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ABSTRACT: We describe algorithms and their implementations that calculate local and global Schur indices of ordinary irreducible characters of finite groups and cyclotomic algebras over abelian number fields. Side benefits include functions for cyclotomic reciprocity calculations and for calculating the defect group associated with an ordinary irreducible character of a finite group. These functions are available in `GAP` via the package `wedderga`, versions 4.7 and higher.

1. INTRODUCTION. This article summarizes the approach that has been implemented for computing Schur indices in `GAP` [7] offered by the `GAP` package `wedderga` since version 4.7 [2]. It especially reports on some of the improvements that have been added since their first implementation in version 4.6, which was described in [9]. These improvements have already had an impact on new research; see, for example, [1].

The Schur index is a fundamental invariant for algebras and representation theory when working over nonalgebraically closed fields. The Schur index of a finite-dimensional division algebra D is the square root of its dimension over its center, i.e., $m(D) := \sqrt{[D : Z(D)]}$, so it can be seen as a measure of its noncommutativity. Each simple component of a finite-dimensional semisimple algebra is a matrix algebra over a division algebra, and the Schur index of one of these simple components is the Schur index of its division algebra part. The Schur index of an irreducible character χ of a finite group G over a field F is the Schur index of the simple component of FG that corresponds naturally to χ. The Schur index is also equal to the degree of the minimal extension E of the field of character values $F(\chi)$ needed for there to exist an irreducible representation affording χ whose matrix entries will lie in E. The latter description is the reason Schur index calculation plays an essential role in many applications of group representation theory.

Currently in `wedderga`, Schur index calculation for irreducible characters of finite groups can only be done over cyclotomic number fields. This is not due to limitations on the field implementations in `GAP`, but rather to the mathematics involved in the algorithms. (A completely separate `wedderga` function is available for computing Schur indices of quaternion algebras whose center is $\mathbb{Q}$ [9].)

Note added September 14, 2023: The latest `wedderga` package distributed with GAP is version 4.10.4. Version 4.10 includes a significant performance improvement with the addition of the global splitting

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and character descent Functions described in Section 7.3 of the `wedderga` manual, which are due to Ángel del Río and myself. The global splitting functions are designed to quickly identify crossed product algebras with trivial Schur indices, and they help to reduce others to a state where the local Schur index algorithms listed in Section 7.4 and 7.5, and described in greater detail here, are needed.

2. SCHUR INDEX COMPUTATION. Starting with an irreducible character χ of a finite group G, and a cyclotomic number field F, one objective of the `wedderga` package is to provide software for describing the algebraic structure of the simple component FGe of the group algebra FG for which $\chi(FGe) \neq 0$. It is a consequence of the Brauer–Witt theorem (see [16]) that any simple component of the group ring of a finite group over a cyclotomic number field will be Morita equivalent to a cyclotomic algebra. (By *cyclotomic algebra* we mean a crossed product algebra whose factor set takes values that are roots of unity.) It follows then that the simple component FGe corresponding to $\chi \in \mathrm{Irr}(G)$ can be expressed as $M_r(B_\chi)$, where B_χ is a cyclotomic algebra whose center is the field of character values $F(\chi)$. The Wedderburn decomposition functions in `wedderga` produce these cyclotomic crossed product algebras using the algorithm described in [12].

We also know that FGe is simple, so the cyclotomic algebra B_χ is a matrix ring over a division algebra D whose Schur index is $m_F(\chi)$. From the character degree $\chi(1)$ we know the dimension of FGe over its center, so the problem comes down to calculating $m_F(\chi)$ and identifying D up to isomorphism over $F(\chi)$. Since $F(\chi)$ is an algebraic number field, the isomorphism type of D as an $F(\chi)$-algebra is determined by its list of local invariants, one for each prime $\mathcal{P}$ of the number field $F(\chi)$, and almost all of these local invariants are 1; see [13]. By Benard–Schacher theory [4], for our division algebras the $\mathcal{P}$-local invariants have the same lowest-terms denominator for all primes $\mathcal{P}$ lying over a fixed rational prime p, including the case where $p = \infty$. The Schur index $m_F(\chi)$ is the least common multiple of these denominators, which are known as *p-local indices*. So it suffices to compute the p-local index for each rational prime p. Benard–Schacher theory also tells us the p-local index of D can be greater than 1 only at primes p dividing $|G|$, and greater than 1 at ∞ only when 4 divides $|G|$. In the end, this finite list of p-local indices comes quite close to identifying our division algebra up to ring isomorphism, so `wedderga` gives the identification of the division algebra in the form of a record containing the center, the global Schur index, and the list of all of its nontrivial p-local indices. There is no function in `wedderga` that isolates the division algebra as an actual algebra in `GAP`, but our new implementation does offer functions that can be used to isolate the cyclotomic algebra part of the simple component as an algebra with structure constants.

3. ALGORITHMS FOR LOCAL INDEX COMPUTATION. The local index of χ at ∞ can be computed directly from the values of χ using the Frobenius–Schur indicator. Our implementation uses this as a default, but uses an arithmetic shortcut that is even faster when our simple component FGe is presented as a cyclic cyclotomic algebra [9]. In this special situation, the implementation makes use of arithmetic shortcuts for the local index at 2 or an odd rational prime that are based on Janusz's lemma [10, Lemma 3.1].

These shortcut algorithms have been improved since the first release to make better use of the cyclotomic reciprocity functions in `wedderga`. These functions are designed to calculate the residue degree f, splitting degree g, and ramification index e of a cyclotomic extension of the form $F(\zeta_n)/F$ at a prime p. An additional function is offered that computes the maximal subextension of $F(\zeta_n)/F$ that splits completely at the prime p. Our routines for these rely on a careful determination of the element of the Galois group that induces the Frobenius automorphism at p. If $F \subseteq E \subseteq F(\zeta_n)$, then $E(\zeta_n) = F(\zeta_n)$, so a simple manipulation of cyclotomic reciprocity values for $F(\zeta_n)/F$ and for $F(\zeta_n)/E$ at p will produce the cyclotomic reciprocity values for the extension E/F at p (for more details, see [9]).

When the cyclotomic algebra presentation for FGe is not that of a cyclic cyclotomic algebra, the algorithm defaults to a procedure for calculating the p-local index of χ with the same mathematical basis as the one initially developed by Bill Unger and Gabriele Nebe for `MAGMA` [5].

Step 1: (Brauer–Witt search) For each prime q dividing $\chi(1)$, find a minimal subgroup (i.e., a Schur group) H and $\xi \in \mathrm{Irr}(H)$ that isolates the q-part of the p-local index of χ.

Step 2: (p-modular characters) If the p-defect group of ξ is cyclic, use Benard's theorem on characters in blocks with cyclic defect group [3] to obtain the p-local index of ξ.

Step 3: (Dyadic Schur groups) If the p-defect group of $\xi \in \mathrm{Irr}(H)$ is not cyclic, then it will be the case that $p = 2$, and one can apply Riese and Schmid's classification of dyadic Schur groups (see [15] and [14]) to obtain the 2-local index of ξ.

Details concerning this procedure were described in [9]. In practice we take advantage of the fact that `wedderga`'s presentation of FGe as a cyclotomic algebra gives us a natural extension $F(\zeta_n)$ over which the algebra splits. Our cyclotomic reciprocity functions allow us to extend the field F to the maximal p-split subextension E of F inside $F(\zeta_n)$. In this case the p-local index of χ over F is the same as its p-local index over E. We then recalculate the Wedderburn decomposition of EG and isolate the simple component corresponding to χ again. Sometimes this simple component already is presented as a cyclic cyclotomic algebra and the arithmetic shortcuts will get the job done. At worst the defining group H for this component is metabelian, has 3 generators modulo the derived subgroup, and its defining character ξ is faithful. These are the H and ξ we require for Step 2.

One of the biggest improvements since the initial release has been the addition of functions that enable the user to precisely determine the defect group at p for an ordinary character χ. (The author is indebted to Frieder Ladisch for pointing out the possibility for this improvement.) The new routine makes use of the ordinary character half of Brauer's min-max theorem, as presented in [11, (4.4)]. First the defect classes for the p-block containing χ are found, then we find a class for which the defect group has minimal order. The theory guarantees that these defect groups of minimal order are the defect groups of the block [11, (4.5)]. All of this is done using only the ordinary character table and so it is reasonably efficient even for fairly large groups.

An accurate decision on whether or not the defect group is cyclic means we have to complete at most one of Steps 2 and 3, both of which can be quite expensive. For example, since our defining group H is solvable, we can apply [8], which tells us that when $p = 2$ and the defect group is abelian, the 2-local index is 1. For odd primes p, Step 2 will be conclusive when we know the defect group is cyclic. So Step 3 only becomes necessary when $p = 2$ and the defect group of the 2-block of H containing ξ is nonabelian.

In the latest implementation improvements were made to the final step, which applies a Theorem of Yamada (see [6, Theorem 9.2]) to calculate the p-local index $m_F(\chi)$ from the result $m_{\mathbb{Q}(\chi)}(\chi)$ arising from Benard's theorem. Again, these improvements resulted from making more direct use of the cyclotomic reciprocity values for the prime p. The author is indebted to Andreas Bächle and Inneke Van Gelder for pointing out some discrepancies in the first implementation that led to these improvements.

SUPPLEMENT. The online supplement contains version 4.10.5 of `wedderga`.

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