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Computing free bases for projective modules

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ABSTRACT. The *QuillenSuslin* package for *Macaulay2* provides the ability to compute a free basis for a projective module over a polynomial ring with coefficients in $\mathbb{Q}, \mathbb{Z}$, or $\mathbb{Z}/p$ for a prime integer p. A brief description of the underlying algorithm and the related tools are given.

INTRODUCTION. In 1955, J-P. Serre in [FAC] posed the following question: do there exist finitely generated projective $k[x_1,\ldots,x_n]$ modules, with k a field, which are not free? This question was known as "Serre's Problem" and remained open for 21 years until it was resolved independently by D. Quillen and A.A. Suslin in 1976.

Quillen–Suslin Theorem (1976 [Qui, Sus])**.** *If k a field and $S = k[x_1,\ldots,x_n]$, then every finitely generated projective S-module is free.*

However, these proofs were not entirely constructive, and it was not until the early 1990s that papers such as [FG, LS, LW] began giving fully constructive versions of the proof. In 1992, A. Logar and B. Sturmfels [LS] published the algorithmic proof of the Quillen-Suslin Theorem that forms the basis for the methods in *QuillenSuslin*. Logar and Sturmfels describe, via the technique of completion of unimodular rows, how to construct a free generating set for a projective module over $\mathbb{C}[x_1,\ldots,x_n]$. One can extend these constructive techniques to work over more general coefficient rings such as $\mathbb{Q}, \mathbb{Z}$, and $\mathbb{Z}/p\mathbb{Z}$, for p a prime integer. Descriptions of some of these more general techniques, along with applications to areas such as systems control theory, can be found in [Fa2, Ch. 2] and [FQ, Ch. 3]. These algorithms were used to create a QuillenSuslin package [Fa1] for the computer algebra system *Maple*. We have implemented these algorithms, with some modifications, in our *QuillenSuslin* package for *Macaulay2* [M2].

PRELIMINARIES. In this section, we reduce the statement of the Quillen-Suslin Theorem to a more concrete matrix theoretic problem concerning the completion of unimodular rows over polynomial rings to square invertible matrices. Throughout, R is a commutative ring and M denotes a finitely generated R-module. We say that M is a ***projective*** R-module if it is a direct summand of a free module. Similarly, we define a slightly stronger notion by saying that M is ***stably free*** if there exists some $m \geq 0$ such that $M \oplus R^m$ is free. A module M is stably free if and only if it is isomorphic to the kernel of a surjective R-linear map $\phi : R^n \to R^m$ for some $m \leq n$. Since ϕ surjects onto a free module, we know that this map splits and admits a right inverse $\psi : R^m \to R^n$ so that $\phi\psi = \mathrm{id}_{R^m}$. Therefore a matrix representing ϕ is right invertible, and we call such a right invertible matrix over R ***unimodular***. Using this terminology, it is not difficult to show that the Quillen-Suslin Theorem as stated above is equivalent to the following matrix theoretic statement about unimodular matrices.

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QuillenSuslin version 1.7.

Quillen–Suslin Theorem (restatement [LS, Theorem 1.1]). *If the ring R is a principal ideal domain, $S = R[x_1, \ldots, x_n]$ and $U \in \mathrm{Mat}_{m\times n}(R)$ is a unimodular matrix over S with $m \leq n$, then there exists a unimodular matrix $V \in \mathrm{Mat}_{n\times n}(S)$ such that*

$$UV = \left[\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{array}\right].$$

A matrix V as in the above theorem is said to ***solve the unimodular matrix problem for*** U. Given such a V, the first m rows of the invertible matrix V^{-1} are the same as the original matrix U. Therefore, proving the Quillen-Suslin Theorem is equivalent to showing that any unimodular matrix can be completed to a square invertible matrix over the polynomial ring. As it turns out, it suffices to show that the ***unimodular row problem*** can be solved; the Quillen–Suslin Theorem holds for unimodular row vectors [LS]. For more details, we refer the interested reader to the excellent book [Lam].

THE LOGAR-STURMFELS ALGORITHM. Before describing the general algorithm, we mention that *QuillenSuslin* contains several shortcut methods as described in [Fa2, §2.2]. These methods allow us to quickly solve the unimodular row problem for a row satisfying certain properties, often allowing us to avoid the worst-case general algorithm. These shortcut methods are automatically used as soon as they are applicable during the methods `computeFreeBasis`, `completeMatrix`, `qsAlgorithm`, and `qsIsomorphism`.

The main idea behind the Logar-Sturmfels algorithm is to iteratively reduce the number of variables involved in a unimodular row $\boldsymbol{f}$ one by one, eventually obtaining a unimodular row $\tilde{\boldsymbol{f}}$ over the coefficient ring, which is a PID. We can then use a simple algorithm based on the Smith normal form of $\tilde{\boldsymbol{f}}$ to construct a final unimodular matrix U so that $\tilde{\boldsymbol{f}}U = [1\ 0\ \cdots\ 0]$. Multiplying together all of the matrices used during the process, one can construct a unimodular matrix that solves the unimodular row problem for the original row $\boldsymbol{f}$.

The process of eliminating a variable from a unimodular matrix is organized into three main steps: the ***normalization step***, the ***local loop***, and the ***patching step***. Below is a brief description of each step, as well as a demonstration of the corresponding commands in the *QuillenSuslin* package. We work $S = \mathbb{Z}[x,y]$ and consider the unimodular row $\boldsymbol{f} = [x^2\,,\, 2y+1\,,\, x^5y^2+y]$.

```
Macaulay2, version 1.6
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : needsPackage "QuillenSuslin";

i2 : S = ZZ[x,y];

i3 : f = matrix {{x^2,2*y+1,x^5*y^2+y}};

             1       3
o3 : Matrix S  <--- S
```

We can use the command `isUnimodular` to check that this row is indeed unimodular over S.

```
i4 : isUnimodular f

o4 = true
```

In order to eliminate the variable y, we will construct a unimodular matrix U so that the product $\boldsymbol{f}U$ is the same as $\boldsymbol{f}$ with y replaced by 0. We first demonstrate the normalization step.

Normalization Step. Since Horrocks' Theorem requires a monic polynomial, we must first construct a unimodular matrix U_1 and an invertible change of variables $x_i \leftrightarrow X_i$ so that the first entry of $\boldsymbol{f}U_1$ is monic in X_n. The normalization step is based on the following result, which has been slightly restated for our purposes.

Lemma ([VS, Lemma 10.6])**.** *Let R be a Noetherian ring, n and m natural numbers, $S = R[x_1,\dots,x_n]$, $m \geq \dim R + 2$, and $\boldsymbol{f} = [f_1\ f_2\ \cdots\ f_m]$ a unimodular row over S. There exists an $m \times m$ unimodular matrix U over S, and an invertible change of variables $x_i \leftrightarrow X_i$, so that after applying the change of variables the first entry of $\boldsymbol{f}U$ is monic in X_n when viewed as a polynomial in $R[X_1,\dots,X_{n-1}][X_n]$.*

A constructive version of this result is implemented in the method `changeVar`, and is used as in the following example.

```
i5 : (U1,subs,invSubs) = changeVar(f,{x,y})
o5 = (| 1 0 0 |, | y x |, | y x |)
      | 0 1 0 |
      | 0 0 1 |
o5 : Sequence
i6 : f = sub(f*U1,subs)
o6 = | y2 2x+1 x2y5+x |
             1       3
o6 : Matrix S  <--- S
```

Since the first entry of the row $\boldsymbol{f}$ was already monic in x, the method simply returned a permutation of the variables interchanging x and y so that the first entry of the new row would be monic in y. Now that the first entry of the row is monic in the variable we are trying to eliminate, we may proceed to the local loop.

Local Loop. The purpose of the local loop is to compute a collection of *local solutions* to the unimodular row problem for $\boldsymbol{f}$. The local loop is based on the following result of Horrocks.

Horrocks' Theorem ([Rot, Prop. 4.98])**.** *Consider the polynomial ring $B[y]$, where B is a local ring, and let $\boldsymbol{f} = [f_1\ f_2\ \cdots\ f_m]$ be a unimodular row over $B[y]$. If some f_i is monic in y, then there exists a unimodular $m \times m$ matrix U over $B[y]$ so that $\boldsymbol{f}U = [1\ 0\ \cdots\ 0]$.*

In order to eliminate the last variable x_n in a unimodular row over a polynomial ring $R[x_1,\dots,x_n]$, the local loop proceeds in the following way: First set $I = (0)$ in $A = R[x_1,\dots,x_{n-1}]$. While $I \neq A$, the i-th iteration of the loop is

(1) Find a maximal ideal $\mathfrak{m}_i$ in A containing I.
(2) Apply Horrocks' Theorem to the row $\boldsymbol{f}$, viewed as a unimodular row over $A_{\mathfrak{m}_i}[x_n]$, to find a unimodular matrix L_i over $A_{\mathfrak{m}_i}[x_n]$ that solves the unimodular row problem for $\boldsymbol{f}$ (we call this L_i a ***local solution to the unimodular row problem for $\boldsymbol{f}$***).
(3) Let d_i denote the common denominator for all of the elements in the matrix U_i.
(4) Set $I = I + (d_i)$.

If $I \neq A$, then we repeat the loop. Otherwise we are able to stop and go on to the patching step. Since we are creating a strictly ascending chain of ideals $(d_1) \subset (d_1, d_2) \subset \cdots$ in the Noetherian ring A, this loop must terminate in a finite number of steps with $(d_1, \ldots, d_k) = A$ for some integer k.

In our example, we use the method `getMaxIdeal` to first find an arbitrary maximal ideal in $\mathbb{Z}[x]$, and we set $\mathfrak{m}_1 = (2, x)$. Using the method `horrocks`, we can compute a unimodular matrix L_1 over $\left(\mathbb{Z}[x]_{(2,x)}\right)[y]$ so that $\boldsymbol{f} L_1 = [1\ 0\ 0]$.

```
i7 : A = ZZ[x];
i8 : m1 = getMaxIdeal(ideal(0_A),{x})
o8 = ideal (2, x)
o8 : Ideal of A
i9 : L1 = horrocks(f,y,m1)
o9 = | 0        1          0                |
     | 1/(2x+1) -y2/(2x+1) (-x2y5-x)/(2x+1) |
     | 0        0          1                |
                     3              3
o9 : Matrix (frac S)  <--- (frac S)
```

Since $d_1 = 2x+1$ is a common denominator for the entries of L_1 and $(2x+1) \neq \mathbb{Z}[x]$, we use `getMaxIdeal` again to find a maximal ideal containing $2x+1$, and we set $\mathfrak{m}_2 = (3, x-1)$. We use `horrocks` a second time to compute a new local solution `L2` with common denominator $d_2 = x$.

```
i10 : m2 = getMaxIdeal(sub(ideal(2*x+1),A),{x})
o10 = ideal (- x + 1, 3)
o10 : Ideal of A
i11 : L2 = horrocks(f,y,m2)
o11 = | -xy3 xy5+1 2x2y3+xy3 |
      | 0    0     1         |
      | 1/x  -y2/x (-2x-1)/x |
                     3              3
o11 : Matrix (frac S)  <--- (frac S)
i12 : sub(ideal(2*x+1,x),S) == ideal(1_S)
o12 = true
```

Since $(d_1, d_2) = (2x+1, x) = \mathbb{Z}[x]$, we are able to exit the local loop and proceed to the patching step.

Patching Step. Loosely speaking, the patching step involves multiplying slight variations of the local solutions $L_1, \ldots, L_k$ together in a clever way so that the product U is a unimodular matrix over the polynomial ring $R[x_1, \ldots, x_n]$ and multiplying $\boldsymbol{f}$ times U is equivalent to evaluating $\boldsymbol{f}$ when $x_n = 0$, thereby eliminating one of the variables in the row $\boldsymbol{f}$. For more details, see [LS, p. 235].

Following along with our example, we use the method `patch` applied to our list $\{L_1, L_2\}$ of local solutions and we specify that y is the variable that we want to eliminate.

```
i13 : U = patch({L1,L2},y)
o13 = | -32x6y5+1                  0 8x5y3              |
      | 16x5y7-8x4y7+4x3y7+2xy2-y2 1 -4x4y5+2x3y5-x2y5 |
      | -4xy2                      0 1                  |
              3       3
o13 : Matrix S  <--- S
i14 : f*U
```

```
o14 = | 0 2x+1 x |
              1       3
o14 : Matrix S  <--- S
```

We can see that multiplying $\boldsymbol{f}$ by the unimodular matrix `U` is equivalent to evaluating $\boldsymbol{f}$ when $y = 0$ (keeping in mind that the variables x and y were interchanged during the normalization step).

CORE METHODS IN THE QUILLENSUSLIN PACKAGE. The method `qsAlgorithm` automates all of the above computations for computing a solution to the unimodular matrix problem, and automatically applies the shortcut methods in [Fa2] when possible. We demonstrate the use of `qsAlgorithm` by finding a solution to the unimodular row problem for the row $\boldsymbol{f} = [y^2, 2x+1, x^2y^5+x]$ given earlier.

```
i15 : U = qsAlgorithm f
o15 = | 2x2y3 -2x2y5+1 -2x4y8-2x3y3 |
      | 1     -y2      -x2y5-x      |
      | -2    2y2      2x2y5+2x+1   |
              3       3
o15 : Matrix S  <--- S
i16 : det U == -1
o16 = true
i17 : f*U
o17 = | 1 0 0 |
              1       3
o17 : Matrix S  <--- S
```

The package also contains a method `completeMatrix` that completes a unimodular matrix over a polynomial ring to a square invertible matrix. Again we demonstrate its use on the unimodular row $\boldsymbol{f}$.

```
i18 : C = completeMatrix f
o18 = {0} | y2 2x+1   x2y5+x |
      {2} | 1  -2x2y3 0      |
      {7} | 0  2      1      |
              3       3
o18 : Matrix S  <--- S
i19 : det C == -1
o19 = true
```

The previous two methods, `qsAlgorithm` and `completeMatrix`, also work over Laurent polynomial rings of the form $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with $k = \mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for p a prime integer. The algorithm in the Laurent polynomial case is due to Park and is described in [Par, p. 215].

Finally, we give an example to demonstrate the methods `computeFreeBasis` and `qsIsomorphism`. The method `computeFreeBasis` computes a free generating set for the given projective module and `qsIsomorphism` computes an isomorphism between a free module and the given projective module. In the following example, we define $K = \ker(\boldsymbol{f})$, which we can verify is indeed a projective $\mathbb{Z}[x,y]$-module by using the command `isProjective`.

```
i20 : K = ker f
o20 = image | 2x+1 2x3y3+x2y3 2x2y5-1 -x2y5-x |
            | -y2  x          y2      0       |
            | 0    -2x-1      -2y2    y2      |
```

```
                                3
o20 : S-module, submodule of S
i21 : isProjective K
o21 = true
i22 : mingens K
o22 = | -2x-1 2x3y3+x2y3 x2y5+x |
      | y2    x          0      |
      | 0     -2x-1      -y2    |
              3       3
o22 : Matrix S  <--- S
i23 : syz mingens K
o23 = {2} | x    |
      {6} | -y2  |
      {7} | 2x+1 |
              3       1
o23 : Matrix S  <--- S
i24 : B = computeFreeBasis K
o24 = | 2x3y3+x2y3 2x2y5-1 |
      | x          y2      |
      | -2x-1      -2y2    |
              3       2
o24 : Matrix S  <--- S
i25 : syz B == 0
o25 = true
i26 : image B == K
o26 = true
```

From this output, we can see that the command `mingens` does not produce a free generating set for K, while `computeFreeBasis` produces a set of 2 generators for K with no relations, demonstrating that K is free. We can also use the method `qsIsomorphism` described above to obtain an explicit isomorphism $\varphi\colon \mathbb{Z}[x,y]^2 \to K$.

```
i27 : phi = qsIsomorphism K
o27 = {2} | 0 0 |
      {6} | 1 0 |
      {7} | 0 1 |
      {7} | 0 0 |
o27 : Matrix
i28 : source phi
       2
o28 = S
o28 : S-module, free
i29 : target phi
o29 = image | 2x+1 2x3y3+x2y3 2x2y5-1 -x2y5-x |
            | -y2  x          y2      0       |
            | 0    -2x-1      -2y2    y2      |
                               3
o29 : S-module, submodule of S
```

```
i30 : isIsomorphism phi
o30 = true
```

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REFERENCES.

[Fa1] A. Fabiańska, *QuillenSuslin Package for Maple*, available at wwwb.math.rwth-aachen.de/QuillenSuslin/.

[Fa2] _____, *Algorithmic analysis of presentations of groups and modules*, PhD Thesis, RWTH Aachen University, July 31, 2009, available at wwwb.math.rwth-aachen.de/QuillenSuslin/DissertationAF.pdf.

[FAC] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278 (French).

[FQ] A. Fabiańska and A. Quadrat, *Applications of the Quillen-Suslin theorem to multidimensional systems theory*, Gröbner bases in control theory and signal processing, Radon Ser. Comput. Appl. Math., vol. 3, Walter de Gruyter, Berlin, 2007, pp. 23–106.

[FG] N. Fitchas and A. Galligo, *Nullstellensatz effectif et conjecture de Serre (théorème de Quillen-Suslin) pour le calcul formel*, Math. Nachr. **149** (1990), 231–253 (French, with English summary).

[Lam] T.Y. Lam, *Serre's problem on projective modules*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006.

[LW] R.C. Laubenbacher and C.J. Woodburn, *A new algorithm for the Quillen-Suslin theorem*, Beiträge Algebra Geom. **41** (2000), no. 1, 23–31.

[LS] A. Logar and B. Sturmfels, *Algorithms for the Quillen-Suslin theorem*, J. Algebra **145** (1992), no. 1, 231–239.

[M2] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at www.math.uiuc.edu/Macaulay2/.

[Par] Hyungju Park, *Symbolic computation and signal processing*, J. Symbolic Comput. **37** (2004), no. 2, 209–226.

[Qui] Daniel Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171.

[Rot] J.J. Rotman, *An introduction to homological algebra*, 2nd ed., Universitext, Springer, New York, 2009.

[Sus] A.A. Suslin, *Projective modules over polynomial rings are free*, Dokl. Akad. Nauk SSSR **229** (1976), no. 5, 1063–1066 (Russian).

[VS] L. N. Vaserštein and A. A. Suslin, *Serre's problem on projective modules over polynomial rings, and algebraic K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 5, 993–1054, 1199 (Russian).

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