Computing quasidegrees of A-graded modules

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ABSTRACT: We describe the main functions of the Macaulay2 package Quasidegrees.m2. The purpose of this package is to compute the quasidegree set of a finitely generated $\mathbb{Z}^d$-graded module presented as the cokernel of a monomial matrix. We provide examples with motivation coming from $A$-hypergeometric systems.

1. INTRODUCTION. Throughout, $R = k[x_1, \ldots, x_n]$ is a $\mathbb{Z}^d$-graded polynomial ring over a field $k$ and $m = (x_1, \ldots, x_n)$ denotes the homogeneous maximal ideal in $R$. Let $M = \bigoplus_{\beta \in \mathbb{Z}^d} M_{\beta}$ be a $\mathbb{Z}^d$-graded $R$-module. The true degree set of $M$ is

$$tdeg(M) = \{\beta \in \mathbb{Z}^d \mid M_{\beta} \neq 0\}.$$ 

The quasidegree set of $M$, denoted $qdeg(M)$, is the Zariski closure in $\mathbb{C}^d$ of $tdeg(M)$.

The purpose of the Macaulay2 package Quasidegrees.m2 (provided as an online supplement) is to compute the quasidegree set of a finitely generated $\mathbb{Z}^d$-graded module presented as the cokernel of a monomial matrix. By a monomial matrix, we mean a matrix where each entry is either zero or a monomial in $R$. The initial motivation for Quasidegrees.m2 was to compute the quasidegree sets of certain local cohomology modules supported at $m$ of $\mathbb{Z}^d$-graded $R$-modules, so there are some methods in the package specific to local cohomology. Recall that the $i$-th local cohomology module of $M$ with support at the ideal $I \subset R$ is the $i$-th right derived functor of the left exact $I$-torsion functor

$$\Gamma_I(M) = \{m \in M \mid I^tm = 0 \text{ for some } t \in \mathbb{N}\}$$

on the category of $R$-modules.

By the vanishing theorems of local cohomology [Eisenbud 1995], the quasidegree sets of the local cohomology modules supported at $m$ of $M$ can be seen as measuring how far the module is from being Cohen–Macaulay. From the $A$-hypergeometric systems point of view, the quasidegree set of the non-top local cohomology modules supported at $m$ of $R/I_A$, where $I_A$ is the toric ideal associated

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Quasidegrees.m2 version 1.0
to $A$ in $R$, determine the parameters $\beta$ where the $A$-hypergeometric system $H_A(\beta)$ has rank higher than expected (see Section 3).

2. QUASIDEGREES. The main function of Quasidegrees.m2 is quasidegrees, which computes the quasidegree set of a module that is presented by a monomial matrix.

We use the idea of standard pairs of monomial ideals to compute the quasidegree set of a $\mathbb{Z}^d$-graded $R$-module. Given a monomial $x^u$ and a subset $Z \subset \{x_1, \ldots, x_n\}$, the pair $(x^u, Z)$ indexes the monomials $x^u \cdot x^v$ where $\text{supp}(x^v) \subset Z$. A standard pair of a monomial ideal $I \subset R$ is a pair $(x^u, Z)$ satisfying:

1. $\text{supp}(x^u) \cap Z = \emptyset$.
2. All of the monomials indexed by $(x^u, Z)$ are outside of $I$.
3. $(x^u, Z)$ is maximal in the sense that $(x^u, Z) \not\subseteq (x^v, Y)$ for any other pair $(x^v, Y)$ satisfying the first two conditions.

To compute the quasidegree set of $M$ we first find a monomial presentation of $M$ so that $M$ is the cokernel of a monomial matrix $\phi$. We then compute the standard pairs of the ideals generated by the rows of $\phi$ and to each standard pair we associate the degrees of the corresponding variables. Algorithm 1 below is implemented in Quasidegrees.m2. The input is an $R$-module presented by a monomial matrix $\phi : R^s \to R^t$.

As in Macaulay2, we write the degree of the $k$-th factor of $R^t$ next to the $k$-th row of the matrix $\phi$.

In the Macaulay2 implementation of the algorithm, we represent the output as a list of pairs $(u, Z)$ with $u \in \mathbb{Q}^d$ and $Z \subset \mathbb{Q}^d$, where the pair $(u, Z)$ represents the plane

$$u + \sum_{v \in Z} C \cdot v.$$

**Input:** $R$-module $M$ presented by monomial matrix $\phi = \alpha_i[x_{j_i}^u x_{j_k}^v] : R^s \to R^t$

**Output:** qdeg($M$)

$Q = \emptyset$

for $1 \leq k \leq t$

$SP = \{\text{standard pairs of } (c_{k,1} x_{j_{k,1}}^u, c_{k,2} x_{j_{k,2}}^u, \ldots, c_{k,s} x_{j_{k,s}}^u)\}$

$Q = Q \cup \{\text{deg}(x^u) + \alpha_k + \sum_{x_i \in F} C \cdot \text{deg}(x_i) \mid (x^u, Z) \in SP\}$

end for

return $Q$

**Algorithm 1.** Compute qdeg($M$).
The union of these planes over all such pairs in the output is the quasidegree set of \( M \).

The following is an example of Quasidegrees.m2 computing the quasidegree set of an \( R \)-module:

\[
i1 : R=\mathbb{Q}[x,y,z] \\
o1 = R \\
o1 : PolynomialRing \\
i2 : I=\text{ideal}(x*y,y*z) \\
o2 = \text{ideal} (x*y, y*z) \\
o2 : Ideal of R \\
i3 : M=R^1/I \\
o3 = \text{cokernel} | xy yz | \\
o3 : R-module, quotient of R \\
i4 : Q = \text{quasidegrees } M \\
o4 = \{\{0, \{1\}\}, \{0, \{1,1\}\}\}
\]

The above example displays a caveat of quasidegrees in that there may be some redundancies in the output. By a redundancy, we mean when one plane in the output is contained in another. The redundancy above is clear:

\[
\text{qdeg}(\mathbb{k}[x,y,z]/\langle xy,yz \rangle) = \mathbb{C} = \{z_1 + z_2 \in \mathbb{C} \mid z_1, z_2 \in \mathbb{C}\}.
\]

The function \text{removeRedundancy} gets rid of redundancies in the list of planes:

\[
i5 : \text{removeRedundancy } Q \\
o5 = \{\{0, \{1\}, \{1\}\}\}
\]

3. QUASIDEGREES AND HYPERGEOMETRIC SYSTEMS. In this section, we discuss the motivation for Quasidegrees.m2 and the methods therein which aid us in our studies. Let \( A = [a_1 \ a_2 \ \cdots \ a_n] \) be an integer \((d \times n)\)-matrix with \( \mathbb{Z} A = \mathbb{Z}^d \) and such that the cone over its columns is pointed. There is a natural \( \mathbb{Z}^d \)-grading of \( R \) by the columns of \( A \) given by \( \text{deg}(x_j) = a_j \), the \( j \)-th column of \( A \). A module that is homogeneous with respect to this grading is said to be \( A \)-graded. By the assumptions on \( A \), \( R \) is positively graded by \( A \), that is, the only polynomials of degree 0 are the constants. Given such a matrix \( A \) and a polynomial ring \( R \) in \( n \) variables, the method \text{toGradedRing} gives \( R \) an \( A \)-grading. For example, let

\[
A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -2 \end{pmatrix}.
\]
We make the $A$-graded polynomial ring $\mathbb{Q}[x_1, x_2, x_3, x_4, x_5]$:

$$i6 : A = \text{matrix}\{\{1,1,1,1,1\}, \{0,0,1,1,0\}, \{0,1,1,0,-2\}\}$$

$$o6 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & -2
\end{pmatrix}
$$

$$o6 : \text{Matrix} \ \mathbb{Z} \ \text{<--} \ \mathbb{Z}$$

$$i7 : R = \mathbb{Q}[x_1..x_5]$$

$$o7 = R$$

$$o7 : \text{PolynomialRing}$$

$$i8 : R = \text{toGradedRing}(A, R)$$

$$o8 = R$$

$$o8 : \text{PolynomialRing}$$

$$i9 : \text{describe} \ R$$

$$o9 = \mathbb{Q}[x_1, x_2, x_3, x_4, x_5, \text{Degrees} => \{\{1\}, \{1\}, \{1\}, \{1\}, \{1\}\},$$

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

$$\text{Heft} => \{1, 2:0\}, \text{MonomialOrder} => \{\text{MonomialSize} => 32\}, \text{DegreeRank} => 3$$

$$\{\text{GRevLex} => \{5:1\}\}$$

$$\{\text{Position} => \text{Up}\}$$

The toric ideal associated to $A$ in $R$ is the binomial ideal

$$I_A(x^u - x^v : Au = Av).$$

The method $\text{toricIdeal}$ computes the toric ideal associated to $A$ in the ring $R$. We continue with the $A$ and $R$ from the above example and compute the toric ideal $I_A$ associated to $A$ in $R$:

$$i10 : I = \text{toricIdeal}(A, R)$$

$$o10 = \text{ideal}(x^u - x^v : Au = Av)$$

$$o10 : \text{Ideal of} \ R$$

We now introduce $A$-hypergeometric systems. Given a matrix $A \in \mathbb{Z}^{d \times n}$ as above and a $\beta \in \mathbb{C}^d$, the $A$-hypergeometric system with parameter $\beta \in \mathbb{C}^d$ [Saito et al. 2000], denoted $H_A(\beta)$, is the system of partial differential equations:

$$\frac{\partial |u|}{\partial x^v} \phi(x) = \frac{\partial |u|}{\partial x^u} \phi(x) \quad \text{for all } u, v, Au = Av,$n

$$\sum_{j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_j} \phi(x) = \beta_i \phi(x), \quad \text{for } i = 1, \ldots, d.$$
Such systems are sometimes called \textit{GKZ-hypergeometric systems}. The function \texttt{gkz} in the Macaulay2 package \texttt{Dmodules} computes this system as an ideal in the Weyl algebra. The \textit{rank} of \( H_A(\beta) \) is

\[
\text{rank}(H_A(\beta)) = \dim \{ \text{germs of holomorphic solutions of } H_A(\beta) \text{ near a generic nonsingular point} \}.
\]

The function \texttt{holonomicRank} in \texttt{Dmodules} computes the rank of an \( A \)-hypergeometric system. In general, rank is not a constant function of \( \beta \). Denote \( \text{vol}(A) \) to be \( d! \) times the Euclidean volume of \( \text{conv}(A \cup \{0\}) \), the convex hull of the columns of \( A \) and the origin in \( \mathbb{R}^d \). The following theorem gives the parameters \( \beta \) for which \( \text{rank}(H_A(\beta)) > \text{vol}(A) \):

\textbf{Theorem 3.1} [Matusevich et al. 2005]. \textit{Let} \( H_A(\beta) \) \textit{be an} \( A \)-\textit{hypergeometric system with parameter} \( \beta \). \textit{If} \( \beta \in \text{qdeg}(\bigoplus_{i=0}^{d-1} H^i_m(R/I_A)) \) \textit{then} \( \text{rank}(H_A(\beta)) > \text{vol}(A) \). \textit{Otherwise,} \( \text{rank}(H_A(\beta)) = \text{vol}(A) \).

Since Theorem 3.1 was the initial motivation for Quasidegrees.m2, the package has a method \texttt{quasidegreesLocalCohomology} (abbreviated \texttt{qlc}) to compute the quasidegree set of the local cohomology modules \( H^i_m(R/I_A) \). If the input is an integer \( i \) and the \( R \)-module \( R/I_A \), then the method computes \( \text{qdeg}(H^i_m(R/I_A)) \). If the input is only the module \( R/I_A \), the method computes the quasidegree set in Theorem 3.1.

We use graded local duality to compute the local cohomology modules of a finitely generated \( A \)-graded \( R \)-module supported at the maximal ideal \( m \):

\textbf{Theorem 3.2} (graded local duality [Bruns and Herzog 1993; Miller 2002]). \textit{Given an} \( A \)-\textit{graded} \( R \)-\textit{module} \( M \), \textit{there is an} \( A \)-\textit{graded vector space isomorphism}

\[
\text{Ext}^{n-i}_R(M, R)_\alpha \cong \text{Hom}_k(H^i_m(M)_{-\alpha - \varepsilon_A}, k),
\]

\textit{where} \( m = \langle x_1, \ldots, x_n \rangle \) \textit{and} \( \varepsilon_A = \sum_{j=1}^{n} a_j \).

The algorithm implemented for \texttt{quasidegreesLocalCohomology} is essentially Algorithm 1 applied to the Ext-modules of \( M \) with the additional twist of \( \varepsilon_A \) coming from local duality. For our purposes, we exploit the fact that the higher syzygies of \( R/I_A \) are generated by monomials in \( R^m \) (see [Miller and Sturmfels 2005], Chapter 9).

Continuing our running example, we use \texttt{quasidegreesLocalCohomology} to compute the quasidegree set of \( \bigoplus_{i=0}^{d-1} H^i_m(R/I_A) \):

\begin{verbatim}
i11 : M=R^1/I
o11 = cokernel | x_1x_3-x_2x_4 x_1x_4^2-x_3^2x_5
x_1^2x_4-x_2x_3x_5 x_1^3-x_2^2x_5 |
\end{verbatim}

\begin{verbatim}
o11 : R-module, quotient of R
\end{verbatim}
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\[ i_{12} : \text{quasidegreesLocalCohomology M} \]
\[ o_{12} = \{\{0|1\}, \{1|0\}\} \]
\[ |0| |0| \]
\[ |1| |-2| \]
\[ o_{12} : \text{List} \]

Thus

\[ \text{qdeg} \left( \bigoplus_{i=0}^{d-1} H_{m}^{i} (R/I_{A}) \right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + C \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}. \] (1)

As a check, we use the methods gkz and holonomicRank from the package Dmodules to compute rank\((H_{A}(0))\) and rank\((H_{A}(\beta))\) for two different \(\beta\) in (1) and demonstrate a rank jump:

\[ i_{13} : \text{holonomicRank gkz(A,\{0,0,0\})} \quad \text{-- vol A in this case} \]
\[ o_{13} = 4 \]
\[ i_{14} : \text{holonomicRank gkz(A,\{0,0,1\})} \]
\[ o_{14} = 5 \]
\[ i_{15} : \text{holonomicRank gkz(A,\{3/2,0,-2\})} \]
\[ o_{15} = 5 \]

SUPPLEMENT. The online supplement contains version 1.0 of Quasidegrees.m2.

REFERENCES.


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