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ABSTRACT: We introduce the package `arrangements` for the software CoCoA. This package provides a data structure and the necessary methods for working with hyperplane arrangements. In particular, the package implements methods to generate several known families of arrangements, to perform operations on them, and to calculate various invariants associated to them.

1. INTRODUCTION. An arrangement of hyperplanes is a finite collection of codimension one affine subspaces in a finite dimensional vector space. Associated to these spaces, there is a plethora of algebraic, combinatorial and topological invariants. Arrangements are easily defined but they lead to deep and beautiful results connecting various area of mathematics. We refer the reader to [Orlik and Terao 1992] for a comprehensive account of this subject.

One of the main goals in the study of hyperplane arrangements is to decide whether a given invariant is combinatorially determined, and, if so, to express it explicitly in terms of the intersection lattice of the arrangement.

We describe the new package `arrangements` for CoCoA [CoCoA; CoCoALib; Abbott and Bigatti 2018]). This package computes several combinatorial invariants (like the lattice of intersections and its flats, the Poincaré, the characteristic and the Tutte polynomials) and algebraic ones (like the Orlik–Terao and the Solomon–Terao ideals) of hyperplane arrangements. Moreover, several functions for the class of free hyperplane arrangements are implemented. In addition, this package also allows computations with multiarrangements. Finally, several known families of arrangements (like classic reflection arrangements, Shi arrangements, Catalan arrangements, Shi–Catalan arrangements, graphical arrangements and signed graphical ones) can be easily constructed: in CoCoA type `?ArrFamily` for the complete list. Some of the functions that compute combinatorial invariants rely on the CoCoA package `posets`, which we implemented for this purpose.

We introduce the package `arrangements` via several examples. Specifically, in Section 2 we first recall the definitions of various combinatorial invariants of

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`arrangements` version 1.0 for CoCoA-5.2.4

a given arrangement and then describe how to compute them. In Section 3, we describe how to work with free hyperplane arrangements, and in Section 4 how to define the Orlik–Terao and Solomon–Terao ideals. Finally, in Section 5 we describe the class of multiarrangements with particular emphasis on the free ones.

This package is part of the official release CoCoA-5.2.4, and has been used during the tutorials of the Hokkaido Summer Institute 2018 course “Hyperplane arrangements and computations with CoCoA” held at Hokkaido University from the 13th to the 17th of August 2018.

2. Combinatorics of arrangements. Let V be a vector space of dimension l over a field K. Fix a system of coordinates $(x_1, \dots, x_l)$ of V^*. We denote by $S = S(V^*) = K[x_1, \dots, x_l]$ the symmetric algebra. A finite set of affine hyperplanes $\mathcal{A} = \{H_1, \dots, H_n\}$ in V is called a *hyperplane arrangement*.

For each hyperplane H_i we fix a polynomial $\alpha_i \in S$ such that $H_i = \alpha_i^{-1}(0)$, and let

$$Q(\mathcal{A}) = \prod_{i=1}^{n} \alpha_i .$$

An arrangement $\mathcal{A}$ is called *central* if each H_i contains the origin of V. In this case, the polynomial $\alpha_i \in S$ is linear homogeneous, and hence $Q(\mathcal{A})$ is a homogeneous polynomial of degree n.

The operation of *coning* allows one to transform any arrangement $\mathcal{A}$ of V with n hyperplanes into a central arrangement $c\mathcal{A}$ with $n+1$ hyperplanes in a vector space of dimension $l+1$; see [Orlik and Terao 1992].

Notice that in CoCoA to compute the cone of an arrangement $\mathcal{A}$, the homogenizing variable needs to be already present in the ring in which the equation of $\mathcal{A}$ is defined. For example, we can construct the cone of the *Shi arrangement of type* A as follows:

```
/**/ use S ::= QQ[x, y, z, w];
/**/ A := ArrShiA(S, 3); A;
[x-y, x-z, y-z, x-y-1, x-z-1, y-z-1]
/**/ ArrCone(A, w);
[x-y, x-z, y-z, x-y-w, x-z-w, y-z-w, w]
```

Let $L(\mathcal{A}) = \left\{\bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\right\}$ be the *intersection poset* of $\mathcal{A}$. Define a partial order on $L(\mathcal{A})$ by $X \leq Y$ if and only if $Y \subseteq X$, for all $X, Y \in L(\mathcal{A})$. Note that this is the reverse inclusion. In addition, if $\mathcal{A}$ is central, $L(\mathcal{A})$ is a geometric lattice. The elements of $L(\mathcal{A})$ are called *flats* of $\mathcal{A}$. Define a rank function on $L(\mathcal{A})$ by $\mathrm{rk}(X) = \mathrm{codim}(X)$. The poset $L(\mathcal{A})$ plays a fundamental role in the study of hyperplane arrangements; in fact it determines the combinatorics of the arrangement.

We can compute the flats in the intersection lattice of the *reflection arrangement of type D* in the following way:

```
/**/ use S ::= QQ[x, y, z];
/**/ A := ArrTypeD(S, 3); A;
[x-y,  x+y,  x-z,  x+z,  y-z,  y+z]
/**/ ArrFlats(A);
[[ideal(0)],
 [ideal(x-y), ideal(x+y), ideal(x-z), ideal(x+z),
  ideal(y-z), ideal(y+z)],
 [ideal(x, y), ideal(x-z, y-z), ideal(x+z, y+z),
  ideal(x-z, y+z),  ideal(x+z, y-z), ideal(x, z),
  ideal(y, z)],
 [ideal(x, y, z)]]
```

In the rest of the section, we will introduce the Poincaré polynomial, the characteristic polynomial and the Tutte polynomial, and the restriction of an arrangement $\mathcal{A}$. Notice that, contrary to the operation of coning, in CoCoA these operations introduce new variables that do not need to be already present in the ring in which the equation of $\mathcal{A}$ is defined.

Let $\mu : L(\mathcal{A}) \to \mathbb{Z}$ be the *Möbius function* of $L(\mathcal{A})$ defined by

$$\mu(X) = \begin{cases} 1 & \text{for } X = V, \\ -\sum_{Y<X} \mu(Y) & \text{if } X > V. \end{cases}$$

The *Poincaré polynomial* of $\mathcal{A}$ is defined by

$$\pi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\mathrm{rk}(X)},$$

and it satisfies the formula

$$\pi(c\mathcal{A}, t) = (t + 1)\pi(\mathcal{A}, t).$$

We now verify the previous result for the *Shi arrangement of type A*.

```
/**/ use S ::= QQ[x, y, z, w];
/**/ A := ArrShiA(S, 3);
/**/ pi_A := ArrPoincarePoly(A); pi_A;
9*t^2 +6*t +1
/**/ cA := ArrCone(A, w);
/**/ pi_cA := ArrPoincarePoly(cA); pi_cA;
9*t^3+15*t^2+7*t+1
/**/ t := indet(RingOf(pi_A), 1);
/**/ pi_cA = (1+t)*pi_A;
```

```
true
```

For any flat $X \in L(\mathcal{A})$ define the *localization* of $\mathcal{A}$ to X as the subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}.$$

Similarly, define the *restriction* of $\mathcal{A}$ to X as the arrangement $\mathcal{A}^X$ in X,

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \backslash \mathcal{A}_X \text{ and } X \cap H \neq \varnothing\}.$$

The *characteristic polynomial* of $\mathcal{A}$ is

$$\chi(\mathcal{A}, t) = t^l \pi(\mathcal{A}, -t^{-1}) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim(X)}.$$

The characteristic polynomial is characterized by the recursive relation

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A} \backslash H, t) - \chi(\mathcal{A}^H, t),$$

for any $H \in \mathcal{A}$. See [Orlik and Terao 1992, Corollary 2.57] for more details.

We verify the previous result for $\mathcal{A}^{[-1,2]}$ the *Shi–Catalan arrangement of type* A.

```
/**/ use S ::= QQ[x, y, z];
/**/ A := ArrShiCatalanA(S, 3, [-1, 2]); A;
[x-y, x-z, y-z, x-y-1, x-z-1, y-z-1, x-y+1, x-y+2, x-z+1,
 x-z+2, y-z+1, y-z+2]
/**/ ArrLocalization(A, [x-y, x-z]);
[x-y,  x-z,  y-z]
/**/ A_minusH := ArrDeletion(A, x-y-1); A_minusH;
[x-y, x-z, y-z, x-z-1, y-z-1, x-y+1, x-y+2, x-z+1, x-z+2,
 y-z+1, y-z+2]
/**/ A_restrH := ArrRestriction(A, x-y-1); A_restrH;
[y[1]-y[2]+1, y[1]-y[2], y[1]-y[2]-1, y[1]-y[2]+2,
 y[1]-y[2]+3]
/**/ ArrCharPoly(A) = ArrCharPoly(A_minusH) -
 ArrCharPoly(A_restrH);
true
```

For $i = 0, \ldots, l$ we define the *i-th Betti number* $b_i(\mathcal{A})$ to be the coefficients of $\chi(\mathcal{A}, t)$ as in the formula

$$\chi(\mathcal{A}, t) = \sum_{i=0}^{l} (-1)^i b_i(\mathcal{A}) t^{l-i}.$$

The following statement is the combination of three different results from [Crapo and Rota 1970], [Orlik and Solomon 1980] and [Zaslavsky 1975], and it describes

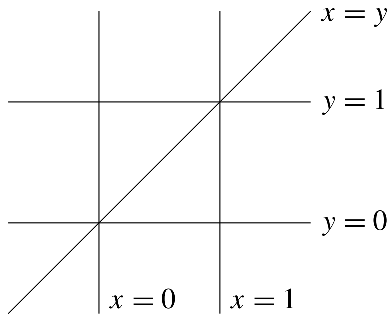

Figure 1. A line arrangement in $\mathbb{R}^2$.

the connection between the characteristic polynomial in combinatorics, and geometrical and topological aspects of arrangements.

Theorem 2.1. (1) *If $\mathcal{A}$ is an arrangement in $\mathbb{F}_q^l$ (vector space over a finite field $\mathbb{F}_q$), then $|\mathbb{F}_q^l \setminus \bigcup_{H\in\mathcal{A}} H| = \chi(\mathcal{A}, q)$.*

(2) *If $\mathcal{A}$ is an arrangement in $\mathbb{C}^l$, then the topological i-th Betti number of the complement of $\mathcal{A}$ is $b_i(\mathbb{C}^l \setminus \bigcup_{H\in\mathcal{A}} H) = b_i(\mathcal{A})$.*

(3) *If $\mathcal{A}$ is an arrangement in $\mathbb{R}^l$, then $|\chi(\mathcal{A}, -1)|$ is the number of chambers and $|\chi(\mathcal{A}, 1)|$ is the number of bounded chambers.*

Using the previous statements, we can compute the Betti numbers, the number of chambers and the number of bounded chambers of the arrangement in Figure 1.

```
/**/ use S ::= QQ[x, y];
/**/ A := [x, x-1, y, y-1, x-y];
/**/ ArrBettiNumbers(A);
[1, 5, 6]
/**/ NumChambers(A);
12
/**/ NumBChambers(A);
2
```

Associated to each hyperplane arrangement, we can naturally define a third polynomial. The *Tutte polynomial* of $\mathcal{A}$ is

$$T_{\mathcal{A}}(x, y) = \sum_{\substack{\mathcal{B}\subseteq\mathcal{A} \\ \mathcal{B}\text{ central}}} (x-1)^{\mathrm{rk}(\mathcal{A})-\mathrm{rk}(\mathcal{B})}(y-1)^{|\mathcal{B}|-\mathrm{rk}(\mathcal{B})}.$$

As shown in [Ardila 2007], it turns out that the Tutte and the characteristic polynomials are related by

$$\chi(\mathcal{A}, t) = (-1)^{\mathrm{rk}(\mathcal{A})} t^{l-\mathrm{rk}(\mathcal{A})} T_{\mathcal{A}}(1-t, 0).$$

We verify the previous result for the *reflection arrangement of type* D. Notice that here, since the Tutte and the characteristic polynomials live in different rings, we need to construct a ring homomorphism, with the command `PolyAlgebraHom`, to check the required equality.

```
/**/ use S ::= QQ[x, y, z];
/**/ A := ArrTypeD(S, 3);
/**/ Tutte_A := ArrTuttePoly(A); Tutte_A;
t[1]^3+t[2]^3+3*t[1]^2+4*t[1]*t[2]+3*t[2]^2+2*t[1]+2*t[2]
/**/ char_A := ArrCharPoly(A); char_A;
t^3-6*t^2+11*t-6
/**/ QQt1t2 := RingOf(Tutte_A); QQt := RingOf(char_A);
/**/ t := indet(QQt, 1);
/**/ phi := PolyAlgebraHom(QQt1t2, QQt, [1-t, 0]);
/**/ char_A = (-1)^3*t^(dim(S)-3)*phi(Tutte_A);
true
```

3. Free hyperplane arrangements.

In the theory of hyperplane arrangements, the freeness of an arrangement is a very important algebraic property. In fact, freeness implies several interesting geometric and combinatorial properties of the arrangement itself. See, for example, [Terao 1980; Yoshinaga 2014; Abe 2016; Bigatti et al. 2019; Palezzato and Torielli 2018].

We use

$$\mathrm{Der}_V = \left\{ \sum_{i=1}^{l} f_i \partial_{x_i} \mid f_i \in S \right\}$$

to denote the S-module of *polynomial vector fields* on V (or S-derivations). Let $\delta = \sum_{i=1}^{l} f_i \partial_{x_i} \in \mathrm{Der}_V$. If $f_1, \ldots, f_l$ are homogeneous polynomials of degree d in S, then δ is said to be *homogeneous of polynomial degree* d, and we write $\mathrm{pdeg}(\delta) = d$.

For any central arrangement $\mathcal{A}$ we define the *module of vector fields logarithmic tangent* to $\mathcal{A}$ (logarithmic vector fields) by

$$D(\mathcal{A}) = \{\delta \in \mathrm{Der}_V \mid \delta(\alpha_i) \in \langle \alpha_i \rangle S, \text{ for all } i\}.$$

The module $D(\mathcal{A})$ is a graded S-module and we have

$$D(\mathcal{A}) = \{\delta \in \mathrm{Der}_V \mid \delta(Q(\mathcal{A})) \in \langle Q(\mathcal{A}) \rangle S\}.$$

Definition 3.1. We say a central arrangement $\mathcal{A}$ is *free with exponents* $(e_1, \ldots, e_l) \in \mathbb{N}^l$ if and only if $D(\mathcal{A})$ is a free S-module and there exists a basis $\delta_1, \ldots, \delta_l \in D(\mathcal{A})$ such that $\mathrm{pdeg}(\delta_i) = e_i$, or equivalently $D(\mathcal{A}) \cong \bigoplus_{i=1}^{l} S(-e_i)$.

Let $\delta_1, \dots, \delta_l \in D(\mathcal{A})$. Then $\det(\delta_i(x_j))$ is divisible by $Q(\mathcal{A})$. One of the most famous characterizations of freeness is due to Saito [1980] and it uses the determinant of the coefficient matrix of $\delta_1, \dots, \delta_l$.

Theorem 3.2 (Saito's criterion). *Let $\delta_1, \dots, \delta_l \in D(\mathcal{A})$. Then the following facts are equivalent*:

(1) *$D(\mathcal{A})$ is free with basis $\delta_1, \dots, \delta_l$, i.e., $D(\mathcal{A}) = S \cdot \delta_1 \oplus \cdots \oplus S \cdot \delta_l$.*

(2) $\det(\delta_i(x_j)) = c\,Q(\mathcal{A})$, *where $c \in K \setminus \{0\}$.*

(3) *$\delta_1, \dots, \delta_l$ are linearly independent over S and $\sum_{i=1}^{l} \operatorname{pdeg}(\delta_i) = n$.*

Given a simple graph G, we can define the *graphical arrangement* $\mathcal{A}(G)$; see [Orlik and Terao 1992]. Stanley [2007], showed that $\mathcal{A}(G)$ is free if and only if G is a chordal graph. See also [Suyama and Tsujie 2019] and [Suyama et al. 2019] for more general results.

We verify this result for a given graphical arrangement.

```
/**/ use S ::= QQ[x, y, z, w];
/**/ G := [[1, 2], [1, 3], [1, 4], [2, 4], [3, 4]];
/**/ A := ArrGraphical(S, G);
/**/ ArrDerModule(A);
matrix( /*RingWithID(18935, "QQ[x, y, z, w]")*/
 [[1, 0, 0, 0],
  [1, x-y, 0, 0],
  [1, x-z, x*z-z^2-x*w+z*w, x*y-y*z-x*w+z*w],
  [1, x-w, 0, x*y-x*w-y*w+w^2]])
/**/ IsArrFree(A);
true
/**/ ArrExponents(A);
[0, 1, 2, 2]
/**/ B := ArrDeletion(A, x-w);
/**/ IsArrFree(B);
false
```

4. ALGEBRAS. Orlik and Terao [1994] introduced a commutative analogue of the Orlik–Solomon algebra in order to answer a question of Aomoto related to cohomology groups of a certain "twisted" de Rham chain complex. The crucial difference between the Orlik–Solomon algebra and Orlik–Terao algebra is not just the difference between the exterior algebra and symmetric algebra, but rather the fact that the Orlik–Terao algebra actually captures the "coefficients" of the dependencies among the hyperplanes.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central arrangement in V and $\Lambda \subseteq \{1, \ldots, n\}$. If $\operatorname{codim}\bigl(\bigcap_{i\in\Lambda} H_i\bigr) < |\Lambda|$, then we say that Λ is *dependent*. If Λ is dependent, then there exist $c_i \in K$ such that

$$\sum_{i\in\Lambda} c_i \alpha_i = 0.$$

Definition 4.1. Let $R = K[y_1, \ldots, y_n]$. For each dependent set $\Lambda = \{i_1, \ldots, i_k\}$, let $r_\Lambda = \sum_{j=1}^{k} c_{i_j} y_{i_j} \in R$. Define now

$$f_\Lambda = \partial(r_\Lambda) = \sum_{j=1}^{k} c_{i_j} (y_{i_1} \cdots \hat{y}_{i_j} \cdots y_{i_k}),$$

where $\hat{y}_{i_j}$ means that the variable y_{i_j} is omitted, and let I be the ideal of R generated by the f_Λ. This ideal is called the *Orlik–Terao ideal* of $\mathcal{A}$. The *Orlik–Terao algebra* $\mathrm{OT}(\mathcal{A})$ is the quotient R/I. The *Artinian Orlik–Terao algebra* $\mathrm{AOT}(\mathcal{A})$ is the quotient of $\mathrm{OT}(\mathcal{A})$ by the square of the variables.

These algebras and their Betti diagrams give us a lot of information on the given arrangement, for example about its *formality*; see for example [Schenck and Tohăneanu 2009].

We can construct the Orlik–Terao ideal, its Artinian version and the Betti diagram of the Orlik–Terao algebra of the *Braid arrangement* as follows:

```
/**/ use S ::= QQ[x, y, z];
/**/ A := ArrBraid(S, 3);
/**/ OT_A := OrlikTeraoIdeal(A); OT_A;
ideal(y[1]*y[2]-y[1]*y[3]+y[2]*y[3])
/**/ PrintBettiDiagram(RingOf(OT_A)/OT_A);
        0    1
---------------
  0:    1    -
  1:    -    1
---------------
Tot:    1    1
/**/ ArtinianOrlikTeraoIdeal(A);
ideal(y[1]*y[2]-y[1]*y[3]+y[2]*y[3],  y[1]^2,  y[2]^2,
     y[3]^2)
```

In [Abe et al. 2018], the authors introduced a new algebra associated to a central hyperplane arrangement. This algebra can be considered as a generalization of the coinvariant algebras in the setting of hyperplane arrangements and it contains the cohomology rings of regular nilpotent Hessenberg varieties.

Definition 4.2. Let $\mathcal{A}$ be a central arrangement in V and $f \in S$ a homogeneous polynomial. Then the ideal

$$\mathfrak{a}(\mathcal{A}, f) = \{\delta(f) \mid \delta \in D(\mathcal{A})\}$$

is called the *Solomon–Terao ideal* of $\mathcal{A}$ with respect to f. The *Solomon–Terao algebra* of $\mathcal{A}$ with respect to f is the quotient $\mathrm{ST}(\mathcal{A}, f) = S/\mathfrak{a}(\mathcal{A}, f)$.

We can construct the Solomon–Terao ideal of the *reflection arrangement of type D* with respect to f, the sum of the square of the variables, as follows:

```
/**/ use S ::= QQ[x, y, z];
/**/ A := ArrTypeD(S, 3);
/**/ f := x^2+y^2+z^2;
/**/ SolomonTeraoIdeal(A, f);
ideal(2*x^2+2*y^2+2*z^2, 6*x*y*z,
  2*x^2*y^2-2*y^4+2*x^2*z^2-2*z^4)
```

5. Multiarrangements of hyperplanes. A *multiarrangement* is a pair $(\mathcal{A}, m)$ of an arrangement $\mathcal{A}$ with a map $m : \mathcal{A} \to \mathbb{Z}_{\geq 0}$, called the *multiplicity*. An arrangement $\mathcal{A}$ can be identified with a multiarrangement with constant multiplicity $m \equiv 1$, and it is sometimes called a *simple arrangement*. Define $Q(\mathcal{A}, m) = \prod_{i=1}^{n} \alpha_i^{m(H_i)}$ and $|m| = \sum_{i=1}^{n} m(H_i)$. With this notation, the main object is the *module of vector fields logarithmic tangent* to $\mathcal{A}$ with multiplicity m (logarithmic vector field) defined by

$$D(\mathcal{A}, m) = \{\delta \in \mathrm{Der}_V \mid \delta(\alpha_i) \in \langle \alpha_i \rangle^{m(H_i)} S, \text{ for all } i\}.$$

The module $D(\mathcal{A}, m)$ is a graded S-module. In general, in contrast to the case of simple arrangements, $D(\mathcal{A}, m)$ does not coincide with

$$\{\delta \in \mathrm{Der}_V \mid \delta(Q(\mathcal{A})) \in \langle Q(\mathcal{A}, m)\rangle S\}.$$

Definition 5.1. Let $\mathcal{A}$ be a central arrangement. The multiarrangement $(\mathcal{A}, m)$ is said to be *free with exponents* $(e_1, \ldots, e_l)$ if and only if $D(\mathcal{A}, m)$ is a free S-module and there exists a basis $\delta_1, \ldots, \delta_l \in D(\mathcal{A}, m)$ such that $\mathrm{pdeg}(\delta_i) = e_i$, or equivalently $D(\mathcal{A}, m) \cong \bigoplus_{i=1}^{l} S(-e_i)$.

As for simple arrangements, if $\delta_1, \ldots, \delta_l \in D(\mathcal{A}, m)$, then $\det(\delta_i(x_j))$ is divisible by $Q(\mathcal{A}, m)$. Moreover, we can generalize Theorem 3.2; see [Ziegler 1989].

Theorem 5.2 (generalized Saito's criterion). *Let $\delta_1, \ldots, \delta_l \in D(\mathcal{A}, m)$. Then the following are equivalent*:

(1) *$D(\mathcal{A}, m)$ is free with basis $\delta_1, \ldots, \delta_l$, i.e., $D(\mathcal{A}, m) = S \cdot \delta_1 \oplus \cdots \oplus S \cdot \delta_l$.*

(2) $\det(\delta_i(x_j)) = c\,Q(\mathcal{A}, m)$, *where $c \in K \setminus \{0\}$.*

(3) *$\delta_1, \dots, \delta_l$ are linearly independent over S and* $\sum_{i=1}^{l} \operatorname{pdeg}(\delta_i) = |m|$.

Given a simple arrangement $\mathcal{A}$ and H one of its hyperplanes, we can naturally define *Ziegler's multirestriction* (see [Ziegler 1989]) as the multiarrangement $(\mathcal{A}^H, m^H)$, where the function $m^H : A^H \to \mathbb{Z}_{>0}$ is defined by

$$X \in \mathcal{A}^H \mapsto \#\{H' \in \mathcal{A} \mid H' \supset X\} - 1.$$

Theorem 5.3 [Ziegler 1989]. *Let $\mathcal{A}$ be a central arrangement. If $\mathcal{A}$ is free with exponents $(1, e_2, \dots, e_l)$, then $(\mathcal{A}^H, m^H)$ is free with exponents $(e_2, \dots, e_l)$, for any $H \in \mathcal{A}$.*

In general, the converse is false. However, we have the following:

Theorem 5.4 [Yoshinaga 2004]. *Assume $l \geq 4$. Let $\mathcal{A}$ be a central arrangement and $H \in \mathcal{A}$. Then $\mathcal{A}$ is free with exponents $(1, e_2, \dots, e_l)$ if and only if the following conditions are satisfied*:

(1) *$\mathcal{A}$ is locally free along H, i.e., $\mathcal{A}_X$ is free for any $X \in L(\mathcal{A})$ with $X \subset H$ and $X \neq \varnothing$.*

(2) *Ziegler's multirestriction $(\mathcal{A}^H, m^H)$ is a free multiarrangement with exponents $(e_2, \dots, e_l)$.*

We can construct Ziegler's multirestriction of a given arrangement and check its freeness as follows:

```
/**/ use S ::= QQ[x, y, z];
/**/ A := [x, y, z, x-y, x-y-z, x-y+2*z];
/**/ A_1 := MultiArrRestrictionZiegler(A, z); A_1;
[[y[1], 1], [y[2], 1], [y[1]-y[2], 3]]
/**/ IsMultiArrFree(A_1);
true
/**/ MultiArrDerModule(A_1);
matrix( /*RingWithID(18, "QQ[y[1], y[2]]")*/
 [[y[1]*y[2], y[1]^3],
  [y[1]*y[2], 3*y[1]^2*y[2]-3*y[1]*y[2]^2+y[2]^3]])
/**/ MultiArrExponents(A_1);
[2, 3]
/**/ ArrExponents(A);
[1, 2, 3]
```

SUPPLEMENT. The online supplement contains version 1.0 of `arrangements` for CoCoA-5.2.4.

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