A package for computations with sparse resultants

GIOVANNI STAGLIANÒ

```
gap> g:= SymmetricGroup( 4 );
Sym( [ 1 .. 4 ] )
gap> tbl:= CharacterTable( g );;  HasIrr( tbl );
false
gap> tblmod2:= CharacterTable( tbl, 2 );
BrauerTable( Sym( [ 1 .. 4 ] ), 2 )
gap> tblmod2 = CharacterTable( tbl, 2 );
true
gap> tblmod2 = BrauerTable( tbl, 2 );
true

gap> libtbl:= CharacterTable( "M" );
CharacterTable( "M" )
gap> CharacterTableRegular( libtbl, 2 );
BrauerTable( "M" )
gap> BrauerTable( libtbl, 2 );
fail
gap> CharacterTable( "Symmetric", 4 );
CharacterTable( "Sym(4)" )
gap> ComputedBrauerTables( tbl );
[ , BrauerTable( Sym( [ 1 .. 4 ] ), 2 ) ]
```

```
i5 : betti(t,Weights=>{1,0})

          0 1  2  3 4
o5 = total: 1 4 13 14 4
        0: 1  .  .  . .
        1: . 2  2  4 2
        2: . 2  5  6 .
        3: . .  4  . 2
        4: . .  .  . 4
        5: . .  2  . .

o5 : BettiTally
i6 : betti(t,Weights=>{0,1})

          0 1  2  3 4
o6 = total: 1 4 13 14 4
        0: 1  .  .  . .
        1: . 2  2  4 2
        2: . 2  5  6 .
        3: . .  4  . 2
        4: . .  .  . 4
        5: . .  2  . .

o6 : BettiTally
i7 : t1 = betti(t,Weights=>{1,1})

          0 1  2  3 4
o7 = total: 1 4 13 14 4
        0: 1 .  .  . .
        1: . .  .  . .
        2: . .  .  . .
        3: . 2  .  . .
        4: . .  .  . .
        5: . 2  .  . .
        6: . .  1  . .
        7: . .  8  6 .
        8: . .  4  8 4

o7 : BettiTally
i8 : peek t1

o8 = BettiTally{(0, {0, 0}, 0) => 1 }
               (1, {2, 2}, 4) => 2
               (1, {3, 3}, 6) => 2
               (2, {3, 7}, 10) => 2
               (2, {4, 4}, 8) => 1
               (2, {4, 5}, 9) => 4
               (2, {5, 4}, 9) => 4
               (2, {7, 3}, 10) => 2
               (3, {4, 7}, 11) => 4
               (3, {5, 5}, 10) => 6
               (3, {7, 4}, 11) => 4
               (4, {5, 7}, 12) => 2
               (4, {7, 5}, 12) => 2
```

```
ring r1 = 32003,(x,y,z),ds;
int a,b,c,t=11,5,3,0;
poly f = x^a+y^b+z^(3*c)+x^(c+2)*y^(c-1)+x^
         x^(c-2)*y^c*(y^2+t*x)^2;
option(noprot);
timer=1;
ring r2 = 32003,(x,y,z),dp;
poly f=imap(r1,f);
ideal j=jacob(f);
vdim(std(j));
==> 536
vdim(std(j+f));
==> 195
timer=0;  // reset timer
```

A package for computations with sparse resultants

GIOVANNI STAGLIANÒ

ABSTRACT: We introduce the *Macaulay*2 package *SparseResultants*, which provides general tools for computing sparse resultants, sparse discriminants, and hyperdeterminants. We give some background on the theory and briefly show how the package works.

INTRODUCTION. The classical Macaulay resultant [1903] (also called the dense resultant) of a system of $n + 1$ polynomial equations in n variables characterizes the solvability of the system, and therefore it is a fundamental tool in computer algebra. However, it is a large polynomial, since it depends on all coefficients of the equations. If we restrict attention to *sparse* polynomial equations, that is, to polynomials which involve only monomials lying in a small set, then we can replace the *dense* resultant with the *sparse* resultant.

The sparse resultant generalizes not only the dense resultant but, for specific choices of the set of monomials, we can obtain other types of classical resultants, such as for instance the *Dixon resultant* [1909] and the *hyperdeterminant* [Cayley 1845; Gelfand et al. 1992]. In the last decades, sparse resultants have received a lot of interest, both from a theoretical point of view (see, e.g., [Gelfand et al. 2008; Sturmfels 1994; Cattani et al. 1998; D'Andrea and Sombra 2015]) and from more computational and applied aspects (see, e.g., [Emiris and Mourrain 1999; Canny and Emiris 2000; Sturmfels 2002; D'Andrea 2002; Jeronimo et al. 2004; Cox et al. 2005; Jeronimo et al. 2009]).

Using the computer program Macaulay2, dense resultants can be calculated using the package *Resultants* [Staglianò 2018], while sparse resultants can be calculated using the new package *SparseResultants*. We point out that in the latter most of the algorithms implemented are based on elimination via Gröbner basis methods. The main defect of this approach is that even when the input polynomials have numerical coefficients, in the calculation all the coefficients are replaced by variables. However, this approach suffices for a number of applications, as we try to show in the following.

This short paper is organized as follows. In Section 1, we review the general theory of sparse resultants (Sections 1A and 1B) and related topics such as the sparse discriminants (Section 1C) and the hyperdeterminants (Section 1D). We focus on the computational aspects used in the package *SparseResultants*. In Section 2, we illustrate how this package works with the help of some examples.

1. AN OVERVIEW OF SPARSE ELIMINATION.   In this section we give some background on the theory of sparse resultants, sparse discriminants, and hyperdeterminants. For details and proofs we refer mainly to [Gelfand et al. 2008, Chapters 8, 9, 13, and 14] and [Cox et al. 2005, Chapter 7]; other references are [Sturmfels 1993; Ottaviani 2013].

1A. *Sparse mixed resultant.* Let $R = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ring of complex Laurent polynomials in n variables. The set of monomials in R is identified with $\mathbb{Z}^n$ by associating to $x^\omega = x_1^{\omega_1} \cdots x_n^{\omega_n} \in R$ the exponent vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n$. If $\mathcal{A}$ is a finite subset of $\mathbb{Z}^n$, we denote by $\mathbb{C}^{\mathcal{A}}$ the space of polynomials in R involving only monomials from $\mathcal{A}$, that is, of polynomials of the form $\sum_{\omega \in \mathcal{A}} a_\omega x^\omega$.

Let $\mathcal{A}_0, \ldots, \mathcal{A}_n$ be $n+1$ finite subsets of $\mathbb{Z}^n$ satisfying the following conditions:

(1) Each $\mathcal{A}_i$ generates $\mathbb{R}^n$ as an affine space.

(2) The union of the sets $\mathcal{A}_i$ generates $\mathbb{Z}^n$ as a $\mathbb{Z}$-module.

Let $\mathcal{Z}_{\mathcal{A}_0, \ldots, \mathcal{A}_n} \subset \prod_{i=0}^n \mathbb{C}^{\mathcal{A}_i}$ be the Zariski closure in the product $\prod_{i=0}^n \mathbb{C}^{\mathcal{A}_i}$ of the set

$$\left\{ (f_0, \ldots, f_n) \in \prod_{i=0}^n \mathbb{C}^{\mathcal{A}_i} : \text{there exists } x \in (\mathbb{C}^*)^n \text{ such that } f_0(x) = \cdots = f_n(x) = 0 \right\}, \tag{1-1}$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $f_i(x) = \sum_{\omega \in \mathcal{A}_i} a_{i,\omega} x^\omega$, for $i = 0, \ldots, n$.

Proposition-Definition 1.1 [Gelfand et al. 2008, Chapter 8, §1]. Under the above assumptions, the variety $\mathcal{Z}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}$ is an irreducible hypersurface in $\prod_{i=0}^n \mathbb{C}^{\mathcal{A}_i}$ that can be defined by an integral irreducible polynomial $\mathrm{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n} \in \mathbb{Z}[(a_{i,\omega}), i = 0, \ldots, n]$ in the coefficients $a_{i,\omega}$ of f_i, for $i = 0, \ldots, n$. Such a polynomial $\mathrm{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}$ is unique up to sign and is called the $(\mathcal{A}_0, \ldots, \mathcal{A}_n)$-resultant (also known as the *sparse* (*mixed*) *resultant*).

The polynomial $\mathrm{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}$ is homogeneous with respect to each group of variables $(a_{i,\omega})$, for $i = 0, \ldots, n$. Moreover, $\mathrm{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}(f_0, \ldots, f_n) = 0$ if the $(n+1)$-tuple $(f_0, \ldots, f_n)$ belongs to (1-1).

Example 1.2. Let $d_0, \ldots, d_n$ be positive integers. For $i = 0, \ldots, n$, let

$$\mathcal{A}_i = \left\{ \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}_{\geq 0}^n : \sum_{j=1}^n \omega_j \leq d_i \right\}.$$

Then the $(\mathcal{A}_0, \ldots, \mathcal{A}_n)$-resultant coincides with the classical (affine) resultant $\mathrm{Res}_{d_0, \ldots, d_n}$, also called the *dense resultant*. Therefore, if $F_i \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ denotes the polynomial obtained by homogenizing $f_i \in \mathbb{C}^{\mathcal{A}_i}$ with respect to a new variable x_0, then $\mathrm{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}(f_0, \ldots, f_n) = 0$ if and only if $F_0, \ldots, F_n$ have a common nontrivial root.

1B. *Sparse unmixed resultant.* Keep the notation and assumptions as above. If all the sets $\mathcal{A}_i$ coincide with each other, that is, $\mathcal{A}_0 = \cdots = \mathcal{A}_n = \mathcal{A}$, then the $(\mathcal{A}_0, \ldots, \mathcal{A}_n)$-resultant is called the $\mathcal{A}$-resultant (also known as the *sparse* (*unmixed*) *resultant*). In this case, we have a useful geometric interpretation that allows us to write down the $\mathcal{A}$-resultant in a compact form. By choosing a numbering $\omega^{(0)}, \ldots, \omega^{(k)}$

of the elements of $\mathcal{A}$, we get a map $\phi_{\mathcal{A}} : (\mathbb{C}^*)^n \to \mathbb{P}^k$ defined by $\phi_{\mathcal{A}}(x) = (\omega^{(0)}(x) : \cdots : \omega^{(k)}(x))$. Let $X_{\mathcal{A}} \subset \mathbb{P}^k$ be the closure of the image of $\phi_{\mathcal{A}}$, which is an irreducible toric variety of dimension n. Then, by taking pull-backs we get an identification between the space of polynomials in $\mathbb{C}^{\mathcal{A}}$ with the space of linear forms on $\mathbb{P}^k$. Moreover, if $f_0, \ldots, f_n \in \mathbb{C}^{\mathcal{A}}$ have a common root in $(\mathbb{C}^*)^n$ then the corresponding linear forms $l_0, \ldots, l_n$ on $\mathbb{P}^k$ define a linear subspace that intersects $X_{\mathcal{A}}$. From this, the following proposition follows directly.

Proposition 1.3 [Gelfand et al. 2008, Chapter 8, §2]. *The polynomial* $\mathrm{Res}_{\mathcal{A}} \in \mathbb{Z}[a_0^{(i)}, \ldots, a_k^{(i)}, i = 0, \ldots, n]$ *coincides with the X-resultant of* $X_{\mathcal{A}} \subset \mathbb{P}^k$. *More precisely, let* $W_{\mathcal{A}} \subset \mathbb{G}(k - n - 1, \mathbb{P}^k)$ *be the Chow hypersurface of the variety* $X_{\mathcal{A}}$, *and let*

$$\psi : \mathbb{P}(\mathbb{C}^{(n+1) \times (k+1)}) \dashrightarrow \mathbb{G}(n, k) \simeq \mathbb{G}(k - n - 1, k)$$

be the natural projection from the projectivization of the space of complex matrices which have the shape $(n + 1) \times (k + 1)$ *to* $\mathbb{G}(n, k)$. *Then we have that* $\mathrm{Res}_{\mathcal{A}}$ *is the polynomial defining the pull-back* $\overline{\psi^{-1}(W_{\mathcal{A}})}$.

Remark 1.4. With the notation of the proposition above, in coordinates, the map ψ is defined by the $(n + 1) \times (n + 1)$ minors of the generic $(n + 1) \times (k + 1)$ matrix of variables

$$\begin{pmatrix} a_0^{(0)} & a_1^{(0)} & \cdots & a_k^{(0)} \\ \vdots & \vdots & \ddots & \vdots \\ a_0^{(n)} & a_1^{(n)} & \cdots & a_k^{(n)} \end{pmatrix}. \tag{1-2}$$

Notably, $\mathrm{Res}_{\mathcal{A}}$ can be expressed as a homogeneous polynomial of degree $\deg(X_{\mathcal{A}})$ in the $(n+1) \times (n+1)$ minors of the matrix (1-2).

Example 1.5. Let $\mathcal{A} = \{(\omega_1, \omega_2) \in \mathbb{Z}^2 : \omega_1 + \omega_2 \leq 2\}$, so that $X_{\mathcal{A}} \subset \mathbb{P}^5$ is the Veronese surface. The $\mathcal{A}$-resultant is a polynomial of degree 12 in 18 variables with 21894 terms. It can be expressed as a polynomial of degree 4 in the Plücker coordinates of $\mathbb{G}(2, 5)$ with 74 terms.

1C. *Sparse discriminant.* We continue by letting $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set of $k + 1$ elements that generate $\mathbb{Z}^n$ as a $\mathbb{Z}$-module, and let $\phi_{\mathcal{A}} : (\mathbb{C}^*)^n \to \mathbb{P}^k$ and $X_{\mathcal{A}} \subset \mathbb{P}^k$ be defined as above. Let $\nabla_{\mathcal{A}} \subset \mathbb{C}^{\mathcal{A}}$ be the Zariski closure of the set

$$\left\{ f \in \mathbb{C}^{\mathcal{A}} : \text{there exists } x \in (\mathbb{C}^*)^n \text{ such that } f(x) = \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0 \right\}. \tag{1-3}$$

Proposition-Definition 1.6 [Gelfand et al. 2008, Chapter 9, §1]. The projectivization $\mathbb{P}(\nabla_{\mathcal{A}}) \subset \mathbb{P}^k$ of the variety $\nabla_{\mathcal{A}}$ coincides with the dual variety $X_{\mathcal{A}}^\vee$ of $X_{\mathcal{A}}$. In the case where $X_{\mathcal{A}}^\vee$ is a hypersurface, an integral irreducible polynomial $\mathrm{Disc}_{\mathcal{A}}$ defining it (which is unique up to sign) is called the $\mathcal{A}$-discriminant (also known as the *sparse discriminant*).

Thus the $\mathcal{A}$-discriminant (when it exists) is a homogeneous polynomial $\mathrm{Disc}_{\mathcal{A}} \in \mathbb{Z}[a_\omega, \omega \in \mathcal{A}]$, and $\mathrm{Disc}_{\mathcal{A}}(f) = 0$ for each polynomial f belonging to (1-3).

Example 1.7. Let $d \geq 1$ and let $\mathcal{A} = \big\{(\omega_1, \ldots, \omega_n) \in \mathbb{Z}_{\geq 0}^n : \sum_{j=1}^n \omega_j \leq d\big\}$. Then the $\mathcal{A}$-discriminant coincides with the classical (affine) discriminant Disc_d, also called the *dense discriminant*. Therefore, if $F \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ denotes the polynomial obtained by homogenizing $f \in \mathbb{C}^{\mathcal{A}}$ with respect to a new variable x_0, then $\mathrm{Disc}_{\mathcal{A}}(f) = 0$ if and only if the hypersurface $\{F = 0\} \subset \mathbb{P}^n$ is not smooth.

Remark 1.8 ("Cayley trick", [Gelfand et al. 2008, Chapter 9, Proposition 1.7]). Let $\mathcal{A}_0, \ldots, \mathcal{A}_n \subset \mathbb{Z}^n$ be finite subsets satisfying the assumptions in Section 1A. Let $\mathcal{A} \subset \mathbb{Z}^n \times \mathbb{Z}^n$ be defined by

$$\mathcal{A} = (\mathcal{A}_0 \times \{0\}) \cup (\mathcal{A}_1 \times \{e_1\}) \cup \cdots \cup (\mathcal{A}_n \times \{e_n\}),$$

where the e_i are the standard basis vectors of $\mathbb{Z}^n$. Thus a polynomial $f \in \mathbb{C}^{\mathcal{A}}$ has the form

$$f_0(x) + \sum_{i=1}^n y_i \, f_i(x) \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

where $f_i \in \mathbb{C}^{\mathcal{A}_i}$. We have the following relation (up to sign), known as the "Cayley trick":

$$\mathrm{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}(f_0, \ldots, f_n) = \mathrm{Disc}_{\mathcal{A}}\bigg(f_0(x) + \sum_{i=1}^n y_i \, f_i(x)\bigg). \tag{1-4}$$

1D. *Hyperdeterminant.* An important special type of sparse discriminant is the determinant (or hyperdeterminant) of multidimensional matrices, which was introduced by Cayley [1845] (see also [Gelfand et al. 2008, Chapter 14] and [Ottaviani 2013]). Let f be a multilinear form in r groups of variables $x_0^{(1)}, \ldots, x_{k_1}^{(1)}; \ldots; x_0^{(r)}, \ldots, x_{k_r}^{(r)}$, that is

$$f = \sum_{0 \leq i_\iota \leq k_\iota} a_{i_1, \ldots, i_r} x_{i_1}^{(1)} \cdots x_{i_r}^{(r)}.$$

Let $\mathcal{A} \subset \mathbb{Z}^{(k_1+1)+\cdots+(k_r+1)}$ denote the set of exponent vectors that can occur in such a form f. Notice that to give f is equivalent to giving an r-dimensional matrix

$$M_f = (a_{i_1, \ldots, i_r})_{0 \leq i_\iota \leq k_\iota}$$

of shape $(k_1 + 1) \times \cdots \times (k_r + 1)$. The determinant of shape $(k_1 + 1) \times \cdots \times (k_r + 1)$ is defined to be the $\mathcal{A}$-discriminant, that is, for a form f as above, we have

$$\det(M_f) = \mathrm{Disc}_{\mathcal{A}}(f).$$

One sees that the variety $X_{\mathcal{A}}$ is the image of the Segre embedding of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$. Therefore, the hypersurface in $\mathbb{P}(\mathbb{C}^{(k_1+1) \times \cdots \times (k_r+1)})$ defined by the determinant of shape $(k_1 + 1) \times \cdots \times (k_r + 1)$ is the dual variety of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$. Notice also that we have $\det(M_f) = 0$ if and only if the hypersurface

$$\{f = 0\} \subset \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$$

is not smooth.

The next two basic results have been proved in [Gelfand et al. 2008, Chapter 14, Theorems 1.3 and 2.4].

Theorem 1.9 [Gelfand et al. 2008]. *The determinant of shape* $(k_1 + 1) \times \cdots \times (k_r + 1)$ *exists* (*that is the dual variety of* $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$ *is a hypersurface*) *if and only if*

$$2 \max_{1 \leq j \leq r} (k_j) \leq \sum_{j=1}^{r} k_j. \tag{1-5}$$

Theorem 1.10 [Gelfand et al. 2008]. *Denote by* $N(k_1, \ldots, k_r)$ *the degree of the determinant of shape* $(k_1 + 1) \times \cdots \times (k_r + 1)$ *when* (1-5) *is satisfied, and let* $N(k_1, \ldots, k_r) = 0$ *otherwise. We have*

$$\sum_{k_1,\ldots,k_r \geq 0} N(k_1, \ldots, k_r) z_1^{k_1} \cdots z_r^{k_r} = \frac{1}{\left(1 - \sum_{i=2}^{r}(i-2)e_i(z_1, \ldots, z_r)\right)^2},$$

where $e_i(z_1, \ldots, z_r)$ *is the* i*-th elementary symmetric polynomial.*

Remark 1.11 [Gelfand et al. 2008, Chapter 4, Propositions 1.4 and 1.8]. The determinant of shape $(k_1 + 1) \times \cdots \times (k_r + 1)$ is invariant under the action of $\mathrm{SL}(k_1 + 1) \times \cdots \times \mathrm{SL}(k_r + 1)$ on the space of matrices of shape $(k_1 + 1) \times \cdots \times (k_r + 1)$. It is also invariant under permutations of the dimensions, that is, if $M = (a_{i_1,\ldots,i_r})$ is a matrix of shape $(k_1 + 1) \times \cdots \times (k_r + 1)$ and σ is a permutation of indices $1, \ldots, r$, denoting by $\sigma(M)$ the matrix of shape $(k_{\sigma^{-1}(1)} + 1) \times \cdots \times (k_{\sigma^{-1}(r)} + 1)$, whose $(i_1, \ldots, i_r)$-th entry is equal to $a_{i_{\sigma(1)},\ldots,i_{\sigma(r)}}$, we have $\det(\sigma(M)) = \det(M)$.

There are at least two important cases where determinants can be computed without resorting to elimination. We briefly recall them in 1D1 and 1D2.

1D1. *Schläfli's method.* Let M be an r-dimensional matrix of shape $(k_1+1) \times \cdots \times (k_r+1)$ corresponding to a multilinear form $f \in \mathbb{C}[x_0^{(1)}, \ldots, x_{k_1}^{(1)}; \ldots; x_0^{(r)}, \ldots, x_{k_r}^{(r)}]$. Assume that there exist both the determinants of shapes $(k_1+1) \times \cdots \times (k_r+1)$ and $(k_1+1) \times \cdots \times (k_{r-1}+1)$. We can interpret the r-dimensional matrix M as an $(r-1)$-dimensional matrix $\tilde{M}(x_0^{(r)}, \ldots, x_{k_r}^{(r)})$ of shape $(k_1 + 1) \times \cdots \times (k_{r-1} + 1)$ whose entries are linear forms in the variables $x_0^{(r)}, \ldots, x_{k_r}^{(r)}$; in other words, we can see f as a polynomial $\tilde{f} \in (\mathbb{C}[x_0^{(r)}, \ldots, x_{k_r}^{(r)}])[x_0^{(1)}, \ldots, x_{k_1}^{(1)}; \ldots; x_0^{(r-1)}, \ldots, x_{k_{r-1}}^{(r-1)}]$. Let

$$F_M = F_M(x_0^{(r)}, \ldots, x_{k_r}^{(r)}) = \det(\tilde{M}(x_0^{(r)}, \ldots, x_{k_r}^{(r)})),$$

which is a homogeneous polynomial in $x_0^{(r)}, \ldots, x_{k_r}^{(r)}$, and let $\mathrm{Disc}(F_M)$ be the (classical) discriminant of F_M. Then we have the following:

Theorem 1.12 [Gelfand et al. 2008; Schläfli 1852]. *The polynomial* $\mathrm{Disc}(F_M)$ *is divisible by the determinant* $\det(M)$. *Moreover if the shape of* M *is one of*

$$m \times m \times 2, \quad m \times m \times 3, \quad 2 \times 2 \times 2 \times 2, \quad with \ m \geq 2, \tag{1-6}$$

then we have $\mathrm{Disc}(F_M) = \det(M)$.

The method above turns out to be very effective; however it was conjectured in [Gelfand et al. 2008, p. 479], and later proved in [Weyman and Zelevinsky 1996], that the shapes in (1-6) are the only ones for which the method gives the determinant exactly.

1D2. *Determinants of boundary shape.* For an $(r+1)$-dimensional matrix M of shape $(k_0+1) \times (k_1+1) \times$ $\cdots \times (k_r + 1)$, we say that it is of *boundary shape* if the inequality (1-5) is an equality. Without loss of generality, we can assume that $k_0 = \max_{0 \le j \le r}(k_j)$, so that $k_0 = k_1 + \cdots + k_r$. Let $f \in \mathbb{C}[x_0^{(0)}, \ldots, x_{k_0}^{(0)}; \ldots; x_0^{(r)}, \ldots, x_{k_r}^{(r)}]$ be the corresponding multilinear form of such a matrix M. Thinking of f as a linear polynomial in

$$(\mathbb{C}[x_0^{(1)}, \ldots, x_{k_1}^{(1)}; \ldots; x_0^{(r)}, \ldots, x_{k_r}^{(r)}])[x_0^{(0)}, \ldots, x_{k_0}^{(0)}],$$

we can interpret M as a list of $k_0 + 1$ multilinear forms $f_0, \ldots, f_{k_0}$ in the r groups of variables

$$x_0^{(1)}, \ldots, x_{k_1}^{(1)}; \ldots; x_0^{(r)}, \ldots, x_{k_r}^{(r)}.$$

A simple consequence of the "Cayley trick" (see [Gelfand et al. 2008, Chapter 3, Corollary 2.8]) gives the following:

Proposition 1.13 [Gelfand et al. 2008]. *The determinant of an $(r+1)$-dimensional matrix M of boundary shape $(k_0 + 1) \times \cdots \times (k_r + 1)$ coincides with the resultant of the multilinear forms $f_0, \ldots, f_{k_0}$, that is,* $\det(M) = 0$ *if and only if the system of multilinear equations* $f_0(x) = \cdots = f_{k_0}(x) = 0$ *has a nontrivial solution on* $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$. *In other words, the determinant of shape $(k_0 + 1) \times \cdots \times (k_r + 1)$ coincides with the X-resultant of the Segre embedding of* $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$.

Remark 1.14. The determinant of a matrix M of boundary shape $(k_0+1) \times \cdots \times (k_r+1)$ can be explicitly expressed as the determinant of an ordinary square matrix of order $(k_0 + 1)!/(k_1! \cdots k_r!)$ whose entries are linear forms in the entries of M; see [Gelfand et al. 2008, Chapter 14, Theorem 3.3].

2. SPARSE RESULTANTS IN *Macaulay2*. In this section, we describe some of the functions implemented in the package *SparseResultants*. For more details and examples, we refer to its documentation.

One of the main functions is `sparseResultant`, which via elimination techniques calculates sparse mixed resultants $\mathrm{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}$ (see Section 1A) and sparse unmixed resultants $\mathrm{Res}_{\mathcal{A}}$ (see Section 1B). This function can be called in two ways. The first one is to pass a list of $n + 1$ matrices $A_0, \ldots, A_n$ over $\mathbb{Z}$ and with n rows to represent the sets $\mathcal{A}_0, \ldots, \mathcal{A}_n \subset \mathbb{Z}^n$ (it is enough to pass just one matrix A in the unmixed case). Then the output will be another function that takes as input $n + 1$ polynomials $f_i = \sum_{\omega \in \mathcal{A}_i} a_{i,\omega} x^\omega$, for $i = 0, \ldots, n$, and returns their sparse resultant. An error is thrown if the polynomials f_i do not have the correct form. Roughly, this returned function is a container for the general expression of the sparse resultant (possibly written out in a compact form as in Proposition 1.3) and for the rule to evaluate it at the $n + 1$ polynomials f_i. The second way to call `sparseResultant` is to pass directly the polynomials f_i. This is equivalent to forming the matrices A_i whose columns are given by $\{\omega \in \mathbb{Z}^n : \text{the coefficient in } f_i \text{ of } x^\omega \text{ is } \neq 0\}$ (see the function `exponentsMatrix`) and then proceeding as described above.

As an example we now calculate a particular type of sparse unmixed resultant, known as the *Dixon resultant* (see [Sturmfels 1993, Section 2.4] and [Cox et al. 2005, Chapter 7, §2, Exercise 10]; see also the classical reference [Dixon 1909]).

Example 2.1. Consider the following system of three bihomogeneous polynomials of bidegree $(2, 1)$ in the two groups of variables (x_0, x_1), (y_0, y_1):

$$c_{1,1}x_1^2 y_1 + c_{1,2}x_1x_2 y_1 + c_{1,3}x_2^2 y_1 + c_{1,4}x_1^2 y_2 + c_{1,5}x_1x_2 y_2 + c_{1,6}x_2^2 y_2 = 0,$$
$$c_{2,1}x_1^2 y_1 + c_{2,2}x_1x_2 y_1 + c_{2,3}x_2^2 y_1 + c_{2,4}x_1^2 y_2 + c_{2,5}x_1x_2 y_2 + c_{2,6}x_2^2 y_2 = 0, \qquad (2\text{-}1)$$
$$c_{3,1}x_1^2 y_1 + c_{3,2}x_1x_2 y_1 + c_{3,3}x_2^2 y_1 + c_{3,4}x_1^2 y_2 + c_{3,5}x_1x_2 y_2 + c_{3,6}x_2^2 y_2 = 0.$$

Putting $x_2 = y_2 = 1$ we get a system of three nonhomogeneous polynomials in two variables $(x, y) = (x_1, y_1)$, of which we can calculate the sparse (unmixed) resultant. This polynomial is homogeneous of degree 12 in the 18 variables $c_{1,1}, \ldots, c_{3,6}$ with 20791 terms, which vanishes if and only if (2-1) has a nontrivial solution. The time for this computation is less than one second (on a standard laptop).

```
$ M2 --no-preload
Macaulay2, version 1.17
i1 : needsPackage "SparseResultants";

i2 : R = ZZ[c_(1,1)..c_(3,6)][x,y];

i3 : f = (c_(1,1)*x^2*y+c_(1,2)*x*y+c_(1,3)*y+c_(1,4)*x^2+c_(1,5)*x+c_(1,6),
         c_(2,1)*x^2*y+c_(2,2)*x*y+c_(2,3)*y+c_(2,4)*x^2+c_(2,5)*x+c_(2,6),
         c_(3,1)*x^2*y+c_(3,2)*x*y+c_(3,3)*y+c_(3,4)*x^2+c_(3,5)*x+c_(3,6));

i4 : A = exponentsMatrix f
o4 = | 0 0 1 1 2 2 |
     | 0 1 0 1 0 1 |
             2       6
o4 : Matrix ZZ  <--- ZZ

i5 : time Res = sparseResultant A;
     -- used 0.241391 seconds
o5 : SparseResultant (sparse unmixed resultant associated to | 0 0 1 1 2 2 |)
                                                             | 0 1 0 1 0 1 |

i6 : time U = Res f;
     -- used 0.574002 seconds

i7 : (first degree U, # terms U)
o7 = (12, 20791)
```

Another function, sparseDiscriminant, calculates sparse discriminants $\text{Disc}_{\mathcal{A}}$ (see Section 1C). This function works similarly to the previous one. In particular, it accepts as input either a matrix representing the exponent vectors of a (Laurent) polynomial or the polynomial directly.

Example 2.2. Using the Cayley trick (1-4), we express the dense resultant of three generic ternary forms of degrees 1, 1, 2 (which is a special type of sparse mixed resultant) as a sparse discriminant. The calculation time is less than one second.

```
i8 : clearAll;

i9 : K = ZZ[a_0..a_2,b_0..b_2,c_0..c_5], Rx = K[x_1,x_2];

i10 : f = (a_0+a_1*x_1+a_2*x_2,
          b_0+b_1*x_1+b_2*x_2,
          c_0+c_1*x_1+c_2*x_2+c_3*x_1^2+c_4*x_1*x_2+c_5*x_2^2);

i11 : Rxy = K[x_1,x_2,y_1,y_2], f' = (sub(f_0,Rxy), sub(f_1,Rxy), sub(f_2,Rxy));

i12 : time sparseResultant(f_0,f_1,f_2) ==
          -sparseDiscriminant(f'_0 + y_1*f'_1 + y_2*f'_2)
      -- used 0.746274 seconds
o12 = true
```

A derived function of `sparseDiscriminant` is `determinant` (or simply `det`), which calculates determinants of multidimensional matrices (see Section 1D). However for this last one, more specialized algorithms are also available and automatically applied.

Example 2.3. We calculate the determinant of a generic four-dimensional matrix of shape $2 \times 2 \times 2 \times 2$ (see also [Huggins et al. 2008]). This polynomial is homogeneous of degree 24 in the 16 variable entries of the matrix and it has 2894276 terms. The approach for this calculation is to apply (1-6) recursively. The calculation time is about 10 minutes, but it takes much less time if we specialize the entries of the matrix to be random numbers.

```
i13 : M = genericMultidimensionalMatrix {2,2,2,2}
o13 = {{{{a       , a       }, {a       , a       }}, {{a       , a       }, ...
         0,0,0,0   0,0,0,1     0,0,1,0   0,0,1,1        0,1,0,0   0,1,0,1     ...

o13 : 4-dimensional matrix of shape 2 x 2 x 2 x 2 over ZZ[a       , a       , ...
                                                          0,0,0,0   0,0,0,1   ...

i14 : time D = det M;
      -- used 634.773 seconds

i15 : (first degree D, # terms D)
o15 = (24, 2894276)
```

Example 2.4. Here we take A and B to be random matrices of shapes $2 \times 2 \times 2 \times 4$ and $4 \times 2 \times 5$, respectively. We calculate the convolution $A * B$ (see [Gelfand et al. 2008, p. 449]), which is a matrix of shape $2 \times 2 \times 2 \times 2 \times 5$. Then we verify a formula proved in [Dionisi and Ottaviani 2003] for $\det(A * B)$, which generalizes the Cauchy–Binet formula in the multidimensional case. The approach for the calculation of the determinant of shape $4 \times 2 \times 5$ is using Proposition 1.13, while the determinants of shapes $2 \times 2 \times 2 \times 4$ and $2 \times 2 \times 2 \times 2 \times 5$ are calculated using Remark 1.14. The calculation time is less than one second.

```
i16 : K = ZZ/33331;

i17 : A = randomMultidimensionalMatrix({2,2,2,4},CoefficientRing=>K);
o17 : 4-dimensional matrix of shape 2 x 2 x 2 x 4 over K

i18 : B = randomMultidimensionalMatrix({4,2,5},CoefficientRing=>K);
o18 : 3-dimensional matrix of shape 4 x 2 x 5 over K

i19 : time det(A * B) == (det A)^5 * (det B)^6
      -- used 0.535271 seconds
o19 = true
```

SUPPLEMENT.   The online supplement contains version 1.1 of `SparseResultants`.

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