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**The relative canonical resolution:
Macaulay2-package, experiments and conjectures**

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ABSTRACT: This short note provides a quick introduction to relative canonical resolutions of curves on rational normal scrolls. We present our *Macaulay*2 package that computes the relative canonical resolution associated to a curve and a pencil of divisors. We end with a list of conjectural shapes of relative canonical resolutions. In particular, for curves of genus $g = n \cdot k + 1$ and pencils of degree k for $n \geq 1$, we conjecture that the syzygy divisors on the Hurwitz scheme $\mathcal{H}_{g,k}$ constructed by Deopurkar and Patel (*Contemp. Math.* **703** *(2018) 209–222*) all have the same support.

1. RELATIVE CANONICAL RESOLUTIONS.

The *relative canonical resolution* is the minimal free resolution of a canonically embedded curve C inside a rational normal scroll. Every such scroll is swept out by linear spaces parametrized by pencils of divisors on C.

Studying divisors on moduli spaces reveals certain aspects of the global geometry of these spaces. A famous example for odd genus g is the *Koszul divisor* on the moduli space of curves $\mathcal{M}_g$ (see [Hirschowitz and Ramanan 1998; Voisin 2005; Farkas 2009]). It can be derived from the minimal free resolution of $C \subset \mathbb{P}^{g-1}$. Set-theoretically the Koszul divisor consists of curves such that the minimal free resolution of the canonical model has extra syzygies at a certain step. In [Bujokas and Patel 2015; Deopurkar and Patel 2015; 2018], the relative canonical resolution was used to define similar *syzygy divisors* on Hurwitz spaces $\mathcal{H}_{g,k}$, parametrizing pairs of curves of genus g and pencils of divisors of degree k (equivalently, covers of $\mathbb{P}^1$ of degree k by curves of genus g). We also refer to [Farkas 2018] for divisors on Hurwitz spaces.

We will briefly summarize the connection between pencils of divisors on canonical curves and rational normal scrolls in order to define the relative canonical resolution. The following definition and statements can be found in [Harris 1981, §3] and [Schreyer 1986, §1]. Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus g, and let

$$g_k^1 = \{D_\lambda\}_{\lambda \in \mathbb{P}^1} \subset |D|$$

be a pencil of divisors of degree k. If we denote by $\overline{D_\lambda} \subset \mathbb{P}^{g-1}$ the linear span of the divisor, then

$$X := \bigcup_{\lambda \in \mathbb{P}^1} \overline{D_\lambda} \subset \mathbb{P}^{g-1}$$

is a $(k-1)$-dimensional rational normal scroll of degree $\deg(X) := f = g - k + 1$.

Definition 1.1. Let $e_1 \geq e_2 \geq \cdots \geq e_d \geq 0$ be integers, $\mathscr{E} = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_d)$, and let $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ be the corresponding $\mathbb{P}^{d-1}$-bundle. A *rational normal scroll* of type $(e_1, \ldots, e_d)$ is the image of

$$j : \mathbb{P}(\mathscr{E}) \to \mathbb{P}H^0(\mathbb{P}(\mathscr{E}), \mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)) = \mathbb{P}^r.$$

Note that $r = f + d - 1$ with $f = e_1 + \cdots + e_d \geq 2$. Conversely if X is a rational normal scroll of degree f containing a canonical curve, then the ruling on X cuts out a pencil of divisors $\{D_\lambda\} \subset |D|$ such that $h^0(C, \omega_C \otimes \mathcal{O}_C(D)^{-1}) = f$.

Example 1.2. We consider a nonhyperelliptic canonically embedded curve $C \subset \mathbb{P}^3$ of genus 4. The curve C is a complete intersection of a quadric surface Q and a cubic surface S. If C admits exactly two pencils of degree 3 (which is also the maximal number), then the quadric Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. By Bézout's theorem, the two rulings of lines on Q cut out the two pencils of degree 3 on C, and conversely, the quadric is the scroll of type $(1, 1)$ swept out by either of these pencils. If C admits only one pencil of degree 3, then the quadric Q is isomorphic to a cone (i.e., a quadric of rank 3) and coincides with the scroll of type $(0, 2)$ swept out by the unique pencil.

In [Harris 1981] it is shown that the variety X defined above is a nondegenerate d-dimensional variety of minimal degree $\deg X = f = r - d + 1 = \operatorname{codim} X + 1$. If $e_1, \ldots, e_d > 0$, then

$$j : \mathbb{P}(\mathscr{E}) \to X \subset \mathbb{P}H^0(\mathbb{P}(\mathscr{E}), \mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)) = \mathbb{P}^r$$

is an isomorphism. Otherwise, it is a resolution of singularities. Since $R^i j_* \mathcal{O}_{\mathbb{P}(\mathscr{E})} = 0$ for $i > 0$, it is convenient to consider $\mathbb{P}(\mathscr{E})$ instead of X for cohomological considerations.

It is known (see, e.g., [Eisenbud and Harris 1987b]) that the Picard group $\operatorname{Pic}(\mathbb{P}(\mathscr{E}))$ is generated by the class of the ruling $R = [\pi^* \mathcal{O}_{\mathbb{P}^1}(1)]$ and the hyperplane class $H = [j^* \mathcal{O}_{\mathbb{P}^r}(1)]$ with intersection products

$$H^d = f, \quad H^{d-1} \cdot R = 1, \quad R^2 = 0.$$

Hence, we will write a line bundle $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)$ in the form

$$\mathcal{O}_{\mathbb{P}(\mathscr{E})}(aH + bR) = \pi^*(\mathcal{O}_{\mathbb{P}^1}(b))(aH).$$

Theorem 1.3 [Schreyer 1986, Corollary 4.4]. *Let C be a curve with a complete base point free g_k^1, and let $\mathbb{P}(\mathscr{E})$ be the projective bundle associated to the scroll X swept out by the g_k^1.*

(a) *$C \subset \mathbb{P}(\mathscr{E})$ has a resolution $F_\bullet$ of type*

$$0 \to \pi^* N_{k-2}(-kH) \to \pi^* N_{k-3}((-k+2)H) \to \cdots \to \pi^* N_1(-2H) \to \mathcal{O}_{\mathbb{P}(\mathscr{E})} \to \mathcal{O}_C \to 0$$

with

$$N_i = \bigoplus_{j=1}^{\beta_i} \mathscr{O}_{\mathbb{P}^1}(a_j^{(i)}) \quad \text{and} \quad \beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}.$$

(b) *The complex $F_\bullet$ is self-dual, i.e., $\mathscr{H}om(F_\bullet, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-kH+(f-2)R)) \cong F_\bullet$.*

The resolution $F_\bullet$ above is called the *relative canonical resolution*. The degree of the bundles N_i is known.

Proposition 1.4 [Bopp and Hoff 2015, Proposition 2.9]. *The degree of the bundle N_i of rank $\beta_i = \frac{k}{i+1}(k-2-i)\binom{k-2}{i-1}$ in the relative canonical resolution $F_\bullet$ is*

$$\deg(N_i) = \sum_{j=1}^{\beta_i} a_j^{(i)} = (g-k-1)(k-2-i)\binom{k-2}{i-1}.$$

Since the rank and degree of the syzygy bundles N_i over $\mathbb{P}^1$ are known, the main object of investigation is the *splitting type*.

Remark 1.5. Casnati and Ekedahl [1996] generalized the relative canonical resolution to finite Gorenstein covers $\pi : X \to Y$ of degree k. They define a relative resolution of $X \subset \mathbb{P}(\mathscr{E}_T)$, where $\mathscr{E}_T$ is the Tschirnhausen bundle on Y defined by the short exact sequence

$$0 \to \mathscr{O}_Y \to \pi_*(\mathscr{O}_X) \to \mathscr{E}_T^\vee \to 0.$$

Note that for a cover $C \xrightarrow{k:1} \mathbb{P}^1$, $\mathscr{E}_T = \mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^1}(2)$, where $\mathscr{E}$ is the bundle associated to (C, g_k^1) as in Theorem 1.3. The twists and hence the splitting types of the syzygy bundles in a resolution of $C \subset \mathbb{P}(\mathscr{E}_T)$ also differ from the ones in the relative canonical resolution of $C \subset \mathbb{P}(\mathscr{E})$. Indeed, following the proof of [Casnati and Ekedahl 1996, Step B, p. 445], for each i, the twist in the i-th syzygy bundle in a resolution of $C \subset \mathbb{P}(\mathscr{E}_T)$ differs by exactly $2 \cdot (i+1)$ from the ones given in our definition. Hence, we can deduce the degrees of the bundles in this relative resolution of $C \subset \mathbb{P}(\mathscr{E}_T)$ from Proposition 1.4. These degrees have also been computed directly in [Deopurkar and Patel 2018].

Definition 1.6. We say that a bundle on $\mathbb{P}^1$ of the form $N = \bigoplus_{j=1}^{\beta} \mathscr{O}_{\mathbb{P}^1}(a_j)$ is *balanced* if

$$\max_{i,j}|a_j - a_i| \leq 1.$$

Equivalently, the bundle N is balanced if $h^1(\mathbb{P}^1, \mathscr{E}nd(N)) = 0$. The relative canonical resolution is called balanced if all bundles N_i occurring in the resolution are balanced.

Remark 1.7. The locus of curves inside $\mathscr{H}_{g,k}$ that have a balanced relative canonical resolution forms an open subset of $\mathscr{H}_{g,k}$ which might be empty. Hence, to show that a generic relative canonical resolution is balanced for fixed values (g, k) it is sufficient to examine a single balanced example.

Remark 1.8. The scroll associated to a general element in $\mathscr{H}_{g,k}$ is always balanced by [Ballico 1989] and [Harris 1981]. The sublocus inside $\mathscr{H}_{g,k}$ parametrizing covers such that the associated scroll is

unbalanced defines a divisor on $\mathscr{H}_{g,k}$ precisely if g is a multiple of $(k-1)$. This divisor is called the *Maroni divisor* (for more details on the Maroni divisor see, e.g., [van der Geer and Kouvidakis 2017] and [Deopurkar and Patel 2015]).

On the other hand, knowing the splitting type of the syzygy bundles in the relative canonical resolution for generic elements in $\mathscr{H}_{g,k}$ one can study the sublocus inside $\mathscr{H}_{g,k}$ consisting set-theoretically of curves for which a certain syzygy bundle has nongeneric splitting type. This yields interesting subvarieties which also turn out to be divisors in some cases (see [Deopurkar and Patel 2018]). Similar to Koszul divisors on the moduli space $\mathscr{M}_g$, the study of the divisors obtained from the relative canonical resolution sheds light on the global geometry of the Hurwitz space.

2. MACAULAY2 PACKAGE. The Macaulay2 package [RelativeCanonicalResolution] includes various useful functions to do experiments with k-gonal canonical curves and the relative canonical resolution of those curves. We will briefly explain how functions in this package construct g-nodal k-gonal canonical curves of genus g.

The main idea is that we start with a rational normalization of the desired curve and a degree k map from the normalization to $\mathbb{P}^1$. In the next step we choose g pairs of points $\{P_i, Q_i\}$ for $i = 1, \ldots, g$ on the normalization, and we glue the points in each pair to each other. If $\mathscr{L}$ is a line bundle of degree k on a g-nodal curve C with rational normalization $\nu : \mathbb{P}^1 \to C$, then $\mathscr{L}$ is given as $\nu(\mathscr{L}) \cong \mathscr{O}_{\mathbb{P}^1}(k)$ together with gluing data between the residue class fields

$$\frac{a_i}{b_i} : \mathscr{O}_{\mathbb{P}^1}(k) \otimes \Bbbk(P_i) \to \mathscr{O}_{\mathbb{P}^1}(k) \otimes \Bbbk(Q_i), \ i = 1, \ldots, g.$$

Let $S = \Bbbk[s, t]$ be the coordinate ring of $\mathbb{P}^1$. We start over by choosing two forms $f, h \in S_k$ of degree k and g points $R_i = (R_i^{(0)} : R_i^{(1)}) \in \mathbb{P}^1$ such that for all $i = 1, \ldots, g$ the determinant

$$\det \begin{pmatrix} f & R_i^{(0)} \\ h & R_i^{(1)} \end{pmatrix} = l_i^0 \cdot l_i^{(1)} \cdot r_i$$

has at least two linear factors $l_i^{(0)}$ and $l_i^{(1)}$. Note that this step might be hard to perform over a field $\Bbbk$ of characteristic 0 and we therefore work over a finite field. We compute $2g$ points $P_i = V(l_i^{(0)})$ and $Q_i = V(l_i^{(1)})$ as the vanishing loci of these linear forms. We want to define multipliers $\{a_i, b_i\}_{i=1,\ldots,g}$ such that

$$b_i \cdot f(P_i) = a_i \cdot h(Q_i) \text{ and } b_i \cdot h(P_i) = a_i \cdot f(Q_i) \quad \text{for } i = 1, \ldots, g.$$

By construction, we can choose $\{a_i, b_i\}_{i=1,\ldots,g}$ to be $b_i = 1$ and $a_i = \frac{f(P_i)}{f(Q_i)} = \frac{h(P_i)}{h(Q_i)}$. If we define

$$q_i := \det \begin{pmatrix} s & P_i^{(0)} \\ t & P_i^{(1)} \end{pmatrix} \cdot \det \begin{pmatrix} s & Q_i^{(0)} \\ t & Q_i^{(1)} \end{pmatrix} \quad \text{for } i = 1, \ldots, g,$$

then a basis of $H^0(C, \omega_C)$ is given by

$$\left\{ s_j := \prod_{i=1, i \neq j}^{g} q_i \right\}_{j=1,\dots,g}.$$

This basis $\{s_j\}_{j=1,\dots,g}$ can furthermore be modified in such a way that the scroll defined by the line bundle of degree k will have a "normalized" form, i.e., the $2 \times (g - k + 1)$ matrix defining the scroll will consist of blocks of the form

$$\begin{pmatrix} t_i & t_{i+2} \\ t_{i+1} & t_{i+3} \end{pmatrix},$$

where $T = \Bbbk[t_0, \dots, t_{g-1}]$ is the coordinate ring of $\mathbb{P}^{g-1}$.

In the package [RelativeCanonicalResolution] we also provide a function that describes the generators of the ideal of C in terms of elements of the Cox ring of the scroll $\mathbb{P}(\mathscr{E})$.

Remark 2.1. There is an explicit identification

$$H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR)) \cong H^0(\mathbb{P}^1, (S_a\mathscr{E})(b)) \text{ for } a \geq 0,$$

where $S_a\mathscr{E}$ is the a-th symmetric power of the vector bundle $\mathscr{E}$ (see [Schreyer 1986, (1.3)]). This gives a description of the coordinate ring

$$R_{\mathbb{P}(\mathscr{E})} = \bigoplus_{a,b \in \mathbb{Z}} H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(aH + bR))$$

of $\mathbb{P}(\mathscr{E})$ as the Cox ring $\Bbbk[v, w, \varphi_0, \dots, \varphi_{d-1}]$ equipped with bigrading $\deg v = \deg w = (1, 0)$ and $\deg \varphi_i = (e_1 - e_{i+1}, 1)$.

Finally the relative canonical resolution of $C \subset \mathbb{P}(\mathscr{E})$ can be computed by successively picking syzygies in correct degrees.

Example 2.2. We compute a nodal 6-gonal canonical curve of genus 9.

```
i1 : loadPackage("RelativeCanonicalResolution")
i2 : g=9; -- the genus
i3 : k=6; -- the degree of the pencil
i4 : n=10000; -- characteristic: next prime number after n
i5 : Ican=canCurveWithFixedScroll(g,k,n); -- the canonical curve
i6 : (dim Ican,genus Ican, degree Ican)
o6 = (2, 9, 16)
i7 : betti(res(Ican,DegreeLimit=>1))
            0  1  2  3
o7 = total: 1 15 35 21
         0: 1  .  .  .
         1: . 21 64 70
```

Next we compute the ideal of C inside the Cox ring of the scroll $\mathbb{P}(\mathscr{E})$.

```
i8 : Jcan=curveOnScroll(Ican,g,k); -- the curve inside the scroll
i9 : RX=ring Jcan; -- the bigraded Cox ring of the scroll
        ZZ
o9 = -----[pp , pp , pp , pp , pp , v,w]
     10007    0    1    2    3    4
```

We compute the relative canonical resolution:

```
i10 : T=ring Ican; -- the canonical ring
i11 : H=basis({1,1},RX); -- a basis of H^0(PE, OO_PE(H))
i12 : phi=map(RX,T,H)
i13 : Ican==preimage_phi(Jcan)
o13 = true
i14 : lengthRes=2; -- a lengthlimit for the resolution on the scroll
```

With respect to the total degree, the Betti table of the relative canonical resolution has the following form:

```
-- the relative canonical resolution:
i15 : betti(resX=resCurveOnScroll(Jcan,g,lengthRes))

             0  1  2  3  4
o15 = total: 1  9 16  9  1
          0: 1  .  .  .  .
          1: .  .  .  .  .
          2: .  6  2  .  .
          3: .  3 12  3  .
          4: .  .  2  6  .
          5: .  .  .  .  1
```

The scroll cut out by the g_6^1 on C has the following normalized determinantal representation:

```
i16 : X=preimage_phi(ideal 0_RX); -- the ideal of the scroll
i17 : repX=matrix{{t_0,t_2,t_4,t_6},{t_1,t_3,t_5,t_7}}
o17 = | t_0 t_2 t_4 t_6 |
      | t_1 t_3 t_5 t_7 |
i18 : minors(2,repX)==X
o18 = true
```

Remark 2.3. By o15, we see that the second syzygy bundle N_2 is unbalanced in our example. Although this single example does not show that the generic relative canonical resolution is unbalanced for this case, one can show that this is indeed the generic form (see [Bopp and Hoff 2017]).

3. EXPERIMENTS AND CONJECTURES.

Database of experiments. Using our Macaulay2 package [RelativeCanonicalResolution] we have computed the relative canonical resolution for various cases. For nonhyperelliptic, generic curves of genus $g \leq 23$ with a pencil of degree $3 \leq k \leq \min\{g-1, 14\}$, all expected Betti tables are listed on the webpage [Blug et al. 2018].

The web page was set up with the help of Sascha Blug. All the experiments that led to Betti tables in [Blug et al. 2018] were performed over a finite field. If the examples for certain values (g, k) yield a balanced relative canonical resolution, then by semi-continuity one can conclude that this is indeed the general behavior (even for complex algebraic curves).

Since changing the characteristic for the unbalanced cases did not change the shape of the Betti tables, we believe that the Betti tables in [Blug et al. 2018] reflect the generic behavior. In general, we do not have a proof of this statement. However, it has been determined in some cases whether the first bundle N_1 is balanced (see [Bopp and Hoff 2015] and [Bujokas and Patel 2015]). For several cases our examples

lead to a conjecture that certain higher syzygy bundles in the relative canonical resolution are unbalanced. Most of these cases remain mysterious.

Syzygy divisors on Hurwitz spaces. Deopurkar and Patel used the relative canonical resolution to describe new effective divisors on the Hurwitz scheme $\mathcal{H}_{g,k}$. If the degree k divides $g-1$, it is shown in [Bujokas and Patel 2015] that the relative canonical resolution for a generic element in $\mathcal{H}_{g,k}$ is totally balanced, and hence, the locus μ_i, corresponding set-theoretically to covers in $\mathcal{H}_{g,k}$ for which the i-th syzygy bundle N_i is unbalanced, has expected codimension 1. Deopurkar and Patel [2018] give these syzygy divisors $\mu_1, \dots, \mu_{k-3}$ a scheme structure and compute their classes in a partial compactification of the Hurwitz scheme $\widetilde{\mathcal{H}}_{g,k}$. In their main theorem, they represent the divisor classes $[\mu_i]$ in terms of certain tautological classes κ, ζ and δ (see [Deopurkar and Patel 2018, §2] for the precise definition of those classes).

Theorem 3.1 [Deopurkar and Patel 2018, Theorem 1.1]. *Suppose k divides $g-1$. Let i be an integer with $1 \le i \le k-3$. The locus $\mu_i \subset \widetilde{\mathcal{H}}_{g,k}$ is an effective divisor whose class in $\mathrm{Pic}_{\mathbb{Q}}(\widetilde{\mathcal{H}}_{g,k})$ is given by*

$$[\mu_i] = A_i \cdot \big(6(gk - 6g + k + 6) \cdot \zeta - k(k-12) \cdot \kappa - k^2 \cdot \delta\big),$$

where

$$A_i := \frac{(k-2)(k-3)}{6(i+1)(k-i-1)} \cdot \binom{k-4}{i-1}^2.$$

Note that all the classes $[\mu_i]$ are proportional. The same phenomenon appears for classes of divisorial Brill–Noether loci in the moduli space $\overline{\mathcal{M}}_g$ (see [Eisenbud and Harris 1987a] and [Harris and Mumford 1982]). For the divisorial Brill–Noether classes it is known that these classes are supported on different sets, and in [Deopurkar and Patel 2018] the authors conjecture that this also happens for the syzygy divisors on $\mathcal{H}_{g,k}$.

We come to a different conclusion. Computing various examples of curves and their relative canonical resolution for $(g,k) \in \{(6,13),(7,15),(8,17),(6,19)\}$ over a field of small characteristic $p \le 500$ we found the following pattern which we conjecture to be true in general. Note that the probability of a random computed example to end up in a certain codimension 1 locus is roughly $\frac{1}{p}$.

Conjecture 1. *Let n and k be integers, and $g - 1 = n \cdot k$. Let i be an integer with $1 \le i \le k-3$. For a general element $(C, g_k^1) \in \mu_i \subset \widetilde{\mathcal{H}}_{g,k}$, let N_j be the j-th syzygy bundle in the relative canonical resolution of C with $1 \le j \le k-3$. Then N_j is unbalanced and the splitting type of N_j is*

$$N_j = \mathcal{O}_{\mathbb{P}^1}\big((n-1)(j+1) - 1\big)^{\oplus \binom{k-4}{j-1}} \oplus \mathcal{O}_{\mathbb{P}^1}\big((n-1)(j+1)\big)^{\mathrm{rk}\, N_j - 2 \cdot \binom{k-4}{j-1}} \oplus \mathcal{O}_{\mathbb{P}^1}\big((n-1)(j+1) + 1\big)^{\oplus \binom{k-4}{j-1}}.$$

In particular, all the effective divisors μ_i are supported on the same set.

Remark 3.2. One can easily check that the number A_i in Theorem 3.1 is precisely

$$A_i = \frac{1}{6k} \cdot \mathrm{rk}\, N_i \cdot \binom{k-4}{i-1}.$$

Conjecture 1 predicts that the factor $\binom{k-4}{i-1}$ of A_i also measures how unbalanced the bundle N_i is.

Remark 3.3. If $(g-1) \neq n \cdot k$ then one can still consider the jumping loci set-theoretically defined as the subset of $\widetilde{\mathcal{H}_{g,k}}$ consisting of covers such that the i-th syzygy bundle in the relative canonical resolution does not have generic splitting type. As in the divisorial case, one could ask if all those loci are supported on the same set. Experiments using our package [RelativeCanonicalResolution] show that there are several examples where these jumping loci have different support.

Further conjectures. We state several conjectures concerning the shape of relative canonical resolutions. This has partly also been discussed in [Bopp and Hoff 2015]. We refer to [Arbarello et al. 1985] for basics about Brill–Noether theory. Recall that the Brill–Noether number is defined as $\rho(g,k,r) := g - (r+1)(g-k+r)$ for integers g, k and r.

Conjecture 2. *Let $C \subset \mathbb{P}^{g-1}$ be a general canonical curve, and let k be a positive integer such that $\rho := \rho(g,k,1) \geq 0$, and let g_k^1 be a general pencil in $W_k^1(C)$. Then for bundles $N_i = \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_j^{(i)})$, $i = 2, \ldots, \lceil \frac{k-3}{2} \rceil$, there is the bound*

$$\max_{j,l} |a_j^{(i)} - a_l^{(i)}| \leq \min\{g-k-1, i+1\}.$$

This bound is furthermore sharp in the following sense. Given two integers $k \geq 3$ and $2 \leq i \leq \lceil \frac{k-3}{2} \rceil$, there exists an integer g such that the general canonical curve C of genus g has an i-th syzygy bundle N_i in the relative canonical resolution, associated to a general pencil in $W_k^1(C)$, that satisfies $\max_{j,l} |a_j^{(i)} - a_l^{(i)}| = \min\{g-k-1, i+1\}$. In particular, if $g-k = 2$, the relative canonical resolution is balanced.

Remark 3.4. Conjecture 2 in the case $g - k = 2$ says that the bundles in the relative canonical resolution are of the form

$$N_i = \mathcal{O}_{\mathbb{P}^1}^{\oplus i \cdot \binom{g-4}{i+1}} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus (g-4-i) \cdot \binom{g-4}{g-3-i}}.$$

Note that the Betti numbers $i \cdot \binom{k-2}{i+1}$ appearing in the conjecture are the Betti numbers of a rational normal curve of degree $k - 2$.

We also verified Conjecture 3 for $g \leq 23$.

Conjecture 3. *For a general cover $C \to \mathbb{P}^1$ in $\mathcal{H}_{g,k}$ with $\rho(g,k,1) \leq 0$, the bundle N_1 is balanced.*

SUPPLEMENT. The online supplement contains version 1.0 of [RelativeCanonicalResolution].

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