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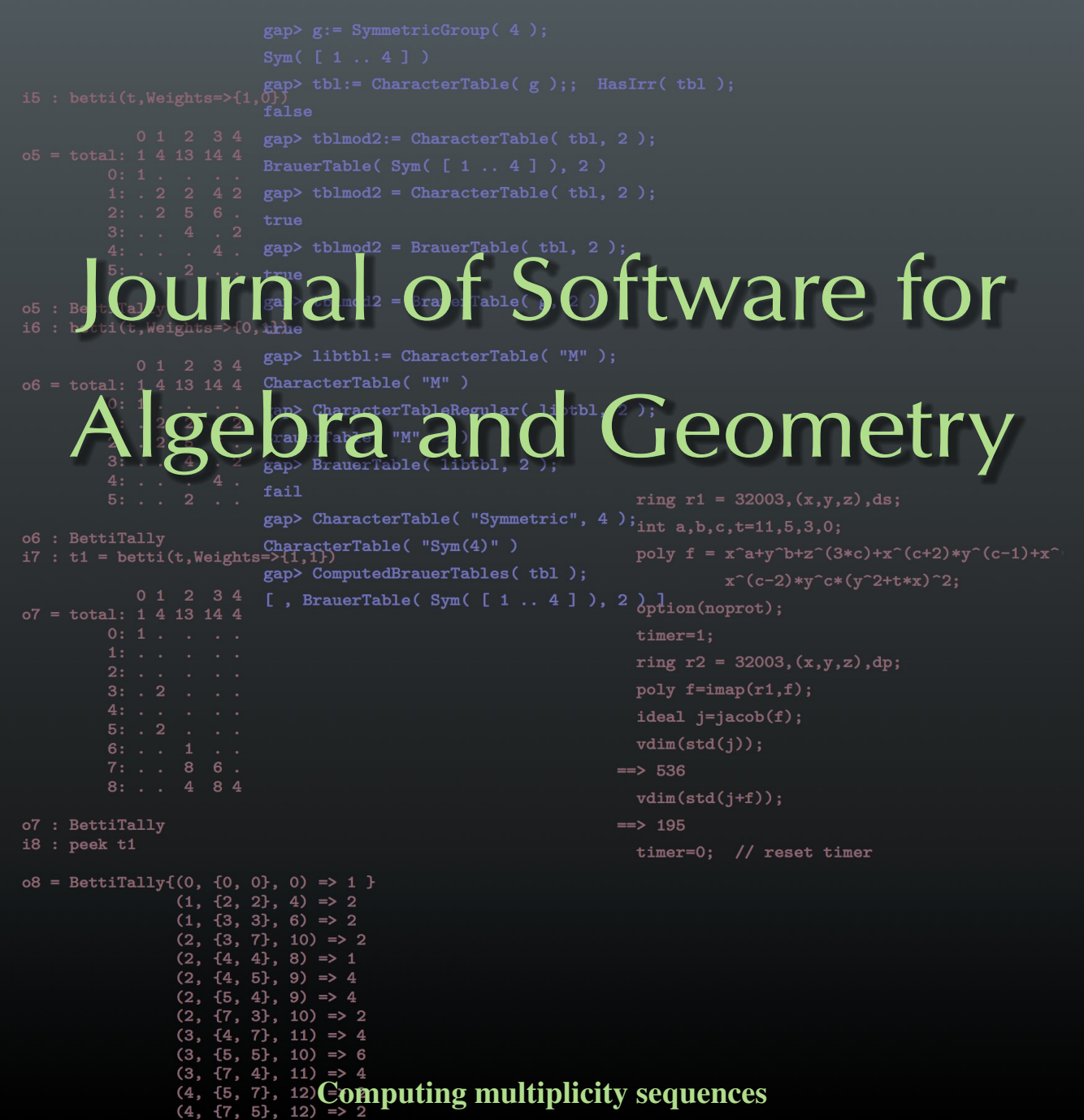

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ABSTRACT: The `MultiplicitySequence` package for `Macaulay2` computes the multiplicity sequence of a graded ideal in a standard graded ring over a field, as well as several invariants of monomial ideals related to integral dependence. We discuss two strategies that were implemented for computing multiplicity sequences: one via the bivariate Hilbert polynomial, and the other via the technique of general elements.

1. INTRODUCTION. Let $(R, \mathfrak{m}, k)$ be a d-dimensional Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let I be an R-ideal. If I is $\mathfrak{m}$-primary, then the *Hilbert–Samuel multiplicity* of I is defined as the degree of the standard graded algebra

$$\operatorname{gr}(I) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1},$$

i.e., the normalized leading coefficient of its Hilbert polynomial. This classical numerical invariant has been the base of several important results in commutative algebra and algebraic geometry. For example, a classical result of Rees states that the Hilbert–Samuel multiplicity gives an effective criterion for deciding whether two ideals have the same integral closure, provided R is formally equidimensional [14]. Rees' theorem is of fundamental importance in singularity theory as it is a key component in the proof of Teissier's *principle of specialization of integral dependence* (PSID), which provides a fiberwise numerical criterion for a family of hypersurfaces with isolated singularities to be equisingular [15].

The *j-multiplicity* and *ε-multiplicity* are extensions of the Hilbert–Samuel multiplicity to arbitrary ideals. These multiplicities were originally introduced in [1] and [8], respectively, in large part to extend Rees' theorem to the non-$\mathfrak{m}$-primary case. Such extensions were obtained in [4] and [16], but with the requirement of having to localize at all prime ideals of R.

The *multiplicity sequence* of an arbitrary ideal I in R is a sequence of $d+1$ nonnegative integers corresponding to the leading coefficients of the second sum transform of the bivariate Hilbert polynomial of the standard bigraded algebra

$$\mathcal{G} := \operatorname{gr}(\mathfrak{m}\operatorname{gr}(I)) = \bigoplus_{i,j=0}^{\infty} \frac{\mathfrak{m}^i I^j + I^{j+1}}{\mathfrak{m}^{i+1} I^j + I^{j+1}}. \tag{1-1}$$

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`MultiplicitySequence` version 0.7

This sequence is a particular case of the multiplicities defined by Kleiman and Thorup [9, §8], and it was also considered by Gaffney and Gassler [5] in the analytic case, and by Achilles and Manaresi [2] in our general setting. In the recent work [13], Polini, Trung, Ulrich, and Validashti extended Rees' theorem to arbitrary ideals without the need for localizations, by using multiplicity sequences. More precisely, they show that if R is formally equidimensional, then ideals $I \subseteq J$ have the same integral closure if and only if their multiplicity sequences agree (the forward direction was previously obtained by Ciupercă [3]). Furthermore, the authors of [13] develop a PSID for arbitrary ideals using the multiplicity sequence, demonstrating the importance of this invariant.

The main goal of the `MultiplicitySequence` package in `Macaulay2` [10] is to compute the multiplicity sequence of graded ideals in standard graded rings over a field. Two strategies have currently been implemented for doing so: the first one is based on the definition via the bivariate Hilbert polynomial of $\mathcal{G}$, and the second strategy is based on the technique of general elements; see [2, Theorem 4.1]. Finally, the package also includes a number of methods related to multiplicities and integral dependence, which have been adapted to the case of monomial ideals.

2. Multiplicity sequence.

Throughout, we keep the same notation as in the introduction. For a module M, $\lambda(M)$ denotes the length of M.

Associated bigraded ring. The second sum transform of the *bivariate Hilbert polynomial* of $\mathcal{G}$ is the polynomial $P(m,n)$ that agrees with

$$h(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} \lambda(\mathcal{G}_{i,j}), \quad \text{where } \mathcal{G}_{i,j} = \frac{\mathfrak{m}^i I^j + I^{j+1}}{\mathfrak{m}^{i+1} I^j + I^{j+1}} \tag{2-1}$$

for $m, n \gg 0$. The polynomial $P(m,n)$ can be written in the form

$$P(m,n) = \sum_{i=0}^{d} \frac{c_i(I)}{(d-i)!i!} m^{d-i} n^i + (\text{lower degree terms}),$$

with $c_i(I) \in \mathbb{Z}_{\geq 0}$ for $i = 0, \ldots, d$ [17].

Definition 2.1. The sequence $c_0(I), \ldots, c_d(I)$ is called the *multiplicity sequence* of I.

One has $c_i(I) = 0$ if $i < d - \dim R/I$ or $i > \ell(I)$, where $\ell(I) := \dim \operatorname{gr}(I) \otimes_R k$ is the *analytic spread* of I [2, Proposition 2.3]. Moreover, $c_d(I)$ equals the j-multiplicity of I. In particular, if I is $\mathfrak{m}$-primary, then $c_d(I)$ is the Hilbert–Samuel multiplicity of I while $c_i(I) = 0$ for $i \neq d$.

For purposes of `Macaulay2` computation, we take the local ring R to be of the form $A_{\mathfrak{n}}$, where A is a standard graded algebra over a field and $\mathfrak{n}$ is its irrelevant ideal (note that lengths of graded modules do not change under localizing at $\mathfrak{n}$). We now describe our first strategy for computing the multiplicity sequence.

Strategy 2.2. Given an ideal I, we compute the bigraded algebra $\mathcal{G}$ using `tangentNormalCone` (which iteratively calls `normalCone`). Subsequently, the method `hilbertSequence` extracts the relevant coefficients of the Hilbert polynomial $P(m,n)$ of $\mathcal{G}$ from the Hilbert series of $\mathcal{G}$.

Strategy 2.2 is the default strategy for computing the multiplicity sequence, and is executed whenever `multiplicitySequence` is called without specifying any options. We illustrate its use in the following example:

```
Macaulay2, version 1.17
i1 : needsPackage "MultiplicitySequence";
i2 : S = QQ[a..e]/(ideal(a-b,c)*ideal(c,d,e));
i3 : I = ideal"a2-bd,b4,e3";
i4 : multiplicitySequence I
o4 = HashTable{2 => 3 }
               3 => 12
i5 : hilbertSequence tangentNormalCone I
o5 =     0 1 2 3
       +----------
     3 | . . 3 12
     2 | . . 2 .
     1 | . 1 . .
     0 | . . . .
```

In the output `o4` above, the multiplicity sequence is displayed as a hash table, indicating that $c_2(I) = 3$ and $c_3(I) = 12$. The coefficients of the Hilbert polynomial of $\mathcal{G}$ are displayed in `o5` as a 2-dimensional table, whose top row is precisely the multiplicity sequence of I.

The most time-consuming step in Strategy 2.2 is that of computing (a presentation of) $\mathcal{G}$ — the Hilbert series and coefficient extraction are comparatively fast. For convenience, this expensive step is cached upon completion, so later calls to `multiplicitySequence` for a given ideal are nearly instant.

General elements. Our second strategy is based on Theorem 2.3 below which uses the method of general elements. For a local ring S, we denote by $e(S)$ the Hilbert–Samuel multiplicity of its maximal ideal.

Theorem 2.3 [13, Remark 2.3]. *Suppose R is equidimensional and catenary with infinite residue field. For any $i \geqslant 0$ and general elements $x_1, \ldots, x_i$ of I, one has*

$$c_i(I) = \sum_{\mathfrak{p}} \lambda\left(\frac{R_{\mathfrak{p}}}{(x_1, \ldots, x_{i-1})R_{\mathfrak{p}} : I^\infty + x_i R_{\mathfrak{p}}}\right) e(R/\mathfrak{p}), \tag{2-2}$$

where the sum ranges over the set of prime ideals

$$\left\{\mathfrak{p} \in V(I) \mid \operatorname{ht}\mathfrak{p} = i,\ \mathfrak{p} \supset (x_1, \ldots, x_{i-1}) : I^\infty\right\}, \tag{2-3}$$

and by convention the colon ideal $(x_1, \ldots, x_{i-1}) : I^\infty$ is 0 if $i = 0$ and is $0 : I^\infty$ if $i = 1$.

In view of Theorem 2.3, one could compute $c_i(I)$ by choosing general elements $x_1, \ldots, x_\ell \in I$, and then computing the various lengths and multiplicities in (2-2). However, this necessitates localizing at all the primes $\mathfrak{p}$ appearing above, which is undesirable for `Macaulay2` computation. Thus we take a different approach, as explained below.

Strategy 2.4. Via Theorem 2.3, we identify $c_i(I)$ with $e(R/J_i)$ for a suitable R-ideal J_i, and the latter can be computed in `Macaulay2` using a combination of `degree` and `normalCone` (in particular, avoiding localizations). The ideal J_i is constructed as follows: first, compute the minimal primes of the ideal $(x_1, \ldots, x_{i-1}) : I^\infty + (x_i)$. Next, set K to be the intersection of these minimal primes that do not contain I.

Finally, define $J_i := ((x_1, \ldots, x_{i-1}) : I^\infty + x_i) : K^\infty$. To see that $c_i(I) = e(R/J_i)$, note that we may identify the set of primes (2-3) as

$$\begin{aligned}\{\mathfrak{p} \in V(I) \mid \operatorname{ht}\mathfrak{p} = i,\ \mathfrak{p} \supset (x_1, \ldots, x_{i-1}) : I^\infty\} \\ &= \{\mathfrak{p} \in V((x_1, \ldots, x_{i-1}) : I^\infty + x_i) \mid \operatorname{ht}\mathfrak{p} = i\} \cap V(I) \\ &= \{\mathfrak{p} \in V((x_1, \ldots, x_{i-1}) : I^\infty + x_i) \mid \operatorname{ht}\mathfrak{p} = i\} \setminus (\operatorname{Spec}(R) \setminus V(I)) \\ &= \{\mathfrak{p} \in V(J_i) \mid \operatorname{ht}\mathfrak{p} = i\}.\end{aligned}$$

Then by the associativity formula for Hilbert–Samuel multiplicity, the sum in (2-2), taken over the last set above, is precisely $e(R/J_i)$.

We illustrate the use of Strategy 2.4. Note that the index i for $c_i(I)$ is specified here.

```
i6 : R = QQ[a..d];
i7 : I = ideal "a2,ab,b3,ad - bc,c2-bd";
i8 : multiplicitySequence(3, I, Strategy => "generalElements") -- c_3(I)
o8 = 5
i9 : multiplicitySequence(4, I, Strategy => "generalElements") -- c_4(I)
o9 = 7
```

For specific values of i, Strategy 2.4 may be faster than Strategy 2.2. However, for computing the entire multiplicity sequence, Strategy 2.2 tends to outperform Strategy 2.4, hence our choice of Strategy 2.2 as the default strategy.

As noted before, for $i = d$, the coefficient $c_d(I)$ is equal to the j-multiplicity of I, which has been studied by several authors; see, e.g., [6; 7; 11; 12]. We isolate this case in the method `jMultiplicity`, which is based on code written by H. Schenck and J. Validashti.

3. METHODS FOR MONOMIAL IDEALS. Monomial ideals carry combinatorial structure which often allows for special algorithms. The `MultiplicitySequence` package contains a few methods dedicated to monomial ideals, such as `newtonPolyhedron`, `monomialReduction`, and specialized algorithms for `analyticSpread` and `jMultiplicity` in the case of monomial ideals. These methods utilize the Newton polyhedron of a monomial ideal and scale much more efficiently than general methods. For comparison, we show the difference in timings for some of these methods:

```
i10 : I = monomialIdeal"ab2,bc3,cd4,da5";
i11 : elapsedTime jMultiplicity I^3
-- 0.874315 seconds elapsed
o11 = 9639
i12 : elapsedTime jMultiplicity ideal I^3
-- 456.039 seconds elapsed
o12 = 9639
i13 : elapsedTime analyticSpread I^5
-- 0.515529 seconds elapsed
o13 = 4
i14 : elapsedTime analyticSpread ideal I^5
-- 42.4524 seconds elapsed
o14 = 4
```

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SUPPLEMENT. The online supplement contains version 0.7 of `MultiplicitySequence`.

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