Journal of Software for Algebra and Geometry

```
i5 : betti(t,Weights=>{1,0})

            0  1   2   3 4
o5 = total: 1  4  13  14 4
        0:  1  .   .   . .
        1:  .  2   2   4 2
        2:  .  2   5   6 .
        3:  .  .   4   . 2
        4:  .  .   .   4 .
        5:  .  .   2   . .

o5 : BettiTally
i6 : betti(t,Weights=>{0,1})

            0  1   2   3 4
o6 = total: 1  4  13  14 4
        0:  1  .   .   . .
        1:  .  2   2   4 2
        2:  .  2   5   6 .
        3:  .  .   4   . 2
        4:  .  .   .   4 .
        5:  .  .   2   . .

o6 : BettiTally
i7 : t1 = betti(t,Weights=>{1,1})

            0  1   2   3 4
o7 = total: 1  4  13  14 4
        0:  1  .   .   . .
        1:  .  .   .   . .
        2:  .  .   .   . .
        3:  .  2   .   . .
        4:  .  .   .   . .
        5:  .  2   .   . .
        6:  .  .   1   . .
        7:  .  .   8   6 .
        8:  .  .   4   8 4

o7 : BettiTally
i8 : peek t1

o8 = BettiTally{(0, {0, 0}, 0) => 1 }
               (1, {2, 2}, 4) => 2
               (1, {3, 3}, 6) => 2
               (2, {3, 7}, 10) => 2
               (2, {4, 4}, 8) => 1
               (2, {4, 5}, 9) => 4
               (2, {5, 4}, 9) => 4
               (2, {7, 3}, 10) => 2
               (3, {4, 7}, 11) => 4
               (3, {5, 5}, 10) => 6
               (3, {7, 4}, 11) => 4
               (4, {5, 7}, 12) => 2
               (4, {7, 5}, 12) => 2
```

```
gap> g:= SymmetricGroup( 4 );
Sym( [ 1 .. 4 ] )
gap> tbl:= CharacterTable( g );;  HasIrr( tbl );
false
gap> tblmod2:= CharacterTable( tbl, 2 );
BrauerTable( Sym( [ 1 .. 4 ] ), 2 )
gap> tblmod2 = CharacterTable( tbl, 2 );
true
gap> tblmod2 = BrauerTable( tbl, 2 );
true
gap> tblmod2 = BrauerTable( tbl, 2 );

gap> libtbl:= CharacterTable( "M" );
CharacterTable( "M" )
gap> CharacterTableRegular( libtbl, 2 );
BrauerTable( "M" )
gap> BrauerTable( libtbl, 2 );
fail
gap> CharacterTable( "Symmetric", 4 );
CharacterTable( "Sym(4)" )
gap> ComputedBrauerTables( tbl );
[ , BrauerTable( Sym( [ 1 .. 4 ] ), 2 ) ]
```

```
ring r1 = 32003,(x,y,z),ds;
int a,b,c,t=11,5,3,0;
poly f = x^a+y^b+z^(3*c)+x^(c+2)*y^(c-1)+x^
         x^(c-2)*y^c*(y^2+t*x)^2;
option(noprot);
timer=1;
ring r2 = 32003,(x,y,z),dp;
poly f=imap(r1,f);
ideal j=jacob(f);
vdim(std(j));
==> 536
vdim(std(j+f));
==> 195
timer=0;  // reset timer
```

The TestIdeals package for Macaulay2

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ABSTRACT: We describe a *Macaulay2* package for computations in prime characteristic commutative algebra. This includes those for Frobenius powers and roots, p^{-e}-linear and p^e-linear maps, singularities defined in terms of these maps, different types of test ideals and modules, and ideals compatible with a given p^{-e}-linear map.

1. INTRODUCTION. This paper describes methods for computing objects and numerical invariants in prime characteristic commutative algebra, implemented in the package *TestIdeals* for the computer algebra system [Macaulay2]. A ring R of prime characteristic p comes equipped with the Frobenius endomorphism

$$F : R \to R \ \text{ given by } F(x) = x^p,$$

which is the basis for many constructions and methods. Notably, the Frobenius endomorphism can be used to detect whether a ring is regular [Kunz 1969], and further, to quantify how far a ring is from being regular, measuring the severity of a singularity.

In this direction, two notable applications of the Frobenius endomorphism are the theory of tight closure (see [Hochster and Huneke 1990; Hochster 2007] for an introduction) and the resulting theory of test ideals (see the survey [Schwede and Tucker 2012]). These methods are used by a wide group of commutative algebraists and algebraic geometers.

The *TestIdeals* package was started during a *Macaulay2* development workshop in 2012, hosted by Wake Forest University. The package, at that time called *PosChar*, aimed to provide a unified and efficient set of tools to study singularities in characteristic $p > 0$, and in particular, collecting and implementing several algorithms that had been described in research papers. Since then, *PosChar* has been split into two

packages, *TestIdeals* and [FrobeniusThresholds] (see [Hernández et al. 2019]), and much more functionality has been added by many contributors, during several more *Macaulay2* development workshops.[1]

Starting at least with Kunz [1969] and Fedder [1983], it has been known that the Frobenius endomorphism offers *effective* methods for measuring singularities in positive characteristic. However, the algorithms that form the basis of this package are the methods for computing Frobenius roots, ⋆-closures, and parameter test ideals. These first appeared in [Katzman 2008; 2010; Blickle et al. 2008; 2009]. Another algorithm in [Katzman and Schwede 2012] for computing prime ideals compatible with a given p^{-e}-linear map was implemented and used to produce the examples in that paper. The methods for computing test ideals and test modules that were used implicitly in papers such as [Blickle et al. 2010; Katzman et al. 2009; Schwede and Tucker 2014a] became implementable via the Frobenius roots functionality. Algorithms for computing F-pure thresholds from [Hernández 2014; 2015; Hernández and Teixeira 2017] form some of the key methods in the forthcoming *FrobeniusThresholds* package.

2. Frobenius powers and Frobenius roots.

Let R denote a commutative ring of prime characteristic p.

Definition 2.1. Given an ideal $I \subseteq R$ and an integer $e \geq 0$, we define the p^e-*th Frobenius power of I*, denoted $I^{[p^e]}$, to be the ideal generated by the p^e-th powers of all elements of I.

It is easy to see that if I is generated by $g_1, \ldots, g_\ell$, then $I^{[p^e]}$ is generated by $g_1^{p^e}, \ldots, g_\ell^{p^e}$.

Definition 2.2. Given an ideal $I \subseteq R$ and an integer $e \geq 0$, we define the p^e-*th Frobenius root of I*, denoted $I^{[1/p^e]}$, to be the smallest ideal J such that $I \subseteq J^{[p^e]}$, if such an ideal exists.

Frobenius roots always exist in polynomial and power series rings, and in F-finite regular rings (see [Blickle et al. 2008, §2] and [Katzman 2008, §5]). Below is an example of a computation of a Frobenius root in a polynomial ring. In Section 2A we describe the main ideas behind such computations.

```
i1 : R = ZZ/5[x,y,z];

i2 : I = ideal(x^6*y*z + x^2*y^12*z^3 + x*y*z^18);

o2 : Ideal of R

i3 : frobeniusPower(1/5, I)

              2   3
o3 = ideal (x, y , z )

o3 : Ideal of R
```

2A. *The mathematics behind the computation of Frobenius roots.*

We can also describe Frobenius roots as follows: In a (sufficiently local) regular ring, we have an identification of R with its canonical module ω_R. On the other hand, the Grothendieck dual to the e-iterated Frobenius map[2] $R \to R^{1/p^e}$

---

[1] Development workshops hosted by the University of California, Berkeley (2014, 2017), Boise State University (2015), and the University of Utah (2016).

[2] In a reduced ring R, it is often convenient to express the Frobenius endomorphism $R \to R$ as the inclusion $R \hookrightarrow R^{1/p}$, so that the source and target are distinguished.

provides a map

$$T : \omega_{R^{1/p^e}} \to \omega_R, \tag{2-1}$$

which, using the identification $R \cong \omega_R$, gives us a map

$$T : R^{1/p^e} \to R.$$

It is not difficult to prove that $T(I^{1/p^e}) = I^{[1/p^e]}$, where $I^{1/p^e} \subseteq R^{1/p^e}$ is simply the ideal of p^e-th roots of the elements of I. In the case that

$$R = \mathbb{K}[x_1, \ldots, x_d],$$

where $\mathbb{K}$ is a perfect field, R^{1/p^e} is a free R-module with basis consisting of the monomials

$$\boldsymbol{x}^{\lambda/p^e} = x_1^{\lambda_1/p^e} \cdots x_d^{\lambda_d/p^e},$$

where $0 \leq \lambda_i \leq p^e - 1$. Furthermore, the map $T : R^{1/p^e} \to R$ is simply the projection defined as follows:

$$T(x_1^{\lambda_1/p^e} \cdots x_d^{\lambda_d/p^e}) = \begin{cases} 1 & \text{if } \lambda_i = p^e - 1 \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

Using this, it is not difficult to see that if

$$f^{1/p^e} = \sum_{\lambda} f_{\lambda} \boldsymbol{x}^{\lambda/p^e}, \tag{2-2}$$

where λ runs over the tuples $(\lambda_1, \ldots, \lambda_d)$ with $0 \leq \lambda_j \leq p^e - 1$, and $f_{\lambda} \in R$, then $(f)^{[1/p^e]} = T((f)^{1/p^e})$ is the ideal generated by the f_{λ}. We can then compute $I^{[1/p^e]}$ for more general I by linearity: if $I = (f_1, \ldots, f_m)$, then $I^{[1/p^e]} = \sum(f_i)^{[1/p^e]}$.

Complexity. The computation of Frobenius roots is the main component behind many of the methods in *TestIdeals*. Hence, it is important to understand how this is implemented, and its computational complexity.

The computation of Frobenius roots of ideals reduces to the case of principal ideals, and its complexity grows linearly with the number of generators of the ideal. Furthermore the calculation of $(f)^{[1/p^e]}$ reduces to finding the terms in the right-hand side of (2-2), which essentially amounts to taking each term in f, computing the p^e-th root of its coefficient, and dividing the monomial exponent vector by p^e with remainder. Hence the complexity of computing $(f)^{[1/p^e]}$ is proportional to the number of terms in f, and is independent of its degree. We emphasize that the calculation of Frobenius roots *does not involve the calculation of Gröbner bases.*

2B. ***Dual to Frobenius on quotient rings and Frobenius on top local cohomology.*** For *any* reduced ring R with finite e-iterated Frobenius map identified with $R \hookrightarrow R^{1/p^e}$, we always have a map

$$T_R : \omega_{R^{1/p^e}} \to \omega_R.$$

At a maximal ideal $\mathfrak{m}$ of height d, this map is Matlis dual to the Frobenius map on $H^d_{\mathfrak{m}}(R)$. Hence we can study local cohomology by studying this map. If $R = S/I$, where S is a polynomial ring over a finite field, we can implement this map [Fedder 1983; Katzman 2008]. We briefly explain the case of a hypersurface here.

If $I = (f)$, again $\omega_R \cong R = S/I$, and the action of T_R can be computed on S. In fact, if $\bar{J} \subseteq R \cong \omega_R$ has preimage $J \subseteq S$, then setting $u = f^{p^e-1}$, we have that $T_R(\bar{J}^{1/p^e})$ is the image of $(uJ)^{[1/p^e]}$ in R. More generally, for non-hypersurfaces, the analog to u is chosen to be an element that maps onto the generator of the cyclic R-module

$$\frac{(I^{[p^e]} : I) \cap (\Omega^{[p^e]} : \Omega)}{I^{[p^e]}},$$

where Ω is an ideal of S with $I \subseteq \Omega$ and whose image in R is isomorphic to the canonical module of R; see [Katzman 2008] for details. Here we compute two examples.

```
i1 : S = ZZ/5[x,y,z];

i2 : f = x^3 + y^3 + z^3;

i3 : u = f^(5 - 1);

i4 : frobeniusRoot(1, ideal u)

o4 = ideal (z, y, x)

o4 : Ideal of S

i5 : S = ZZ/7[x,y,z];

i6 : f = x^3 + y^3 + z^3;

i7 : u = f^(7 - 1);

i8 : frobeniusRoot(1, ideal u)

o8 = ideal 1

o8 : Ideal of S
```

The above example has shown that the Frobenius map on top local cohomology of the cone over the Fermat elliptic curve is injective in characteristic 7 (the dual map $\omega_{R^{1/p}} \to \omega_R$ is surjective) and not injective in characteristic 5 (the dual map $\omega_{R^{1/p}} \to \omega_R$ is not surjective).

2C. *A generalization of Frobenius powers and roots.* Let R be an F-finite regular ring. We can extend the definition of Frobenius powers as follows.

Definition 2.3 [Hernández et al. 2018]. Let $I \subseteq R$ be an ideal.

(a) If n is a positive integer with base p expansion $n = d_0 + d_1 p + \cdots + d_r p^r$, we define

$$I^{[n]} = I^{d_0}(I^{d_1})^{[p]} \cdots (I^{d_r})^{[p^r]}.$$

(b) If t is a nonnegative rational number of the form $t = a/p^e$, we define $I^{[t]} = (I^{[a]})^{[1/p^e]}$.

(c) If t is any nonnegative rational number, and $\{a_n/p^{e_n}\}_{n \geq 1}$ is a sequence of rational numbers converging to t from above, we define $I^{[t]}$ to be the stable value of the nondecreasing chain of ideals $\{I^{[a_n/p^{e_n}]}\}_{n \geq 1}$.

```
i1 : p = 3;

i2 : R = ZZ/p[x,y];

i3 : m = monomialIdeal(x, y);

o3 : MonomialIdeal of R

i4 : I = m^5;

o4 : MonomialIdeal of R

i5 : t = 3/5 - 1/(5*p^3);

i6 : frobeniusPower(t, I)

            2        2
o6 = ideal (y , x*y, x )

o6 : Ideal of R

i7 : frobeniusPower(t - 1/p^5, I)

o7 = ideal (y, x)

o7 : Ideal of R
```

The generalized Frobenius powers of an ideal are test ideals (see Section 4) of sufficiently general linear combinations of generators of that ideal. Thus, the ideals computed above are test ideals of generic quintics in two variables in characteristic $p = 3$, and these computations suggest that $3/5 - 1/(5p^3)$ is a higher F-jumping exponent of such quintics, which is indeed the case for any $p \equiv 3 \bmod 5$ (see [Hernández et al. 2020]). To illustrate, compare the computations above with the following.

```
i8 : S = ZZ/3[a..f,x,y];

i9 : G = a*x^5 + b*x^4*y + c*x^3*y^2 + d*x^2*y^3 + e*x*y^4 + f*y^5;

i10 : testIdeal(t, G)

             2        2
o10 = ideal (y , x*y, x )

o10 : Ideal of S

i11 : testIdeal(t - 1/p^5, G)

o11 = ideal (y, x)

o11 : Ideal of S
```

Caveat. The computations of Frobenius powers with exponents whose denominators are not powers of the characteristic p are often very slow, and in several instances it is more efficient to introduce more variables and compute the test ideal of a "very general" linear combination of the generators of the ideal, as above. The code used for computations of Frobenius powers needs to undergo significant optimization.

2D. *Frobenius roots of submodules of free modules.* Let S be a polynomial ring or power series ring. Given a submodule M of the free module S^k, there is a smallest submodule N of S^k that contains M, for which $M \subseteq N^{[p^e]}$. Here, $N^{[p^e]}$ is the submodule of S^k generated by the vectors of N with all coordinates raised to the p^e-th power. (See [Katzman and Zhang 2014].) Extending the previous definitions, we call this N the p^e-th Frobenius root of M, and denote it by $M^{[1/p^e]}$.

Example 2.4. Let $R = \mathbb{Z}/p\mathbb{Z}[a, b, c, d]$, and consider the ideals $\mathfrak{m} = (a, b, c, d)$ and

$$I = (a, b) \cap (a, c) \cap (c, d) \cap (c + d, a^3 + bd^2).$$

Then R/I is a 3-dimensional non–Cohen–Macaulay ring. Matlis duality applied to $H^2_{\mathfrak{m}}(R/I)$ with its natural Frobenius map yields a p^{-1}-linear map U on $\text{Ext}^2_R(R/I, R)$.

```
i1 : R = ZZ/2[a,b,c,d];

i2 : I = intersect(ideal(a,b),ideal(a,c),ideal(c,d),ideal(c+d,a^3+b*d^2));

o2 : Ideal of R

i3 : f = inducedMap(R^1/I, R^1/frobenius(I));

o3 : Matrix

i4 : E2 = Ext^2(f, R^1)

o4 = {-8}  | a4+abc2+abcd a2b             |
     {-10} | a2cd3        a3cd+a3d2+bcd3 |

o4 : Matrix

i5 : target E2

o5 = cokernel {-8}  | 0     a2 b2c2 |
              {-10} | c2+d2 d4 a4d2 |

                                     2
o5 : R-module, quotient of R

i6 : source E2

o6 = cokernel {-4} | 0   a  bc  |
              {-5} | c+d d2 a2d |

                                     2
o6 : R-module, quotient of R
```

The Frobenius map on $H^2_{\mathfrak{m}}(R/I)$ is injective if and only if the image of $(\text{Image } U)^{[1/p]}$ in $\text{Ext}^2_R(R/I, R)$ as computed above is the whole of R^2.

```
i7 : U = matrix entries E2;

            2       2
o7 : Matrix R  <--- R

i8 : A = image matrix entries relations source E2;

i9 : frobeniusRoot(1, image U)

o9 = image {-2} | 1 0 0 |
           {0}  | 0 d a |

                                 2
o9 : R-module, submodule of R
```

The calculation above shows that R/I is not F-injective (see Section 3). Moreover, it shows that $H^2_{\mathfrak{m}}(R/I) \cong \text{Ann}_{E^2} A^t$, where E is the injective hull of $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$, and the Frobenius map on $H^2_{\mathfrak{m}}(R/I)$ induced from Frobenius on R is given by $U^t\Theta$, where Θ is the induced Frobenius on E. The submodule of nilpotent elements in $H^2_{\mathfrak{m}}(R/I)$ is given by $\text{Ann}_{E^2} B^t$, where B is the smallest submodule of R^2 containing Image $A + \text{Image } U$ such that $UB \subseteq B^{[p]}$. The method `ascendModule` can be used to calculate B (see a detailed description of the similar method `ascendIdeal` in Section 4).

```
i10 : B = ascendModule(1, A, U)

o10 = image | 0   a  bc  |
            | c+d d2 a2d |

                                  2
o10 : R-module, submodule of R
```

3. F-SINGULARITIES.   The *TestIdeals* package includes methods for determining if a ring is F-injective, F-pure, F-rational, or F-regular.

3A. *F-injectivity.*

Definition 3.1. A local ring $(R, \mathfrak{m})$ is called *F-injective* if the map $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R^{1/p})$ is injective for all $i > 0$. An arbitrary ring is called *F-injective* if its localization at each prime ideal of R is F-injective.

The function `isFInjective` determines whether the ring $R = S/I$ is F-injective, where S is a polynomial ring.

```
i1 : R = ZZ/7[x,y,z]/(x^3 + y^3 + z^3);
i2 : isFInjective R
o2 = true
i3 : R = ZZ/5[x,y,z]/(x^3 + y^3 + z^3);
i4 : isFInjective R
o4 = false
```

Equivalently, a ring is F-injective if the maps on the cohomology of the dualizing complex

$$h^{-i}\omega^{\bullet}_{R^{1/p}} \to h^{-i}\omega^{\bullet}_R$$

surject for all i. Note that $h^{-i}\omega^{\bullet}_R \cong \mathrm{Ext}^{\dim S - i}(R, S)$, the latter of which, *Macaulay2* readily computes. The algorithm used by `isFInjective` works by checking the surjectivity of the dual Frobenius map

$$\mathrm{Ext}^{\dim S - i}(R^{1/p}, S) \to \mathrm{Ext}^{\dim S - i}(R, S).$$

We begin by computing the map $R \to R^{1/p}$ using [PushForward]. Next, the algorithm computes $\mathrm{Ext}^i(\_, S)$ applied to the map from the previous step. Then R is F-injective precisely when the cokernel of $\mathrm{Ext}^i(\_, S)$ is trivial for i.

The Frobenius action on top local cohomology (dual to ω_R) is usually computed in a different (faster) way than the other cohomologies, and this is modified by the `CanonicalStrategy` option. The default value for this option is `Katzman`, which instead of using the *PushForward* package, relies on the fact that we already know how to compute the Frobenius action on the canonical module, as described in Section 2B.

The performance of the algorithm can be improved if the ring of interest is nice enough. If the ring is Cohen–Macaulay, then setting `AssumeCM => true` (the default value is `false`) lets the algorithm check the Frobenius action only on top cohomology (which is typically much faster, as explained above). When studying a reduced ring, setting `AssumeReduced => true` (the default value) avoids computing the bottom local cohomology, and when studying a normal ring, setting `AssumeNormal => true` (the default is `false`) avoids computing the bottom two local cohomologies.

By default the algorithm checks for F-injectivity at all points of Spec R. However, one can choose to check F-injectivity only at the origin by setting the option `AtOrigin` to `true`.

```
i1 : R = ZZ/7[x,y,z]/((x - 1)^5 + (y + 1)^5 + z^5);

i2 : isFInjective R -- R is not globally F-injective...

o2 = false

i3 : isFInjective(R, AtOrigin => true) -- but is F-injective at the origin

o3 = true
```

3B. *F-regularity.*

Definition 3.2 [Hochster and Huneke 1990; Hara and Watanabe 2002]. A ring R is called *strongly F-regular* if the (big) test ideal $\tau(R)$ is equal to R. Likewise a pair (R, f^t) is called *strongly F-regular* if $\tau(R, f^t) = R$.

The command `isFRegular` checks whether a ring or pair is strongly F-regular. Below are two examples, one of which is F-regular and the other is not.

```
i1 : R = ZZ/5[x,y,z]/(x^2 + y*z);

i2 : isFRegular R

o2 = true

i3 : R = ZZ/7[x,y,z]/(x^3 + y^3 + z^3);

i4 : isFRegular R

o4 = false
```

We can also check whether a pair (R, f^t) is F-regular.

```
i1 : R = ZZ/5[x,y];

i2 : f = y^2 - x^3;

i3 : isFRegular(1/2, f)

o3 = true

i4 : isFRegular(5/6, f)

o4 = false

i5 : isFRegular(4/5, f)

o5 = false

i6 : isFRegular(4/5 - 1/100000, f)

o6 = true
```

All of these checks are done by actually computing the test ideal, as described in Section 4.

If the input ring is $\mathbb{Q}$-Gorenstein, then in each of the cases above, the output is a boolean indicating if the ring is strongly F-regular. If the input ring is not $\mathbb{Q}$-Gorenstein, then the algorithm can be used to determine if a ring is strongly F-regular, but cannot prove that a ring is not strongly F-regular (this latter functionality can, however, be enabled by setting `QGorensteinIndex => infinity`).

In the case that R is $\mathbb{Q}$-Gorenstein, the algorithm works by computing the test ideal τ of the ring (or the pair) using `testIdeal` and checking whether $\tau = R$. In the non-$\mathbb{Q}$-Gorenstein case, the algorithm checks for strong F-regularity by computing better and better approximations of the test ideal, and checking whether any of these is the unit ideal. To compute approximations of the test ideal, the algorithm computes a test element c with `testElement` and then uses `frobeniusRoot` to compute the e-th root

of $c(I^{[p^e]} : I)$; appropriate modifications are made for pairs. If at any step the approximation is the unit ideal, then the algorithm returns `true`. Otherwise the algorithm continues checking for each e until a specified limit is reached. The default limit is 2, and can be changed using the option `DepthOfSearch`.

A number of options can be used to speed up the performance of some of the internal functions. The option `AssumeDomain` can be used if R is an integral domain, `FrobeniusRootStrategy` chooses a strategy for internal `frobeniusRoot` calls, `MaxCartierIndex` sets the maximum Gorenstein index to search for when working with a $\mathbb{Q}$-Gorenstein ambient ring, and `QGorensteinIndex` allows the user to specify the $\mathbb{Q}$-Gorenstein index of the ring.

The default behavior of `isFRegular` is that it checks for strong F-regularity globally. If the option `AtOrigin` is set to `true`, the algorithm will only check F-regularity at the origin, by checking whether the computed test ideal is in the irrelevant ideal. Below are examples for both a ring and a pair.

```
i1 : R = ZZ/7[x,y,z]/((x - 1)^3 + (y + 1)^3 + z^3);
i2 : isFRegular R -- R is not globally F-regular...
o2 = false
i3 : isFRegular(R, AtOrigin => true) -- but is F-regular at the origin
o3 = true
i4 : R = ZZ/13[x,y];
i5 : f = (y - 2)^2 - (x - 3)^3;
i6 : isFRegular(5/6, f) -- (R,f^(5/6)) is not F-regular...
o6 = false
i7 : isFRegular(5/6, f, AtOrigin => true) -- but is F-regular at the origin
o7 = true
```

3C. *F-purity.*

Definition 3.3. A ring R is called *F-pure* if the inclusion $R \hookrightarrow R^{1/p^e}$ is a pure map, i.e., the tensor of this map with any R-module remains injective. If $R^{1/p}$ is a finite R-module, this is equivalent to requiring that the inclusion $R \hookrightarrow R^{1/p}$ split as a map of R-modules.

The function `isFPure` checks whether a ring is F-pure. Either a ring or a defining ideal can be input, as seen in the following example.

```
i1 : R = ZZ/5[x,y,z]/(x^2 + y*z);
i2 : isFPure R
o2 = true
i3 : R = ZZ/7[x,y,z]/(x^3 + y^3 + z^3);
i4 : isFPure R
o4 = true
i5 : S = ZZ/2[x,y,z];
i5 : isFPure ideal(y^2 - x^3)
o5 = false
i6 : isFPure ideal(z^2 - x*y*z + x*y^2 + x^2*y)
o6 = true
```

The algorithm works by applying Fedder's criterion [1983], which states that a local ring $(R, \mathfrak{m})$ is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$. When `AtOrigin` is set to `true`, the algorithm checks F-purity only at the origin, by explicitly checking the above containment. When `AtOrigin` is set to `false`, which is its default value, the algorithm computes the non F-pure locus, by applying `frobeniusRoot` to $I^{[p]} : I$. If the non F-pure locus is the whole ring, the algorithm returns `true`.

3D. *F-rationality.*

Definition 3.4. Suppose that R is a Cohen–Macaulay ring and that $T_R : \omega_{R^{1/p}} \to \omega_R$ is the canonical dual to the Frobenius. We say that R has *F-rational singularities*, or simply that *R is F-rational*, if there are no nonzero proper submodules M of ω_R such that $T_R(M^{1/p}) \subseteq M$.

The command `isFRational` checks if a ring is F-rational.

```
i1 : S = ZZ/3[a,b,c,d,t];
i2 : M = matrix{{ a^2 + t^4, b, d }, { c, a^2, b^3 - d }};
              2        3
o2 : Matrix S  <--- S
i3 : I = minors(2, M);
o3 : Ideal of S
i4 : R = S/I;
i5 : isFRational R
o5 = true
```

The algorithm used by `isFRational` first checks if the ring is Cohen–Macaulay, unless the option `AssumeCM` is set to `true`. If the ring is not Cohen–Macaulay then `false` is returned. Next, the algorithm computes the test module $M \subseteq \omega_R$ and checks to see if $\omega_R \subseteq M$; see the next section for the description of a test module.

The options `AssumeDomain` and `FrobeniusRootStrategy` can be used to improve the speed of the `testModule` computation. By default, these options are set to `false` and `Substitution`, respectively. Finally, if `AtOrigin` is set to `true`, then F-rationality is checked only at the origin.

4. TEST IDEALS.   In this section, we explain how to compute parameter test modules, parameter test ideals, test ideals, and HSLG modules.[3]

4A. *Parameter test modules.* Given a reduced ring R of finite type over a perfect field k, the Frobenius map $R \hookrightarrow R^{1/p^e}$ is dual to

$$T : \omega_{R^{1/p^e}} \to \omega_R.$$

As in Section 2B, we can represent the canonical module $\omega_R \subseteq R$ as an ideal, we can write $R = S/I$, and so we can find an element $u \in S^{1/p^e}$ representing the map $T : \omega_{R^{1/p^e}} \to \omega_R$; see Section 2B or [Katzman 2008].

---

[3]HSLG modules can be used to give a scheme structure to the F-injective or F-pure locus.

Definition 4.1. The *parameter test submodule* is the smallest submodule $M \subseteq \omega_R$ (and hence ideal of R, since $M \subseteq \omega_R \subseteq R$) that agrees generically with ω_R and that satisfies

$$T(M^{1/p^e}) \subseteq M$$

for some $e > 0$ (or equivalently for all $e > 0$).

Using *Macaulay2*, we can compute this using the `testModule` command as follows.

```
i1 : R = ZZ/5[x,y,z]/(x^4 + y^4 + z^4);
i2 : N = testModule R;
i3 : N#0
           2           2        2
o3 = ideal (z , y*z, x*z, y , x*y, x )

o3 : Ideal of R
i4 : N#1
o4 = ideal 1
o4 : Ideal of R
```

The output of `testModule` is a sequence, consisting of three items:

(1) The test module itself, given as an ideal of R.

(2) The canonical module that contains the test module, given as an ideal of R. (Note the representation of the canonical module as an ideal is not unique; it is only unique up to isomorphism, hence it is important to return this module as well).

(3) The element u described above (not displayed above, since it takes a lot of space).

Note since this ring is Gorenstein, the canonical module is simply represented as the unit ideal. Here is another example, where the ring is not Gorenstein.

```
i1 : R = ZZ/5[x,y,z]/(y*z, x*z, x*y);
i2 : N = testModule R;
i3 : N#0
           2   2   2
o3 = ideal (z , y , x )
o3 : Ideal of R
i4 : N#1
o4 = ideal (y + 2z, x - z)
o4 : Ideal of R
```

We briefly explain how this is computed: First, we find a *test element*.

Remark 4.2 (computation of test elements). We recall that, roughly, an element of the Jacobian ideal that is not contained in any minimal prime is a test element [Hochster 2007]. We compute test elements by computing random partial derivatives (and linear combinations thereof) until we find an element that does not vanish at all the minimal primes. This method is much faster than computing the entire

Jacobian ideal. If it is known that the ring is a domain, setting `AssumeDomain => true` can speed this up further.

After the test element c has been identified, we pull back the ideal ω_R to an ideal $J \subseteq S$. Next, we compute the following ascending sequence of ideals where u represents $T : \omega_{R^{1/p^e}} \to \omega_R$ as above:

$$
\begin{aligned}
J_0 &:= cJ \\
J_1 &:= J_0 + (uJ_0)^{[1/p]} \\
J_2 &:= J_1 + (uJ_1)^{[1/p]} = J_0 + (uJ_0)^{[1/p]} + (u^{1+p}J_0)^{[1/p^2]} \\
J_3 &:= J_2 + (uJ_2)^{[1/p]} = J_0 + (uJ_0)^{[1/p]} + (u^{1+p}J_0)^{[1/p^2]} + (u^{1+p+p^2}J_0)^{[1/p^3]} \\
&\vdots
\end{aligned}
\tag{4-1}
$$

As soon as this ascending sequence of ideals stabilizes, we are done. In fact, because this strategy is used in several contexts, the user can call it directly for a chosen ideal J and u with the function `ascendIdeal` (this is done for test ideals below).

We can also compute parameter test modules of pairs (ω_R, f^t) with $t \in \mathbb{Q}_{\geq 0}$. This is done by modifying the element u when the denominator of t is not divisible by p. When t has p in its denominator, we rely on the fact (see [Blickle et al. 2008; Schwede and Tucker 2014a]) that

$$T(\tau(\omega_R, f^a)) = \tau(\omega_R, f^{a/p}),$$

$$\texttt{frobeniusRoot}(1, \mathtt{u} * \mathtt{I}_1) = \mathtt{I}_2,$$

where the second line roughly explains how this is accomplished internally. Here $\mathtt{I}_1$ is $\tau(\omega_R, f^a)$ pulled back to S and, likewise, $\mathtt{I}_2$ defines $\tau(\omega_R, f^{a/p})$ modulo the defining ideal of R.

Remark 4.3 (optimizations in `ascendIdeal` and other `testModule` computations). Throughout the computations described above, we very frequently use the following fact:

$$(f^p \cdot J)^{[1/p]} = f \cdot (J^{[1/p]}).$$

To access this enhancement, one should try to pass functions like `ascendIdeal` and `frobeniusRoot` the elements and their exponents (see the documentation). In particular, when computing the p^e-th Frobenius root of an ideal of the form $f^n \cdot J$, one should not multiply out f^n and J, nor even raise f to the n-th power, but rather call `frobeniusRoot(e, n, f, J)`.

4B. *Parameter test ideals.* The parameter test ideal is simply the annihilator of $\omega_R/\tau(\omega_R)$. In other words, it is

$$(\tau(\omega_R) : \omega_R).$$

This can also be described as

$$\bigcap_I (I^* : I),$$

where I varies over ideals defined by a partial system of parameters, and I^* denotes its tight closure. This latter description is not computable, however.

Example 4.4. In this example we repeat the calculation in [Katzman 2008, §9].

```
i1 : R = ZZ/2[a..e];

i2 : E = matrix {{a, b, b, e}, {d, d, c, a}};

              2       4
o2 : Matrix R  <--- R

i3 : I = minors(2, E);

o3 : Ideal of R

i4 : S = R/I;

i5 : J = parameterTestIdeal S

o5 = ideal (c + d, b, a)

o5 : Ideal of S

i6 : J = substitute(J, R);

o6 : Ideal of R

i7 : mingens(J + I)

o7 = | c+d b a de |

              1       4
o7 : Matrix R  <--- R
```

4C. *Test ideals.* For an F-finite reduced ring $R = S/I$, where S is a regular ring, the (big) test ideal[4] of R is the smallest ideal J in R, not contained in any minimal prime, such that for all $e > 0$ and all $\phi \in \text{Hom}_R(R^{1/p^e}, R)$, we have

$$\phi(J^{1/p^e}) \subseteq J.$$

In the case that R is Gorenstein, $\text{Hom}_R(R^{1/p^e}, R)$ is a cyclic R^{1/p^e}-module generated by Φ_e, which corresponds with the map T above based on the identification $\omega_R \cong R$ [Blickle and Schwede 2013]. More generally, if R is $\mathbb{Q}$-Gorenstein with index not divisible by p, then for at least sufficiently divisible $e > 0$, such a generating Φ_e still exists.

If such a Φ_e exists, it can be identified to a generator of the module

$$(I^{[p^e]} : I)/I^{[p^e]}$$

by Fedder's lemma [1983]. Hence we can find a corresponding[5] $u \in I^{[p^e]} : I$. In this case, if $c \in S$ is the preimage of a test element of R, then setting $I_0 = cR$, it follows that

$$\tau(R) = \texttt{ascendIdeal(e, u, I}_0\texttt{)},$$

where $\texttt{ascendIdeal}$ is the method explained above, in (4-1). The e here means all Frobenius roots are taken as multiples of e. In other words, we first compute $\texttt{I}_0 + (\texttt{u} \cdot \texttt{I}_0)^{1/\texttt{p}^{[e]}}$. Then we compute

$$\texttt{I}_0 + (\texttt{u} \cdot \texttt{I}_0)^{1/\texttt{p}^{[e]}} + (\texttt{u}^{\texttt{p}^e+1} \cdot \texttt{I}_0)^{1/\texttt{p}^{[2e]}}$$

etc.

---

[4]The notion of test ideals was originally introduced in [Hochster and Huneke 1990] in the context of tight closure in finitely generated modules, whereas our notion of test ideals arises from tight closure in ("big") modules which are possibly not finitely generated. Confusingly, big test ideals are included in (small) test ideals.

[5]This is done via the function `QGorensteinGenerator`.

Here is an example (a $\mathbb{Z}/3\mathbb{Z}$-quotient, where $3 \mid (7-1)$), where exactly this logic occurs.

```
i1 : T = ZZ/7[x,y];

i2 : S = ZZ/7[a,b,c,d];

i3 : f = map(T, S, {x^3, x^2*y, x*y^2, y^3});

i4 : I = ker f;

i5 : R = S/I;

i6 : testIdeal R

o6 = ideal 1

o6 : Ideal of R
```

However, the term u constructed above can be quite complicated if $e > 1$ (which happens exactly when $(p-1)K_R$ is not Cartier). For instance, even in the above example we have an extremely complex u:

```
i7 : toString QGorensteinGenerator(1, R)

o7 = a^2*b^6*c^12+a^3*b^4*c^13+a^3*b^5*c^11*d+a^4*b^3*c^12*d+a^5*b*c^13*d+b^
     12*c^6*d^2+a^3*b^6*c^9*d^2+a^4*b^4*c^10*d^2+a^5*b^2*c^11*d^2+a^6*c^12*d
     ^2+b^13*c^4*d^3+a*b^11*c^5*d^3+a^2*b^9*c^6*d^3+a^4*b^5*c^8*d^3+a^5*b^3*
     c^9*d^3+a^6*b*c^10*d^3+a*b^12*c^3*d^4+a^2*b^10*c^4*d^4+a^3*b^8*c^5*d^4+
     a^4*b^6*c^6*d^4+a^5*b^4*c^7*d^4+a^6*b^2*c^8*d^4+a^7*c^9*d^4+a*b^13*c*d^
     5+a^2*b^11*c^2*d^5+a^3*b^9*c^3*d^5+a^4*b^7*c^4*d^5+a^5*b^5*c^5*d^5+a^6*
     b^3*c^6*d^5+a^7*b*c^7*d^5+a^2*b^12*d^6+a^3*b^10*c*d^6+a^4*b^8*c^2*d^6+a
     ^5*b^6*c^3*d^6+a^6*b^4*c^4*d^6+a^7*b^2*c^5*d^6+a^8*c^6*d^6+a^4*b^9*d^7+
     a^5*b^7*c*d^7+a^6*b^5*c^2*d^7+a^7*b^3*c^3*d^7+a^8*b*c^4*d^7+a^6*b^6*d^8
     +a^7*b^4*c*d^8+a^8*b^2*c^2*d^8+a^9*c^3*d^8+a^8*b^3*d^9+a^9*b*c*d^9+a^10*d^10
```

Therefore, we use a different strategy if either $(p-1)K_R$ is not Cartier or, more generally, if R is $\mathbb{Q}$-Gorenstein of index divisible by p. In these situations, this alternate strategy typically appears to be faster. We rely on the observation (see [Blickle et al. 2015]) that

$$\tau(\omega_R, K_R) \cong \tau(R).$$

In fact, by embedding $\omega_R \subseteq R$, we can compute g so that $\tau(\omega_R, K_R) = g \cdot \tau(R)$. We can therefore find $\tau(R)$ if we can find $\tau(\omega_R, K_R)$. Next, if K_R is $\mathbb{Q}$-Cartier with $nK_R = \mathrm{div}_R(f)$ for some $f \in R$ and $n > 0$, then

$$\tau(\omega_R, K_R) = \tau(\omega_R, f^{1/n}).$$

Thus we directly compute $\tau(\omega_R, f^{1/n})$ via the command `testModule(1/n, f)`. Consider the following example, a μ_3-quotient, which uses the logic described above.

```
i1 : T = ZZ/3[x,y];

i2 : S = ZZ/3[a,b,c,d];

i3 : f = map(T, S, {x^3, x^2*y, x*y^2, y^3});

o3 : RingMap T <--- S

i4 : I = ker f;

o4 : Ideal of S
```

```
i5 : R = S/I;
i6 : testIdeal R
o7 = ideal 1
o7 : Ideal of R
```

Remark 4.5 (nongraded caveats). It frequently happens that $(I^{[p^e]} : I)/I^{[p^e]}$ is principal but *Macaulay2* cannot identify the principal generator (since *Macaulay2* cannot always find minimal generators of non-graded ideals or modules). The same thing can happen when computing the element u corresponding to the map $T : \omega_{R^{1/p}} \to \omega_R$, as described in Section 2B. In such situations, instead of a single u, *Macaulay2* will produce $u_1, \ldots, u_n$ (all multiples of u, and u is a linear combination of the u_i). Instead of computing the ideal

$$(u \cdot J)^{[1/p]},$$

we compute

$$(u_1 \cdot J)^{[1/p]} + \cdots + (u_n \cdot J)^{[1/p]},$$

which will produce the same answer.

We can similarly use the `testIdeal` command to compute test ideals of pairs, $\tau(R, f^t)$, and even mixed test ideals, $\tau(R, f_1^{t_1} \cdots f_n^{t_n})$.

4D. *HSLG module; computing F-pure submodules of rank-1 Cartier modules.* Again, take the maps $T^e : \omega_{R^{1/p^e}} \to \omega_R$ we have considered throughout this section. From [Hartshorne and Speiser 1977; Lyubeznik 1997; Gabber 2004], we have the theorem that the descending images

$$\omega_R \supseteq T(\omega_{R^{1/p}}) \supseteq \cdots \supseteq T^e(\omega_{R^{1/p^e}}) \supseteq T^{e+1}(\omega_{R^{1/p^{e+1}}}) \supseteq \cdots \tag{4-2}$$

stabilize for $e \gg 0$. The function `FPureModule` computes the stable submodule in this chain, called the *HSLG module*, and returns a sequence containing the following items:

(1) The HSLG module.

(2) The canonical module, embedded (nonuniquely) as an ideal.

(3) The element u representing the map on the canonical module (see Section 2B).

(4) The value of $e > 0$ at which the descending images in (4-2) stabilize. This is sometimes also called the HSLG number of the canonical module as a Cartier module.

```
i1 : R = ZZ/3[x,y,z]/(x^3 + y^4 + z^5);
i2 : L = FPureModule R;
i3 : L#0
                   2        2   3
o3 = ideal (y*z, x*z, y , x*y, x , z )
o3 : Ideal of R
i4 : L#1
o4 = ideal 1
o4 : Ideal of R
```

```
i5 : L#3
o5 = 1
i6 : R = ZZ/3[x,y,z]/(x^3 + y^4 + z^5);
i7 : L = FPureModule R;
i8 : L#0
                  2        2   3
o8 = ideal (y*z, x*z, y , x*y, x , z )
o8 : Ideal of R
i9 : L#1
o9 = ideal 1
o9 : Ideal of R
i10 : L#3
o10 = 1
```

More generally, for any ideal J with a map $\phi : J^{1/p^e} \to J$, we have that the images

$$J \supseteq \phi(J^{1/p^e}) \supseteq \phi^2(J^{1/p^{2e}}) \supseteq \cdots$$

stabilize as well (in fact, the analogous result even holds for modules [Gabber 2004]).

4E. *Computing the level of a polynomial.* Another interesting invariant that can be calculated using this package is the so-called *level* of a polynomial; more precisely:

Definition 4.6. Let K be an F–finite field of prime characteristic p, and let $f \in R = K[x_1, \ldots, x_d]$. We define the *level of f* as the smallest possible integer e where the descending chain

$$R = (f^{p^0-1})^{[1/p^0]} \supseteq (f^{p-1})^{[1/p]} \supseteq (f^{p^2-1})^{[1/p^2]} \supseteq \cdots \supseteq (f^{p^i-1})^{[1/p^i]} \supseteq \cdots$$

stabilizes.

This invariant was introduced in [Alvarez-Montaner et al. 2005]. It is also essentially the same data as the HSLG number of the pair (R, f^1) as computed in (4) above (it is that number $+1$). One interesting particular case is when $K = \mathbb{F}_p$ and f is the defining homogeneous polynomial of a hyperelliptic curve of genus g; when $g = 1$, it was proved in [Boix et al. 2015] that the corresponding elliptic curve defined by f is ordinary if and only if its level is 1, (equivalently, if and only if $(R/(f))_{(x,y,z)}$ is F-pure,) and supersingular if and only if its level is 2. When the genus is at least 2, level 2 is a necessary (but not sufficient) condition for the curve for being ordinary [Blanco-Chacón et al. 2018]; in this case, one also has that, if the curve is supersingular, then its level has to be at least 3, so the level can always distinguish these two properties in any genus. We illustrate these results by means of the following examples.

```
i1 : R = ZZ/2[x,y,z];
i2 : f = x^3 + y^2*z + y*z^2;
i3 : frobeniusPower(1/2, ideal f)
o3 = ideal (z, y, x)
o3 : Ideal of R
```

```
i4 : u = f^3;
i5 : frobeniusPower(1/4, ideal u)
o5 = ideal (z, y, x)
o5 : Ideal of R
i6 : ((FPureModule(1, f))#3) + 1
o6 = 2
```

This shows that this elliptic curve has level 2; hence it is supersingular. The next example shows a nonordinary hyperelliptic curve of genus 2 with level 2.

```
i1 : R = ZZ/11[x,y,z];
i2 : f = y^2*z^3 - x^5 - 2*z^5;
i3 : frobeniusPower(1/11, ideal f^10)
            2      3
o3 = ideal (z , x*z, x )
o3 : Ideal of R
i4 : frobeniusPower(1/121, ideal f^120)
            2      3
o4 = ideal (z , x*z, x )
o4 : Ideal of R
i5 : ((FPureModule(1, f))#3) + 1
o5 = 2
```

5. IDEALS COMPATIBLE WITH A GIVEN p^{-e}-LINEAR MAP. Throughout this section, let R denote a polynomial $\mathbb{K}[x_1, \ldots, x_n]$. In this section we address the following question: given an R-linear map $\phi : R^{1/p^e} \to R$, what are the ideals $I \subseteq R$ such that $\phi(I^{1/p^e}) \subseteq I$? We call these ideals ϕ-compatible.

Recall that $\operatorname{Hom}_R(R^{1/p^e}, R)$ is a principal R^{1/p^e}-module generated by a *trace map* $T \in \operatorname{Hom}_R(R^{1/p^e}, R)$, constructed as in Section 2A (see [Fedder 1983, Lemma 1.6] and [Brion and Kumar 2005, Example 1.3.1]).

We can now write our given ϕ as multiplication by some u^{1/p^e} followed by T and it is not hard to see that an ideal $I \subseteq R$ is ϕ-compatible if and only if $uI \subseteq I^{[p^e]}$.

Theorem 5.1. *If ϕ is surjective, there are finitely many ϕ-compatible ideals, consisting of all possible intersections of ϕ-compatible prime ideals* (see [Kumar and Mehta 2009, Schwede 2009, Sharp 2007, Enescu and Hochster 2008]). *In general, there are finitely many ϕ-compatible prime ideals not containing* $\sqrt{\operatorname{Image} \phi}$ (see [Katzman and Schwede 2012]).

The method `compatibleIdeals` produces the finite set of ϕ-compatible prime ideals in the second claim of Theorem 5.1.

```
i1 : R = ZZ/3[u,v];
i2 : u = u^2*v^2;
i3 : compatibleIdeals u
o3 = {ideal v, ideal (v, u), ideal u}
o3 : List
```

The defining condition $uI \subseteq I^{[p^e]}$ for ϕ-compatible ideals allows us to think of these in a dual form: write $\mathfrak{m} = (x_1, \ldots, x_n)$, $E = E_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}) = H^n_{\mathfrak{m}}(R)$, and let $\Theta : E \to E$ be the p^e-linear map[6] induced from the Frobenius map on R. If $\psi = u\Theta$, then $\psi \operatorname{Ann}_E I \subseteq \operatorname{Ann}_E I$ if and only if $uI \subseteq I^{[p^e]}$. Thus finding all R-submodules of E compatible with $\psi = u\Theta$ also amounts to finding all ϕ-compatible ideals, where $\phi = T \circ u^{1/p^e}$.

Example 5.2. We return to Example 4.4. In [Katzman 2008, §9] it is shown that there is a surjection $\operatorname{Ann}_E I \to H^2_{\mathfrak{m}}(R/I)$ which is compatible with the induced p^1-linear map on $H^2_{\mathfrak{m}}(R/I)$ and the p^1-linear map $u\Theta$ on $\operatorname{Ann}_E I$, where u is computed as follows.

```
i1 : R = ZZ/2[a..f];
i2 : E = matrix {{a, b, b, e}, {d, d, c, a}};

            2         4
o2 : Matrix R  <--- R
i3 : I = minors(2, E);
o3 : Ideal of R
i4 : S = R/I;
i5 : isCohenMacaulay S
o5 = true
```

In [Katzman 2008] it is shown that since R/I is Cohen–Macaulay, u can be taken as the generator of the cyclic module $(I^{[p]} : I) \cap (\Omega^{[p]} : \Omega)/I^{[p]}$ where $\Omega \subseteq R$ is an ideal whose image in R/I is a canonical module of that ring.

```
i6 : omega = canonicalIdeal S
o6 = ideal (e, d, a)
o6 : Ideal of S
i7 : omega = substitute(omega, R) + I;
o7 : Ideal of R
i8 : u = intersect((frobenius I):I, (frobenius omega) : omega);
o8 : Ideal of R
i9 : u = compress((gens u) % gens(frobenius I))
o9 = | a3bc+a3bd+a2cde+abcde+abd2e+b2d2e+cd2e2+d3e2 |

            1         1
o9 : Matrix R  <--- R
i10 :  u = first first entries u

       3       3       2                      2     2 2      2 2     3 2
o10 = a b*c + a b*d + a c*d*e + a*b*c*d*e + a*b*d e + b d e + c*d e  + d e
o10 : R
```

Now we can compute all annihilators of R-submodules of E stable under the p^1-linear map uT.

```
i11 : L = compatibleIdeals u;
i12 : print \ L;

          3       3       2                      2     2 2      2 2     3 2
   ideal(a b*c + a b*d + a c*d*e + a*b*c*d*e + a*b*d e + b d e + c*d e  + d e )
```

---

[6]That is, Θ is additivive and $\Theta(ra) = r^{p^e}\Theta(a)$ for all $a \in E$ and $r \in R$.

```
               2
ideal (a + b, a  + d*e)

ideal (e, a, d)

ideal (e, d, a, c)

ideal (e, d, c, b, a)

ideal (e, d, b, a)

ideal (e, b, a)

ideal (e, b, a, c + d)
                           2
ideal (c + d, a + b, b  + d*e)

ideal (d, a)

ideal (d, a, c)

ideal (d, c, b, a)

ideal (d, b, a)
```

We can also compute all annihilators of R-submodules of $H^2_{\mathrm{m}}(R/I)$ stable under the p^1-linear map induced from the Frobenius map on R/I.

```
i13 : print \ unique apply(L, J -> J : omega);
        3       3       2                       2       2 2       2 2       3 2
ideal(a b*c + a b*d + a c*d*e + a*b*c*d*e + a*b*d e + b d e + c*d e  + d e )
                   2
ideal (a + b, b  + d*e)

ideal (e, d, a)

ideal 1

ideal (e, b, a)

ideal (e, c + d, b, a)
                           2
ideal (c + d, a + b, b  + d*e)

ideal (d, a)

ideal (d, c, a)

ideal (d, c, b, a)

ideal (d, b, a)
```

6. FUTURE PLANS. In [Katzman and Zhang 2014], the algorithms behind the methods in Section 5 were extended to compute prime annihilators of submodules of Artinian modules compatible with a given p^e-linear map. This would require, among other things, a faster implementation of our method for finding Frobenius roots of submodules of free modules.

On the other hand, it should be possible to compute test ideals of pairs $(R, \mathfrak{a}^t)$ even when $\mathfrak{a}$ is not principal. One strategy to do this is outlined in [Schwede and Tucker 2014b] although certain improvements can be made.

We hope to achieve all these things during a future *Macaulay2* workshop. We also want to bring to the reader's attention the package *FrobeniusThresholds*, which computes F-pure thresholds, F-thresholds, F-jumping numbers and more!

SUPPLEMENT.   The online supplement contains version 1.01 of the *TestIdeals* package.

REFERENCES.

[Alvarez-Montaner et al. 2005] J. Alvarez-Montaner, M. Blickle, and G. Lyubeznik, "Generators of D-modules in positive characteristic", *Math. Res. Lett.* **12**:4 (2005), 459–473. MR Zbl

[Blanco-Chacón et al. 2018] I. Blanco-Chacón, A. F. Boix, S. Fordham, and E. S. Yilmaz, "Differential operators and hyperelliptic curves over finite fields", *Finite Fields Appl.* **51** (2018), 351–370. MR Zbl

[Blickle and Schwede 2013] M. Blickle and K. Schwede, "p^{-1}-linear maps in algebra and geometry", pp. 123–205 in *Commutative algebra*, edited by I. Peeva, Springer, 2013. MR Zbl

[Blickle et al. 2008] M. Blickle, M. Mustaţă, and K. E. Smith, "Discreteness and rationality of F-thresholds", *Michigan Math. J.* **57** (2008), 43–61. MR Zbl

[Blickle et al. 2009] M. Blickle, M. Mustaţă, and K. E. Smith, "F-thresholds of hypersurfaces", *Trans. Amer. Math. Soc.* **361**:12 (2009), 6549–6565. MR Zbl

[Blickle et al. 2010] M. Blickle, K. Schwede, S. Takagi, and W. Zhang, "Discreteness and rationality of F-jumping numbers on singular varieties", *Math. Ann.* **347**:4 (2010), 917–949. MR Zbl

[Blickle et al. 2015] M. Blickle, K. Schwede, and K. Tucker, "F-singularities via alterations", *Amer. J. Math.* **137**:1 (2015), 61–109. MR Zbl

[Boix et al. 2015] A. F. Boix, A. De Stefani, and D. Vanzo, "An algorithm for constructing certain differential operators in positive characteristic", *Matematiche* (*Catania*) **70**:1 (2015), 239–271. MR

[Brion and Kumar 2005] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics **231**, Birkhäuser, Boston, 2005. MR Zbl

[Enescu and Hochster 2008] F. Enescu and M. Hochster, "The Frobenius structure of local cohomology", *Algebra Number Theory* **2**:7 (2008), 721–754. MR Zbl

[Fedder 1983] R. Fedder, "F-purity and rational singularity", *Trans. Amer. Math. Soc.* **278**:2 (1983), 461–480. MR Zbl

[FrobeniusThresholds] J. Bruce, D. Hernández, K. Schwede, D. Smolkin, P. Teixeira, and E. Witt, "FrobeniusThresholds: A package for calculations of F-thresholds.", Macaulay2 package Version 2.0, available at https://github.com/Macaulay2/M2/tree/master/M2/Macaulay2/packages.

[Gabber 2004] O. Gabber, "Notes on some t-structures", pp. 711–734 in *Geometric aspects of Dwork theory*, edited by A. Adolphson et al., Walter de Gruyter, Berlin, 2004. MR Zbl

[Hara and Watanabe 2002] N. Hara and K.-I. Watanabe, "F-regular and F-pure rings vs. log terminal and log canonical singularities", *J. Algebraic Geom.* **11**:2 (2002), 363–392. MR Zbl

[Hartshorne and Speiser 1977] R. Hartshorne and R. Speiser, "Local cohomological dimension in characteristic p", *Ann. of Math.* (2) **105**:1 (1977), 45–79. MR Zbl

[Hernández 2014] D. J. Hernández, "*F*-pure thresholds of binomial hypersurfaces", *Proc. Amer. Math. Soc.* **142**:7 (2014), 2227–2242. MR Zbl

[Hernández 2015] D. J. Hernández, "*F*-invariants of diagonal hypersurfaces", *Proc. Amer. Math. Soc.* **143**:1 (2015), 87–104. MR Zbl

[Hernández and Teixeira 2017] D. J. Hernández and P. Teixeira, "*F*-threshold functions: syzygy gap fractals and the two-variable homogeneous case", *J. Symbolic Comput.* **80**:part 2 (2017), 451–483. MR Zbl

[Hernández et al. 2018] D. J. Hernández, P. Teixeira, and E. E. Witt, "Frobenius powers", 2018. arXiv

[Hernández et al. 2019] D. J. Hernández, K. Schwede, P. Teixeira, and E. E. Witt, "The FrobeniusThresholds package for Macaulay2", preprint, 2019. arXiv

[Hernández et al. 2020] D. J. Hernández, P. Teixeira, and E. E. Witt, "Frobenius powers of some monomial ideals", *J. Pure Appl. Algebra* **224**:1 (2020), 66–85. MR Zbl

[Hochster 2007] M. Hochster, "Foundations of tight closure theory", lecture notes, University of Michigan, 2007, available at http://www.math.lsa.umich.edu/~hochster/711F07/fndtc.pdf.

[Hochster and Huneke 1990] M. Hochster and C. Huneke, "Tight closure, invariant theory, and the Briançon–Skoda theorem", *J. Amer. Math. Soc.* **3**:1 (1990), 31–116. MR Zbl

[Katzman 2008] M. Katzman, "Parameter-test-ideals of Cohen–Macaulay rings", *Compos. Math.* **144**:4 (2008), 933–948. MR Zbl

[Katzman 2010] M. Katzman, "Frobenius maps on injective hulls and their applications to tight closure", *J. Lond. Math. Soc.* (2) **81**:3 (2010), 589–607. MR Zbl

[Katzman and Schwede 2012] M. Katzman and K. Schwede, "An algorithm for computing compatibly Frobenius split subvarieties", *J. Symbolic Comput.* **47**:8 (2012), 996–1008. MR Zbl

[Katzman and Zhang 2014] M. Katzman and W. Zhang, "Annihilators of Artinian modules compatible with a Frobenius map", *J. Symbolic Comput.* **60** (2014), 29–46. MR Zbl

[Katzman et al. 2009] M. Katzman, G. Lyubeznik, and W. Zhang, "On the discreteness and rationality of *F*-jumping coefficients", *J. Algebra* **322**:9 (2009), 3238–3247. MR Zbl

[Kumar and Mehta 2009] S. Kumar and V. B. Mehta, "Finiteness of the number of compatibly split subvarieties", *Int. Math. Res. Not.* **2009**:19 (2009), 3595–3597. MR Zbl

[Kunz 1969] E. Kunz, "Characterizations of regular local rings of characteristic *p*", *Amer. J. Math.* **91** (1969), 772–784. MR Zbl

[Lyubeznik 1997] G. Lyubeznik, "*F*-modules: applications to local cohomology and *D*-modules in characteristic $p > 0$", *J. Reine Angew. Math.* **491** (1997), 65–130. MR

[Macaulay2] D. R. Grayson and M. E. Stillman, "Macaulay2: a software system for research in algebraic geometry", available at http://www.math.uiuc.edu/Macaulay2.

[PushForward] C. Raicu, "PushForward: push forwards of finite ring maps", Macaulay2 package, version 0.1, available at https://github.com/Macaulay2/M2/tree/master/M2/Macaulay2/packages.

[Schwede 2009] K. Schwede, "*F*-adjunction", *Algebra Number Theory* **3**:8 (2009), 907–950. MR Zbl

[Schwede and Tucker 2012] K. Schwede and K. Tucker, "A survey of test ideals", pp. 39–99 in *Progress in commutative algebra 2*, edited by C. Francisco et al., Walter de Gruyter, Berlin, 2012. MR Zbl

[Schwede and Tucker 2014a] K. Schwede and K. Tucker, "On the behavior of test ideals under finite morphisms", *J. Algebraic Geom.* **23**:3 (2014), 399–443. MR Zbl

[Schwede and Tucker 2014b] K. Schwede and K. Tucker, "Test ideals of non-principal ideals: computations, jumping numbers, alterations and division theorems", *J. Math. Pures Appl.* (9) **102**:5 (2014), 891–929. MR Zbl

[Sharp 2007] R. Y. Sharp, "Graded annihilators of modules over the Frobenius skew polynomial ring, and tight closure", *Trans. Amer. Math. Soc.* **359**:9 (2007), 4237–4258. MR Zbl

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Computing Schur complexes

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ABSTRACT: We describe a *Macaulay2* package for computing Schur complexes. This package expands on the `ChainComplexOperations` package by David Eisenbud.

1. INTRODUCTION. Let R be a commutative ring. The goal of this article is to describe the *Macaulay2* package `SchurComplexes`, which computes the Schur complex $S_\lambda(F)$ associated to a bounded complex F of finitely generated free R-modules and a partition λ.

Schur complexes are a simultaneous generalization of the symmetric and exterior power operations on complexes. The notion of a Schur complex was introduced by Nielsen [1978] in the characteristic 0 setting, and it was generalized to the characteristic free setting by Akin, Buchsbaum and Weyman in [Akin et al. 1982].

The importance of such operations on complexes is illustrated by Walker's recent proof of the weak Buchsbaum–Eisenbud–Horrocks conjecture [Walker 2017, Theorem 2.4] in which exterior and symmetric squares of complexes play a crucial role. Walker's breakthrough work led to Eisenbud's implementation of the second exterior and symmetric power for complexes in the *Macaulay2* package `ChainComplexOperations`. The work suggests that properties of Schur complexes should be further developed. Our goal here is to expand on Eisenbud's package by implementing the construction of an arbitrary Schur complex.

In Section 2, we provide some background on Schur complexes, following the detailed treatment in Weyman's book [2003, Section 2.4]. In particular, we recall the "straightening algorithm" of [Akin et al. 1982] which expresses a $\mathbb{Z}/2$-graded tableau as a $\mathbb{Z}$-linear combination of so-called "standard" $\mathbb{Z}/2$-graded tableaux; the implementation of this algorithm is the key component of the `SchurComplexes` package. Section 3 contains some examples of computations using `SchurComplexes`.

2. BACKGROUND ON SCHUR COMPLEXES.   This section closely follows [Weyman 2003, Section 2.4]. Let $F = (F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_d)$ be a bounded complex of finitely generated free R-modules. Denote by F_{even} (resp. F_{odd}) the direct sum of the even (resp. odd) degree components of F, and choose bases $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ of F_{odd} and F_{even}, respectively, which are unions of bases of the F_i.

Exterior powers of complexes.   Fix a positive integer r, and let $T^r(F)$ denote the r-th tensor power of F. $T^r(F)$ may be equipped with a Σ_r-action in the following way:

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_r) = \pm x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(r)},$$

where the x_i are homogeneous elements of F, and the sign is determined by declaring that transposing the elements x_i and x_j contributes the sign $(-1)^{|x_i||x_j|}$. By [Weyman 2003, page 74], the Σ_r-action is compatible with the differential on $T^r(F)$. Let

$$\varepsilon : T^r(F) \to T^r(F)$$

denote the R-linear antisymmetrization map

$$(x_1 \otimes \cdots \otimes x_r) \mapsto \sum_{\sigma \in \Sigma_r} (-1)^{\text{sign}(\sigma)} \sigma \cdot (x_1 \otimes \cdots \otimes x_r).$$

Definition 2.1. The r-th exterior power $\bigwedge^r F$ is defined to be the subcomplex $\varepsilon(T^r(F))$ of $T^r(F)$.

Remark 2.2. Suppose F is concentrated in even degrees, and let I_r denote the R-submodule of $T^r(F)$ spanned by elements of the form

$$x_1 \otimes \cdots \otimes x \otimes x \otimes \cdots \otimes x_r.$$

By [Akin et al. 1982, Section 1.2], ε determines a split injection $T^r(F)/I_r \to T^r(F)$, and so Definition 2.1 recovers the usual definition of an exterior power in this case. As noted in [loc. cit.], $\bigwedge^r F$ coincides with the antisymmetric tensors in $T^r(F)$ (i.e., those elements $v \in T^r(F)$ such that $\sigma \cdot v = (-1)^{\text{sign}(\sigma)} v$ for all $\sigma \in \Sigma_r$) when $\text{char}(R) \neq 2$.

There is a canonical R-linear embedding

$$\iota : \bigoplus_i D_i(F_{\text{odd}}) \otimes (T^{r-i}(F_{\text{even}})/I_{r-i}) \hookrightarrow T^r(F)$$

whose image is precisely $\bigwedge^r F$. Here, $D_i(F_{\text{odd}}) := \text{Sym}_i((F_{\text{odd}})^*)^*$, the i-th divided power of F_{odd} (where $(-)^*$ denotes the R-linear dual). We now describe this embedding in detail.

Let I denote the two-sided ideal of $T(F_{\text{even}})$ generated by elements of the form $x \otimes x$, where $x \in F$. By Remark 2.2, the composition $T(F_{\text{even}}) \hookrightarrow T(F) \xrightarrow{\varepsilon} T(F)$ factors through $T(F_{\text{even}})/I$ and induces an embedding

$$\iota_\wedge : T(F_{\text{even}})/I \hookrightarrow T(F_{\text{even}}).$$

By [Roby 1963, Proposition IV.5], there is an embedding of R-algebras

$$\iota_D : D(F_{\text{odd}}) \to T(F_{\text{odd}})$$

such that

$$\iota_D(x^{(j)}) = \underbrace{x \otimes \cdots \otimes x}_{j \text{ copies}}$$

for all $x \in F_{\text{odd}}$, where the target is equipped with the shuffle product. The embedding ι is defined on each summand $D_i(F_{\text{odd}}) \otimes (T^{r-i}(F_{\text{even}})/I_{r-i})$ by

$$\sum_{\sigma \in \Sigma_{i,r-i}} (-1)^{\text{sign}(\sigma)} \sigma \cdot (\iota_D \otimes \iota_\Lambda),$$

where $\Sigma_{i,r-i} \subseteq \Sigma_r$ denotes the set of $(i, r-i)$ shuffles.

Example 2.3. Let $x, y \in R$, and take F to be the Koszul complex

$$R \xleftarrow{\ (x \ \ y)\ } R^{\oplus 2} \xleftarrow{\ \binom{-y}{x}\ } R$$

on x and y, lying in homological degrees 0, 1, and 2. Then $\bigwedge^2(F)$ is the complex

$$R^{\oplus 2} \xleftarrow{\ \begin{pmatrix} y & x & 0 & x \\ 0 & y & x & -y \end{pmatrix}\ } R^{\oplus 4} \xleftarrow{\ \begin{pmatrix} 2x & 0 \\ -y & x \\ 0 & -2y \\ -y & -x \end{pmatrix}\ } R^{\oplus 2},$$

lying in homological degrees 1, 2, and 3.

The complex $\bigwedge F := \bigoplus_{r \geq 0} \bigwedge^r F$ is equipped with a product

$$\mu : \overset{r_1}{\bigwedge} F \otimes \overset{r_2}{\bigwedge} F \to \overset{r_1+r_2}{\bigwedge} F$$

and a coproduct

$$\Delta : \overset{r}{\bigwedge} F \to \bigoplus_{r_1+r_2=r} \overset{r_1}{\bigwedge} F \otimes \overset{r_2}{\bigwedge} F.$$

For explicit formulas for μ and Δ, we refer the reader to the proof of [Weyman 2003, Proposition 2.4.1].

Schur complexes. Let r be a positive integer, and let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be a partition of r, where $\lambda_i \geq \lambda_{i+1}$. We will encode partitions with Young diagrams. For example, the partition $(3, 2, 2)$ of 7 corresponds to the diagram

Let $c_1, \ldots, c_t$ denote the lengths of the columns of λ.

Definition 2.4. The *Schur complex* $\mathbf{S}_\lambda(F)$ is the quotient $\left(\bigwedge^{c_1} F \otimes \cdots \otimes \bigwedge^{c_t} F \right)/R$, where R is the sum of submodules

$$\overset{c_1}{\bigwedge} F \otimes \cdots \otimes \overset{c_{a-1}}{\bigwedge} F \otimes R_{a,a+1} \otimes \overset{c_{a+2}}{\bigwedge} F \otimes \cdots \otimes \overset{c_t}{\bigwedge} F.$$

Here, $R_{a,a+1}$ is the submodule spanned by the images of the compositions

$$\Theta(a,u,v;F): \overset{u}{\bigwedge} F \otimes \overset{c_a-u+c_{a+1}-v}{\bigwedge} F \otimes \overset{v}{\bigwedge} F \xrightarrow{1\otimes\Delta\otimes1} \overset{u}{\bigwedge} F \otimes \overset{c_a-u}{\bigwedge} F \otimes \overset{c_{a+1}-v}{\bigwedge} F \otimes \overset{v}{\bigwedge} F \xrightarrow{\mu\otimes\mu} \overset{c_a}{\bigwedge} F \otimes \overset{c_{a+1}}{\bigwedge} F$$

for $u+v < c_{a+1}$.

Remark 2.5. Our definition of the Schur complex differs from the one in [Weyman 2003, Section 2.4] in that the roles of the rows and columns are swapped. In other words, Weyman's definition of the Schur complex of F with respect to λ is recovered by applying Definition 2.4 to the complex F and the conjugate partition λ^*, i.e., the result of transposing the rows and columns of λ.

Example 2.6. Of course, if $\lambda = (1, \ldots, 1)$, $S_\lambda(F) = \bigwedge^r F$. If $\lambda = (r)$, $S_\lambda(F) = \mathrm{Sym}_r(F)$ (see [Weyman 2003, Section 2.4] for the definition of the symmetric power of a complex).

The basis of F chosen above determines a basis $\mathcal{B}_i$ for each $\bigwedge^{c_i} F$: namely, the images under ι of elements of the form

$$\{e_1^{(j_1)} \cdots e_m^{(j_l)} \otimes f_{k_1} \wedge \cdots \wedge f_{k_{c_i-(j_1+\cdots+j_l)}}\}$$

(from now on, we will tacitly identify these elements with their images under ι). The set $\mathcal{S} := \{b_1 \otimes \cdots \otimes b_t : b_i \in \mathcal{B}_i\}$ therefore gives an R-linear spanning set for $S_\lambda(F)$. We will write elements of $\mathcal{S}$ as $\mathbb{Z}/2$-*graded Young tableaux of shape* λ, i.e., functions

$$T : \{1, \ldots, r\} \to \{-m, \ldots, -1\} \cup \{1, \ldots, n\},$$

where, as above, $m = \mathrm{rank}(F_{\mathrm{odd}})$ and $n = \mathrm{rank}(F_{\mathrm{even}})$. Here, divided power factors correspond to negative values, and exterior factors correspond to positive values. For instance, if $\lambda = (3, 3, 1)$ and $m = 2 = n$, the element $(e_2^{(2)} \otimes f_1) \otimes (e_1 \otimes f_1) \otimes (f_1 \wedge f_2)$ in $S_{(3,3,1)}(F)$ corresponds to the function

l	1	2	3	4	5	6	7
$T(l)$	-2	-2	1	-1	1	1	2

which we express as the following Young tableau:

$$T = \begin{array}{|c|c|c|}\hline -2 & -1 & 1 \\\hline -2 & 1 & 2 \\\cline{1-2} 1 \\\cline{1-1}\end{array}$$

We will call such a tableau *standard* if

(A) the columns increase from top to bottom, with equality possible only for negative values, and

(B) the rows increase from left to right, with equality possible only for positive values.

Remark 2.7. Our definition of a standard tableau is the transpose of Weyman's [2003, Definition 1.1.12(c)] (cf. Remark 2.5).

For instance, the tableau T above is standard. The tableaux

-2	-1	1
-2	-1	2
-3		

-2	-1	1
1	1	2
1		

-2	-1	-1
-2	1	2
1		

are nonstandard.

Proposition 2.8 [Weyman 2003, Proposition 2.4.2]. *The standard $\mathbb{Z}/2$-graded tableaux of shape λ form an R-linear basis of $S_\lambda(F)$.*

We compute the differentials in $S_\lambda(F)$ with respect to this basis in the `SchurComplexes` package. It is therefore essential for us to implement an algorithm for writing the image of a standard tableau under the differential in $S_\lambda(F)$ as a linear combination of standard tableaux. The proof of [Weyman 2003, Proposition 2.4.2] explains such an algorithm: the "straightening algorithm" of [Akin et al. 1982]. We now discuss this algorithm in detail.

The straightening algorithm. As in the previous subsection, $\lambda = (\lambda_1, \ldots, \lambda_s)$ denotes a partition of a positive integer r with column lengths $c_1, \ldots, c_t$. The straightening algorithm is a process for writing a $\mathbb{Z}/2$-graded tableau in the spanning set $\mathcal{S}$ of $S_\lambda(F)$ described above as a $\mathbb{Z}$-linear combination of standard tableaux. Here is how it works.

Let T be a tableau in $\mathcal{S}$. We make the following observations:

- If a column of T contains a repeated positive entry, $T = 0$ in $S_\lambda(F)$.

- Since the divided power (resp. exterior) algebra is commutative (resp. skew commutative), rearranging the columns in T so that they satisfy (A) only changes the element of $S_\lambda(F)$ represented by T up to a sign.

With these facts in mind, we recall the straightening algorithm:

• **Input**: A tableau $T \in \mathcal{S}$.

• **Step 1:** Denote by T' the result of rearranging the columns of T so that they satisfy (A), and let $\sigma \in \{\pm 1\}$ denote the resulting sign.

• **Step 2:** If T' satisfies (B), output $\sigma T'$. Otherwise, choose the topmost row, say the w-th row, with a "violation" of (B). Let $T'(i, j)$ denote the entry with horizontal coordinate i and vertical coordinate j in the Young diagram, counting from the top-left corner. So $T'(i, j)$ is the entry in the i-th column and the j-th row. Choose the smallest index a such that either $T'(a, w) > T'(a + 1, w)$ or $T'(a, w) = T'(a + 1, w) < 0$. Then, choose the smallest index w' such that $T'(a + 1, w) < T'(a + 1, w' + 1)$; if no such index exists, set $w' = c_{a+1}$.

• **Step 3:** Set $u := w - 1$ and $v := c_{a+1} - w'$. Define

(1) $V_1 \in \bigwedge^u F$ to be the element corresponding to the first u entries in the a-th column of T',

(2) $V_2 \in \bigwedge^{c_a-u+c_{a+1}-v} F$ to be the element corresponding to the last $c_a - u$ entries in the a-th column of T' followed by the first $c_{a+1} - v = w'$ entries of the $(a+1)$-st column of T',

(3) $V_3 \in \bigwedge^v F$ to be the element corresponding to the last v entries in the $(a+1)$-st column of T'.

For $k \in \{1, \ldots, a-1, a+2, \ldots, s\}$, define U_k to be the element of $\bigwedge^{c_k} F$ which corresponds to the k-th column of T'. Recall that

$$L := (1 \otimes \Theta(a, u, v; F) \otimes 1)(U_1 \otimes \cdots \otimes U_{a-1} \otimes V_1 \otimes V_2 \otimes V_3 \otimes U_{a+2} \otimes \cdots \otimes U_s)$$

is 0 in $S_\lambda(F)$, where $\Theta(a, u, v; F)$ is as in Definition 2.4. L is a $\mathbb{Z}$-linear combination of tableaux in $\mathcal{S}$ which contains T' with coefficient 1. If each tableau in the sum $T' - L$ is standard, output $\sigma(T' - L)$. Otherwise, repeat this algorithm on each tableau in $\sigma(T' - L)$, keeping track of the coefficients.

The key observation is that each tableau appearing in the linear combination $T' - L$ from Step 3 is strictly "smaller" than T', in the sense described in [Weyman 2003, Section 1.1], and so the algorithm does indeed terminate.

Example 2.9. Let's apply the straightening algorithm to the tableau

$$T = \begin{array}{|c|c|c|} \hline -3 & 2 & -1 \\ \hline -2 & 1 & 3 \\ \hline -2 & 3 \\ \cline{1-2} \end{array}$$

- **Step 1:** The middle column needs to be rearranged. Since $f_2 f_1 f_3 = -f_1 f_2 f_3$ in $\bigwedge^3 F_{\text{even}}$, we have

$$T' = \begin{array}{|c|c|c|} \hline -3 & 1 & -1 \\ \hline -2 & 2 & 3 \\ \hline -2 & 3 \\ \cline{1-2} \end{array}$$

and $\sigma = -1$.

- **Step 2:** T' is not standard. Here, $w = 1$, $a = 2$, and $w' = 1$.

- **Step 3:** Here, $u = 0$ and $v = 1$. We have $U_1 = e_3 e_2^{(2)}$, $V_2 = e_1 \otimes f_1 \wedge f_2 \wedge f_3$, and $V_3 = f_3$ (since $u = 0$, V_1 plays no role), so

$$L = (1 \otimes \Theta(2, 0, 1; F))(e_3 e_2^{(2)} \otimes (e_1 \otimes f_1 \wedge f_2 \wedge f_3) \otimes f_3).$$

In this case, $\Theta(2, 0, 1)$ is the composition

$$\left(\bigwedge^4 F\right) \otimes F \xrightarrow{\Delta \otimes 1} \left(\bigwedge^3 F \otimes F\right) \otimes F \xrightarrow{\text{id} \otimes \mu} \bigwedge^3 F \otimes \bigwedge^2 F.$$

By the proof of [Weyman 2003, Proposition 2.4.1(a)], the relevant component of the coproduct

$$\Delta : \bigwedge^4 F \to \bigwedge^3 F \otimes F$$

in $\Theta(2, 0, 1; F)$ is:

$$D_1(F_{\text{odd}}) \otimes \bigwedge^3 F_{\text{even}} \xrightarrow{\Delta_D \otimes \Delta_\Lambda} \bigoplus_{i=0}^{1} D_i(F_{\text{odd}}) \otimes D_{1-i}(F_{\text{odd}}) \otimes \bigwedge^{3-i} F_{\text{even}} \otimes \bigwedge^{i} F_{\text{even}}$$

$$\xrightarrow{\ \tau\ } \bigoplus_{i=0}^{1} D_i(F_{\text{odd}}) \otimes \bigwedge^{3-i} F_{\text{even}} \otimes D_{1-i}(F_{\text{odd}}) \otimes \bigwedge^{i} F_{\text{even}},$$

where Δ_D and Δ_Λ are the coproducts on the ordinary divided power and exterior algebra, and τ transposes the middle factors and multiplies by the sign $(-1)^{(1-i)(3-i)}$.

Note: one might think that, since the elements of $\bigwedge^{3-i} F_{\text{even}}$ have even degree, transposing the middle two factors should not introduce a sign. But recall that we are considering $D_{1-i}(F_{\text{odd}}) \otimes \bigwedge^{3-i} F_{\text{even}}$ as a submodule of $T^4(F)$ via the embedding ι, which shuffles together the elements of $D_{1-i}(F_{\text{odd}})$ and $\bigwedge^{3-i} F_{\text{even}}$; this is why it is necessary to multiply by $(-1)^{(1-i)(3-i)}$. The general rule here is: when one transposes the factors of $D_s(F_{\text{odd}}) \otimes \bigwedge^t F_{\text{even}}$, one must introduce the sign (-1)-st $= \text{sign}((1\ 2 \cdots s+t)^t)$.

Applying the formula for Δ_Λ in [Weyman 2003, Section 1.1, page 3], one gets

$$L = -(e_3 e_2^{(2)}) \otimes (f_1 \wedge f_2 \wedge f_3) \otimes (e_1 \otimes f_3) - (e_3 e_2^{(2)}) \otimes (e_1 \otimes f_1 \wedge f_3) \otimes (f_2 \wedge f_3)$$
$$+ (e_3 e_2^{(2)}) \otimes (e_1 \otimes f_2 \wedge f_3) \otimes (f_1 \wedge f_3),$$

and therefore

$$T = \begin{array}{|c|c|c|}\hline -3 & -1 & 2 \\\hline -2 & 1 & 3 \\\hline -2 & 3 \\\cline{1-2}\end{array} \ - \ \begin{array}{|c|c|c|}\hline -3 & -1 & 1 \\\hline -2 & 2 & 3 \\\hline -2 & 3 \\\cline{1-2}\end{array}$$

Both of these tableaux are standard, so we're done.

3. Examples of computations using the package SchurComplexes.

The SchurComplexes package has two main functions:

- `straightenTableau`, which applies the straightening algorithm to a tableau.

- `schurComplex`, which computes the Schur complex of a bounded complex of finitely generated free modules.

Using the function `straightenTableau`. We apply `straightenTableau` to the tableau in Example 2.9. First, we load the package:

```
i1 : loadPackage "SchurComplexes.m2"
```

We encode the tableau T in a hash table:

```
i2 : T = new HashTable from {(1,1) => -3, (1,2) => -2, (1,3) => -2,
(2,1) => 2, (2,2) => 1, (2,3) => 3, (3,1) => -1, (3,2) => 3}
```

We encode the partition $(3, 3, 2)$ in a list:

```
i3 : lambda = {3,3,2}
```

Now, we apply `straightenTableau` to the pair (`T`, `lambda`):

```
i4 : straightenTableau(T, lambda)

o4 = HashTable{HashTable{(1, 1) => -3} => 1 }
                         (1, 2) => -2
                         (1, 3) => -2
                         (2, 1) => -1
                         (2, 2) =>  1
                         (2, 3) =>  3
                         (3, 1) =>  2
                         (3, 2) =>  3
               HashTable{(1, 1) => -3} => -1
                         (1, 2) => -2
                         (1, 3) => -2
                         (2, 1) => -1
                         (2, 2) =>  2
                         (2, 3) =>  3
                         (3, 1) =>  1
                         (3, 2) =>  3
```

The output is a hash table which assigns a coefficient to each standard tableau in the linear combination comprising the straightening of *T*. Notice that the output agrees with the calculation in Example 2.9.

Using the function `schurComplex`**.** Let $R = \mathbb{Q}[x_{i,j}]$, where $1 \leq i \leq 2$ and $1 \leq j \leq 4$, and let

$$F : R^{\oplus 4} \xrightarrow{(x_{i,j})} R^{\oplus 2}$$

denote the generic 2×4 matrix, considered as a complex concentrated in degrees 1 and 0. By [Weyman 2003, Exercise 6.34(d)], $S_{(3)}(F) = \mathrm{Sym}_3(F)$ has nonzero homology only in degree 0. We now use the `schurComplex` function to compute $S_{(3)}(F)$ and verify this fact.

We first load the package, fix our ground ring *R*, and define our complex *F*:

```
i1 : loadPackage "SchurComplexes.m2"
i2 : R = QQ[x11,x21,x12,x22,x13,x23,x14,x24];
i3 : M = genericMatrix(R,x11,2,4);
i4 : F = new ChainComplex;
i5 : F.ring = R; F#0 = target M; F#1 = source M; F.dd#1 = M;
```

The function `schurComplex` takes as input a `ChainComplex` and a `List` which encodes the partition. Let's define our partition and compute $S_{(3)}(F)$:

```
i6 : lambda = {3};
i7 : S = schurComplex(lambda,F)

      4       12       12       4
o7 = R   <-- R    <-- R    <-- R

      0       1        2        3
```

Finally, let's check that $S_{(3)}(F)$ has trivial homology in degrees greater than 0:

```
i8 : apply((length S)+1,i->reduceHilbert hilbertSeries HH_i(S))

           4      0  0  0
o8 = {--------, -, -, -}
            5    1  1  1
       (1 - T)
```

REFERENCES.

[Akin et al. 1982] K. Akin, D. A. Buchsbaum, and J. Weyman, "Schur functors and Schur complexes", *Adv. in Math.* **44**:3 (1982), 207–278. MR Zbl

[Nielsen 1978] A. H. Nielsen, "Tensor functors of complexes", preprint 15, Univ. Aarhus, 1978. Zbl

[Roby 1963] N. Roby, "Lois polynomes et lois formelles en théorie des modules", *Ann. Sci. École Norm. Sup.* (3) **80** (1963), 213–348. MR Zbl

[Walker 2017] M. E. Walker, "Total Betti numbers of modules of finite projective dimension", *Ann. of Math.* (2) **186**:2 (2017), 641–646. MR Zbl

[Weyman 2003] J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics **149**, Cambridge University Press, 2003. MR Zbl

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