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ABSTRACT: We introduce the *Macaulay*2 package `LinearTruncations` for finding and studying the truncations of a multigraded module over a standard multigraded ring that have linear resolutions.

1. INTRODUCTION AND PRELIMINARIES. Castelnuovo–Mumford regularity is a fundamental invariant in commutative algebra and algebraic geometry. Roughly speaking, it measures the complexity of a module or sheaf. Let S be a polynomial ring with the standard grading and let M be a finitely generated S-module. In this case, Castelnuovo–Mumford regularity is typically defined in terms of either the graded Betti numbers of M or the vanishing of local cohomology modules $H^i_{\mathfrak{m}}(M)$, where $\mathfrak{m}$ is the maximal homogeneous ideal of S. Eisenbud and Goto [1984] showed that the Castelnuovo–Mumford regularity of M is the minimum degree where the truncation of M has a linear resolution.

Extensions of Castelnuovo–Mumford regularity were introduced for bigraded modules by Hoffman and Wang [2004], independently for multigraded modules by Maclagan and Smith [2004], then later in a more general setting by Botbol and Chardin [2017]. The multigraded regularity of a module is a region in $\mathbb{Z}^r$ rather than an integer. For a polynomial ring with a standard $\mathbb{Z}^r$-grading, multigraded regularity is invariant under positive translations and thus can be described by its minimal elements. An affirmative answer to the following open question would reduce this to a finite computation.

Question 1.1. Can the minimal elements of the regularity of M be bounded in terms of S and the Betti numbers of M?

In analogy to the singly graded case, one may ask about the relation between multigraded Castelnuovo–Mumford regularity and the multidegrees where the truncation of a module has a linear resolution, which we call the *linear truncation region*. (See Definitions 1.2 and 1.3.) In the multigraded setting these regions can differ, but a bound on the linear truncations would answer Question 1.1 by [Eisenbud et al. 2015, Proposition 4.11]. Our goal is to compute the minimal elements of the linear truncation region within a specified finite region of $\mathbb{Z}^r$.

We introduce the *LinearTruncations* package for [Macaulay2], which provides tools for studying the resolutions of truncations of modules over rings with standard multigradings. Given a module and a bounded range of multidegrees, our package can identify all linear truncations in the range. The algorithm

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`LinearTruncations` version 1.0

uses a search function that is also applicable to other properties of modules described by sets of degrees. The examples here were computed using version 1.18 of *Macaulay*2 and version 1.0 of *LinearTruncations*.

In Section 2, we describe the main algorithms of this package, `findRegion` and `linearTruncations`. In Section 3, we discuss the relation between the linear truncation region and the multigraded regularity and we introduce `regularityBound` and `linearTruncationsBound` as faster methods for calculating subsets of the multigraded regularity and linear truncation regions, respectively.

To set our notation, let k be a field and let

$$S = k[x_{i,j} \mid 1 \le i \le r,\ 0 \le j \le n_i]$$

be a $\mathbb{Z}^r$-graded polynomial ring with $\deg x_{ij} = \boldsymbol{e}_i$, the i-th standard basis vector in $\mathbb{Z}^r$, for all j (so that S is the coordinate ring of a product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ of projective spaces). The function `multigradedPolynomialRing` produces such rings:

```
i1 : needsPackage "LinearTruncations"
o1 = LinearTruncations
o1 : Package
i2 : S = multigradedPolynomialRing {1,2}
o2 = S
o2 : PolynomialRing
i3 : degrees S
o3 = {{1, 0}, {1, 0}, {0, 1}, {0, 1}, {0, 1}}
o3 : List
```

Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. For a multidegree $\boldsymbol{d} = (d_1, \ldots, d_r) \in \mathbb{Z}^r$, write $\bar{\boldsymbol{d}}$ for the total degree $d_1 + \cdots + d_r$ of $\boldsymbol{d}$ and $M_{\ge \boldsymbol{d}}$ for the truncation $\bigoplus_{\boldsymbol{d}' \ge \boldsymbol{d}} M_{\boldsymbol{d}'}$ of M at $\boldsymbol{d}$, where $\boldsymbol{d}' \ge \boldsymbol{d}$ if this inequality is true for each coordinate.

Definition 1.2. A homogeneous chain complex

$$0 \leftarrow G_0 \leftarrow G_1 \leftarrow \cdots \leftarrow G_k \leftarrow 0$$

of free S-modules is *linear* if $G_0 \simeq \bigoplus S(-\boldsymbol{d})$ for some $\boldsymbol{d} \in \mathbb{Z}^r$ and for each free summand $S(-\boldsymbol{d}')$ of G_i we have $\bar{\boldsymbol{d}}' = \bar{\boldsymbol{d}} + i$.

The function `isLinearComplex` checks this condition. To print the degrees appearing in the complex, use `supportOfTor`:

```
i4 : B = irrelevantIdeal S
o4 = ideal (x    x   , x    x   , x    x   , x    x   , x    x   , x    x   )
             0,1 1,2   0,0 1,2   0,1 1,1   0,0 1,1   0,1 1,0   0,0 1,0
o4 : Ideal of S
i5 : F = res comodule B
      1      6      9      5      1
o5 = S  <-- S  <-- S  <-- S  <-- S  <-- 0
     0      1      2      3      4      5
o5 : ChainComplex
```

```
i6 : netList supportOfTor F

     +------+------+
o6 = |{0, 0}|      |
     +------+------+
     |{1, 1}|      |
     +------+------+
     |{2, 1}|{1, 2}|
     +------+------+
     |{2, 2}|{1, 3}|
     +------+------+
     |{2, 3}|      |
     +------+------+

i7 : isLinearComplex F

o7 = false
```

Definition 1.3. The *linear truncation region* of M is

$$\{\boldsymbol{d} \mid M_{\geq \boldsymbol{d}} \text{ has a linear resolution with generators in degree } \boldsymbol{d}\} \subset \mathbb{Z}^r.$$

Remark 1.4. Our definitions imply that the nonzero entries in the differential matrices of a linear resolution will have total degree 1. We also require that the generators have degree $\boldsymbol{d}$ so that a linear resolution for $M_{\geq \boldsymbol{d}}$ implies the existence of a linear resolution for $M_{\geq \boldsymbol{d}'}$ whenever $\boldsymbol{d}' \geq \boldsymbol{d}$. We can thus describe the linear truncation region by giving its minimal elements.

2. FINDING LINEAR TRUNCATIONS. Eisenbud, Erman, and Schreyer [Eisenbud et al. 2015] proved that the linear truncation region of M is nonempty. In particular it contains the output of the function `coarseMultigradedRegularity` from their package [TateOnProducts]. However, in general this degree is neither a minimal element itself nor greater than all the minimal elements. (See Example 2.1.)

The function `linearTruncations` searches for multidegrees where the truncation of M has a linear resolution by calling the function `findRegion`, which implements Algorithm 1. Since we do not know of a bound on the total degree of the minimal elements in the linear truncation region given the Betti numbers of M, `linearTruncations` is not guaranteed to produce all generators as a module over the semigroup $\mathbb{N}^r$. By default it searches above the componentwise minimum of the degrees of the generators of M and below the degree with all coordinates equal to $c+1$, where c is the output of `regularity`. Otherwise the range is taken as a separate input.

Example 2.1. Let $S = k[x_{0,0}, x_{0,1}, x_{0,2}, x_{1,0}, x_{1,1}, x_{1,2}, x_{1,3}]$ be the Cox ring of $\mathbb{P}^2 \times \mathbb{P}^3$. For each $d \geq 2$, let $\phi_d : S(-d,-d)^6 \to S(0,-d)^2 \oplus S(-d,0)^4$ be given by

$$\begin{pmatrix} x_{0,0}^d & x_{0,1}^d & x_{0,2}^d & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{0,1}^d & x_{0,0}^d & x_{0,2}^d \\ x_{1,0}^d & 0 & 0 & x_{1,0}^d & 0 & 0 \\ 0 & x_{1,1}^d & 0 & 0 & x_{1,1}^d & 0 \\ 0 & 0 & x_{1,2}^d & 0 & 0 & x_{1,2}^d \\ 0 & 0 & 0 & x_{1,3}^d & 0 & 0 \end{pmatrix},$$

and define $M^{(d)} := \operatorname{coker} \phi_d$. The `coarseMultigradedRegularity` of $M^{(3)}$ is $\{3, 3\}$, the `regularity` of $M^{(3)}$ is 5, and $\{3, 3\}$ and $\{8, 2\}$ are minimal elements of the linear truncation region. Since $\{8, 2\}$ is

Input : a module M, a Boolean function f that takes M as input, and a range $(\boldsymbol{a}, \boldsymbol{b})$
Output : minimal elements between $\boldsymbol{a}$ and $\boldsymbol{b}$ where M satisfies f

```
A := ∅
K := {a}
while K ≠ ∅ do
    d := first element of K
    K = K \ {d}
    if d ∉ A + ℕ^r then
        if M satisfies f at d then
            A = A ∪ {d}
        else
            for 1 ≤ i ≤ r do
                if d + e_i ≤ b then
                    K = K ∪ {d + e_i}
return minimal elements of A
```

Algorithm 1. `findRegion`

not below $\{5+1, 5+1\}$, it will not be returned by the `linearTruncations` function with the default options:

```
i8 : (S,E) = productOfProjectiveSpaces{2,3};

i9 : d = 3;

i10 : M = coker(map(S^{{0,-d},{0,-d},{-d,0},{-d,0},{-d,0},{-d,0}},
        S^{{-d,-d},{-d,-d},{-d,-d},{-d,-d},{-d,-d},{-d,-d}},
        {{x_(0,0)^d,x_(0,1)^d,x_(0,2)^d,0,0,0}, {0,0,0,x_(0,1)^d,x_(0,0)^d,x_(0,2)^d},
        {-x_(1,0)^d,0,0,-x_(1,0)^d,0,0}, {0,-x_(1,1)^d,0,0,-x_(1,1)^d,0},
        {0,0, -x_(1,2)^d,0,0, -x_(1,2)^d}, {0,0,0, -x_(1,3)^d,0,0}}));

i11 : linearTruncations M

o11 = {{3, 3}}

o11 : List

i12 : linearTruncations({{0,0},{8,6}},M)

o12 = {{3, 3}, {8, 2}}

o12 : List
```

Based on the computations from $M^{(d)}$ for $2 \le d \le 10$ we expect that for $d \ge 2$ the module $M^{(d)}$ will have `coarseMultigradedRegularity` equal to $\{d, d\}$, with $\{d, d\}$ and $\{3d-1, d-1\}$ both minimal elements of the linear truncation region.

At each step of Algorithm 1 the set A contains degrees satisfying f and the set K contains the minimal degrees remaining to be checked. There are options to initialize A and K differently — degrees in A will be assumed to satisfy f, and degrees below those in K will be excluded from the search (and thus assumed not to satisfy f). Supplying such prior knowledge can decrease the length of the computation by limiting the number of times the algorithm calls f.

The pseudocode in Algorithm 1 masks the fact that A and K are stored as monomial ideals in a temporary singly graded polynomial ring. Similarly, the function `findMins` will convert a list of multidegrees to a monomial ideal in order to calculate its minimal elements via a Gröbner basis.

3. Relation to regularity. As discussed above, the minimal element of the linear truncation region of a singly graded module agrees with its Castelnuovo–Mumford regularity, which can be determined from its Betti numbers. In the multigraded case these concepts are still linked, but their relationship is more complicated. For instance, the following inclusion is strict:

Theorem 3.1. *If $H^0_B(M) = 0$, then the linear truncation region of M is a subset of the multigraded regularity region* $\operatorname{reg} M$ *of M, as defined in* [Maclagan and Smith 2004].

Proof. See [Berkesch et al. 2020, Theorem 2.9] or [Eisenbud et al. 2015, Proposition 4.11]. □

Unfortunately the multigraded Betti numbers of M do not determine either its regularity or its linear truncations. However, the functions `regularityBound` and `linearTruncationsBound` compute subsets of these regions using only the twists appearing in the minimal free resolution of M. In many examples they produce the same outputs as `multigradedRegularity` (from the package *VirtualResolutions* [Almousa et al. 2020]) and `linearTruncations`, respectively, without computing sheaf cohomology or truncating the module.

The algorithms for `linearTruncationsBound` and `regularityBound` are based on the following theorem (from [Bruce et al. 2021]):

Theorem 3.2. *If $H^0_B(M) = 0$ and $H^1_B(M) = 0$ then*

$$\bigcap_{\operatorname{Tor}_i(M,k)_{\boldsymbol{b}}\neq 0} \bigcup_{\sum \lambda_j = i} [\boldsymbol{b} - \lambda_1 \boldsymbol{e}_i - \cdots - \lambda_r \boldsymbol{e}_r + \mathbb{N}^r]$$

is a subset of the `linearTruncations` *of M, and*

$$\bigcap_{\operatorname{Tor}_i(M,k)_{\boldsymbol{b}}\neq 0} \bigcup_{\sum \lambda_j = i-1} [\boldsymbol{b} - \mathbf{1} - \lambda_1 \boldsymbol{e}_i - \cdots - \lambda_r \boldsymbol{e}_r + \mathbb{N}^r]$$

is a subset of the `multigradedRegularity` *of M.*

The function `partialRegularities` calculates the Castelnuovo–Mumford regularity in each component of a multigrading.

Remark 3.3. In the bigraded case, Theorem 3.2 implies that $\boldsymbol{d}$ is in `linearTruncations M` if $\boldsymbol{d} \geq$ `partialRegularities M` and $\bar{\boldsymbol{d}} \geq$ `regularity M`.

For some modules, `linearTruncationsBound` gives a proper subset of the linear truncations:

```
i13 : S = multigradedPolynomialRing 2;

i14 : M = coker(map(S^{{-1,0},{0,-1},{0,-1}},S^{{-1,-1},{-1,-1}},
          {{x_(1,0),x_(1,1)},{-x_(0,0),0},{0,-x_(0,1)}}));

i15 : multigraded betti res M

            0    1
o15 = 1: a+2b    .
      2:    .  2ab

i16 : linearTruncations M

o16 = {{0, 2}, {1, 1}}

i17 : linearTruncationsBound M

o17 = {{1, 1}}
```

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SUPPLEMENT. The online supplement contains version 1.0 of `LinearTruncations`.

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