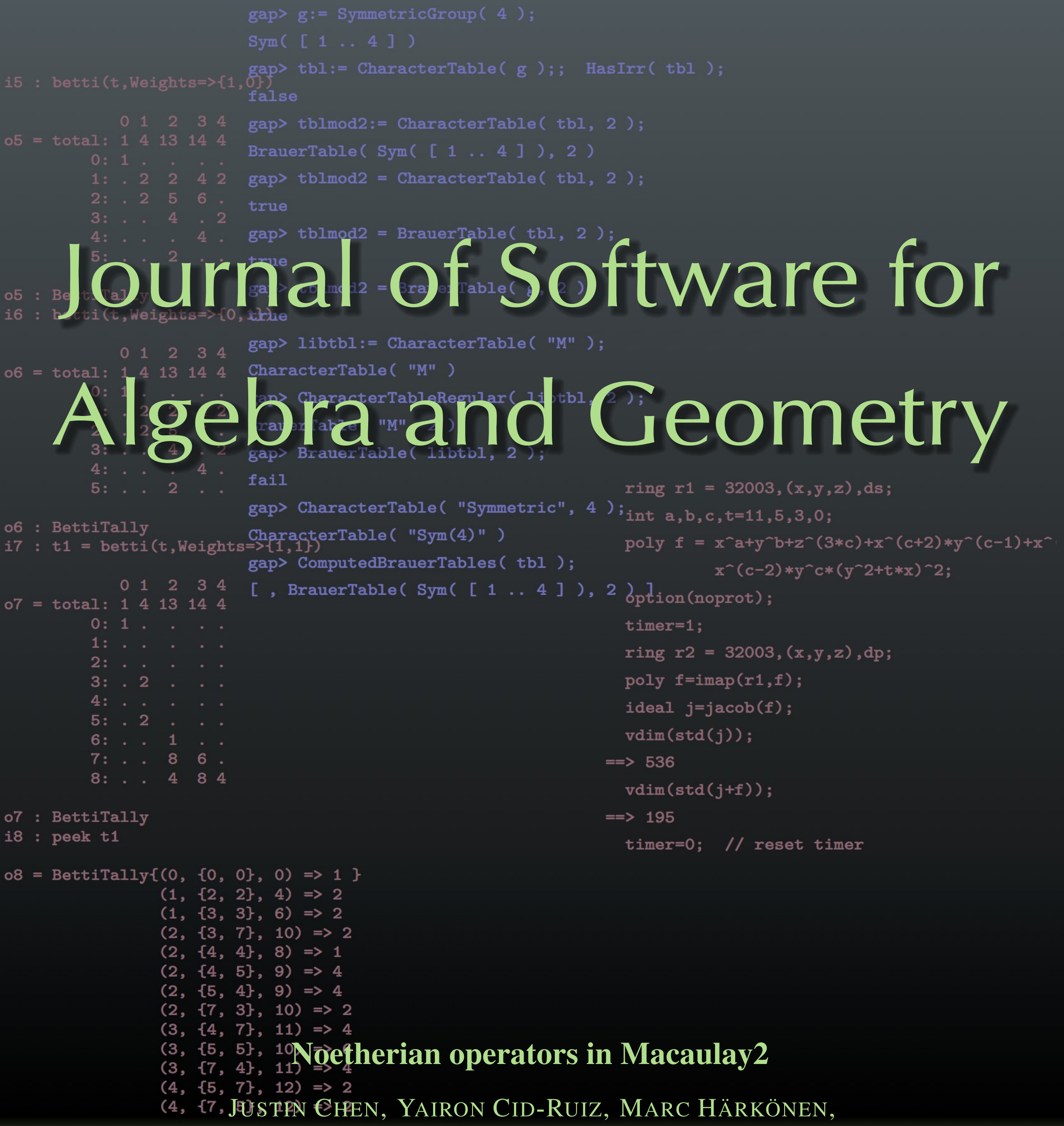

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Noetherian operators in Macaulay2

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ABSTRACT: A primary module over a polynomial ring can be described by an algebraic variety and a finite set of Noetherian operators, which are vectors of differential operators with polynomial coefficients. We implement both symbolic and numerical algorithms to produce such a description in various scenarios, as well as routines for studying affine schemes and coherent sheaves through the prism of Noetherian operators and Macaulay dual spaces.

1. INTRODUCTION. The idea of describing ideals in polynomial rings via systems of differential operators has been brewing since the beginning of the twentieth century. Macaulay [1916] brought forth the notion of an *inverse system*, a system of differential conditions that describes a *modular system* (a system of polynomials, or a polynomial ideal, in the modern language).

It was apparent to the contemporaries of Macaulay that a finite number of differential conditions should suffice to describe a 0-dimensional affine or projective scheme. Gröbner [1938] derived explicit characterizations for ideals that are primary to a rational maximal ideal [Gröbner 1970, p. 174–178]. Moreover, he suggested that the same program can be carried out for any primary ideal [Gröbner 1952, §1].

Despite this early algebraic interest, a complete description of primary ideals and modules in terms of differential operators was first obtained by analysts in the fundamental principle of Ehrenpreis and Palamodov [Ehrenpreis 1970; Palamodov 1970]. At the core of this fundamental principle, one has the following theorem by Palamodov:

Theorem 1.1 (Palamodov). *Let R be a polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$ over the complex numbers, $P \subseteq R$ be a prime ideal, and $Q \subseteq R^k$ be a P-primary R-module. Then there exist vectors of differential operators $A_1, \dots, A_m \in R\langle \partial_{x_1}, \dots, \partial_{x_n}\rangle^k$ such that $Q = \{f \in R^k \mid A_i \bullet f \in P \text{ for } 1 \le i \le m\}$.*

Following the terminology of Palamodov, the differential operators $A_1, \dots, A_m$ are commonly called *Noetherian operators for the P-primary submodule $Q \subseteq R^k$*. Subsequent algebraic and computational approaches to characterize primary ideals and modules with the use of differential operators have been given in [Brumfiel 1978; Oberst 1999; Damiano et al. 2007; Cid-Ruiz 2021; Chen et al. 2022; Cid-Ruiz et al. 2021; Chen and Cid-Ruiz 2022; Ait El Manssour et al. 2021].

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`NoetherianOperators` version 2.2.1

The purpose of this note is to present a Macaulay2 software package that implements the algorithms for Noetherian operators introduced in [Chen et al. 2022; Cid-Ruiz et al. 2021; Chen and Cid-Ruiz 2022; Ait El Manssour et al. 2021] as well as the algorithms for (Macaulay) dual spaces addressed in [Krone and Leykin 2017a; 2017b; Krone 2013]. While some of these algorithms rely on exact symbolic computation, the others employ numerical approximations using paradigms of numerical algebraic geometry.

2. Computing Noetherian operators from modules.

The main method within the package `NoetherianOperators` is `noetherianOperators`, which contains implementations of symbolic algorithms. To represent Noetherian operators, this package introduces a new type called `DiffOp`, representing elements in $R\langle\partial_{x_1},\dots,\partial_{x_n}\rangle$. The type `DiffOp` is a wrapper around `Vector`, with the added feature that instances of `DiffOp` in $R\langle\partial_{x_1},\dots,\partial_{x_n}\rangle^k$ operate on elements of R^k.

The symbolic backbone of the default Noetherian operator computation routine rests on [Chen and Cid-Ruiz 2022, Theorem 3.2], which can be seen as a "representation theorem" that parametrizes primary modules via three closely related objects (points in the punctual Quot scheme, differentially closed vector spaces, and bisubmodules of the Weyl–Noether module).

For a prime ideal P of codimension c, let $\mathbb{F}$ be the field of fractions of the integral domain R/P. Up to a linear change of coordinates, we may (and do) assume that $\{x_{c+1},\dots,x_n\}$ is a maximal independent set of variables modulo P, to simplify notation.

We now recall the steps of [Chen and Cid-Ruiz 2022, Algorithm 4.1]. The main idea of this algorithm is to reduce the study of arbitrary P-primary modules over R to a zero-dimensional setting over the function field $\mathbb{F}$. This reduction is made by parametrizing P-primary modules with the punctual Quot scheme $\mathrm{Hilb}^m(\mathbb{F}[[y_1,\dots,y_c]]^k)$. This is a quasiprojective scheme over the function field $\mathbb{F}$. Its classical points are $\mathbb{F}[[y_1,\dots,y_c]]$-submodules $V\subseteq\mathbb{F}[[y_1,\dots,y_c]]^k$ of colength m. If $k=1$, the punctual Quot scheme is known as the *punctual Hilbert scheme*. For more details regarding punctual Hilbert and Quot schemes the reader is referred to [Iarrobino 1972; Baranovsky 2000]. We define the inclusion map

$$\gamma: R\hookrightarrow \mathbb{F}[y_1,\dots,y_c]\qquad \begin{array}{ll} x_i\mapsto y_i+\overline{x_i}, & \text{for } 1\le i\le c,\\ x_j\mapsto \overline{x_j}, & \text{for } c+1\le j\le n,\end{array}$$

where $\overline{x_i}$ denotes the class of x_i in $\mathbb{F}$, $1\le i\le n$. With this, we can give the explicit bijective correspondence

$$\begin{array}{ccc} \left\{\begin{array}{c} P\text{-primary } R\text{-submodules of } R^k\\ \text{with multiplicity } m \text{ over } P\end{array}\right\} & \longleftrightarrow & \left\{\text{points in } \mathrm{Quot}^m(\mathbb{F}[[y_1,\dots,y_c]]^k)\right\},\\ Q & \longrightarrow & V=\gamma(Q)+\langle y_1,\dots,y_c\rangle^m\mathbb{F}[[y_1,\dots,y_c]]^k,\\ Q=\gamma^{-1}(V) & \longleftarrow & V.\end{array}$$

After computing the point $V\subseteq\mathrm{Quot}^m(\mathbb{F}[[y_1,\dots,y_c]]^k)$ corresponding to a P-primary ideal Q of multiplicity m over P, the inverse system $V^\perp$ of V is computed. Lastly, an $\mathbb{F}$-basis of $V^\perp$ is lifted to a set of Noetherian operators for the module Q.

While this is the default strategy used to compute Noetherian operators, it can also be explicitly called by specifying `Strategy => "PunctualQuot"`, as shown below:

```
i1 : needsPackage "NoetherianOperators";
i2 : S = QQ[x_1, x_2, x_3];
i3 : Q = image matrix{{x_1, x_2^2, 0}, {x_3, x_3^2, x_2^2-x_1*x_3}}
o3 = image | x_1 x_2^2 0            |
           | x_3 x_3^2 x_2^2-x_1x_3 |

                             2
o3 : S-module, submodule of S
i4 : associatedPrimes comodule Q
              2
o4 = {ideal(x  - x x )}
              2    1 3
o4 : List
i5 : noetherianOperators(Q, Strategy => "PunctualQuot")
o5 = {| -x_3 |}
      |  x_1 |
o5 : List
```

Nonprimary modules can be studied via *differential primary decompositions* [Chen and Cid-Ruiz 2022, Algorithm 4.9], implemented in differentialPrimaryDecomposition. The output is a list of pairs whose first entry is an associated prime and the second entry is a list of Noetherian operators corresponding to that primary component. Our implementation returns the minimal number of Noetherian operators required to describe a module, namely the *arithmetic multiplicity* of the module (amult) [Cid-Ruiz and Sturmfels 2021, Theorem 4.6]:

```
i6 : Q = image matrix{{x_1^2, x_2*x_1, 0},{x_3^2,0,x_1^2}};
i7 : amult Q
o7 = 8
i8 : netList differentialPrimaryDecomposition Q
     +--------------+--------------------------------------------+
o8 = |ideal x       |{| 1 |}                                     |
     |       1      | | 0 |                                      |
     +--------------+--------------------------------------------+
     |ideal (x , x )|{| dx_1 |, | x_3^2dx_1^2 |, | x_3^2dx_1^3 |}|
     |        2   1 | |   0  |  |      -2     |  |    -6dx_1   | |
     +--------------+--------------------------------------------+
     |ideal (x , x )|{| 0 |, |   0  |, |   0  |, |     0    |}   |
     |        3   1 | | 1 |  | dx_1 |  | dx_3 |  | dx_1dx_3 |    |
     +--------------+--------------------------------------------+
```

One application of Noetherian operators is in solving systems of PDE with constant coefficients. The fundamental principle of Ehrenpreis and Palamodov asserts that every distributional solution to a system of PDE can be represented as an integral of exponential-polynomial solutions with respect to suitable measures supported on algebraic varieties. The exponential-polynomial solutions correspond to Noetherian operators, and the varieties correspond to the varieties of the associated primes of the module in question. See [Ait El Manssour et al. 2021] and the function solvePDE for more details on this viewpoint.

3. COMPUTING NOETHERIAN OPERATORS FROM IDEALS. In this section we discuss additional algorithms that can be used to compute a set of Noetherian operators for primary ideals. Specifying a value for the option Strategy in the method noetherianOperators allows the user to choose which symbolic algorithm to use. The method numericalNoetherianOperators implements a numerical algorithm, which can deal with approximate input. In addition to the algorithm described in Section 2, we implement the following three algorithms:

(1) *Symbolic algorithm via dual spaces*: this algorithm computes Noetherian operators as bases of Macaulay dual spaces.

(2) *Numerical algorithm via interpolation*: this algorithm interpolates Noetherian operators from their specializations at several general points sampled on the underlying variety.

(3) *Hybrid symbolic/numerical*: this approach optimizes the approach in (1) by using information obtained from applying the numerical algorithm (2) at one general point on the underlying variety.

We now discuss each algorithm and illustrate its use in the package. Throughout the article, let $\mathbb{K}$ denote a field of characteristic zero, $R = \mathbb{K}[x_1, \dots, x_n]$ a polynomial ring over $\mathbb{K}$, and $P \subseteq R$ a prime ideal in R. We typically use Q to denote a P-primary ideal, and I to denote a general (not necessarily primary) ideal which has P as a minimal prime.

3.1. ***Symbolic algorithm via dual spaces.*** The next algorithm to compute a set of Noetherian operators is a direct approach which reduces the problem to linear algebra. For convenience, write $\boldsymbol{t}$ for a maximal set of independent variables modulo P, and $\boldsymbol{x} := \{x_1, \dots, x_n\} \setminus \boldsymbol{t}$ as the dependent variables. Then in the localization $S := \mathbb{K}(\boldsymbol{t})[\boldsymbol{x}]$ of R, the extension of I to S is zero-dimensional. If now I is a zero-dimensional P-primary ideal, then a set of Noetherian operators for I is the same as a basis for the dual space of I at P (as will be discussed in Section 5). This in turn can be computed as the kernel of a *Macaulay matrix*, which is a matrix over R/P with columns indexed by differential monomials and rows indexed by elements of I, whose entries are the result of applying a differential monomial to an element of I. As the numbers of rows and columns increase, the kernel eventually stabilizes, at which point the result is returned.

This is the default strategy used to compute Noetherian operators, when the input is a pair of ideals, the second of which should be a minimal prime of the first. It can also be explicitly called by specifying `Strategy => "MacaulayMatrix"`:

```
i9 : needsPackage "K3Carpets";
i10 : I = carpet(2, 2, Characteristic => 0);
o10 : Ideal of QQ[x ..x , y ..y ]
                   0   2   0   2
i11 : R = ring I;
i12 : noetherianOperators(I, Strategy => "MacaulayMatrix")
o12 = {| 1 |, | 2y_1dy_0+y_2dy_1 |}
o12 : List
```

Calling `noetherianOperators` with a minimal prime ideal P of I as the second argument will compute a set of Noetherian operators for the P-primary component of I. If I is unmixed, then the result of applying this method to all associated primes is a differential primary decomposition of I.

```
i13 : (P1, P2) = (radical I, ideal(R_0 + R_1));
i14 : J = intersect(I, P2^2);
o14 : Ideal of R
i15 : noetherianOperators(J, P1)
o15 = {| 1 |, | 2y_1dy_0+y_2dy_1 |}
o15 : List
```

```
i16 : noetherianOperators(J, P2)
o16 = {| 1 |, | dx_0 |}
o16 : List
```

3.2. ***Numerical algorithm via interpolation.*** We also provide algorithms to compute Noetherian operators purely from numerical data, bypassing the need to compute Gröbner bases. This is based on computing a set of *specialized Noetherian operators*, i.e., the result of evaluating (at some point) all polynomial coefficients in a set of Noetherian operators. The key observation is that one can obtain a set of specialized Noetherian operators by suitably slicing the variety [Chen et al. 2022, Theorem 4.1]. More precisely, for a P-primary ideal $Q \subseteq \mathbb{C}[\boldsymbol{t}, \boldsymbol{x}]$ and a point $p = (\boldsymbol{t}_0, \boldsymbol{x}_0) \in V(P)$, a minimal set of specialized Noetherian operators corresponds to a basis of the dual space of the zero-dimensional ideal $Q + (\boldsymbol{t} - \boldsymbol{t}_0)$ at the point p. The function `specializedNoetherianOperators` can be used to perform this computation.

```
i17 : p = point{{1,1,1,1,1,1}};
i18 : specializedNoetherianOperators(I, p, DependentSet => {R_1, R_3, R_4})
o18 = {| 1 |, | 2dy_0+dy_1 |}
o18 : List
```

Once a set of specialized Noetherian operators has been computed at a single general point, subsequent computations at other points can be sped up as the monomial support of a valid set of Noetherian operators is known (this fact also underlies the hybrid approach in Section 3.3). After specialized Noetherian operators are computed at sufficiently many general points on the variety, the original set of Noetherian operators can be recovered from their specializations via interpolation of rational functions; see [Chen et al. 2022, Algorithm 5].

This is the preferred strategy when the input is inexact, although it can also be used for exact input, as shown here. Note that the value of `DependentSet` must be specified:

```
i19 : numericalNoetherianOperators(I, DependentSet => {R_1, R_3, R_4})
-- warning: experimental computation over inexact field begun
--          results not reliable (one warning given per session)
           1y
             1
o19 = {1, ----dy  + 1dy }
          .5y   0      1
             2
o19 : List
```

By default, `Bertini` is used to sample points on $V\left(\sqrt{I}\right)$. The user can specify their own sampling function with the option `Sampler`. The sampler should be a function that takes an integer n and the ideal I as input, and returns a `List` of n points on the variety.

```
i20 : needsPackage"NumericalImplicitization";
i21 : numericalNoetherianOperators(I, DependentSet => {R_1, R_3, R_4},
Sampler => (n,I) -> apply(n, i -> point sub(matrix realPoint radical I, CC)))
           y
            1
o21 = {1, ----dy  + 1dy }
          .5y   0      1
             2
o21 : List
```

3.3. ***Hybrid symbolic/numerical.*** In Section 3.1, Noetherian operators are found by computing the kernel of a Macaulay matrix with entries in the function field $\mathbb{K}(\boldsymbol{t})$, but computations in this field can be expensive. The numerical approach in Section 3.2 instead specializes the independent variables to random values. This allows computations to be done in $\mathbb{K}$ (typically with $\mathbb{K} = \mathbb{C}$) which is cheaper but the result consists of specializations of the Noetherian operators. A hybrid approach can combine the best of both strategies. In essence, the information revealed from running the numerical algorithm at a single point (without performing interpolation) can be used to trim the Macaulay matrix down to a smaller (optimal) size, without changing the kernel. For more details, we refer the interested reader to [Chen et al. 2022, Section 4.1].

Specifying `Strategy => "Hybrid"` with the method `noetherianOperators` calls this strategy. On larger examples, this strategy can greatly outperform the approach in Section 3.1.

```
i22 : noetherianOperators(I, Strategy=>"Hybrid")
o22 = {| 1 |, | 2y_1dy_0+y_2dy_1 |}
o22 : List
```

As in the numerical algorithm, the user may also specify a sampling function to find a general point.

4. Computing modules from Noetherian operators. In this section, we discuss a procedure that can be seen as the inverse of the process of computing a set of Noetherian operators. First, note that for any R-bisubmodule $\mathcal{E} \subseteq R\langle\partial_{x_1}, \ldots, \partial_{x_n}\rangle^k$, the set

$$\{f \in R^k \mid A \bullet f \in P \text{ for all } A \in \mathcal{E}\}$$

is always a P-primary R-submodule of R^k (see [Cid-Ruiz 2021, Proposition 3.5]). We now consider the following problem:

Problem 4.1. Given an R-bisubmodule $\mathcal{E} \subseteq R\langle\partial_{x_1}, \ldots, \partial_{x_n}\rangle^k$, compute (generators for) the P-primary submodule $\{f \in R^k \mid A \bullet f \in P \text{ for all } A \in \mathcal{E}\}$.

This is accomplished by [Chen and Cid-Ruiz 2022, Algorithm 4.3]. The idea is to use the explicit maps provided in [Chen and Cid-Ruiz 2022, Theorem 3.2] in inverse order to how they appear in [Chen and Cid-Ruiz 2022, Algorithm 4.1] (i.e., as discussed in Section 2). It should be noted that our implementation solves the following effective version of the problem above:

Problem 4.1′. Given $A_1, \ldots, A_m \in R\langle\partial_{x_1}, \ldots, \partial_{x_n}\rangle^k$, compute the P-primary submodule

$$\{f \in R^k \mid A \bullet f \in P \text{ for all } A \in \mathcal{E}\},$$

where $\mathcal{E} \subseteq R\langle\partial_{x_1}, \ldots, \partial_{x_n}\rangle^k$ is the R-bisubmodule generated by $A_1, \ldots, A_m$.

The function `getIdealFromNoetherianOperators` implements [Cid-Ruiz et al. 2021, Algorithm 8.2]. Below we show an example for $k = 1$, in which, given a P-primary ideal Q, we compute a set of Noetherian operators for Q, and then we recover Q from its Noetherian operators along with P. In general, this process may result in a different set of generators for Q.

```
i23 : R = QQ[x_1,x_2,x_3];
i24 : Q = ideal(x_1^2, x_2^2, x_3^2, x_1*x_2 + x_1*x_3 +x_2*x_3);
o24 : Ideal of R
i25 : L = noetherianOperators Q
o25 = {| 1 |, | dx_1 |, | dx_2 |, | dx_3 |, | dx_1dx_2-dx_1dx_3 |, | dx_1...
o25 : List
i26 : Q' = getIdealFromNoetherianOperators(L, radical Q)
              2   2                       2
o26 = ideal (x , x , x x  + x x  + x x , x )
              3   2   1 2    1 3    2 3   1
o26 : Ideal of R
i27 : Q == Q'
o27 = true
```

5. Dual spaces and local Hilbert functions. In Section 3, dual spaces were used in service of computing Noetherian operators. However, dual spaces can also directly provide information about polynomial ideals. Suppose P is the maximal ideal corresponding to a $\mathbb{K}$-rational point $p \in \mathbb{K}^n$, and $I \subseteq R$ is an ideal with $p \in V(I)$. The *dual space of I at P* is

$$D_P[I] := \{A \in \mathbb{K}[\partial_{x_1}, \dots, \partial_{x_n}] \mid (A \bullet f)(p) = 0 \text{ for all } f \in I\}.$$

The dual space is a subspace of the space of differential operators on R with constant coefficients and finite support which uniquely determines IR_P, where R_P denotes the localization of R at P.

The following dual space algorithms work with numerical data, for example if $\mathbb{K} = \mathbb{C}$ and the point associated to P is known only approximately. This is in contrast to symbolic algorithms relying on Gröbner bases. The methods described in this section were previously part of a package titled `NumericalHilbert` which has now been incorporated into `NoetherianOperators` due to an overlap in functionality.

If P is a minimal prime of I, then the dual space is finite-dimensional, and a basis of the dual space is a set of Noetherian operators for the P-primary component of I. Otherwise the dual space is infinite-dimensional, and only a truncation up to a specified degree can be computed. The methods `zeroDimensionalDual` and `truncatedDual` compute a basis for the dual space in these two cases. As in Section 3.1, these dual spaces are computed as kernels of Macaulay matrices. From a basis of the dual space truncated to degree d, it is straightforward to compute the local Hilbert function of R/I up to degree d, and this is implemented as applying `hilbertFunction` to the generators of a dual space.

```
i28 : R = CC[x_1,x_2];
i29 : I = ideal{x_1^2 + x_2^2 - 4, (x_1 - 1)^2};
o29 : Ideal of R
i30 : p = point{{1.0, 1.7320508}};
i31 : D = zeroDimensionalDual(p, I)
o31 = | 1 -1.73205x_1+x_2 |
o31 : DualSpace
i32 : apply(3, i -> hilbertFunction(i, D))
o32 = {1, 1, 0}
o32 : List
```

Another way of truncating the dual space of a positive-dimensional ideal is with respect to a local elimination order instead of a degree order. In [Krone and Leykin 2017a], this is referred to as an *eliminating dual space*. Assume $\{x_{c+1}, \dots, x_n\}$ is a maximal set of independent variables for R/I, and choose an order that eliminates $V = \{x_1, \dots, x_c\}$. An eliminating dual for I with respect to V truncated to degree d, denoted $E^d_P[I, V]$, is defined as the set of dual operators whose lead terms with respect to the monomial order have degree at most d in the variables V, and is computed with the method `eliminatingDual`. When $V = \{x_1\}$, such a truncated dual space allows one to find the dual space of the colon ideal $I : x_1$ directly (without requiring the potentially expensive symbolic computation of finding $I : x_1$):

$$E^d_P[I : x_1, \{x_1\}] = x_1 \cdot E^{d+1}_0[I, \{x_1\}]$$

where $x_1 \cdot A$ represents the right action of $x_1 \in R$ on $A \in \mathbb{K}[\partial_{x_1}, \dots, \partial_{x_n}]$ (for example $x_1 \cdot \partial^2_{x_1} = 2\partial_{x_1}$). Representations of these colon ideals are needed in [Krone and Leykin 2017a, Algorithm 5.1] for identifying embedded primes on a curve.

Using [Krone 2013, Algorithm 23], computing truncated dual spaces up to a certain degree provides a numerical algorithm for finding a full set of generators for the initial ideal of I with respect to a local degree order, and this is implemented by `gCorners`. In the process, a standard basis can be computed by specifying `StandardBasis => true`.

```
i33 : gCorners(point p, I)
o33 = | x_2 x_1^2 |

              1      2
o33 : Matrix R  <--- R
```

Finding the initial ideal via approximate numerical methods is an essential part of `isPointEmbedded`, which implements a numerical algorithm for the detection of an embedded component developed in [Krone and Leykin 2017b, Algorithm 4.2].

Supplement. The online supplement contains version 2.2.1 of `NoetherianOperators`.

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