

```

gap> g:= SymmetricGroup( 4 )
Sym( [ 1 .. 4 ] )
gap> tbl:= CharacterTable( g ); HasIrr( tbl );
15 : betti(t,Weights=>{1,0})
false
0 1 2 3 4 gap> tblmod2:= CharacterTable( tbl, 2 );
o5 = total: 1 4 13 14 4 BrauerTable( Sym( [ 1 .. 4 ] ), 2 )
0: 1 . . . .
1: . 2 2 4 2 gap> tblmod2 = CharacterTable( tbl, 2 );
2: . 2 5 6 . true
3: . . 4 . 2
4: . . . 4 . gap> tblmod2 = BrauerTable( tbl, 2 );
5: . . 2 . . true
o5 : BrauerTable( Sym( [ 1 .. 4 ] ), 2 )
16 : betti(t,Weights=>{0,1})
gap> libtbl:= CharacterTable( "M" );
o6 = total: 1 4 13 14 4 CharacterTable( "M" )
0: 1 . . . . gap> CharacterTableRegular( libtbl, 2 );
1: . 2 2 4 2 BrauerTable( "M", 2 )
2: . 2 5 6 . gap> BrauerTable( libtbl, 2 );
3: . . 4 . 2 fail
4: . . . 4 . ring r1 = 32003,(x,y,z),ds;
5: . . 2 . . gap> CharacterTable( "Symmetric", 4 ); int a,b,c,t=11,5,3,0;
o6 : BettiTally CharacterTable( "Sym(4)" ) poly f = x^a+y^b+z^(3*c)+x^(c+2)*y^(c-1)+x^
17 : t1 = betti(t,Weights=>{1,1}) x^(c-2)*y^c*(y^2+t*x)^2;
gap> ComputedBrauerTables( tbl ); [ , BrauerTable( Sym( [ 1 .. 4 ] ), 2 ), ]
o7 = total: 1 4 13 14 4 option(noprot);
0: 1 . . . . timer=1;
1: . . . . ring r2 = 32003,(x,y,z),dp;
2: . . . . poly f=imap(r1,f);
3: . 2 . . . ideal j=jacob(f);
4: . . . . vdim(std(j));
5: . 2 . . . ==> 536
6: . . 1 . . vdim(std(j+f));
7: . . 8 6 . ==> 195
8: . . 4 8 4 timer=0; // reset timer

o7 : BettiTally
o8 = BettiTally{0, {0, 0}, 0} => 1 }
(1, {2, 2}, 4) => 2
(1, {3, 3}, 6) => 2
(2, {3, 7}, 10) => 2
(2, {4, 4}, 8) => 1
(2, {4, 5}, 9) => 4
(2, {5, 4}, 9) => 4
(2, {7, 3}, 10) => 2
(3, {4, 7}, 11) => 4
(3, {5, 5}, 10) => 6
(3, {7, 4}, 11) => 4
(4, {5, 7}, 12) => 2
(4, {7, 5}, 12) => 2

```

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Strongly stable ideals and Hilbert polynomials

DAVIDE ALBERELLI AND PAOLO LELLA

ABSTRACT: The `StronglyStableIdeals.m2` package for Macaulay2 provides a method to compute all saturated strongly stable ideals in a given polynomial ring with a fixed Hilbert polynomial. A description of the main method and auxiliary tools is given.

INTRODUCTION. Strongly stable ideals are a key tool in commutative algebra and algebraic geometry. These ideals have nice combinatorial properties that make them well suited for both theoretical and computational applications. In the case of polynomial rings with coefficients in a field of characteristic zero, the notion of strongly stable ideals coincides with the notion of Borel-fixed ideals. Such ideals are fixed by the action of the Borel subgroup of triangular matrices and play a special role in the theory of Gröbner bases because initial ideals in generic coordinates are of this type [Galligo 1974].

In the context of parameter spaces of algebraic varieties, Galligo’s theorem says that each component and each intersection of components of a Hilbert scheme contains at least one point corresponding to a scheme defined by a Borel-fixed ideal. Hence, these ideals are distributed throughout the Hilbert scheme and can be used to study its local structure. To this end, in recent years several authors [Lella and Roggero 2011; 2016; Cioffi and Roggero 2011; Bertone et al. 2013a; 2017a; 2017b] developed algorithmic methods based on the use of strongly stable ideals to construct flat families corresponding to special loci of the Hilbert scheme. In particular, a new open cover of the Hilbert scheme has been defined using strongly stable ideals and the action of the projective linear group [Bertone et al. 2013b; Brachat et al. 2016]. In this construction, the list of all points corresponding to Borel-fixed ideals in a given Hilbert scheme is needed. The main feature of the package `StronglyStableIdeals.m2` is a method to compute this set of points, i.e., the list of all saturated strongly stable ideals in a polynomial ring with a given

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`StronglyStableIdeals.m2` version 1.1

Hilbert polynomial. The method has been theoretically introduced in [Cioffi et al. 2011] and improved in [Lella 2012]. Several other tools are developed and presented in the current paper.

1. STRONGLY STABLE IDEALS. Let us denote by $\mathbb{K}[\mathbf{x}]$ the polynomial ring in $n + 1$ variables $\mathbb{K}[x_0, \dots, x_n]$ with coefficients in a field \mathbb{K} . We assume that $x_0 > x_1 > \dots > x_n$. We use the multi-index notation to describe monomials, i.e., $\mathbf{x}^\alpha := x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ for every $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n+1}$, and we denote by $\mathbb{T}_{n,s}$ the set of monomials of $\mathbb{K}[\mathbf{x}]$ of degree s . For any monomial \mathbf{x}^α , we denote by $\min \mathbf{x}^\alpha$ and $\max \mathbf{x}^\alpha$ the indices of the minimal and maximal variable dividing \mathbf{x}^α .

Following [Green 2010], *increasing* and *decreasing elementary moves* are defined as the multiplications

$$e_i^+(\mathbf{x}^\alpha) := \frac{x_{i-1}}{x_i} \cdot \mathbf{x}^\alpha, \quad i > 0, \quad \text{and} \quad e_j^-(\mathbf{x}^\alpha) := \frac{x_{j+1}}{x_j} \cdot \mathbf{x}^\alpha, \quad j < n.$$

We say that an elementary move $e_i^{+/-}$ is *admissible* for a monomial \mathbf{x}^α if $\alpha_i > 0$, that is, $e_i^{+/-}(\mathbf{x}^\alpha)$ is a monomial of $\mathbb{K}[\mathbf{x}]$.

Definition 1.1. An ideal $I \subset \mathbb{K}[\mathbf{x}]$ is called *strongly stable* if

- (i) I is a monomial ideal;
- (ii) for every $\mathbf{x}^\alpha \in I$ and for every admissible increasing move e_i^+ , the monomial $e_i^+(\mathbf{x}^\alpha)$ is contained in I .

We recall that a strongly stable ideal is a Borel-fixed ideal. We now summarize some properties holding in general for Borel-fixed ideals and useful in this context.

Proposition 1.2 [Green 2010, Section 2]. *Let $I \subset \mathbb{K}[\mathbf{x}]$ be a strongly stable ideal.*

- (i) *The regularity of I is equal to the maximal degree of a generator.*
- (ii) *Let \mathfrak{m} be the irrelevant ideal of $\mathbb{K}[\mathbf{x}]$. Then, $(I : \mathfrak{m}) = (I : x_n)$, so that the ideal I is saturated if no generator involves the last variable x_n .*
- (iii) *The last variable x_n is a regular element for I , i.e., the multiplication by x_n induces the short exact sequence*

$$0 \longrightarrow \frac{\mathbb{K}[\mathbf{x}]}{I}(t-1) \xrightarrow{\cdot x_n} \frac{\mathbb{K}[\mathbf{x}]}{I}(t) \longrightarrow \frac{\mathbb{K}[\mathbf{x}]}{(x_n, I)}(t) \longrightarrow 0.$$

2. HILBERT POLYNOMIALS. The Hilbert polynomial $p(t)$ of a homogeneous ideal $I \subset \mathbb{K}[\mathbf{x}]$ is the numerical polynomial such that for s sufficiently large

$$\dim_{\mathbb{K}} \left(\frac{\mathbb{K}[\mathbf{x}]}{I} \right)_s = \dim_{\mathbb{K}} \left(\frac{\mathbb{K}[\mathbf{x}]_s}{I_s} \right) = \binom{n+s}{n} - \dim_{\mathbb{K}} I_s = p(s).$$

Obviously, not every numerical polynomial is a Hilbert polynomial of some homogeneous ideal. Those being Hilbert polynomials have been completely described by Gotzmann [1978].

Gotzmann's decomposition. *A numerical polynomial $p(t) \in \mathbb{Q}[t]$ is a Hilbert polynomial if, and only if, it can be written as*

$$p(t) = \binom{n+a_1}{a_1} + \binom{n+a_2-1}{a_2} + \cdots + \binom{n+a_r-(r-1)}{a_r}, \quad a_1 \geq \cdots \geq a_r \geq 0. \quad (1)$$

This decomposition is strictly related to Macaulay's decomposition

$$p(t) = \sum_{k=0}^d \left[\binom{t+k}{k+1} - \binom{t+k-m_k}{k+1} \right],$$

where $d = \deg p(t)$. For all $n \geq d+1$ the saturated lexicographic ideal within $\mathbb{K}[x_0, \dots, x_n]$ with Hilbert polynomial $p(t)$ is

$$(x_0, \dots, x_{n-d-2}, x_{n-d-1}^{b_d+1}, x_{n-d-1}^{b_d} x_{n-d}^{b_{d-1}+1}, \dots, x_{n-d-1}^{b_d} x_{n-d}^{b_{d-1}} \cdots x_{n-1}^{b_0}),$$

where

$$b_d = \#\{a_j \mid a_j = d\} = m_d \quad \text{and} \quad b_k = \#\{a_j \mid a_j = k\} = m_k - m_{k+1}, \quad 0 \leq k < d.$$

The description of the lexicographic ideal in terms of Gotzmann's decomposition gives an insight to the following theorem.

Gotzmann's regularity theorem. *The regularity of a saturated ideal $I \subset \mathbb{K}[\mathbf{x}]$ with Hilbert polynomial $p(t)$ is at most r , where r is the number of terms in the decomposition (1) and it is called the **Gotzmann number** of $p(t)$.*

Example 2.1. The package `StronglyStableIdeals.m2` provides the method `isHilbertPolynomial` to determine if a numerical polynomial is a Hilbert polynomial.

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure,
               InverseSystems, LLBases, PrimaryDecomposition,
               ReesAlgebra, TangentCone

i1 : loadPackage "StronglyStableIdeals";
i2 : QQ[t];
i3 : isHilbertPolynomial (4*t)
o3 = true
i4 : isHilbertPolynomial (5*t-6)
o4 = false
```

Gotzmann's and Macaulay's decompositions of a Hilbert polynomial can be computed using `gotzmannDecomposition` and `macaulayDecomposition`. These

methods return the list of terms in the decompositions. The summand $\binom{t+e}{c}$ is constructed with the command `projectiveHilbertPolynomial(c,c-e)`.

```
i5 : gotzmannDecomposition (4*t)
o5 = {P_1, - P_0 + P_1, - 2*P_0 + P_1, - 3*P_0 + P_1, P_0, P_0}
o5 : List
i6 : macaulayDecomposition (4*t)
o6 = {- P_0 + P_1, 7*P_0 - P_1, - P_1 + P_2, - 10*P_0 + 5*P_1 - P_2}
o6 : List
```

Finally, the saturated lexicographic ideal L with Hilbert polynomial $p(t)$ in the polynomial ring $\mathbb{K}[x]$ can be computed with the method `lexIdeal` and its regularity is equal to the Gotzmann number of $p(t)$.

```
i7 : L = lexIdeal (4*t, QQ[x,y,z,w])
o7 = ideal (x, y^5, y^4 z^2)
o7 : Ideal of QQ[x, y, z, w]
i8 : regularity L == gotzmannNumber (4*t)
o8 = true
```

3. THE MAIN ALGORITHM. In this section, we outline the strategy of the main algorithm. This algorithm was firstly described in [Cioffi et al. 2011] and then optimized in [Lella 2012]. The same problem has been previously discussed in [Reeves 1992] and an alternative algorithm was later presented in [Moore and Nagel 2014].

We need to relate the properties of a strongly stable ideal with its Hilbert polynomial. If I is a strongly stable ideal, for each $s \in \mathbb{N}$ the monomial basis of the homogeneous piece I_s of the ideal is a subset of $\mathbb{T}_{n,s}$ closed by increasing elementary moves. We call *Borel sets* such subsets of $\mathbb{T}_{n,s}$ (see Figure 1 for an example). Proposition 1.2(i) implies that the monomial basis of I_s for a saturated strongly stable ideal $I \subset \mathbb{K}[x]$ with Hilbert polynomial $p(t)$ and regularity at most s is a Borel set with $q(s) := \binom{n+s}{n} - p(s)$ elements. Thus, we consider the map

$$\left\{ \begin{array}{l} \text{saturated strongly stable ideals in } \mathbb{K}[x] \text{ with} \\ \text{Hilbert polynomial } p(t) \text{ and regularity } \leq s \end{array} \right\} \hookrightarrow \left\{ \begin{array}{l} \text{Borel sets of } \mathbb{T}_{n,s} \\ \text{with } q(s) \text{ elements} \end{array} \right\}. \quad (2)$$

Moreover, Gotzmann’s regularity theorem suggests considering s equal to the Gotzmann number of $p(t)$ to determine all saturated strongly stable ideals with Hilbert polynomial $p(t)$. Obviously, there are many Borel sets in $\mathbb{T}_{n,s}$ with $q(s)$ elements not corresponding to an ideal with Hilbert polynomial $p(t)$. To identify the image of the previous map, we recall a definition and a proposition by Mall.

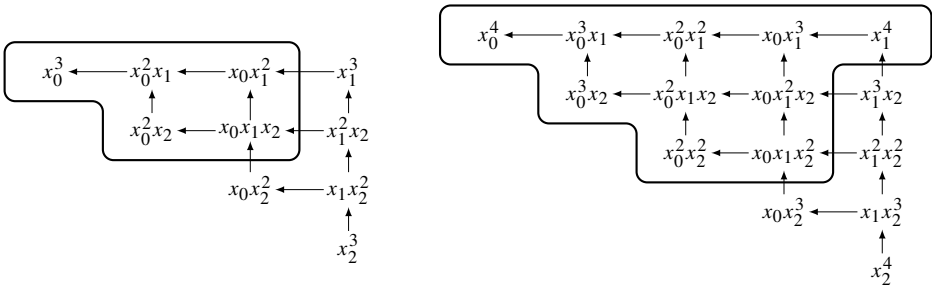


Figure 1. The Borel sets defined in $\mathbb{T}_{2,3}$ and $\mathbb{T}_{2,4}$ by the ideal $(x_0^2, x_0 x_1, x_1^4) \subset \mathbb{K}[x_0, x_1, x_2]$.

Definition 3.1 [Mall 1997, Definition 2.7]. Let $B \subset \mathbb{T}_{n,s}$ be a Borel set. The set $B^{(i)} := \{\mathbf{x}^\alpha \in B \mid \min \mathbf{x}^\alpha = n - i\}$ is called the i -growth class of B . The sequence $\text{gv}(B) := (|B^{(0)}|, \dots, |B^{(n)}|)$ is called the growth vector of B .

Proposition 3.2 [Mall 1997, Proposition 3.2]. Let $I \subset \mathbb{K}[\mathbf{x}]$ be a strongly stable ideal generated by the monomials of a Borel set $B \subset \mathbb{T}_{n,s}$ and let $p(t)$ be its Hilbert polynomial. Then,

$$p(t) = \binom{n+t}{n} - \sum_{k=0}^n |B^{(k)}| \binom{k+t-s}{k}, \quad \text{for all } t \geq s. \tag{3}$$

We can use this result to determine the growth vector of a Borel set $B \subset \mathbb{T}_{n,s}$ starting from the Hilbert polynomial. The i -th difference polynomial of $p(t)$ is

$$(\Delta^i p)(t) = (\Delta^{i-1} p)(t) - (\Delta^{i-1} p)(t-1) = \binom{n+t-i}{n-i} - \sum_{k=i}^n |B^{(k)}| \binom{k+t-s-i}{k-i}.$$

Evaluating these identities at $t = s$, we obtain the linear system

$$\begin{cases} \sum_{k=0}^n |B^{(k)}| = \binom{n+s}{n} - p(s), \\ \vdots \\ \sum_{k=i}^n |B^{(k)}| = \binom{n+s-i}{n-i} - (\Delta^i p)(s), \\ \vdots \\ |B^{(n)}| = \binom{s}{0} - (\Delta^n p)(s), \end{cases} \tag{4}$$

whose solution is

$$|B^{(i)}| = \sum_{k=i}^n |B^{(k)}| - \sum_{k=i+1}^n |B^{(k)}| = \binom{n+s-i-1}{n-i} - (\Delta^i p)(s) + (\Delta^{i+1} p)(s), \quad i < n,$$

and $|B^{(n)}| = 1$ (recall that $(\Delta^i p)(t) \equiv 0$ for $i > \deg p(t)$ and $\deg p(t) < n$). Let us call the growth vector of $p(t)$ in degree s the solution of the linear system (4) and let us denote it by $\text{gv}_s(p(t))$.

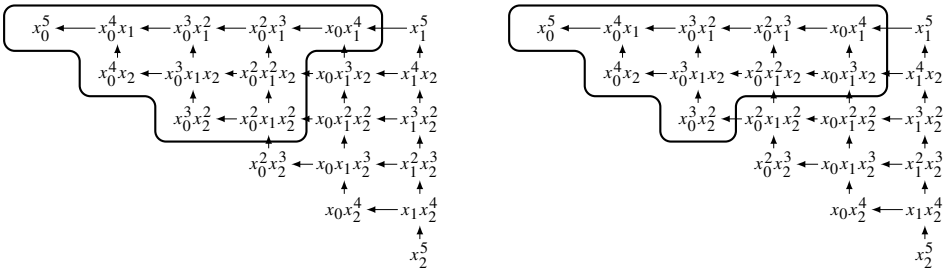


Figure 2. Borel sets in $\mathbb{T}_{2,5}$ corresponding to the saturated strongly stable ideals $(x_0^3, x_0^2 x_1, x_0 x_1^4)$ (on the left) and $(x_0^3, x_0^2 x_1^2, x_0 x_1^3)$ (on the right) in 3 variables with Hilbert polynomial $t + 6$ and regularity at most 5.

Proposition 3.3 (cf. [Lella 2012, Theorem 3.3]). *Let $p(t)$ be a Hilbert polynomial. There is a bijective map*

$$\left\{ \begin{array}{l} \text{saturated strongly stable ideals} \\ \text{in } \mathbb{K}[\mathbf{x}] \text{ with Hilbert polynomial} \\ p(t) \text{ and regularity } \leq s \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{Borel sets of } \mathbb{T}_{n,s} \\ \text{with } q(s) \text{ elements and} \\ \text{growth vector } \text{gv}_s(p(t)) \end{array} \right\}, \tag{5}$$

$$I \longrightarrow \text{monomial basis of } I_s,$$

$$\text{saturation of } (B) \longleftarrow B.$$

In order to determine the Borel sets of Proposition 3.3, we use a recursive algorithm based on Proposition 1.2(iii). Indeed, if $I \subset \mathbb{K}[x_0, \dots, x_n]$ is a strongly stable ideal with Hilbert polynomial $p(t)$ and B is the associated Borel set in $\mathbb{T}_{n,s}$, then the subset $B' = \{\mathbf{x}^\alpha \in B \mid \min \mathbf{x}^\alpha > n\} \subset B$ is a Borel set in $\mathbb{T}_{n-1,s}$ corresponding to the strongly stable ideal $I' = (x_n, I) \cap \mathbb{K}[x_0, \dots, x_{n-1}] \subset \mathbb{K}[x_0, \dots, x_{n-1}]$ with Hilbert polynomial $(\Delta p)(t)$.

Example 3.4. We want to determine the set of strongly stable ideals in the polynomial ring $\mathbb{K}[x_0, x_1, x_2]$ with regularity at most 5 defining schemes with Hilbert polynomial $p(t) = t + 6$. The Gotzmann number of $p(t)$ is 6 and its growth vector in degree 5 is $\text{gv}_5(t + 6) = (5, 4, 1)$. We start considering the set of strongly stable ideals in $\mathbb{K}[x_0, x_1]$ with Hilbert polynomial $\Delta p(t) = 1$ and regularity at most 5 corresponding to Borel sets with growth vector $\text{gv}_5(\Delta p(t)) = (4, 1)$. There is a unique Borel set

$$B' = \{x_0^5, x_0^4 x_1, x_0^3 x_1^2, x_0^2 x_1^3, x_0 x_1^4\}.$$

Since x_1^5 is not contained in B' , a Borel set $B \subset \mathbb{T}_{2,5}$ with growth vector $(5, 4, 1)$ does not contain monomials obtained from x_1^5 by applying decreasing elementary moves, i.e., $x_1^4 x_2, x_1^3 x_2^2, x_1^2 x_2^3, x_1 x_2^4$ and x_2^5 . Hence, we need to select five monomials divisible by both x_0 and x_2 producing a set closed by increasing elementary moves (see Figure 2).

Our package provides the method `stronglyStableIdeals` to compute the set of strongly stable ideals of a given polynomial ring with fixed Hilbert polynomial and bounded regularity.

```
i9 : stronglyStableIdeals (4*t, QQ[x,y,z,w])
o9 = {ideal (x5, y4, z2), ideal (x*z, x*y, x2, y4, z5),
      ideal (x*y, x2, x*z, y4), ideal (x*y, x2, y3)}
o9 : List
i10 : stronglyStableIdeals (4*t, QQ[x,y,z,w], MaxRegularity => 4)
o10 = {ideal (x*y, x2, x*z, y4), ideal (x*y, x2, y3)}
o10 : List
```

4. SEGMENT IDEALS. The transitive closure of the order relation

$$\mathbf{x}^\alpha >_B \mathbf{x}^\beta \iff \mathbf{x}^\beta = e_i^-(\mathbf{x}^\alpha) \quad (6)$$

induces a partial order on the set of monomials of any degree called the *Borel order*. Every graded term ordering is a refinement of this partial order. Since a Borel set B is closed with respect to the Borel order, i.e., $\mathbf{x}^\alpha >_B \mathbf{x}^\beta$, $\mathbf{x}^\beta \in B \Rightarrow \mathbf{x}^\alpha \in B$, it is natural to ask whether there exists a term ordering $<$ with the same property. For instance, for the lexicographic ideal, the graded lexicographic order separates, in each degree, monomials contained in the ideal from those outside. In [Cioffi et al. 2011], several notions of segment ideals are introduced.

Definition 4.1 [Cioffi et al. 2011, Definitions 3.1 and 3.7]. A Borel set $B \subset \mathbb{T}_{n,s}$ is called a *segment* if there exists a term ordering $<$ such that $\mathbf{x}^\alpha > \mathbf{x}^\beta$, for all $\mathbf{x}^\alpha \in B$ and $\mathbf{x}^\beta \in \mathbb{T}_{n,s} \setminus B$.

Let $I \subset \mathbb{K}[\mathbf{x}]$ be a saturated strongly stable ideal.

- (i) I is called a *hilb-segment* if the Borel set $I \cap \mathbb{T}_{n,r}$ is a segment, where r is the Gotzmann number of the Hilbert polynomial of I .
- (ii) I is called a *reg-segment* if the Borel set $I \cap \mathbb{T}_{n,m}$ is a segment, where m is the regularity of I .
- (iii) I is called a *gen-segment* if there exists a term ordering $<$ such that $\mathbf{x}^\alpha > \mathbf{x}^\beta$ for each minimal generator \mathbf{x}^α of degree s of I and for all $\mathbf{x}^\beta \in \mathbb{T}_{n,s} \setminus I_s$.

These notions are very important in the construction of flat families based on properties of Gröbner bases and in general for the study of the Hilbert scheme. The `StronglyStableIdeals.m2` package provides three methods for determining whether a strongly stable ideal may be some type of segment (and, in case, gives the term ordering). These methods use tools of the package `gfanInterface.m2` and the term ordering is given as a weight vector.

```

i11 : sevenPointsP2 = stronglyStableIdeals (7, 3, MaxRegularity => 5)
o11 = {ideal (x2, x2 x5, x3), ideal (x2, x4, x3 x3),
        ideal (x2, x2 x3, x4 x4)}
o11 : List
i12 : for J in sevenPointsP2 list isHilbSegment J
o12 = {(true, {7, 3, 1}), (false, ), (true, {4, 3, 1})}
o12 : List
i13 : for J in sevenPointsP2 list isRegSegment J
o13 = {(true, {7, 3, 1}), (false, ), (true, {4, 3, 1})}
o13 : List
i14 : for J in sevenPointsP2 list isGenSegment J
o14 = {(true, {6, 3, 1}), (true, {4, 3, 1}), (true, {4, 3, 1})}
o14 : List

```

SUPPLEMENT. Version 1.1 of `StronglyStableIdeals.m2` is contained in the [online supplement](#).

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DiffAlg: a Differential algebra package

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ABSTRACT: In this article we present `DiffAlg.m2`, a differential algebra package for Macaulay2. It can perform the following operations: wedge products and exterior differentials of differential forms, contraction and Lie derivatives of differential forms with respect to a vector field and Lie brackets between vector fields.

Given a homogeneous differential operator of degree one D , the lack of an algebraic module structure attached to the kernel or image of D hinders the study of D . The main purpose of `DiffAlg.m2` is to handle these spaces degree-wise.

MOTIVATION AND DESCRIPTION OF THE PACKAGE. Algebraic and differential operations arise naturally when working with differential forms and vector fields, e.g., wedge products and exterior differentials of differential forms, contraction and Lie derivatives of differential forms with respect to a vector field and Lie brackets between vector fields. Some important statements involving these operations include the following:

- (a) A differential r -form ω in the affine space \mathbb{K}^{n+1} descends to the projective space $\mathbb{P}_{\mathbb{K}}^n$ if it satisfies the equation

$$i_R \omega = 0,$$

where \mathbb{K} is a field, R is the radial vector field $R = \sum x_i \frac{\partial}{\partial x_i}$ and i_R denotes the contraction; see [Hartshorne 1977, Theorem 8.13, p. 176].

- (b) If ω is a differential 1-form, then ω defines a foliation in \mathbb{K}^{n+1} if it satisfies the *Frobenius integrability condition* given by the equation

$$\omega \wedge d\omega = 0;$$

see [Suwa 1995, Definition 2.2, p. 823].

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`DiffAlg.m2` version 1.5

- (c) Let ω be an integrable 1-form and $L_X\omega$ be the Lie derivative of ω with respect to a vector field X . Then, the solutions of the equation

$$L_X\omega = 0$$

define all the *infinitesimal automorphisms* of the foliation given by ω ; see [Suwa 1995, Proposition 7.7, p. 845].

- (d) Let ω be an integrable 1-form. Then the *tangent space of the space of foliations* at ω is given by the differential 1-forms η that satisfy the equation

$$\omega \wedge d\eta + d\omega \wedge \eta = 0;$$

see [Cukierman et al. 2009, Section 2.1. p. 709].

- (e) Let D be a *bracket generating distribution*. Some important invariants of D are the ranks of the derived sequence

$$a(p) := \text{rank } D^{(p)} = \text{rank}(D^{(p-1)} + [D, D^{(p-1)}]);$$

see [Tanaka 1970, §1, pp. 8–9].

- (f) A *symplectic structure* in a variety of dimension $2r$ is given by a 2-form ω such that $d\omega = 0$ and $\omega^r \neq 0$; see [Bryant et al. 1991, p. 41].

For a clear understanding of how DiffAlg.m2 deals with such equations, let us fix some notation.

Let $S = \mathbb{K}[x_0, \dots, x_n]$ be the polynomial ring in $n + 1$ variables and let $\Omega = \bigoplus_{r \geq 0} \Omega^r$ be the exterior algebra of differential forms of S over \mathbb{K} . Let $\Omega^r(d)$ denote the space of r -forms with polynomial coefficients of homogeneous degree d , where we assign degree 1 to each x_i and degree 0 to each dx_i .

Therefore $\omega \in \Omega^r(d)$ can be written as

$$\omega = \sum_{\substack{I \subset \{0, \dots, n\} \\ \#I=r}} \sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha|=d}} a_{\alpha, I} x^\alpha dx_I, \quad a_{\alpha, I} \in \mathbb{K}, \quad (1)$$

where, for each I of the form $I = \{i_1, \dots, i_r\} \subset \{0, \dots, n\}$, we let dx_I denote $dx_{i_1} \wedge \dots \wedge dx_{i_r}$, and for each $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$ we denote $|\alpha| := \sum_{i=0}^n \alpha_i$ and $x^\alpha := x_0^{\alpha_0} \dots x_n^{\alpha_n}$.

Let T be the module of vector fields with coefficients in S . Let $T(e)$ denote the homogeneous vector fields with polynomial coefficients of degree e , where we assign degree 1 to each x_i and degree 0 to each $\frac{\partial}{\partial x_i}$ and, analogously to (1), $X \in T(e)$ can be written as

$$X = \sum_{i=0}^n \sum_{\substack{\beta \in \mathbb{N}^{n+1} \\ |\beta|=e}} b_{\beta, i} x^\beta \frac{\partial}{\partial x_i}, \quad b_{\beta, i} \in \mathbb{K}.$$

Current algebraic software systems implement functionality to deal with differential forms and vector fields, but usually scalar parameters $a_{\alpha,I}$ and $b_{\beta,i}$ must be specified as fixed elements in \mathbb{K} . Instead, `DiffAlg.m2` treats homogeneous forms and vector fields in a completely symbolic environment by considering the scalar coefficient rings

$$\mathbb{K}[a_{\alpha,I}] \quad \text{and} \quad \mathbb{K}[b_{\beta,i}].$$

Scalar coefficients can be systematically obtained by looking at the coordinates of differential forms and vector fields written in the standard bases

$$\mathcal{B}_{r,d} = \{x^\alpha dx_I\}_{\substack{\#I=r \\ |\alpha|=d}} \quad \text{and} \quad \mathcal{B}_e = \{x^\beta \frac{\partial}{\partial x_i}\}_{|\beta|=e}$$

of the spaces $\Omega^r(d)$ and $T(e)$, respectively.

Importantly, when using `DiffAlg.m2`, each object is expected to be defined in its own coefficient ring. Then, certain operations, such as contraction or computing the wedge product, involve different input and output rings, producing a modification of the coefficients rings $\mathbb{K}[a_{\alpha,I}]$ or $\mathbb{K}[b_{\beta,i}]$. For greater clarity, consider the following example. Fix $\omega \in \Omega^r(d)$ and $X \in T(e)$ and consider the contraction $i_X \omega \in \Omega^{r-1}(d+e)$. Then, the following will be taking place:

	(ω, X)	\mapsto	$i_X \omega$
Ring	$\mathbb{K}[a_{\alpha,I}] \times \mathbb{K}[b_{\beta,i}]$		$\mathbb{K}[a_{\alpha,I}, b_{\beta,i}]$
Basis	$\mathcal{B}_{r,d} \times \mathcal{B}_e$		$\mathcal{B}_{r-1,d+e}$

As mentioned before, the main purpose of `DiffAlg.m2` is to find algebraic solutions to equations in the context of differential algebra. Equations are treated differently in the linear and nonlinear cases:

- (a) In the linear case, for example $i_R \omega = 0$, `DiffAlg.m2` can compute a basis of the solutions of the equation. Once this is done, it can also compute a generic linear combination of the elements of the basis; see [Example 1](#).
- (b) In the nonlinear case, for example $\omega \wedge d\omega = 0$, the coordinates will be polynomial. In this case, `DiffAlg.m2` would compute the ideal generating the space of solutions. This ideal can be obtained in two different ways: taking coordinates in the basis $\mathcal{B}_{r,d}$ or \mathcal{B}_e , or taking coordinates in the basis $\{dx_I\}$ or $\{\frac{\partial}{\partial x_i}\}$; see [Examples 2 and 4](#).

`DiffAlg.m2` can also be a valuable tool for studying differential operators. The lack of algebraic theory to deal with such objects can be mitigated by nonconclusive computations easily made by `DiffAlg.m2`. As a first example, one could consider computing solutions of a differential operator degree-wise for low degrees.

SOME EXAMPLES.

Example 1. In the following example we obtain a basis of the space of projective differential 2-forms in $\mathbb{P}_{\mathbb{K}}^3$. Then, we define a generic projective differential 2-form to be possibly used in further computations.

```

i1 : loadPackage "DiffAlg";
i2 : R = radial 3
o2 = x  ax  + x  ax  + x  ax  + x  ax
      0  0   1  1   2  2   3  3
o2 : DiffAlgField
i3 : w = newForm(3,2,1,"a");
o3 = (a  x  +a  x  +a  x  +a  x  )dx  dx +(a  x  + a  x  + a  x  + a  x  )dx  dx
      0  0  6  1  12  2  18  3   0  1   1  0   7  1   13  2   19  3   0  2
      + (a  x  + a  x  + a  x  + a  x  )dx  dx + (a  x  + a  x  + a  x  +
        3  0   9  1   15  2   21  3   1  2   2  0   8  1   14  2
        a  x  )dx  dx + (a  x  + a  x  + a  x  + a  x  )dx  dx +(a  x  +a  x  +
        20  3   0  3   4  0   10  1   16  2   22  3   1  3   5  0   11  1
        a  x  + a  x  )dx  dx
        17  2   23  3   2  3
o3 : DiffAlgForm
i4 : pretty ring w
      QQ[i]
o4 = -----[[a  , a  , a  , a  , a  , a  , a  , a  , a  , a  , a  , a  , a  , a  ,
      2          0  1  2  3  4  5  6  7  8  9  10  11  12
      i  + 1
      a  , a  , a  , a  , a  , a  , a  , a  , a  , a  , a  , a  , a  ] [x  , x  ,
      13  14  15  16  17  18  19  20  21  22  23  0  1
      x  , x  ] [dx  , dx  , dx  , dx  ]
      2  3   0  1  2  3
i5 : K = genKer (R _ w, w);
i6 : length K
o6 = 4
i7 : v = linearComb(K,"a")
o7 = (a  x  - a  x  )dx  dx +(- a  x  + a  x  )dx  dx + (a  x  + a  x  )dx  dx +
      0  2   1  3   0  1   0  1   2  3   0  2   0  0   3  3   1  2
      (a  x  - a  x  )dx  dx + (-a  x  - a  x  )dx  dx + (a  x  + a  x  )dx  dx
      1  1   2  2   0  3   1  0   3  2   1  3   2  0   3  1   2  3
o7 : DiffAlgForm
i8 : pretty ring v
      QQ[i]
o8 = -----[[a  , a  , a  , a  ] [x  , x  , x  , x  ] [dx  , dx  , dx  , dx  ]
      2          0  1  2  3   0  1  2  3   0  1  2  3
      i  + 1

```

Let us explain part of the code:

- i2. Creates the radial vector field in four variables. We are denoting the basic field $\partial/\partial x_i$ as ax_i .
- i3. Creates a generic linear 2-form in $\Omega_{\mathbb{K}^4}^2(1)$, with the coefficients indexed as a_i .
- i4. Shows the ring of definition of w .
- i5. Gets a basis (as a Macaulay2 list) of forms in $\Omega_{\mathbb{K}^4}^2(1)$ that descend to projective space. The operation R_w computes the contraction of the differential form w with the vector field R .
- i6. Gets the dimension of $\Omega_{\mathbb{P}^3}^2(1)$ in projective 3-space.
- i7. Defines a generic projective form with coefficients a_i .
- i8. Shows the ring of definition of v .

Example 2. In the finite-dimensional \mathbb{K} -vector space $\Omega^1(d)$, the solutions of the equation $\omega \wedge d\omega = 0$ determine an algebraic variety; its points are the integrable differential 1-forms of degree d . In the following example, we compute the equations of the variety of integrable 1-forms of degree 1 in 3-dimensional space.

It is worth mentioning that, for $n \geq 3$ and $d > 5$, it is an open problem to classify the irreducible components of this varieties; see [Cukierman et al. 2009].

```
i1 : loadPackage "DiffAlg";
i2 : w = newForm (2,1,1,"a")
o2 = (a x + a x + a x )dx + (a x + a x + a x )dx +(a x + a x + a x )dx
      0 0   3 1   6 2   0   1 0   4 1   7 2   1   2 0   5 1   8 2   2
o2 : DiffAlgForm
i3 : moduliIdeal (w ^ (diff w))
o3 = ideal (- a a + a a + a a - a a , - a a + a a + a a - a a ,
            2 3   0 5   1 6   0 7   2 4   1 5   4 6   3 7 ,
            a a - a a + a a - a a )
            5 6   2 7   1 8   3 8
o3 : Ideal of  $\frac{\text{QQ}[i]}{i + 1}$  [a0, a1, a2, a3, a4, a5, a6, a7, a8]
```

About the code:

- i2. Creates a generic linear 1-form in $\Omega_{\mathbb{K}^3}^1(1)$, with coefficients a_i .
- i3. Returns the ideal of the scalar coefficients given by $w \wedge dw = 0$.

Example 3. Let D be a 2-dimensional distribution generated by vector fields X and Y in 5-dimensional space. In the following example we compute the ranks of the derived distributions $D^{(p)}$. We verify that this derived series eventually spans

the entire tangent space. A distribution D satisfying this condition is called *bracket-generating*.

```
i2 : X = newField "x_0^2*ax_0+x_1^2*ax_1+x_2^2*ax_2+x_3^2*ax_3";
i3 : Y = newField "x_5*ax_0+x_4*ax_1+x_3*ax_2+x_2*ax_3+x_1*ax_4+x_0*ax_5";
i4 : D_0 = {X,Y};
i5 : for b in 1..3 do (
  for a in D_(b-1) do (
    D_b = join(D_(b-1),{a|Y,a|X})
  )
);
i6 : {rank dist D_0, rank dist D_1, rank dist D_2, rank dist D_3}
o6 = 2, 3, 5, 6
o6 : List
```

About the code:

- i5. Computes the derived sequence.
- i6. Prints the ranks of the derived series.

Example 4. In the following example, we generate a random rational 1-form of type (1,2) in $\mathbb{P}_{\mathbb{K}}^2$. First, we compute (the dimension of) the space of its integrating factors; see [Suwa 1995, pp. 828–829]. Then, we compute the ideal of its singular locus (the ideal where it vanishes).

```
i2 : w = random logarithmicForm (2,{1,2},"a",Projective => true);
i3 : f = newForm (2,0,3,"a");
i4 : length genKer(w^(diff f) + f*(diff w), f)
o4 = 2
i5 : I = singularIdeal w
o6 = ideal (- 9x x + 63x - 36x x - 54x x - 54x , 9x - 63x x -
           0 1      1      0 2      1 2      2      0      0 1
           27x x - 45x x - 54x , 36x + 81x x + 45x + 54x x + 54x x )
           0 2      1 2      2      0      0 1      1      0 2      1 2
o6 : Ideal of -----[ ] [x , x , x ]
           2      0      1      2
           i + 1
```

SUPPLEMENT. The [online supplement](#) contains version 1.5 of DiffAlg.m2.

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Matroids: a Macaulay2 package

JUSTIN CHEN

ABSTRACT: We give an overview of the Macaulay2 package `Matroids.m2`, which introduces functionality to create and compute with matroids into Macaulay2. Examples highlighting the use of many functions in the package are provided, including applications of matroids to other areas.

INTRODUCTION. A matroid is a combinatorial object which abstracts the notions of (linear algebraic, graph-theoretic) independence. Since their introduction by Whitney [1935], matroids have found diverse applications in combinatorics, graph theory, optimization, and algebraic geometry, in addition to being studied as interesting objects in their own right.

We describe here the Macaulay2 package `Matroids.m2`, which is available at <https://github.com/jchen419/Matroids-M2>. For the reader already familiar with matroids, it provides capabilities to form matroids from a matrix, graph, or ideal; convert between various representations of matroids; create and detect existence of minors; compute Tutte polynomials and Chow rings; as well as applications of matroids to polyhedral and algebraic geometry, commutative algebra, optimization, and even group theory. Each will in turn be illustrated with examples. Virtually all notation and results mentioned below can be found in [Oxley 2011].

One striking feature of matroids is the multitude of distinct ways to define them. This variety of equivalent — or *cryptomorphic* — ways to characterize matroids is a great strength of matroid theory, and one of the reasons for its ubiquity. From the perspective of this package, the key definition is via bases:

Definition. Let E be a finite set, and $\emptyset \neq \mathcal{B} \subseteq 2^E$ a set of subsets of E . The pair (E, \mathcal{B}) is a *matroid* if for any $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 \setminus B_2$, there exists $b_2 \in B_2 \setminus B_1$ with $B_1 \setminus \{b_1\} \cup \{b_2\} \in \mathcal{B}$.

The set E is called the *ground set* of the matroid $M = (E, \mathcal{B})$, and \mathcal{B} is the set of *bases* of M . All bases have the same cardinality, called the *rank* of M . Any subset of a basis is an *independent set*. A subset of E that is not independent is *dependent*.

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`Matroids.m2` version 0.9.7

The minimal (with respect to inclusion) dependent sets are *circuits*. It is easy to see that any of bases, independent sets, dependent sets, and circuits determines the others.

As any subset of an independent set is independent, the set of independent sets of a matroid forms a simplicial complex on E , called the *independence complex* of M , denoted by Δ_M . Via Stanley–Reisner theory, Δ_M corresponds to a squarefree monomial ideal $I_{\Delta_M} := \langle \prod_{i \in C} x_i \mid C \text{ circuit} \rangle$, inside a polynomial ring $k[x_i \mid i \in E]$ (since faces of Δ_M are independent sets, the minimal nonfaces are precisely the minimal dependent sets, i.e., circuits). We call I_{Δ_M} the (circuit) ideal of M : internally, many algorithms in this package work directly with this ideal, to exploit Macaulay2's facility with monomial ideals.

A FIRST EXAMPLE. The most basic way to create a matroid is by specifying the ground set and list of bases:

```
i1 : needsPackage "Matroids";
i2 : M = matroid({a,b,c,d},{a,b},{a,c})
o2 = a matroid of rank 2 on 4 elements
o2 : Matroid
```

This creates a matroid of rank 2 on the ground set $\{a, b, c, d\}$ with two bases. We can peek at the matroid to see more of its internal structure:

```
i3 : peek M
o3 = Matroid{bases => {set {0, 1}, set {0, 2}}}
      cache => CacheTable{...2...}
      groundSet => set {0, 1, 2, 3}
      rank => 2
```

Two things should be noticed: first, `groundSet` is a set of integers $\{0, \dots, 3\}$ (instead of the given list $\{a, b, c, d\}$). Second, the bases consist of a list of subsets of `groundSet`. This convention is by design: internally, the ground set is always identified with the set $\{0, \dots, |E| - 1\}$, and all sets associated to the structure of the matroid are subsets of the ground set. One should think of the integers in `groundSet` as *indices* of the actual elements, so 0 is the index of the first element (in this case a), 1 is the index of the second element, etc.

The actual elements of the user-inputted ground set are not lost though; they have been cached in the `CacheTable`, and can be accessed by using indices as subscripts on M , or all at once with an asterisk:

```
i4 : (M_3, M_{0,1}, M_{set{1,2}}, M_*)
o4 = (d, {a, b}, {b, c}, {a, b, c, d})
```

So far, no attempt has been made to check that M is actually a matroid. We verify this now using the method `isWellDefined` (which internally checks the circuit elimination axiom), and also give a nonexample.

```
i5 : (isWellDefined M, isWellDefined matroid({a,b,c,d},{a,b},{c,d}))
o5 = (true, false)
```

We can obtain plenty of matroid-theoretic information for this example. Recall:

Definition. A *loop* in M is a 1-element circuit, and a *coloop* in M is an element contained in every basis. For $A \subseteq E$, the *rank* of A is the size of the largest independent subset of A , and the *closure* of A is $\bar{A} := \{x \in E \mid \text{rank}(A) = \text{rank}(A \cup \{x\})\}$. A *flat* of M is a closed subset, i.e., $A = \bar{A}$. A *hyperplane* of M is a flat of rank equal to $\text{rank } M - 1$.

```
i6 : (rank M, rank(M, set{0,3}))
o6 = (2, 1)
i7 : (circuits M, independentSets(M, 1))
o7 = ({set {1, 2}, set {3}}, {set {0}, set {1}, set {2}})
i8 : (loops M, coloops M, closure(M, set{2,3}), hyperplanes M)
o8 = ({3}, {0}, set {1, 2, 3}, {set {0, 3}, set {1, 2, 3}})
i9 : flats M -- sorted by increasing size
o9 = {set {3}, set {0, 3}, set {1, 2, 3}, set {0, 1, 2, 3}}
i10 : fVector M -- number of flats of rank i, for 0 <= i <= rank M
o10 = HashTable{0 => 1}
          1 => 2
          2 => 1
```

CONSTRUCTING TYPES OF MATROIDS. The simplest family of matroids is the family of *uniform* matroids, where the set of bases equals all subsets of a fixed size:

```
i11 : U = uniformMatroid(2,4); bases U
o12 = {set {0, 1}, set {0, 2}, set {1, 2}, set {0, 3}, set {1, 3}, set {2, 3}}
```

Another family of fundamental importance is the class of *linear* matroids, which arise naturally from a matrix. The columns of the matrix form the ground set, and a set of column vectors is declared independent if they are linearly independent in the vector space spanned by the columns.

```
i13 : A = matrix{{0,4,-1,6},{0,2/3,7,1}}; MA = matroid A; representationOf MA
o15 = | 0 4   -1 6 |
      | 0 2/3 7  1 |
```

An abstract matroid M is called *representable* or *realizable* over a field k if M is *isomorphic* to a linear matroid over k , where an *isomorphism* of matroids is a bijection between ground sets that induces a bijection on bases. We verify that the matroid M we started with is isomorphic to MA , hence is representable over \mathbb{Q} :

```
i16 : areIsomorphic(M, MA)
o16 = true
```

A matroid can also be constructed by specifying its circuit ideal, which we do for the same M above. Here two matroids are considered equal if they have the same set of bases and same size ground sets; or, equivalently, the identity permutation is an isomorphism between them.

```
i17 : R = QQ[x,y,z,w]; MI = matroid ideal(y*z, w)
o18 = a matroid of rank 2 on 4 elements
i19 : M == MI
o19 = true
```


An important class of representable matroids (over any field) is the class of *graphic* matroids, derived from a graph. If G is an (undirected) graph, then the graphic matroid $M(G)$ has ground set equal to the edge set of G , and circuits given by cycles in G , including loops and parallel edges.

```
i20 : K5 = completeGraph 5; M5 = matroid K5
o21 = a matroid of rank 4 on 10 elements
i22 : #bases M5 -- == n^(n-2) for M(K_n), by Cayley's theorem
o22 = 125
```

In this package, the graphic matroid is created by specifying circuits. This can be done for an abstract matroid as well, using the optional argument `EntryMode => "circuits"` in the constructor function. Regardless of the value of `EntryMode`, the bases are automatically computed upon creation. We recreate the matroid M from before, by specifying its circuits (note the similarity with specifying the circuit ideal):

```
i23 : M == matroid({a,b,c,d},{b,c},{d}}, EntryMode => "circuits")
o23 = true
```

Certain common matroids are close to uniform, in the sense that relatively few subsets of size $\text{rank } M$ are dependent, so the set of *nonbases* (= dependent sets of size $\text{rank } M$) can also be specified:

```
i24 : nb = {{0,2,4},{1,3,4},{1,2,5},{0,3,5},{0,1,6},{2,3,6},{4,5,6}}/set;
i25 : F7 = matroid(toList(0..6), nb, EntryMode => "nonbases")
o25 : a matroid of rank 3 on 7 elements
i26 : (#bases F7, #circuits F7)
o26 = (28, 14)
```

A few specific matroids of theoretical importance are also built-in. Currently these are F_7 , F_7^- , V_8 , V_8^+ , $AG(3, 2)$, R_{10} , and the Pappus and non-Pappus matroids. A library of all matroids on up to eight elements is included as well:

```
i27 : F7 == specificMatroid "fano"
o27 = true
i28 : L7 = allMatroids 7 -- non-isomorphic matroids on 7 elements
o28 = {a matroid of rank 0 on 7 elements, a matroid of rank 1 on 7 elements, ...
i29 : (#L7, #flatten apply(6, allMatroids))
o29 = (306, 70)
```

One can also construct a new matroid from smaller ones by taking *direct sums*: if $M_1 = (E_1, \mathcal{B}_1)$, $M_2 = (E_2, \mathcal{B}_2)$ are matroids, then their direct sum is

$$M_1 \oplus M_2 := (E_1 \sqcup E_2, \{B_1 \sqcup B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}).$$

A matroid that cannot be written as a direct sum of nonempty matroids is called *connected*. Every matroid is a direct sum of connected matroids, its *connected components*, which are unique up to rearrangement:

```
i30 : S = U ++ matroid completeGraph 3
o30 = a matroid of rank 4 on 7 elements
```

```

i31 : C = components S
o31 = {a matroid of rank 2 on 4 elements, a matroid of rank 2 on 3 elements}
i32 : S == C#0 ++ C#1 and C#0 == U and C#1 == matroid completeGraph 3
o32 = true

```

DUALITY AND MINORS. One of the most important features of matroid theory is the existence of a duality. It is straightforward to check that if $M = (E, \mathcal{B})$ is a matroid, then $\{E \setminus B \mid B \in \mathcal{B}\}$ is the set of bases of a matroid on E , called the *dual matroid* of M , denoted by M^* .

```

i33 : D = dual M; (bases M, bases D)
o34 = ({set {0, 1}, set {0, 2}}, {set {2, 3}, set {1, 3}})
i35 : M == dual D
o35 = true

```

Virtually any matroid-theoretic property or operation can be enriched by considering its dual version—for instance, loops of M^* are coloops of M , and circuits of M^* are complements of hyperplanes of M (this is in fact how the method `hyperplanes` works). Another operation is deletion, which dualizes to contraction:

Definition. Let $M = (E, \mathcal{B})$ be a matroid, and $S \subseteq E$. The *restriction* of M to S , denoted $M|_S$, is the matroid on S with bases $\{B \cap S \mid B \in \mathcal{B}, |B \cap S| = \text{rank } S\}$. The *deletion* of S , denoted $M \setminus S$, is the restriction of M to $E \setminus S$. The *contraction* of M by S , denoted M/S , is defined as $(M^* \setminus S)^*$.

```

i36 : N1 = M \ set{3}; (N1_*, bases N1)
o37 = ({a, b, c}, {set {0, 1}, set {0, 2}})
i38 : N2 = M / set{1}; (N2_*, bases N2)
o39 = ({a, c, d}, {set {0}})

```

A *minor* of M is any matroid which can be obtained from M by a sequence of deletions and contractions. It is a fact that any minor of M is of the form $(M/X) \setminus Y$ for disjoint subsets $X, Y \subseteq E$.

```

i40 : minorM5 = minor(M5, set{9}, set{3,5,8}) -- contracts {9}, then deletes {3,5,8}
o40 = a matroid of rank 3 on 6 elements
i41 : (minorM5_*, #bases minorM5)
o41 = ({set {0, 1}, set {0, 2}, set {0, 3}, set {1, 2}, set {1, 4}, set {2, 3}}, 16)

```

Minors can be used to describe many important classes of matroids. For example, a class \mathcal{M} of matroids is said to be *minor-closed* if every minor of a matroid in \mathcal{M} is again in \mathcal{M} . The classes of uniform, k -representable (for any field k), and graphic matroids are all minor-closed. Various classes of matroids can be characterized by their *forbidden* or *excluded* minors: namely the matroids not in the class, but with every proper minor in the class.

Theorem 1 (Tutte 1958a; 1958b; 1959). *Let M be a matroid. We denote by $U_{2,4}$ the uniform matroid of rank 2 on 4 elements, and by F_7 the Fano matroid.*

- (i) M is binary (= representable over \mathbb{F}_2) if and only if M has no $U_{2,4}$ minor (i.e., no minor of M is isomorphic to $U_{2,4}$).

- (ii) M is regular (= representable over any field) if and only if M has no $U_{2,4}$, F_7 , or F_7^* minor.
- (iii) M is graphic if and only if M has no $U_{2,4}$, F_7 , F_7^* , $M(K_5)^*$, or $M(K_{3,3})^*$ minor.

We illustrate this by verifying that $M(K_5)$ is regular (alternatively, note that for any graph G , the signed incidence matrix of any orientation of G represents $M(G)$ over any field):

```
i42 : any({U, F7, dual F7}, forbidden -> hasMinor(M5, forbidden))
o42 = false
```

Every minor of M is in fact of the form $(M/I) \setminus I^*$, where I, I^* are disjoint, I is independent, and I^* is *coindependent* (= independent in M^*). Such a minor has rank equal to that of M/I , which is equal to $\text{rank } M - |I|$. Thus checking existence of a minor N in M can be realized as a two-step process, where the first step contracts independent sets of M of a fixed size down to the rank of N , and the second step deletes coindependent sets down to the size of N .

```
i43 : M4 = matroid completeGraph 4; hasMinor(M5, M4)
o44 = true
i45 : minorM5 == M4
o45 = true
```

Finally, the *Tutte polynomial* $T_M(x, y)$ of a matroid is an invariant which is universal with respect to satisfying a *deletion-contraction recurrence*. It is a bivariate polynomial over \mathbb{Z} which can be defined by the relation

$$T_M(x, y) = T_{M \setminus e}(x, y) + T_{M/e}(x, y), \quad e \in E \text{ not a loop or coloop,}$$

with the initial condition $T_M(x, y) = x^a y^b$ if M consists of a coloops and b loops. Any numerical invariant of matroids which satisfies a (weighted) deletion-contraction recurrence is an evaluation of the Tutte polynomial, up to a scale factor. For instance, the number of bases is equal to $T_M(1, 1)$:

```
i46 : tuttePolynomial M5
o46 = y^6 + 4y^5 + x^4 + 5xy^3 + 10y^4 + 6x^3 + 10xy^2 + 15xy^2 + 15y^3 + 11x^2 + 20xy + 15y^2...
i47 : tutteEvaluate(M5, 1, 1)
o47 = 125
```

For graphic matroids, the Tutte polynomial contains a wealth of information about the graph; e.g., the Tutte polynomial specializes to the chromatic polynomial. Even evaluations at specific points contain nontrivial information: e.g., $T_{M(G)}(2, 1)$ counts the number of spanning forests in G , and $T_{M(G)}(2, 0)$ counts the number of acyclic orientations of G .

```
i48 : (tutteEvaluate(M5, 2, 1), tutteEvaluate(M5, 2, 0), factor chromaticPolynomial K5)
o48 = (291, 120, (x)(x - 4)(x - 3)(x - 2)(x - 1))
```

CONNECTIONS. We now present some connections of matroids to other areas of mathematics. First, polyhedral geometry: let $M = ([n], \mathcal{B})$ be a matroid on $\{1, \dots, n\}$. In Euclidean space \mathbb{R}^n with standard basis $\{e_1, \dots, e_n\}$, define the matroid polytope P_M by taking the convex hull of the indicator vectors of the bases of M :

$$P_M := \text{conv} \left(\sum_{i \in B} e_i \mid B \in \mathcal{B} \right).$$

The matroid polytope can be created as follows:

```
i49 : needsPackage "Polyhedra"; P = convexHull basisIndicatorMatrix M4
o50 = {ambient dimension => 6
      }
      dimension of lineality space => 0
      dimension of polyhedron => 5
      number of facets => 16
      number of rays => 0
      number of vertices => 16
o50 : Polyhedron
```

A theorem of Gelfand, Goresky, MacPherson, and Serganova [Gelfand et al. 1987] classifies the subsets $B \subseteq 2^{[n]}$ which are the bases of a matroid on $[n]$ in terms of the polytope P_M .

Next is optimization: let E be a finite set, and $\mathcal{I} \subseteq 2^E$ a set of subsets that is downward closed: if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$. Let w be a weight function on E , i.e., a function $w : E \rightarrow \mathbb{R}$, extended to $w : 2^E \rightarrow \mathbb{R}$ by setting $w(X) := \sum_{x \in X} w(x)$. Consider the optimization problem (*) of finding a maximal member of \mathcal{I} of maximum weight, with respect to w . One attempt to solve (*) is to apply the greedy algorithm: namely, after having already selected elements $\{x_1, \dots, x_i\}$, choose an element $x_{i+1} \in E$ of maximum weight such that $\{x_1, \dots, x_i, x_{i+1}\} \in \mathcal{I}$, and repeat. It turns out that the greedy algorithm will work if and only if \mathcal{I} is the set of independent sets of a matroid:

Theorem 2 [Borůvka 1926]. *Let E be a finite set, and $\mathcal{I} \subseteq 2^E$. Then \mathcal{I} is the set of independent sets of a matroid on E if and only if \mathcal{I} is downward closed and for all weight functions $w : E \rightarrow \mathbb{R}$, the greedy algorithm successfully solves (*).*

A solution to (*) provided by the greedy algorithm can be obtained using the method `maxWeightBasis` (the weight function w is specified by its list of values on E):

```
i51 : w = {0, log(2), 4/3, 1, -4, 2, pi_RR}; maxWeightBasis(F7, w)
o52 = set {3, 5, 6}
```

Another application to optimization comes from the operation of *matroid union*: if M_1, M_2 are matroids with independent sets $\mathcal{I}_1, \mathcal{I}_2$, then the independent sets of the union are of the form $I_1 \cup I_2$, where $I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2$ (and thus coincides with the direct sum if the ground sets are disjoint).

```
i53 : matroid({a,b,c,d}, {{a},{b},{c}}) + matroid({a,b,c,d}, {{b},{c},{d}}) == U
o53 : true
i54 : F7 + F7 == uniformMatroid(6, 7)
o54 : true
```

Matroid union is an important operation in combinatorial optimization, and is closely related to transversal and matching problems: a matroid is *transversal* if and only if it is a union of rank 1 matroids, and *gammoids* (a class of matroids defined from vertex paths in directed graphs) are the minor-closure of the transversal matroids.

One can also find connections to group theory via the method `getIsos`, which computes all isomorphisms between two matroids. Many interesting groups can be realized as automorphism groups of small matroids:

```
i55 : aut = getIsos(F7, F7)
o55 : {{0, 1, 2, 3, 4, 5, 6}, {1, 0, 2, 3, 4, 6, 5}, {0, 2, 1, 3, 5, 4, 6}, {2, 0, 1, ...
i56 : #aut
o56 : 168
```

The above output is an explicit permutation representation of $\text{Aut}(\mathbb{P}_{\mathbb{F}_2}^2) = \text{PGL}(3, \mathbb{F}_2)$ as a subgroup of S_7 . For a larger example, the automorphism group of the Steiner system $S(5, 6, 12)$ is the Mathieu group M_{12} , a sporadic simple group of order $95040 = 2^6 \cdot 3^3 \cdot 5 \cdot 11$. This in turn is also equal to the automorphism group of the realizable matroid associated to a particular 6×12 matrix over \mathbb{F}_3 ([Oxley 2011], p. 367), and a high-performance computing cluster took just under 2 hours to compute the entire permutation representation of this group inside S_{12} .

For an application to commutative algebra: matroids are closely related to the Cohen–Macaulay property, for symbolic powers of squarefree monomial ideals. Indeed, from [Terai and Trung 2012] we know that if I is a squarefree monomial ideal, then I is the circuit ideal of a matroid if and only if every symbolic power $I^{(n)}$ is Cohen–Macaulay, for $n \geq 1$ (in fact, this is equivalent to requiring just $I^{(3)}$ to be Cohen–Macaulay). As one can quickly check whether an ideal is the ideal of a matroid, this can give a quick proof that a particular symbolic power is Cohen–Macaulay:

```
i57 : M6 = matroid completeGraph 6; L = (irreducibleDecomposition ideal M6)/(P -> P^3);
i59 : try ( alarm 10; I3 = intersect L; ) -- doesn't finish in 10 seconds
i60 : time isWellDefined M6
      -- used 0.359306 seconds
o60 : true
```

Last but not least is algebraic geometry; in particular the emerging field of combinatorial Hodge theory. For a matroid M on ground set E with no loops, one can define a Chow ring associated to M : for a field k , set

$$R := k[x_F \mid F \text{ proper, nonempty flat}]/(I_1 + I_2),$$

$$I_1 := \left(\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F \mid i_1, i_2 \in E \text{ distinct} \right),$$

$$I_2 := (x_F x_{F'} \mid F, F' \text{ incomparable}),$$

where F, F' run over all nonempty proper flats of M . Then R is a standard graded Artinian k -algebra of Castelnuovo–Mumford regularity $r := \text{rank } M - 1$. A result of Adiprasito, Katz, and Huh [Adiprasito et al. 2018] states that R is a Poincaré duality algebra (in particular, is Gorenstein) and has the strong Lefschetz property: for general $l \in R_1$ and $j \leq r/2$, multiplication by l^{r-2j} is an isomorphism $R_j \xrightarrow{\sim} R_{r-j}$. We illustrate the Gorenstein property for the Vamos matroid (which is a smallest matroid not realizable over any field), and conclude by computing the dual socle generator or *volume polynomial* (which generates the Macaulay inverse system of R) for $M(K_4)$:

```

i61 : V = specificMatroid("vamos"); (rank V, #V.groundSet, #bases V, #flats V)
o62 = (4, 8, 65, 79)
i63 : I = idealChowRing V; apply(0..<rank V, i -> hilbertFunction(i, I))
o63 : Ideal of QQ[x_{7}, x_{6}, x_{5}, x_{4}, x_{3}, x_{0}, x_{2}, x_{1}, x_{6, 7}, x_{5, 7}, ...
o64 = (1, 70, 70, 1)
i65 : cogeneratorChowRing M4
o65 = 2t^2_{5} + 2t^2_{4} + 2t^2_{3} + 2t^2_{2} + 2t^2_{1} + 2t^2_{0} - 2t t_{5} - 2t t_{0, 5} + ...

```

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SUPPLEMENT. The [online supplement](#) contains version 0.9.7 of `Matroids.m2`.

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