/U$ as the Kustin-Miller unprojection ring.

**Proposition** ([KM], [PR, Theorem 1.5]). The $R$-algebra $R[T]/U$ is Gorenstein and independent of the choice of $\varphi$ (up to isomorphism).

Following [KM], we now describe the construction of a graded free resolution of $R[T]/U$ from those of $R/I$ and $R/J$. We will refer to this as the **Kustin-Miller complex construction**. Denote by $g = \dim R - \dim R/J$ the codimension of the ideal $J$ of $R$, and suppose $g \geq 4$ (the special cases $g = 2, 3$ may be treated in a similar way). Let $C_J: \frac{R}{J} \leftarrow A_0 \leftarrow A_1 \leftarrow \cdots \leftarrow A_{g-1} \leftarrow A_g \leftarrow 0$ and $C_J: \frac{R}{J} \leftarrow B_0 \leftarrow B_1 \leftarrow \cdots \leftarrow B_{g-1} \leftarrow 0$ be minimal graded free resolutions (self-dual by the Gorenstein property [E, Corollary 21.16]) of $R/J$ and $R/I$ as $R$-modules with $A_0 = B_0 = R$, $A_g = R(-k_2 - \eta)$ and $B_{g-1} = R(-k_1 - \eta)$, where $\eta$ is the sum of the degrees of the variables of $R$.

Consider the complex

$$C_U: \frac{R[T]}{U} \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{g-1} \leftarrow F_g \leftarrow 0$$

with $F_0 = B_0^1$, $F_1 = B_1^1 \oplus A_1^1(k_2 - k_1)$, $F_i = B_i^1 \oplus A_i^1(k_2 - k_1) \oplus B_{i-1}^1(k_2 - k_1)$ for all $2 \leq i \leq g - 2$, $F_{g-1} = A_{g-1}^1(k_2 - k_1) \oplus B_{g-2}^1(k_2 - k_1)$, and $F_g = B_{g-1}^1(k_2 - k_1)$; for an $R$-module $M$, we simply write $M' := M \otimes_R R[T]$. By specifying chain maps $\alpha: C_I \to C_J$, $\beta: C_J \to C_I[-1]$, and a homotopy map (not necessarily a chain map) $h: C_I \to C_I$ (with $\alpha_i: B_i \to A_i$ of degree $0$ and $\beta_i: A_i \to B_{i-1}$ and $h_i: B_i \to B_i$ of degree $k_1 - k_2$ for all $i$) we will define the differentials as

$$f_1 = \begin{bmatrix} b_1 & \beta_1 + T \cdot a_1 \end{bmatrix}, \quad \quad f_2 = \begin{bmatrix} b_2 & \beta_2 - a_2 & h_1 + T \cdot I_1 \\ 0 & -a_2 & -\alpha_1 \end{bmatrix},$$

$$f_i = \begin{bmatrix} b_i & \beta_i - a_i & h_{i-1} + (-1)^i T \cdot I_{i-1} \\ 0 & -a_i & -\alpha_{i-1} \\ 0 & b_{i-1} & \end{bmatrix}$$

for $3 \leq i \leq g - 2$,

$$f_{g-1} = \begin{bmatrix} \beta_{g-1} & h_{g-2} + (-1)^{g-1} T \cdot I_{g-2} \\ -a_{g-1} & -\alpha_{g-2} \\ 0 & b_{g-2} \end{bmatrix}, \quad \quad f_g = \begin{bmatrix} -\alpha_{g-1} + (-1)^g \frac{1}{b_{g-1}} T \cdot a_g \\ 0 & b_{g-1} \end{bmatrix},$$

where $I_i$ denotes the rank($B_i$) \times rank($B_i$) identity matrix. We now discuss the construction of $\alpha, \beta,$ and $h$. Fix $R$-module bases $e_1, \ldots, e_{t_1}$ of $A_1$ and $\hat{e}_1, \ldots, \hat{e}_{t_1}$ of $A_{g-1}$. Write $\Sigma_{i=1}^{t_1} \hat{e}_i \cdot e_i := a_g(1_R)$ and $c_i \cdot 1_R := a_1(e_i)$ for $1 \leq i \leq t_1$ where by the Gorenstein property $c_i, \hat{e}_i \in J$ for all $i$. Denote by $\ell_i, \hat{\ell}_i \in R$ lifts of $\varphi(c_i), \varphi(\hat{e}_i) \in R/I$ respectively. For an $R$-module $A$, we write $A^* = \text{Hom}_R(A, R)$ and for an $R$-basis $f_1, \ldots, f_t$ of $A$ we denote by $f_1^*, \ldots, f_t^*$ the dual basis of $A^*$. Now consider the $R$-homomorphism $A_{g-1}^* \to R = B_{g-1}^1$ defined by $\hat{e}_i^* \mapsto \hat{\ell}_i \cdot 1_R$ which (by self-duality of $C_I$ and $C_J$) extends to a chain map $C_J^* \to C_I^*$. Denote by $\bar{\alpha}: C_I \to C_J$ its dual. The map $\bar{\alpha}_0: B_0 = R \to R = A_0$ is multiplication by an invertible element of $R$, cf. [P, §3]. Set $\alpha = \bar{\alpha}/\bar{\alpha}_0(1_R)$. We obtain $\beta: C_J \to C_I[-1]$ by extending
the map $\beta_1: A_1 \to R = B_0$ given by $e_i \mapsto -\ell_i \cdot 1_R$. Finally, by [KM, p.308], there is a homotopy $h: C_I \to C_I$ with $h_0 = h_{g-1} = 0$ and $\beta_i \alpha_i = h_{i-1} b_i + b_i h_i$ for $1 \leq i \leq g$.

**Theorem ([KM]).** The complex $C_U$ is a graded free resolution of $R[T]/U$ as an $R[T]$-module.

It is important to remark that $C_U$ is not necessarily minimal, although in many examples coming from algebraic geometry it is.

We now describe how to compute $C_U$, as implemented in our Macaulay2 package KustinMiller. First note, that we can determine $\varphi$ via the Macaulay2 commands $\text{Hom}(\text{sub}(J,R/I),(R/I)^{\ast})$ and homomorphism. Furthermore one can extend homomorphisms to chain maps by the command extend.

**Algorithm** (Kustin-Miller complex).

**Input:** Resolutions $C_I$ and $C_J$, denoted as above, for homogeneous ideals $I \subset J$ in a polynomial ring $R$ with $R/I$ and $R/J$ Gorenstein, and $\dim R/J = \dim R/I - 1$.

**Output:** The Kustin-Miller complex $C_U$ associated to $I$ and $J$.

1. Compute $\varphi$ as in the Definition and lift $A_1 \to J \to J/I \xrightarrow{\varphi} R/I$ to $\varphi': A_1 \to B_0$.
2. Compute the dual $C_J^\ast$ of $C_J$ and express the first differential as the product of $a_1$ with a square matrix $Q: A_{g-1}^* \to A_1$.
3. Extend $\varphi' \circ Q: A_{g-1}^* \to B_0 \cong B_{g-1}^\ast$ to a chain map $\alpha^\ast: C_J^\ast \to C_I^\ast$ and dualize to obtain $\tilde{\alpha}: C_I \to C_J$.
   Dividing all differentials of $\tilde{\alpha}$ by the inverse of the entry of $\tilde{\alpha}_0$ yields $\alpha: C_I \to C_J$.
4. Extend $\varphi'$ to a map $C_I \to C_I[-1]$ and multiply the differentials by $-1$ to obtain $\tilde{\beta}: C_J \to C_I[-1]$.
5. Set $h_0 := 0_R$. For $i = 1$ to $g - 1$ do
6: Set $h_i := \beta_i \alpha_i - h_{i-1} b_i$.
7: Using the extend command obtain $h_i$ in the following diagram $B_i \xrightarrow{h_i} B_i$.
8: end for
9: Return the differentials $f_i$ according to the formulas given above.

3. **Applications.** We comment on some applications of the Kustin-Miller complex construction involving the authors. For more examples, see the documentation for the KustinMiller package.

**Cyclic polytopes.** For a polynomial ring $R = k[x_1, \ldots, x_n]$, let $I_d(R)$ denote the Stanley-Reisner ideal of the boundary complex of the cyclic polytope of dimension $d$ with $n$-vertices. As shown in [BP2, Theorem 6.1], the Kustin-Miller complex construction yields a recursion for a minimal resolution of $I_d(R)$. Specifically for $d$ even, apply Algorithm 2 with $T = x_n$ to minimal resolutions $C_J$ and $C_I$ of $I = I_d(k[x_1, \ldots, x_{n-1}])$ and $J = I_{d-2}(k[z,x_2,\ldots,x_{n-2}])$ considered as ideals in $k[z,x_1,\ldots,x_{n-1}]$ and quotient by the ideal $(z)$. For $d$ odd, one can proceed in a similar way.

**Stellar subdivisions.** Suppose $C$ is a Gorenstein* simplicial complex on the variables of $k[x_1, \ldots, x_n]$ and $F$ is a face of $C$. Let $C_F$ be obtained by the stellar subdivision of $C$ with respect to $F$, introducing the new variable $x_{n+1}$. Denote by $I$ the image of the Stanley-Reisner ideal of $C$ in $k[z,x_1,\ldots,x_n]$ and by $J = (z) + 1 : (\prod_{x \in F} x_i)$ the ideal corresponding to the link of $F$. Apply the Algorithm to minimal resolutions of $I$ and $J$ with $T = x_{n+1}$ and quotient by the ideal $(z)$. By [BP1, §5.1], this yields a resolution of the Stanley-Reisner ring of $C_F$. 
Constructions in algebraic geometry. In the paper [NP1], a series of Kustin-Miller unprojections were used to give the first examples of Campedelli algebraic surfaces of general type with algebraic fundamental group $\mathbb{Z}/6$. A similar technique produced in [NP2] seven families of Calabi-Yau 3-folds of high codimension. In both cases, the Kustin-Miller complex construction was used to control the numerical invariants of the new varieties.

Example. Using the package we discuss the Tom example given in [P, §5] passing from a codimension 3 to a codimension 4 ideal. Over the polynomial ring

Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage "KustinMiller";
i2 : R = QQ[x_1..x_4,z_1..z_4];

consider the skew-symmetric matrix

i3 : b2 = matrix{{0,x_1,x_2,x_3,x_4}, {-x_1,0,0,z_1,z_2}, {-x_2,0,0,z_3,z_4},
{-x_3,-z_1,-z_3,0,0},{-x_4,-z_2,-z_4,0,0}}

o3 = | 0 x_1 x_2 x_3 x_4 |
| -x_1 0 0 z_1 z_2 |
| -x_2 0 0 z_3 z_4 |
| -x_3 -z_1 -z_3 0 0 |
| -x_4 -z_2 -z_4 0 0 |

5 5

o3 : Matrix R <--- R

The Buchsbaum-Eisenbud complex

i4 : betti(cI = resBE b2)

o4 = total: 1 5 5 1
 0: 1 . . .
 1: . 5 5 .
 2: . . . 1

o4 : BettiTally

resolves the ideal $I = (b_1) \subset R$ generated by the $4 \times 4$-Pfaffians

i5 : b1 = cI.dd_1

o5 = | z_2z_3 z_1 z_4 -x_4z_3+x_3z_4 x_4z_1-x_3z_2 x_2z_2-x_1z_4 -x_2z_1+x_1z_3 |

1 5

o5 : Matrix R <--- R

of the skew-symmetric matrix $b_2$. Consider the unprojection locus $J$ with Koszul resolution

i6 : J = ideal(z_1..z_4);

o6 : Ideal of R

i7 : betti(cJ = res J)

o7 = total: 1 4 6 4 1
 0: 1 4 6 4 1

o7 : BettiTally
Applying the Algorithm, we obtain the Kustin-Miller resolution of the unprojection ideal $U \subset R[T]$, in this case the ideal of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.

```plaintext
i8 : betti(cU = kustinMillerComplex(cI,cJ,QQ[T]))

0 1 2 3 4
o8 = total: 1 9 16 9 1
    0: 1 . . .
    1: . 9 16 9 .
    2: . . . . 1

o8 : BettiTally
```

with generators

```plaintext
i9 : S = ring cU;
i10 : f1 = cU.dd_1

o10 = | z_2z_3z_1z_4 -x_4z_3+x_3z_2 x_2z_2-x_1z_4 -x_2z_1+x_1z_3 |
    -x_1x_3+Tz_1 -x_1x_4+Tz_2 -x_2x_3+Tz_3 -x_2x_4+Tz_4 |

1 9
```

and syzygy matrix

```plaintext
i11 : f2 = cU.dd_2

o11 = {2} | 0 x_1 x_2 x_3 x_4 0 0 0 0 0 T 0 0 0 0 |
    {2} | -x_1 0 0 z_1 z_2 0 0 -x_1 0 0 x_2 0 T 0 0 0 |
    {2} | -x_2 0 0 z_3 z_4 -x_1 0 0 -x_2 0 0 0 0 0 0 |
    {2} | -x_3 -z_1 -z_3 0 0 0 0 -x_3 -x_3 -x_4 0 0 T 0 |
    {2} | -x_4 -z_2 -z_4 0 0 0 x_3 0 0 0 0 0 0 0 T |
    {2} | 0 0 0 0 0 z_2 z_3 0 z_4 0 0 z_4 0 -x_4 0 x_2 |
    {2} | 0 0 0 0 0 -z_1 0 z_3 0 z_4 0 0 z_4 -z_2 x_4 0 x_2 |
    {2} | 0 0 0 0 0 0 -z_1 -z_2 0 0 z_4 -z_2 x_4 0 0 -x_1 |
    {2} | 0 0 0 0 0 0 0 0 -z_1 -z_2 -z_3 0 -x_3 0 x_1 0 |

9 16
```

The code computing this example and various others related to the applications mentioned above can be found in the documentation of the package KustinMiller.

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