

```

gap> g:= SymmetricGroup( 4 );
Sym( [ 1 .. 4 ] )
gap> tbl:= CharacterTable( g );; HasIrr( tbl );
i5 : betti(t,Weights=>{1,0})
false
      0 1 2 3 4
o5 = total: 1 4 13 14 4
      0: 1 . . . .
      1: . 2 2 4 2
      2: . 2 5 6 .
      3: . . 4 . 2
      4: . . . 4 .
      5: . . 2 . .
gap> tblmod2:= CharacterTable( tbl, 2 );
BrauerTable( Sym( [ 1 .. 4 ] ), 2 )
gap> tblmod2 = CharacterTable( tbl, 2 );
true
gap> tblmod2 = BrauerTable( tbl, 2 );
true
i6 : BettiTally
i6 : betti(t,Weights=>{0,1})
      0 1 2 3 4
o6 = total: 1 4 13 14 4
      0: 1 . . . .
      1: . 2 2 4 2
      2: . 2 5 6 .
      3: . . 4 . 2
      4: . . . 4 .
      5: . . 2 . .
gap> libtbl:= CharacterTable( "M" );
CharacterTable( "M" )
gap> CharacterTableRegular( libtbl, 2 );
BrauerTable( "M", 2 )
gap> BrauerTable( libtbl, 2 );
fail
gap> CharacterTable( "Symmetric", 4 );
CharacterTable( "Sym(4)" )
i7 : t1 = betti(t,Weights=>{1,1})
gap> ComputedBrauerTables( tbl );
[ , BrauerTable( Sym( [ 1 .. 4 ] ), 2 ) ]
      0 1 2 3 4
o7 = total: 1 4 13 14 4
      0: 1 . . . .
      1: . . . . .
      2: . . . . .
      3: . 2 . . .
      4: . . . . .
      5: . 2 . . .
      6: . . 1 . .
      7: . . 8 6 .
      8: . . 4 8 4
      ring r1 = 32003,(x,y,z),ds;
      int a,b,c,t=11,5,3,0;
      poly f = x^a+y^b+z^(3*c)+x^(c+2)*y^(c-1)+x^(
      x^(c-2)*y^c*(y^2+t*x)^2;
      option(noprot);
      timer=1;
      ring r2 = 32003,(x,y,z),dp;
      poly f=imap(r1,f);
      ideal j=jacob(f);
      vdim(std(j));
==> 536
      vdim(std(j+f));
==> 195
      timer=0; // reset timer
o7 : BettiTally
i8 : peek t1
o8 = BettiTally{(0, {0, 0}, 0) => 1 }
      (1, {2, 2}, 4) => 2
      (1, {3, 3}, 6) => 2
      (2, {3, 7}, 10) => 2
      (2, {4, 4}, 8) => 1
      (2, {4, 5}, 9) => 4
      (2, {5, 4}, 9) => 4
      (2, {7, 3}, 10) => 2
      (3, {5, 5}, 10) => 6
      (3, {7, 4}, 11) => 4
      (4, {5, 7}, 12) => 2
      (4, {7, 5}, 12) => 2

```

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Algorithms for computing mixed multiplicities, mixed volumes, and sectional Milnor numbers

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Algorithms for computing mixed multiplicities, mixed volumes, and sectional Milnor numbers

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ABSTRACT: We present a package `MixedMultiplicity` for computing mixed multiplicities of ideals in a Noetherian ring which is either local or a standard graded algebra over a field. This enables us to find mixed volumes of convex lattice polytopes and sectional Milnor numbers of hypersurfaces with an isolated singularity. The algorithms make use of the defining equations of the multi-Rees algebra of ideals, which are obtained by generalising a result of Cox, Lin and Sosa.

1. INTRODUCTION. This article describes the [Macaulay2] package `MixedMultiplicity` which computes the mixed multiplicities of ideals having positive grade in a Noetherian ring, the mixed volume of a collection of convex lattice polytopes, and sectional Milnor numbers of hypersurfaces with an isolated singularity. One of the main steps of the algorithms is the computation of the defining equations of the multi-Rees algebra

$$\mathcal{R}(I_1, \dots, I_s) = R[I_1 t_1, \dots, I_s t_s] = \bigoplus_{a_1, \dots, a_s \geq 0} I_1^{a_1} \cdots I_s^{a_s} t_1^{a_1} \cdots t_s^{a_s} \subseteq R[t_1, \dots, t_s]$$

of ideals I_1, \dots, I_s in a Noetherian ring R . For the case when R is a polynomial ring over any field k and I_1, \dots, I_s are monomial ideals in R , an explicit formula for the defining equations of $\mathcal{R}(I_1, \dots, I_s)$ has been given by D. Cox, K.-N. Lin, and G. Sosa in [Cox et al. 2019]. In Theorem 2.1, we obtain an analogue of their result for any set of ideals I_1, \dots, I_s in a Noetherian ring R such that each I_i has positive grade. The latter condition is always satisfied when R is a domain or for any ideal of positive height in a reduced ring or in a Cohen–Macaulay ring. Using this result, we write a function `multiReesIdeal` to compute the defining ideal of a multi-Rees algebra in Macaulay2. It should be noted that the command `reesIdeal` in the Macaulay2 package `ReesAlgebra` [Eisenbud 2018] is already available to compute the defining ideal of the Rees algebra of a module [Eisenbud et al. 2003], in particular, the defining ideal of a multi-Rees algebra. But, the algorithm presented in this article runs faster in many cases. When the ring is not a domain, the algorithm follows an analogue of the `reesIdeal`, albeit for the multiple ideal setting.

Several authors (see for instance, [Ribbe 1999; Lin and Polini 2014; Sosa 2014; Jabarnejad 2018]) have proposed algorithms to determine the defining equations of the multi-Rees algebra for specific classes of ideals. The algorithm for computing defining equations of $\mathcal{R}(I_1, \dots, I_s)$ helps us to construct algorithms

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`MixedMultiplicity` version 3.0

to compute *mixed multiplicities* (Section 3), *mixed volume* (Section 4) and *sectional Milnor numbers* (Section 5) in the general setting. Observe that to compute mixed multiplicities of ideals one can always assume each ideal to have positive grade using a standard trick (see Remark 3.3).

For any ideal J in a Noetherian ring R , we denote the common length of the maximal R -sequences in J by $\text{grade}(J)$. Let I_0, I_1, \dots, I_r be a set of ideals in a Noetherian ring of dimension $d \geq 1$, which is either local or a standard graded algebra over a field. Further assume that I_0 is primary to the maximal ideal (resp. maximal graded ideal) and $\text{grade}(I_j) > 0$ for all j . Let $\underline{a} = (a_0, a_1, \dots, a_r) \in \mathbb{N}^{r+1}$ with $|\underline{a}| = d - 1$. The function `mixedMultiplicity` computes the mixed multiplicity $e_{\underline{a}}(I_0 | I_1, \dots, I_r)$. Using the results of N. V. Trung, J. K. Verma and B. Teissier, mixed volumes and sectional Milnor numbers can be identified with mixed multiplicities of ideals over polynomial rings. Let Q_1, \dots, Q_n be a collection of lattice polytopes in \mathbb{R}^n . The function `mMixedVolume` computes the mixed volume of Q_1, \dots, Q_n . Let $R = k[x_1, \dots, x_n]$ be a polynomial ring in n variables, \mathfrak{m} be the maximal graded ideal and $f \in R$ be any polynomial. The function `secMilnorNumbers` computes the sectional Milnor numbers by calculating the mixed multiplicities $e(\mathfrak{m}^{[n-i]}, J(f)^{[i]})$, $0 \leq i \leq n - 1$, where $J(f) = (f_{x_1}, \dots, f_{x_n})$ is the ideal generated by the partial derivatives of f . Many researchers, including M. Herrmann et al. [1997], J. K. Verma [1992], and C. D’Cruz [2003], have expressed the multiplicities of Rees algebras, extended Rees algebras, and certain form rings in terms of mixed multiplicities. The `MixedMultiplicity` package is also helpful in this regard. For any unexplained invariants and definitions used in this article, the reader may refer to [Bruns and Herzog 1993], [Eisenbud et al. 2003], and [Huneke and Swanson 2006].

2. DEFINING IDEAL OF MULTI-REES ALGEBRA OF IDEALS. An explicit formula for the defining ideal of the multi-Rees algebra of a finite collection of monomial ideals in a polynomial ring was given in [Cox et al. 2019]. In this section, we generalize their result to find the defining ideal of the multi-Rees algebra of a collection of ideals with positive grade in a Noetherian ring. We use this result to write a `Macaulay2` algorithm to compute the defining ideal when the base ring is a domain. We provide another algorithm for the nondomain case.

Let R be a Noetherian ring and $I_1, \dots, I_s \subseteq R$ be ideals. Suppose that $I_i = \langle f_{ij} \mid j = 1, \dots, n_i \rangle$ for all $i = 1, \dots, s$. Let $\mathcal{R}(I_1, \dots, I_s)$ be the multi-Rees algebra of ideals I_1, \dots, I_s . Consider the set of indeterminates $\underline{Y} = \{Y_{ij} \mid i = 1, \dots, s, j = 1, \dots, n_i\}$ and $\underline{T} = (T_1, \dots, T_s)$ over R . Define an R -algebra homomorphism $R[\underline{Y}] \xrightarrow{\varphi} \mathcal{R}(I_1, \dots, I_s) \subseteq R[\underline{T}]$ such that $\varphi(Y_{ij}) = f_{ij}T_i$, for all $i = 1, \dots, s$, $j = 1, \dots, n_i$ and $\varphi(r) = r$ for all $r \in R$. Then $\mathcal{R}(I_1, \dots, I_s) \simeq R[\underline{Y}]/\ker(\varphi)$. The ideal $\ker \varphi$ is called the defining ideal of $\mathcal{R}(I_1, \dots, I_s)$. We give an explicit description of $\ker(\varphi)$.

Theorem 2.1. *Let R be a Noetherian ring and $I_1, \dots, I_s \subseteq R$ be ideals of positive grade. For each i , consider a generating set $\{f_{ij} \mid j = 1, \dots, n_i\}$ of I_i which contains at least one nonzerodivisor f_{ij} . We set $h = \prod_{i=1}^s f_{ij}$ and set*

$$\Gamma = \langle Y_{ij}f_{ij'} - Y_{ij'}f_{ij} \mid i = 1, \dots, s \text{ and } j, j' \in \{1, \dots, n_i\} \rangle : h^\infty \subseteq R[\underline{Y}].$$

Then $\Gamma \subseteq R[\underline{Y}]$ is the defining ideal of $\mathcal{R}(I_1, \dots, I_s)$.

Proof. Without loss of generality, we may assume that $j_i = 1$ for all $i = 1, \dots, s$ and $h = \prod_{i=1}^s f_{i1}$. Consider the ring homomorphism $\phi : R \rightarrow R[f_{11}^{-1}, f_{21}^{-1}, \dots, f_{s1}^{-1}] \cong R[h^{-1}]$, which induces a natural map $\tilde{\phi} : R[\underline{Y}] \rightarrow R[h^{-1}][\underline{Y}] \cong R[\underline{Y}]_h$. The defining ideal of $\mathcal{R}(\phi(I_1), \dots, \phi(I_s))$ of the ideals $\phi(I_i) = (1, f_{i2}/f_{i1}, \dots, f_{in_i}/f_{i1})$ for $i = 1, \dots, s$, is $J := J_1 + \dots + J_s$ in $R[h^{-1}][\underline{Y}] \cong R[\underline{Y}]_h$, where $J_i := (Y_{i2} - f_{i2}/f_{i1}Y_{i1}, \dots, Y_{in_i} - f_{in_i}/f_{i1}Y_{i1})$. We claim that $\tilde{\phi}^{-1}(J) = \Gamma$. Observe that for all $j \neq j'$, and for all i ,

$$f_{ij}Y_{ij'} - f_{ij'}Y_{ij} = f_{ij} \left(Y_{ij'} - \frac{f_{ij'}}{f_{i1}} Y_{i1} \right) - f_{ij'} \left(Y_{ij} - \frac{f_{ij}}{f_{i1}} Y_{i1} \right) \in J_i.$$

So $\Gamma \subseteq \tilde{\phi}^{-1}(J)$. Now let $r \in \tilde{\phi}^{-1}(J)$. Then $\tilde{\phi}(r) \in J$, i.e.,

$$\frac{r}{1} = \sum_{i=1}^s \sum_{j=2}^{n_i} \frac{a_{ij}}{f_{i1}^{m_{ij}}} \left(Y_{ij} - \frac{f_{ij}}{f_{i1}} Y_{i1} \right)$$

for some $a_{ij} \in R$. Thus we have

$$h^m r \in (f_{i1}Y_{ij} - f_{ij}Y_{i1} \mid 1 \leq i \leq s, 1 \leq j \leq n_i) \subseteq (f_{ij}Y_{ij'} - f_{ij'}Y_{ij} \mid 1 \leq i \leq s, 1 \leq j, j' \leq n_i)$$

for some $m \geq \max\{m_{ij} \mid 1 \leq i \leq s, 1 \leq j \leq n_i\} + 1$. Therefore, $r \in \Gamma$ and hence the claim holds. From [Atiyah and Macdonald 2016, Proposition 3.11 (iii)], we get that $\tilde{\phi}^{-1}(J)$ is the defining ideal of $\mathcal{R}(I_1, \dots, I_s)$, as h^t is a nonzerodivisor on $R[\underline{Y}]/\Gamma$ for all $t \geq 1$. \square

When R is a domain or when a list of nonzerodivisors (one each from the list of ideals with positive grades) is provided by the user, the function `multiReesIdeal` computes the defining ideal of the multi-Rees algebra using Theorem 2.1.

Algorithm (version I: `multiReesIdeal`, set of ideals with positive grade). Let I_1, \dots, I_s be ideals of a Noetherian ring R with grade $I_i > 0$ for all i and let a_1, \dots, a_s be a set of nonzerodivisors, where a_i belongs to the generating set of I_i for all i . When R is a domain, the function picks a_i to be the first element in the generating set of I_i for each i .

Input: The list $W = \{\{I_1, \dots, I_s\}, \{a_1, \dots, a_s\}\}$, or $W = \{I_1, \dots, I_s\}$ if R is a domain.

- (1) Define a polynomial ring S by attaching m indeterminates to the ring R , where m is the sum of the number of generators of all the ideals.
- (2) For each ideal I_i , construct a matrix $M(i)$ whose first row consists of the generators of the ideal and the second row consists of the indeterminates.
- (3) Add the 2×2 minors of these matrices to get an ideal L .
- (4) To get the defining ideal, saturate L with the product of a_i 's.

Output: The defining ideal of the Rees algebra $\mathcal{R}(I_1, \dots, I_s)$.

The elements of the defining ideal are assigned \mathbb{N}^{s+1} degree by the function, where the first \mathbb{N}^s coordinates point to the component of $\mathcal{R}(I_1, \dots, I_s)$ where the element lies and the last coordinate is the

degree of the element. In order to compute the multi-Rees ideal $\mathcal{R}(I, J)$ using the function `reesIdeal`, one needs to enter $I \oplus J$ and this routine is sometimes slower than the `multiReesIdeal` routine.

Example 2.2.

```
i1 : R = QQ[x,y,z];
i2 : I = ideal(x^4+y^2*z^2,x*y^2*z);
i3 : J = ideal(y^3+z^3, x^2*y+x*z^2);
i4 : time multiReesIdeal {I,J};
    -- used 0.0131127 seconds
i5 : transpose gens oo
o5 = {0,-1,-6} | (x2y+xz2)X_2+(-y3-z3)X_3 |
      {-1,0,-8} | xy2zX_0+(-x4-y2z2)X_1 |
i6 : (first entries gens o4)/degree
o6 : {{0, 1, 6}, {1, 0, 8}}
i7 : time multiReesIdeal({I,J},{I_1, J_0});
    -- used 0.0629737 seconds
i8 : M = directSum(module I, module J);
i9 : time reesIdeal M;
    -- used 0.40043 seconds
```

In the following example, our algorithm works faster than the function `reesIdeal`.

Example 2.3.

```
i1 : ZZ/32003[y_0..y_4];
i2 : C = trim monomialCurveIdeal(R, {3, 5, 7, 12});
i3 : time multiReesIdeal (C, C_0);
    -- used 167.118 seconds
i4 : time reesIdeal (C, C_0);
    -- used 295.675 seconds
```

2.1. Routine for the nondomain case. In this section we present an algorithm to find the defining ideal of the Rees algebra using the definitions of Rees algebra. This method does not have any requirements on the grade of the ideals or the domain-ness of the ring, but it seems to be slower than the previous method.

We can construct the Rees algebra of I_i as the kernel of the map $\varphi_i : R[Y_{i1}, \dots, Y_{in_i}] \rightarrow R[T_i]$ where $\varphi_i(Y_{ij}) = f_{ij}T_i$ for $j = 1, \dots, n_i$. Notice that $(\ker \varphi_i)R[\underline{Y}] \subseteq \ker \varphi$. Suppose that ϕ_i is the presentation matrix of I_i . Then the symmetric algebra $\text{Sym}(I_i)$ has a presentation $R[Y_{i1}, \dots, Y_{in_i}]/\mathcal{L}_i$ where $\mathcal{L}_i = I_1([Y_{i1}, \dots, Y_{in_i}] \cdot \phi_i)$. Clearly, $\mathcal{L}_i \subseteq \ker \varphi_i \subseteq \ker \varphi$. So the map φ_i factors through the symmetric algebra $\text{Sym}(I_i)$. Now $\text{Sym}(I_1) \otimes \dots \otimes \text{Sym}(I_s)$ has the presentation $R[\underline{Y}]/(\mathcal{L}_1 + \dots + \mathcal{L}_s)$. Since $\mathcal{L}_i \subseteq \ker \varphi$, the map φ also factors through $\text{Sym}(I_1) \otimes \dots \otimes \text{Sym}(I_s)$. Thus to find the defining ideal of the multi Rees algebra $\mathcal{R}(I_1, \dots, I_s)$ it is enough to find the kernel of the surjective map $\text{Sym}(I_1) \otimes \dots \otimes \text{Sym}(I_s) \rightarrow \mathcal{R}(I_1, \dots, I_s)$.

Algorithm (version II: `multiReesIdeal`, no assumptions). Let I_1, \dots, I_s be ideals in the Noetherian ring R .

Input: The list $W = \{I_1, \dots, I_s\}$.

- (1) For each ideal I_i , compute the presentation $F_i' \xrightarrow{\phi_i} F_i \rightarrow R_i \rightarrow 0$ where ϕ_i is the presentation matrix of I_i .
- (2) Now compute the source symmetric algebra $\text{Sym}(F_1') \otimes \cdots \otimes \text{Sym}(F_s')$ and the target symmetric algebra $\text{Sym}(F_1) \otimes \cdots \otimes \text{Sym}(F_s)$ of the map $\phi_1 \otimes \cdots \otimes \phi_s$.
- (3) Compute the map between the symmetric algebra of the source and target and return the kernel of the above map.

Output: The defining ideal of the Rees algebra $\mathcal{R}(I_1, \dots, I_s)$.

In the following example, the ring U is not a domain and hence the algorithm uses the above method. As expected, the computational time in the case where a nonzerodivisor is given as an optional input is faster than the case where no optional input is given.

Example 2.4.

```
i1 : T = QQ[a,b,c];
i2 : m = matrix{{a,b,c},{b,c,a}};
i3 : U = T/minors(2,m);
i4 : J = ideal vars U;
i5 : time multiReesIdeal J;
-- used 0.0977545 seconds
i6 : time multiReesIdeal (J, a);
-- used 0.0142101 seconds
```

3. COMPUTATION OF MIXED MULTIPLICITIES OF IDEALS. Let I_1, \dots, I_r be ideals of positive height in a local ring (A, \mathfrak{m}) (or a standard graded algebra over a field and \mathfrak{m} be the maximal graded ideal) and let I_0 be an \mathfrak{m} -primary ideal. In [Katz and Verma 1989], the authors prove $\ell(I_0^{u_0} I_1^{u_1} \cdots I_r^{u_r} / I_0^{u_0+1} I_1^{u_1} \cdots I_r^{u_r})$ is a polynomial $P(\underline{u})$, for u_i large, where $\underline{u} = (u_0, \dots, u_r)$. Write this polynomial in the form

$$P(\underline{u}) = \sum_{\substack{\alpha \in \mathbb{N}^{r+1} \\ |\alpha| = t}} \frac{1}{\alpha!} e_\alpha(I_0 | I_1, \dots, I_r) u^\alpha + \text{lower degree terms,}$$

where $t = \deg P(\underline{u})$, $\alpha = (\alpha_0, \dots, \alpha_r) \in \mathbb{N}^{r+1}$, $\alpha! = \prod_{i=0}^r \alpha_i!$ and $|\alpha| = \sum_{i=0}^r \alpha_i$. If $|\alpha| = t$, then $e_\alpha(I_0 | I_1, \dots, I_r)$ are called the *mixed multiplicities* of the ideals I_0, I_1, \dots, I_r .

Theorem 3.1 [Trung and Verma 2007, Theorem 1.2]. *Set*

$$R = R(I_0 | I_1, \dots, I_r) = \bigoplus_{(u_0, u_1, \dots, u_r) \in \mathbb{N}^{r+1}} \frac{I_0^{u_0} I_1^{u_1} \cdots I_r^{u_r}}{I_0^{u_0+1} I_1^{u_1} \cdots I_r^{u_r}}.$$

Assume that $d = \dim A/(0 : I^\infty) \geq 1$, where $I = I_1 \cdots I_r$. Then $\deg P_R(\underline{u}) = d - 1$, where $P_R(\underline{u})$ is the Hilbert polynomial of R .

In [Verma et al. 1994], D. Katz, S. Mandal, and J. K. Verma found a precise formula for the Hilbert polynomial of the quotient of a bigraded algebra over an Artinian local ring. This result can be generalized

to the case of the quotient of a multigraded algebra over an Artinian local ring and we skip the proof as the technique is similar. Let S be an Artinian local ring and $A = S[X_1, \dots, X_r]$ be an \mathbb{N}^r -graded ring over S , where for $1 \leq i \leq r$, $X_i = \{X_i(0), \dots, X_i(s_i)\}$ is a set of indeterminates. Set $\underline{u} = (u_1, \dots, u_r) \in \mathbb{N}^r$ and $|\underline{u}| = u_1 + \dots + u_r$. Then $A = \bigoplus_{\underline{u} \in \mathbb{N}^r} A_{\underline{u}}$, where $A_{\underline{u}}$ is the S -module generated by monomials of the form $P_1 \cdots P_r$, where P_i is a monomial of degree u_i in X_i . An element in $A_{\underline{u}}$ is called multihomogeneous of degree \underline{u} . An ideal $I \subseteq A$ generated by multihomogeneous elements is called a multihomogeneous ideal. Then $R = A/I$ is an \mathbb{N}^r -graded algebra with \underline{u} -graded component $R_{\underline{u}} = A_{\underline{u}}/I_{\underline{u}}$. The Hilbert function of R is defined as $H(\underline{u}) = \lambda(R_{\underline{u}})$, where λ denotes the length as an S -module. Set $\underline{t}^{\underline{u}} = t_1^{u_1} \cdots t_r^{u_r}$. The Hilbert series of R is given by $HS(R, \underline{t}) = \sum_{\underline{u} \in \mathbb{N}^r} \lambda(R_{\underline{u}}) \underline{t}^{\underline{u}}$. Then there exists a polynomial $N(t_1, \dots, t_r) \in \mathbb{Z}[t_1, \dots, t_r]$ so that $HS(R, \underline{t}) = N(t_1, \dots, t_r) / ((1 - t_1)^{s_1+1} \cdots (1 - t_r)^{s_r+1})$.

Theorem 3.2. Write the Hilbert polynomial of R as

$$P(\underline{u}, R) = \sum_{\alpha=0}^{\underline{s}} c_{\alpha} \binom{u_1 + \alpha_1}{\alpha_1} \cdots \binom{u_r + \alpha_r}{\alpha_r}. \quad (1)$$

Then

$$c_{\alpha} = \frac{(-1)^{|\underline{s}-\alpha|}}{(s_1 - \alpha_1)! \cdots (s_r - \alpha_r)!} \cdot \frac{\partial^{|\underline{s}-\alpha|} N}{\partial t_1^{s_1 - \alpha_1} \cdots \partial t_r^{s_r - \alpha_r}} \Big|_{(t_1, \dots, t_r) = \underline{1}}.$$

Note that

$$\binom{u_i + \alpha_i}{\alpha_i} = \frac{1}{\alpha_i!} u_i^{\alpha_i} + \text{lower degree terms}.$$

So if we write $P(\underline{u})$ as in (1), then $c_{\alpha} = e_{\alpha}$ for all $\alpha \in \mathbb{N}^{r+1}$ with $|\alpha| = d - 1$. Therefore, Theorem 3.2 gives an expression for e_{α} .

Remark 3.3. Let I'_0, I'_1, \dots, I'_r denote the images of ideals I_0, I_1, \dots, I_r in the ring $A/(0 : I^{\infty})$, where $I = I_1 \cdots I_r$. Put $R' = R(I'_0 | I'_1, \dots, I'_r)$. Then for \underline{u} large, $P_R(\underline{u}) = P_{R'}(\underline{u})$ (see [Trung and Verma 2007, Theorem 1.2] for details). Therefore, in the case where grade $I_i = 0$ for some i , the user needs to work in the quotient ring $A/(0 : I^{\infty})$ and input the images of the ideals in the quotient ring.

Algorithm. The algorithm for the function `mixedMultiplicity` uses the above ideas. Let I_0, I_1, \dots, I_r be a set of ideals of a Noetherian ring R of dimension $d \geq 1$, where I_0 is primary to the maximal ideal and grade $(I_i) > 0$ for all i ; $\underline{a} = (a_0, a_1, \dots, a_r) \in \mathbb{N}^{r+1}$ with $|\underline{a}| = d - 1$.

Input: The sequence $W = ((I_0, I_1, \dots, I_r), (a_0, a_1, \dots, a_r))$.

- (1) Compute the defining ideal of the multi-Rees algebra using the function `multiReesIdeal` and use it to find the Hilbert series of $R(I_0 | I_1, \dots, I_r)$.
- (2) Extract the powers of $(1 - T_i)$ in the denominator of the Hilbert series.
- (3) Calculate $e_{\underline{a}}$ using the formula given in Theorem 3.2.

Output: The mixed multiplicity $e_{\underline{a}}(I_0 | I_1, \dots, I_r)$.

Example 3.4.

```

i1 : R = QQ[x,y,z,w];
i2 : I = ideal(x*y*w^3,x^2*y*w^2,x*y^3*w,x*y*z^3); m = ideal vars R;
i3 : mixedMultiplicity ((m,I,I,I),(0,1,1,1))
o3 = 6

```

When some ideal has grade zero, the following example explains how to compute the mixed multiplicity by using the fact that $(0 : I^\infty) = (0 : (I^t)^\infty)$ for all $t \geq 1$.

Example 3.5. Let $S = \mathbb{Q}[x, y, z, w]/(xz, yz)$, $\mathfrak{m} = (x, y, z, w)$, and $I = (x, y)$. Notice that $\text{grade } I = 0$, since $I \in \text{Ass } S$.

```

i1 : S = QQ[x,y,z,w]/ideal(x*z, y*z);
i2 : I = ideal(x,y);
i3 : m = ideal vars S;
i4 : L = saturate(sub(ideal 0, S), I);
i5 : T = S/L;
i6 : J = sub(I,T); n = sub(m,T);
i7 : dim T
o7 = 3
i8 : mixedMultiplicity((n,J,J,J),(1,0,1,0))
o8 = 1

```

To calculate mixed multiplicities, the function `mixedMultiplicity` computes the Hilbert polynomial of the graded ring $\bigoplus I_0^{u_0} I_1^{u_1} \cdots I_r^{u_r} / I_0^{u_0+1} I_1^{u_1} \cdots I_r^{u_r}$. In particular, if I_1, \dots, I_r are also \mathfrak{m} -primary ideals, then $e_{(a_0, a_1, \dots, a_r)}(I_0 \mid I_1, \dots, I_r) = e(I_0^{[a_0+1]}, I_1^{[a_1]}, \dots, I_r^{[a_r]})$ (see [Huneke and Swanson 2006, Definition 17.4.3]). So to compute the $(a_0 + 1, a_1, \dots, a_r)$ -th mixed multiplicity of I_0, I_1, \dots, I_r , one needs to enter the sequence (a_0, a_1, \dots, a_r) in the function. The same is illustrated in the following example.

Example 3.6.

```

i1 : R = QQ[x,y,z];
i2 : m = ideal vars R;
i3 : f = z^5 + x*y^7 + x^15;
i4 : I = ideal(apply(0..2, i -> diff(R_i,f)));
i5 : mixedMultiplicity ((m,I),(2,0))
o5 = 1
i6 : mixedMultiplicity ((m,I),(1,1))
o6 = 4

```

4. MIXED VOLUME OF LATTICE POLYTOPES. The Minkowski sum of two polytopes P and Q in \mathbb{R}^n is defined as the polytope $P + Q = \{a + b \mid a \in P, b \in Q\}$. The n -dimensional mixed volume of a collection of n polytopes Q_1, \dots, Q_n in \mathbb{R}^n , denoted by $MV_n(Q_1, \dots, Q_n)$, is the coefficient of $\lambda_1 \cdots \lambda_n$ in $\text{vol}_n(\lambda_1 Q_1 + \cdots + \lambda_n Q_n)$. Given a collection of lattice polytopes in \mathbb{R}^n , Trung and Verma proved that $MV_n(Q_1, \dots, Q_n)$ is equal to a mixed multiplicity of a set of homogeneous ideals.

Corollary 4.1 [Trung and Verma 2007, Corollary 2.5]. *Let Q_1, \dots, Q_n be an arbitrary collection of lattice polytopes in \mathbb{R}^n . Let $R = k[x_0, x_1, \dots, x_n]$ and let \mathfrak{m} be the maximal graded ideal of R . Let M_i be any set of monomials of the same degree in R such that Q_i is the convex hull of the lattice points of their dehomogenized monomials in $k[x_1, \dots, x_n]$. Let I_j be the ideal of R generated by the monomials of M_j . Then $MV_n(Q_1, \dots, Q_n) = e_{(0,1,\dots,1)}(\mathfrak{m} \mid I_1, \dots, I_n)$.*

We use this result to construct an algorithm which calculates the mixed volume of a collection of lattice polytopes. We also give an algorithm which outputs the homogeneous ideal corresponding to the vertices of a lattice polytope.

Let Q be a lattice polytope in \mathbb{R}^n with the set of vertices $\{p_1, \dots, p_r\} \subseteq \mathbb{N}^n$. We first compute the corresponding homogeneous ideal I in the ring $R = k[x_1, \dots, x_{n+1}]$. We write a function `homIdealPolytope` which requires as input the list $W = \{p_1, p_2, \dots, p_r\}$ and produces as output the homogeneous ideal corresponding to the lattice points of Q .

We write a function `mMixedVolume` to calculate the mixed volume of a collection of n lattice polytopes in \mathbb{R}^n . Let Q_1, \dots, Q_n be an arbitrary collection of lattice polytopes in \mathbb{R}^n . Let $R = k[x_1, \dots, x_{n+1}]$ and let I_i be the homogeneous ideal of R such that the polytope Q_i is the convex hull of the lattice points of the dehomogenization of a set of monomials that generates I_i in $k[x_1, \dots, x_n]$, for all i . Each of these homogeneous ideals can be obtained by giving the lattice points of each polytope as input in the function `homIdealPolytope`. The function `mMixedVolume` takes the list $\{I_1, \dots, I_n\}$ as input and produces the mixed volume of Q_1, \dots, Q_n as output. The function can also take the list of lists of vertices of the polytope as input to compute their mixed volume. Since calculating the mixed volume is the same as calculating a mixed multiplicity, the algorithm of the function `mMixedVolume` is similar to the algorithm of the function `mixedMultiplicity`.

Example 4.2. We calculate the mixed volume of a *cross polytope*. An n -cross polytope β_n is the convex hull of the points formed by permuting the coordinates of $(\pm 1, 0, \dots, 0) \in \mathbb{R}^n$:

$$\begin{aligned} \beta_n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid |x_1| + \dots + |x_n| \leq 1\} \\ &= \text{conv}\{(\pm 1, 0, \dots, 0), (0, \pm 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, \pm 1)\}. \end{aligned}$$

The volume of an n -cross polytope is $2^n/n!$ [Betke and Henk 1993, Theorem 2.1] and hence the mixed volume is 2^n . We say that a polytope is a $(0, 1)$ -polytope if its vertex set is a subset of $\{0, 1\}^d$ of the unit cube. In this example, we calculate the mixed volume of a 2-cross polytope and a 2-dimensional $(0, 1)$ -polytope.

```
i1 : A = {(0,1), (1,0), (0,-1), (-1,0)};
i2 : mMixedVolume {A,A}
o2 = 4
i3 : I = homIdealPolytope A;
i4 : B = {(0,0), (0,1), (1,0), (1,1)};
i5 : J = homIdealPolytope B;
i6 : mMixedVolume {I, sub(J, vars ring I)}
o6 = 4
```

The proposed function `mMixedVolume` takes less time to compute the mixed volume of a 3-cross polytope than the existing function `mixedVolume` in the `Polyhedra` package.

```
i1 : needsPackage "Polyhedra";
i2 : Q = crossPolytope 3;
i3 : time mixedVolume {Q,Q,Q};
      -- used 238.277 seconds
----- n-cross polytope
i4 : CP = n -> ( U = (i,p) -> (1..n)/(j -> if j == i then p else 0);
      flatten toList apply(1..n, i -> toList(U(i,1), U(i,-1))) );
i5 : time mMixedVolume {CP(3),CP(3),CP(3)}
      -- used 3.71303 seconds
o5 = 8
```

5. SECTIONAL MILNOR NUMBERS. In this section, we give an algorithm to compute the sectional Milnor numbers. We use Teissier’s observation of identifying the sectional Milnor numbers with mixed multiplicities to achieve this task. Teissier [1973] conjectured that invariance of the Milnor number implies invariance of the sectional Milnor numbers. The conjecture was disproved by Joël Briancon and Jean-Paul Speder. We verify their example using our algorithm.

Suppose the origin is an isolated singular point of a complex analytic hypersurface $H = V(f) \subset \mathbb{C}^{n+1}$. Let f_{z_i} denote the partial derivative of f with respect to z_i . Set

$$\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}\{z_0, z_1, \dots, z_n\}}{(f_{z_0}, f_{z_1}, \dots, f_{z_n})}.$$

The number μ is called the *Milnor number* of the hypersurface H at the origin. Teissier, in his Carge paper [1973], refined the notion of Milnor number by replacing it with a sequence of Milnor numbers of intersections with general linear subspaces. Let (X, x) be a germ of a hypersurface in \mathbb{C}^{n+1} with an isolated singularity. The Milnor number of $X \cap E$, where E is a general linear subspace of dimension i passing through x , is called the i^{th} -sectional Milnor number of X . It is denoted by $\mu^{(i)}(X, x)$. Let $J(f) = (f_{z_0}, \dots, f_{z_n})$ be the Jacobian ideal. In 1973, Teissier proved that the i^{th} -mixed multiplicity, $e(\mathfrak{m}^{[n-i]}, J(f)^{[i]})$, is equal to the i^{th} -sectional Milnor number of the singularity. Here $\mathfrak{m} = (z_0, z_1, \dots, z_n)$.

Let $R = \mathbb{C}[x_1, \dots, x_n]$ be a polynomial ring in n variables, let \mathfrak{m} be the maximal graded ideal and let $f \in R$ be any polynomial with an isolated singularity at the origin. Using Theorem 3.2, one can now calculate the mixed multiplicities of \mathfrak{m} and $J(f)$. We use the ideas in the previous section to write a function `secMilnorNumbers` for computing the first $n - 1$ sectional Milnor numbers. With a polynomial f given as input, the algorithm calculates the Jacobian ideal of f and then using the function `multiReesIdeal`, it finds the defining ideal of $\mathcal{R}(\mathfrak{m}, J(f))$. This helps to compute the Hilbert series of the special fiber $\mathcal{F}(\mathfrak{m}, J(f)) = \mathcal{R}(\mathfrak{m}, J(f)) \otimes_R R/\mathfrak{m}$. Using the formula given in Theorem 3.2, it then calculates the mixed multiplicities. Note that the n^{th} -sectional Milnor number is the Milnor number of the hypersurface f at the origin. So under the extra assumption that the ideal $J(f)$ is \mathfrak{m} -primary, we have $\mu^{(n)}(X, 0) = \dim_{\mathbb{C}} R/J(f)$. Together, the function `secMilnorNumbers` outputs $(\mu^{(0)}(X, 0), \mu^{(1)}(X, 0), \dots, \mu^{(n)}(X, 0))$.

Example 5.1.

```

i1 : R = QQ[x,y,z];
i2 : f = x^2*y+y^2*z+z^3;
i3 : secMilnorNumbers(f)
o3 = HashTable{0 => 1}
      1 => 2
      2 => 4
      3 => 8

```

Joël Briçon and Jean-Paul Speder [1975] considered the family of hypersurfaces $X_t \in \mathbb{C}^3$ defined by $F_t(x, y, z) = z^5 + ty^6z + xy^7 + x^{15} = 0$. They proved that the topological type of X_t is constant whereas the topological type of the section of X_t by a general plane varies. One can verify the example using the methods discussed above. For instance, consider the ideals

$$\mathfrak{m} = (x, y, z) \quad \text{and} \quad J(F_t) = (\partial F_t / \partial x, \partial F_t / \partial y, \partial F_t / \partial z)$$

in the ring $\mathbb{C}[x, y, z]$. In [Goel et al. 2023], an expository version of this article, we show that $e(\mathfrak{m}^{[1]}, J(F_t)^{[2]})$ depends on t although $e(J(F_t))$ is independent of t . The following Macaulay2 session verifies the example given by Briçon and Speder.

```

i1 : QQ[t];
i2 : k = frac oo;
i3 : R = k[x,y,z];
i4 : f = z^5 + t*y^6*z + x*y^7 + x^15;
i5 : secMilnorNumbers (f)
o5 = HashTable{0 => 1 }
      1 => 4
      2 => 26
      3 => 364

i6 : g = z^5 + x*y^7 + x^15;
i7 : secMilnorNumbers (g)
o7 = HashTable{0 => 1 }
      1 => 4
      2 => 28
      3 => 364

```

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