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ExteriorIdeals: a package for computing monomial ideals in an exterior algebra

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ABSTRACT: Let K be a field, V a K-vector space with basis $e_1, \dots, e_n$, and E the exterior algebra of V. We introduce a *Macaulay2* package that allows one to deal with classes of monomial ideals in E. More precisely, we implement in *Macaulay2* some algorithms in order to easily compute stable, strongly stable and lexsegment ideals in E. Moreover, an algorithm to check whether an $(n+1)$-tuple $(1, h_1, \dots, h_n)$ $(h_1 \le n = \dim_K V)$ of nonnegative integers is the Hilbert function of a graded K-algebra of the form E/I, with I a graded ideal of E, is given. In particular, if $H_{E/I}$ is the Hilbert function of a graded K-algebra E/I, the package is able to construct the unique lexsegment ideal I^{lex} such that $H_{E/I} = H_{E/I^{\text{lex}}}$.

1. INTRODUCTION. Monomial ideals are a bridge between algebra and combinatorial algebra. It is well known that, even if such ideals are, in some sense, among the simplest structures in commutative algebra, they are the main objects of combinatorial commutative algebra. Many authors have focused their attention on classes of monomials ideals in an exterior algebra [Aramova et al. 1997; 2000; Crupi and Utano 1999; 2007; Crupi and Ferró 2015; Eisenbud et al. 2003; Crupi 2015; Gasharov 1997; Murai 2011; Shakin 2004; 2005] and on the behavior of certain invariants, such as for instance, the Hilbert function.

In this paper, we introduce `ExteriorIdeals.m2` — a new package written for [Macaulay2] for manipulating special classes of monomial ideals in an exterior algebra of a finite-dimensional vector space over a field. More precisely, the package provides functions to check whether a monomial ideal is stable, strongly stable, or lexsegment, and, respectively, to compute the smallest stable, strongly stable, or lexsegment ideal containing a given monomial ideal. Moreover, given an exterior algebra, the package allows the computation of all the Hilbert sequences of quotients of the exterior algebra. Some utility functions are necessary to simplify and optimize the implementation of the main algorithms, such as the Macaulay expansion, the initial degree of a graded ideal, the support of a monomial and the shadow of a set of monomials. Most of the algorithms must work in an exterior algebra

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ExteriorIdeals.m2 version 1.0

endowed with the lexicographic order, so that such an ordering is forced within routines. Nevertheless, the ideals obtained by our algorithms are made compatible with the native exterior algebra to allow further computations.

2. MATHEMATICAL BACKGROUND. Let K be a field. We denote by

$$E = K\langle e_1, \dots, e_n \rangle$$

the exterior algebra of a K-vector space V with basis $e_1, \dots, e_n$. For any subset $\sigma = \{i_1, \dots, i_d\}$ of $\{1, \dots, n\}$, with $i_1 < i_2 < \cdots < i_d$, we write $e_\sigma = e_{i_1} \wedge \cdots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \varnothing$. The set of monomials in E forms a K-basis of E of cardinality 2^n.

In order to simplify the notation, we write $fg = f \wedge g$ for any two elements f and g in E. An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$. An ideal I is called *graded* if I is generated by homogeneous elements. If I is graded, then $I = \oplus_{j \geq 0} I_j$, where I_j is the K-vector space of all homogeneous elements $f \in I$ of degree j. We denote by $\mathrm{indeg}(I)$ the *initial degree* of I, i.e., the least degree of a homogeneous generator of I.

If I is a graded ideal in E, then the function $H_I : \mathbb{N} \to \mathbb{N}$ given by $H_I(d) = \dim_K I_d$ $(i \geq 0)$ is called the Hilbert function of I.

Now let $e_\sigma = e_{i_1} \cdots e_{i_d} \neq 1$ be a monomial in E. We define

$$\mathrm{supp}(e_\sigma) = \sigma = \{j : e_j \text{ divides } e_\sigma\}, \quad \mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\}.$$

Moreover, we set $\mathrm{m}(e_\sigma) = 0$ if $e_\sigma = 1$.

If M is a set of monomials of degree $d < n$ of E, the set of monomials of degree $d + 1$,

$$\mathrm{Shad}(M) = \{(-1)^{\alpha(\sigma, j)} e_j e_\sigma : e_\sigma \in M,\ j \notin \mathrm{supp}(e_\sigma),\quad j = 1, \dots, n\},$$

where $\alpha(\sigma, j) = |\{r \in \sigma : r < j\}|$, is called the *shadow* of M and is denoted by $\mathrm{Shad}(M)$ [Crupi and Ferró 2015, Definition 2.4].

Definition 2.1. Let I be a monomial ideal of E. I is called *stable* if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$. I is called *strongly stable* if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.

If I is a monomial ideal of E, we denote by $G(I)$ the unique minimal set of monomial generators of I.

Remark 2.2. One can observe that the defining property of a strongly stable ideal needs to be checked only for the set of monomial generators of a monomial ideal. Indeed, let I be a monomial ideal and suppose that for all $e_\sigma \in G(I)$, and for all integers $1 \leq i < j \leq n$ such that $j \in \sigma$, one has $e_i e_{\sigma \setminus \{j\}} \in I$. Then I is strongly stable.

Let $e_\tau \in I$ be a monomial and $1 \le i < j \le n$ be integers such that $j \in \tau$. There exist $e_\sigma \in G(I)$ and a monomial $e_\mu \in E$ such that $e_\tau = e_\sigma e_\mu$ in E.

We distinguish two cases: $j \in \sigma$, $j \in \mu$. If $j \in \sigma$, then $e_i e_{\sigma\setminus\{j\}} \in I$ by assumption, and so $e_i e_{\tau\setminus\{j\}} = e_i e_{\sigma\setminus\{j\}} e_\mu \in I$.

If $j \in \mu$, then $e_i e_{\tau\setminus\{j\}} = e_i e_\sigma e_{\mu\setminus\{j\}} \in I$.

Another class of monomial ideals which plays a relevant role in combinatorial commutative algebra is the class of *lexsegment ideals*. The lexsegment ideals provide an upper bound for the graded Betti numbers of graded ideals with given Hilbert function [Aramova et al. 1997, Theorem 4.4].

Let $\mathrm{Mon}_d(E)$ be the set of all monomials of degree $d \ge 1$ in E. Denote by $>_{\mathrm{lex}}$ the *lexicographic order* on $\mathrm{Mon}_d(E)$, i.e., if $e_\sigma = e_{i_1} e_{i_2} \cdots e_{i_d}$ and $e_\tau = e_{j_1} e_{j_2} \cdots e_{j_d}$ are monomials belonging to $\mathrm{Mon}_d(E)$, with $1 \le i_1 < i_2 < \cdots < i_d \le n$ and $1 \le j_1 < j_2 < \cdots < j_d \le n$, then $e_\sigma >_{\mathrm{lex}} e_\tau$ if $i_1 = j_1, \ldots, i_{s-1} = j_{s-1}$ and $i_s < j_s$ for some $1 \le s \le d$.

Definition 2.3. A nonempty subset M of $\mathrm{Mon}_d(E)$ is called a lexsegment of degree d if for all $v \in M$ and all $u \in \mathrm{Mon}_d(E)$ such that $u >_{\mathrm{lex}} v$, we have that $u \in M$.

Definition 2.4. A monomial ideal I of E is called a lexsegment ideal (lex ideal, for short) if for all monomials $u \in I$ and all monomials $v \in E$ with $\deg u = \deg v$ and $v >_{\mathrm{lex}} u$, we have $v \in I$.

Equivalently, a monomial ideal I in E is called a *lex ideal* if $\mathrm{Mon}_d(I)$ is a lexsegment for all d; $\mathrm{Mon}_d(I)$ is the set of all monomials of degree d in I.

Remark 2.5. Every lex ideal of E is obviously a strongly stable ideal, and consequently a stable ideal.

Now let a and i be two positive integers. Then a has the unique i-th Macaulay expansion [Herzog and Hibi 2011, Lemma 6.3.4]

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

with $a_i > a_{i-1} > \cdots a_j \ge j \ge 1$. We define

$$a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}.$$

We also set $0^{(i)} = 0$ for all $i \ge 1$.

The next theorem describes the possible Hilbert functions of graded K-algebras of the form E/I, with I a graded ideal in E. It is the precise analogue to Macaulay's theorem [Bruns and Herzog 1993; Eisenbud 1995] which describes the possible Hilbert functions of standard graded K-algebras.

Theorem 2.6 [Aramova et al. 1997, Theorem 4.1]. *Let* $(h_1, \dots, h_n)$ *be a sequence of nonnegative integers. Then the following conditions are equivalent*:

(a) $1 + \sum_{i=1}^{n} h_i t^i$ *is the Hilbert series of a graded* K*-algebra* E/I.

(b) $0 < h_{i+1} \le h_i^{(i)},\ 0 < i \le n-1$.

Theorem 2.6 is known as the *Kruskal–Katona theorem*. Its proof points out that if I is a graded ideal of E, then there exists a unique lex ideal of E, usually denoted by I^{lex}, such that $H_{E/I} = H_{E/I^{\text{lex}}}$.

More precisely, if $(1, h_1, \dots, h_n)$ is a sequence of nonnegative integers such that

(i) $h_1 \le n$,

(ii) $0 < h_{i+1} \le h_i^{(i)},\ 0 < i \le n-1$,

then there exists a unique lex ideal J ($\operatorname{indeg} J \ge 1$) of an exterior algebra E with n generators over a field K such that $H_{E/J}(d) = h_d$ ($d = 0, \dots, n$).

If $1 + \sum_{i=1}^{n} h_i t^i$ is the Hilbert series of a graded K-algebra E/I, then the sequence $(1, h_1, \dots, h_n)$ is called the *Hilbert sequence* of E/I.

From the Kruskal–Katona theorem, one can deduce that a sequence of nonnegative integers $(h_0, h_1, \dots, h_n)$ is the Hilbert sequence of a graded K-algebra E/I, with $I \subsetneq E$ a graded ideal of initial degree ≥ 1, if $h_0 = 1$, and (i) and (ii) hold.

From now on, when we speak about Hilbert sequences we refer to Hilbert sequences of quotients of an exterior algebra.

3. EXAMPLES. In this section, we collect some examples in order to describe the algorithms. Our implementation works in any characteristic.

Example 3.1. Given a monomial ideal I in an exterior algebra E, we illustrate how some functions from our package allow one to check whether I is stable, strongly stable, or lex, and to produce stable or strongly stable ideals containing I. The core of the algorithms is based on the fact that the minimal monomial generators of a stable or strongly stable ideal must satisfy the criterion in Definition 2.1 (see Remark 2.2) and on the fact that the shadow of a lexsegment of monomials is again a lexsegment [Herzog and Hibi 2011].

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra,
TangentCone

i1 : loadPackage "ExteriorIdeals"
i2 : E=QQ[e_1..e_5,SkewCommutative=>true]
i3 : I=ideal {e_2*e_3,e_3*e_4*e_5}
o3 = ideal (e e , e e e )
             2 3   3 4 5
```

```
o3 : Ideal of E
i4 : isStableIdeal I
o4 = false
```

The ideal I is not stable. Indeed, the monomial e_1e_2 is not in I even though e_2e_3 is. Hence, by the function `stableIdeal(ideal)`, we compute the smallest stable ideal (Is) containing I:

```
i5 : Is=stableIdeal I
o5 = ideal (e e , e e e , e e , e e e )
             1 2   1 3 4   2 3   3 4 5
o5 : Ideal of E
i6 : isStableIdeal Is
o6 = true
i7 : isStronglyStableIdeal Is
o7 = false
```

The ideal Is is stable but not strongly stable in E. Note that the monomial e_1e_3 is not in Is even though e_2e_3 is.

Using the function `stronglyStableIdeal(ideal)`, we compute the smallest strongly stable ideal (Iss) containing Is, and consequently I:

```
i8 : Iss=stronglyStableIdeal Is
o8 = ideal (e e , e e , e e e , e e , e e e , e e e )
             1 2   1 3   1 4 5   2 3   2 4 5   3 4 5
o8 : Ideal of E
i9 : isStronglyStableIdeal Iss
o9 = true
i10 : Iss2=stronglyStableIdeal I
o10 = ideal (e e , e e , e e e , e e , e e e , e e e )
              1 2   1 3   1 4 5   2 3   2 4 5   3 4 5
o10 : Ideal of E
i11 : Iss==Iss2
o11 = true
```

The ideal Iss is not a lex ideal in E. Indeed, the monomial e_1e_4 does not belong to Iss, but $e_1e_4 >_{\text{lex}} e_2e_3$. One can verify this by the function `isLexIdeal(ideal)`:

```
i12 : isLexIdeal Iss
o12 = false
```

Example 3.2. Letting E be an exterior algebra with n generators over a field K and $h = (h_0, h_1, \dots, h_n)$ be a sequence of nonnegative integers, we describe how one can verify if h is a Hilbert sequence.

The key tools in our algorithm are the functions `isHilbertSequence(list, exterior algebra)` and `lexIdeal(list,exterior algebra)`. The first function verifies if a list of nonnegative integers of length $n+1$ is a Hilbert function; the second one returns a lex ideal of E if and only if the list is a Hilbert sequence. In more detail, if $(h_0, h_1, \dots, h_n)$ is a Hilbert sequence, the lex ideal of E produced by the function `lexIdeal(`$\{h_0, \dots, h_n\}$`, E)` is the unique lex ideal I of E with $H_{E/I}(d) = h_d$ $(d = 0, \dots, n)$. The procedure for the computation of the required lex ideal is based on the constructive proof of Theorem 2.6 (see [Aramova et al. 1997, Theorem 4.1, (b) ⇒ (a)]).

We start with some examples of sequences which are not Hilbert sequences. The property is verified by using either `isHilbertSequence(list,exterior algebra)` or `lexIdeal(list,exterior algebra)`:

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra,
TangentCone

i1 : loadPackage "ExteriorIdeals"
i2 : E=QQ[e_1..e_5,SkewCommutative=>true]
i3 : isHilbertSequence({2,4,3,0,0,0},E)
o3 = false
i4 : isHilbertSequence({0,4,3,0,0,0},E)
o4 : false
i5 : lexIdeal({1,6,3,0,0,0,0},E)
stdio:24:1:(3): error: expected a Hilbert sequence
i6 : lexIdeal({1,5,10,10,5,1,0},E)
stdio:26:1:(3): error: expected a Hilbert sequence
```

Moreover, the next statements provide some examples of the lex ideal produced by a Hilbert sequence. The length of the sequence can be at most $n+1$; if the length is less than $n+1$, then the sequence will be completed by adding zeros on the right.

```
i6 : lexIdeal({1,4,3,0,0,0},E)
o6 = ideal (e , e e , e e , e e , e e e )
             1   2 3   2 4   2 5   3 4 5
o6 : Ideal of E
i7 : lexIdeal({1,4,4},E)
o7 = ideal (e , e e , e e , e e e )
             1   2 3   2 4   3 4 5
o7 : Ideal of E
i8 : lexIdeal({1,5,7,4,0,0},E)
o9 = ideal (e e , e e , e e , e e e e )
             1 2   1 3   1 4   2 3 4 5
o9 : Ideal of E
```

The function `lexIdeal(list,exterior algebra)`, defined above, also plays a relevant role in the next algorithm.

Example 3.3. Given an exterior algebra E and a graded ideal I in E, we illustrate how to obtain the unique lex ideal I^{lex} with the same Hilbert function as I. In more detail, we describe two different methods for computing such a lex ideal.

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra,
TangentCone
```

```
i1 : loadPackage "ExteriorIdeals";
i2 : E=QQ[e_1..e_5,SkewCommutative=>true]
i3 : I=ideal {e_1*e_2*e_3+e_3*e_4*e_5,e_1*e_3+e_4*e_5,e_2*e_3*e_4}
o3 = ideal (e e e  + e e e , e e  + e e , e e e )
             1 2 3    3 4 5   1 3    4 5   2 3 4
o3 : Ideal of E
i4 : hilbSeq=hilbertSequence(I)
o4 = {1, 5, 9, 3, 0, 0}
o4 : List
```

A first way for computing the lex ideal we are looking for is to use the function `lexIdeal(list,exterior algebra)`:

```
i5 : Ilex1=lexIdeal(hilbSeq,E)
o5 = ideal (e e , e e e , e e e , e e e , e e e)
             1 2   1 3 4   1 3 5   1 4 5   2 3 4
o5 : Ideal of E
i6 : isLexIdeal Ilex1
o6 = true
i7 : hilbertSequence(Ilex1)
o7 = {1, 5, 9, 3, 0, 0}
o7 : List
```

and a second one is *via* the new function `lexIdeal(ideal)`, which returns directly the required lex ideal:

```
i8 : Ilex2=lexIdeal(I)
o8 = ideal (e e , e e e , e e e , e e e , e e e)
             1 2   1 3 4   1 3 5   1 4 5   2 3 4
o8 : Ideal of E
i9 : hilbertSequence(Ilex2)
o9 = {1, 5, 9, 3, 0, 0}
o9 : List
```

Finally, our last example is related to the algorithm for the computation of Hilbert sequences.

Example 3.4. Given an exterior algebra E, we illustrate how to get all the Hilbert sequences of quotients of E.

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra,
TangentCone
i1 : loadPackage "ExteriorIdeals";
i2 : E=QQ[e_1..e_4,SkewCommutative=>true]
i3 : hilbSeqs=allHilbertSequences(E)
o3 = {{1, 4, 6, 4, 1}, {1, 4, 6, 4, 0}, {1, 4, 6, 3, 0}, {1, 4, 6,2, 0},
     ------------------------------------------------------------------
     {1, 4, 6, 1, 0}, {1, 4, 6,0, 0}, {1, 4, 5, 2, 0}, {1, 4, 5, 1, 0},
     ------------------------------------------------------------------
     {1, 4, 5, 0, 0}, {1, 4, 4, 1, 0}, {1, 4, 4, 0, 0}, {1,4, 3, 1, 0},
     ------------------------------------------------------------------
     {1, 4, 3, 0, 0}, {1, 4, 2, 0, 0}, {1, 4, 1, 0, 0}, {1, 4, 0, 0, 0},
     ------------------------------------------------------------------
     {1, 3, 3, 1,0}, {1, 3, 3, 0, 0}, {1, 3, 2, 0, 0}, {1, 3, 1, 0, 0},
     ------------------------------------------------------------------
     {1, 3, 0, 0, 0}, {1, 2, 1, 0, 0}, {1, 2,0, 0, 0}, {1, 1, 0, 0, 0},
     ------------------------------------------------------------------
     {1, 0, 0, 0, 0}}
o3 : List
i4 : transpose matrix hilbSeqs
o4 = | 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 |
     | 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 3 3 3 3 3 2 2 1 0 |
     | 6 6 6 6 6 6 5 5 5 4 4 3 3 2 1 0 3 3 2 1 0 1 0 0 0 |
     | 4 4 3 2 1 0 2 1 0 1 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 |
     | 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
             5        25
o4 : Matrix ZZ  <--- ZZ
```

Note that the method `allHilbertSequences` returns an object of type `List`; for a more compact view it could be displayed as a matrix.

4. CONCLUSIONS AND PERSPECTIVES. The algorithms described in the examples above are part of a *Macaulay2* package `ExteriorIdeals.m2`, which has been tested with *Macaulay2* version 1.10. We are confident that this package may prove useful for further applications. Indeed, to the best of our knowledge, it seems that no packages for manipulating monomial ideals in an exterior algebra have been implemented, though functions for computing monomial ideals in a polynomial ring are available in many computer algebra systems (for instance, [CoCoA], [Macaulay2] and [Singular]).

We believe it would be nice to implement such a package for monomial modules over an exterior algebra. This task is currently under investigation by the authors.

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SUPPLEMENT. The online supplement contains version 1.0 of `ExteriorIdeals.m2`.

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