

```

gap> g:= SymmetricGroup( 4 );
Sym( [ 1 .. 4 ] )
gap> tbl:= CharacterTable( g ); HasIrr( tbl );
i5 : betti(t,Weights=>{1,0})
false
      0 1 2 3 4
o5 = total: 1 4 13 14 4
      0: 1 . . .
      1: . 2 2 4 2
      2: . 2 5 6 .
      3: . . 4 . 2
      4: . . . 4 .
      5: . . 2 . .
gap> tblmod2:= CharacterTable( tbl, 2 );
BrauerTable( Sym( [ 1 .. 4 ] ), 2 )
gap> tblmod2 = CharacterTable( tbl, 2 );
true
gap> tblmod2 = BrauerTable( tbl, 2 );
true
o5 : BrauerTable
i6 : betti(t,Weights=>{0,1})
      0 1 2 3 4
o6 = total: 1 4 13 14 4
      0: 1 . . .
      1: . 2 2 4 2
      2: . 2 5 6 .
      3: . . 4 . 2
      4: . . . 4 .
      5: . . 2 . .
gap> libtbl:= CharacterTable( "M" );
CharacterTable( "M" )
gap> CharacterTableRegular( libtbl, 2 );
BrauerTable( "M" )
gap> BrauerTable( libtbl, 2 );
fail
gap> CharacterTable( "Symmetric", 4 );
CharacterTable( "Sym(4)" )
gap> ComputedBrauerTables( tbl );
[ , BrauerTable( Sym( [ 1 .. 4 ] ), 2 ) ]
ring r1 = 32003,(x,y,z),ds;
int a,b,c,t=11,5,3,0;
poly f = x^a+y^b+z^(3*c)+x^(c+2)*y^(c-1)+x^(
x^(c-2)*y^c*(y^2+t*x)^2;
option(noprot);
timer=1;
ring r2 = 32003,(x,y,z),dp;
poly f=imap(r1,f);
ideal j=jacob(f);
vdim(std(j));
==> 536
vdim(std(j+f));
==> 195
timer=0; // reset timer
o7 : BettiTally
i7 : t1 = betti(t,Weights=>{1,1})
      0 1 2 3 4
o7 = total: 1 4 13 14 4
      0: 1 . . .
      1: . . . .
      2: . . . .
      3: . 2 . .
      4: . . . .
      5: . 2 . .
      6: . . 1 .
      7: . . 8 6 .
      8: . . 4 8 4
o7 : BettiTally
i8 : peek t1
o8 = BettiTally{(0, {0, 0}, 0) => 1 }
      (1, {2, 2}, 4) => 2
      (1, {3, 3}, 6) => 2
      (2, {3, 7}, 10) => 2
      (2, {4, 4}, 8) => 1
      (2, {4, 5}, 9) => 4
      (2, {5, 4}, 9) => 4
      (2, {7, 3}, 10) => 2
      (3, {4, 7}, 11) => 4
      (3, {5, 5}, 10) => 6
      (3, {7, 4}, 11) => 4
      (4, {7, 5}, 12) => 2

```

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Tropical computations for toric intersection theory in Macaulay2

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ABSTRACT: We present the Macaulay2 package `TropicalToric` for toric intersection theory computations using tropical geometry.

1. INTRODUCTION. Toric varieties are ubiquitous in algebraic geometry. Their intersection theory was first studied in Fulton and Sturmfels [12], and it has many applications in different contexts, including wonderful and tropical compactifications [7; 10; 34], birational geometry [5; 14; 15], tropical geometry [23], tropical intersection theory [2; 21; 22; 29; 30] and combinatorial Hodge theory [1; 19; 20].

In a certain way, intersection classes of a toric variety with fan Σ can be thought of in terms of balanced subfans of Σ , also referred to as *Minkowski weights*; see [12] or [27, Theorem 6.7.5]. From the structure theorem of tropical geometry [27, Theorem 3.3.5], we know that the tropicalization of a subvariety of a torus $Y \subseteq T^n$ is a balanced fan. A surprising connection between tropical and toric geometry is that the tropicalization of Y is the balanced fan corresponding to the intersection class of the closure of Y inside an “enough refined” toric variety; see [Theorem 3.1](#) for a more precise statement. This fact allows us to compute toric intersection classes starting from the data of the tropicalization.

We present a new package, `TropicalToric`, for Macaulay2 [26]. The package implements toric cycles and intersection products on simplicial toric varieties ([Section 2](#)), and, following the ideas outlined above, allows us to compute the intersection class of an irreducible subvariety of a simplicial toric variety not contained in the toric boundary, from the data of its tropicalization ([Section 3](#)). The tropicalization is performed with the use of the Macaulay2 package `Tropical` [3]. Further, we present some applications to the intersection theory of wonderful compactifications and the moduli space $\overline{M}_{0,n}$ and illustrate an example in a multiprojective space using a theorem of Huh and Katz [20] about characteristic polynomials of realizable matroids.

2. TORIC INTERSECTION THEORY. In this section, we review the basics of toric intersection theory, for more information see [12], [6, Section 12.5] or [27, Section 6.7]. In addition, we showcase how it is implemented in the package.

Let X_Σ be a smooth complete toric variety of dimension n with fan Σ . We denote by $\Sigma(k)$ the cones of Σ of dimension k , with $Z^k(X_\Sigma) = Z_{n-k}(X_\Sigma)$ the group of codimension k cycles and with $A^k(X_\Sigma) = A_{n-k}(X_\Sigma)$ the codimension k Chow group, that is, the group of codimension k cycles modulo

MSC2020: 14Q99, 14T20.

Keywords: tropical geometry, toric geometry, intersection theory, wonderful compactifications.

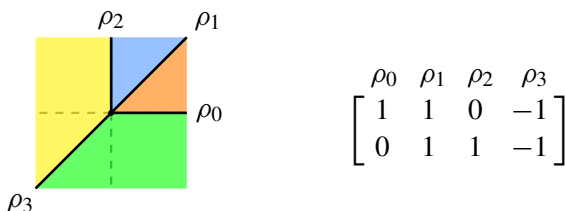
`TropicalToric` version 1.0

rational equivalence. The codimension k Chow group $A^k(X_\Sigma)$ is generated by the set $\{[V(\sigma)] : \sigma \in \Sigma(k)\}$ of classes of orbit closures of codimension k , and the relations in each Chow group can be described in an explicit way; see [12, Proposition 2.1].

We can show (see [11, Chapter 8]) that there is an intersection product $A^k(X_\Sigma) \times A^r(X_\Sigma) \rightarrow A^{k+r}(X_\Sigma)$ that makes $A^*(X_\Sigma) = \bigoplus_{k=0}^n A^k(X_\Sigma)$ into a graded ring, called the *Chow ring* of X_Σ . If we now assume that X_Σ is just complete and simplicial, then the intersection product can be defined on rational cycles, making $A^*(X_\Sigma)_\mathbb{Q} = A^*(X_\Sigma) \otimes \mathbb{Q}$ into a graded ring. The structure of the Chow ring has an explicit description; see, for instance, [27, Theorem 6.7.1].

Our Macaulay2 package implements toric cycles and the intersection product as in [6, Lemma 12.5.2].

Example 2.1. Let X_Σ be the blow-up of \mathbb{P}^2 at one of the coordinate points, where the fan Σ and the primitive ray vectors of its rays are



Let H be the strict transform in X_Σ of a general line in \mathbb{P}^2 and E be the exceptional divisor. The Picard group of X_Σ is generated by the classes of these two divisors $\text{Pic}(X_\Sigma) = \langle [H], [E] \rangle$. With the notation above, we have

$$[V(\rho_0)] = [H] - [E], \quad [V(\rho_1)] = [E], \quad [V(\rho_2)] = [H] - [E], \quad [V(\rho_3)] = [H].$$

Now we verify with our package that the divisor class $[V(\rho_1)]$ has a negative self-intersection.

```
i1 : needsPackage "TropicalToric";
i2 : raysList = {{1,0},{1,1},{0,1},{-1,-1}};
i3 : coneList = {{0,1},{1,2},{2,3},{3,0}};
i4 : X = normalToricVariety (raysList, coneList);
```

Now define the toric cycle $V(\rho_1)$.

```
i5 : E = X_{1}
o5 = X_{1}
o5 : ToricCycle on X
```

We point out that the type `ToricCycle` should not be confused with the type `ToricDivisor` from the `NormalToricVarieties` package. The toric cycle $V(\sigma)$ of the normal toric variety X associated to the cone σ given by a list of rays L is defined with the command `X_L`. For example, `X_{1,2}` or `X_{0}` define toric cycles, whereas `X_1` defines a toric divisor. We are allowed only to multiply a toric cycle with a toric divisor. Now, we finally compute the self intersection of E :

```
i6 : X_1 * E
o6 = - X_{1, 2}
o6 : ToricCycle on X
```

The resulting cycle $-V(\rho_1 + \rho_2)$ is rationally equivalent to E^2 . The negative sign tells us that the self-intersection number of the exceptional divisor is -1 . We can compute the degree of maximal codimension cycles with `degCycle`:

```
i7 : degCycle(-X_{1,2})
o7 = -1
```

3. TROPICAL COMPUTATIONS. In this section, we describe and showcase the algorithm implemented in the main function of the package `classFromTropical` that computes the intersection class of an irreducible subvariety of a smooth toric variety from its tropicalization.

The algorithm is mainly based on the following result, that appears in various versions in the literature; see, for instance, [23, Lemma 2.3], [21, Section 9] or [27, Theorem 6.7.7].

Theorem 3.1. *Let Y be a subvariety of the algebraic torus T^n , and let \bar{Y} be its closure in a toric variety X_Σ such that $|\Sigma| = \text{trop}(Y)$ and Σ is simplicial. Let Σ' be a simplicial completion of the fan Σ , and let $i : X_\Sigma \rightarrow X_{\Sigma'}$ be the induced inclusion. Then, for every maximal cone σ in Σ , we have*

$$m(\sigma) = \deg([i_* (\bar{Y})] \cdot [V(\sigma)]),$$

where $m(\sigma)$ is the multiplicity of σ in $\text{trop}(Y)$.

Now let Y be an irreducible k -dimensional subvariety of an n -dimensional simplicial toric variety X_Σ , and suppose that $Y \cap T^n \neq \emptyset$. Note that in this setting we cannot directly apply [Theorem 3.1](#) since $\text{trop}(Y)$ is not necessarily a subfan of Σ .

In order to compute the class of Y in the Chow ring of X_Σ , we proceed as follows. First, let Σ' be a completion of Σ and $i : X_\Sigma \rightarrow X_{\Sigma'}$ be the induced inclusion. Now let $\tilde{\Sigma}$ be a refinement of Σ' such that it contains a subfan with support the tropicalization of $Y \cap T^n$, and let $\pi : X_{\tilde{\Sigma}} \rightarrow X_{\Sigma'}$ be the induced toric map. From [12, Proposition 2.4], we have an isomorphism $A_k(X_{\Sigma'}) \simeq \text{Hom}(A^k(X_{\Sigma'}), \mathbb{Z})$ mapping a class $[Z]$ to the homomorphism $[Z'] \mapsto \deg([Z] \cdot [Z'])$. Therefore, in order to compute the class $[Y] \in A_k(X_\Sigma)$, it is enough to compute the intersection numbers $\deg([i_* (Y)] \cdot [V(\sigma)])$ for every $\sigma \in \Sigma'(k)$, as the classes $[V(\sigma)]$ generate $A^k(X_{\Sigma'})$. Let Y' be the strict transform of Y in $X_{\tilde{\Sigma}}$. From the projection formula [11, Proposition 2.3 (c)], we have

$$[i_* (Y)] \cdot [V(\sigma)] = \pi_* ([Y']) \cdot [V(\sigma)] = \pi_* ([Y'] \cdot \pi^* ([V(\sigma)])),$$

from which it follows that $\deg([i_* (Y)] \cdot [V(\sigma)]) = \deg([Y'] \cdot \pi^* ([V(\sigma)]))$. These last intersection numbers can be computed from the tropicalization of $Y \cap T^n$ by using [Theorem 3.1](#), since $\deg([Y'] \cdot [V(\sigma')])$ is the multiplicity of the cone $\sigma' \in \tilde{\Sigma}(k)$ in the tropicalization of $Y \cap T^n$.

The algorithm described above, while working on any simplicial toric variety, requires to compute a completion. This can be avoided by requiring the toric variety X_Σ to be smooth. In fact, the only step in which we are really using the completion is when we apply [12, Proposition 2.4] (sometimes called *Kronecker duality*). If the variety X_Σ is smooth, this can be substituted instead with Poincaré duality.

The function `classFromTropical` performs the above algorithm to compute a toric cycle rationally equivalent to a given irreducible subvariety Y of a smooth toric variety X_Σ (by using Poincaré duality). The input of the function consists of the toric variety X_Σ and the ideal I of $Y \cap T^n$ of the Laurent ring of T^n . Since Laurent rings are not implemented in Macaulay2, the actual input will be instead the saturation of I with respect to the product of the variables in the polynomial ring:

```
i2 : X = toricProjectiveSpace 2;
i3 : R = QQ[x,y];
i4 : I = ideal(x+y+1);
i5 : classFromTropical(X,I)
o5 = X
     {0}
o5 : ToricCycle on X
i6 : J = ideal(x*y + x + y);
i7 : classFromTropical(X,J)
o7 = 2*X
     {0}
o7 : ToricCycle on X
```

The function `classFromTropicalCox` allows us to input the ideal of Y in the Cox ring of X_Σ :

```
i8 : R = ring X;
i9 : I = ideal(R_0+R_1+R_2);
i10 : classFromTropicalCox(X,I)
o10 = X
      {0}
o10 : ToricCycle on X
```

4. APPLICATIONS.

4A. Wonderful compactifications. Let \mathcal{A} be an essential hyperplane arrangement of $n + 1$ hyperplanes in \mathbb{P}^d . The intersection lattice $\mathcal{L}(\mathcal{A})$ of \mathcal{A} is isomorphic to the lattice of flats of the underlying matroid M of \mathcal{A} [32, Proposition 3.6]. Fix a building set \mathcal{G} of the lattice of flats of M (see [10, Section 2]), let $\Sigma \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1} \simeq \mathbb{R}^n$ be the Bergman fan of M with respect to \mathcal{G} (see [27, Chapter 4]) and let X_Σ be its associated toric variety. From [27, Proposition 4.1.1], the hyperplane arrangement complement $Y = \mathbb{P}^d \setminus \cup \mathcal{A}$ is naturally isomorphic to a linear subspace of the algebraic torus T^n . Thus we can embed Y inside the toric variety X_Σ and consider its closure \bar{Y} . This compactification coincides with the so-called De Concini–Procesi *wonderful compactification* [7], with respect to the building set \mathcal{G} (see [34, Section 4]). The next result follows from [7, Theorem 3.2]; see also [9, Definition 2.3].

Proposition 4.1. *Let X_1, \dots, X_t be a linear extension of the opposite order of $\mathcal{L}(\mathcal{A})$. The wonderful compactification \bar{Y} is the result of successively blowing up \mathbb{P}^d at (the strict transforms of) X_1, \dots, X_t .*

In [10], Feichtner and Yuzvinsky showed that the cohomology of \bar{Y} agrees with that of X_Σ . Since both varieties are *homology isomorphism schemes* (in the sense of the definition in the appendix of [24]), their Chow rings coincide as well.

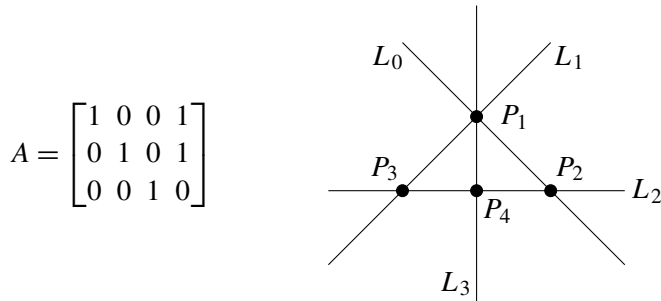
Theorem 4.2 [27, Theorem 6.7.14]. *Let \bar{Y} be a wonderful compactification of a hyperplane arrangement \mathcal{A} with respect to a building set \mathcal{G} , and let Σ be the associated Bergman fan. Then*

$$A^*(\bar{Y}) \simeq A^*(X_\Sigma),$$

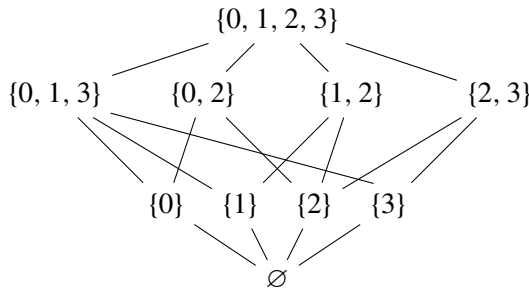
where the above isomorphism is the pullback map induced by the inclusion.

The above theorem allows us to view the intersection classes of a wonderful compactification as intersection classes of the associated toric variety. Thus, we can use our package to perform intersection theory computations on wonderful compactifications.

Example 4.3. Let \mathcal{A} be a line arrangement consisting of 4 lines L_0, L_1, L_2, L_3 in \mathbb{P}^2 given by the equations $x_0 = 0, x_1 = 0, x_2 = 0, x_0 + x_1 = 0$, respectively. Let A be the matrix with columns the normal vectors of the lines L_i , and let P_1, P_2, P_3, P_4 be the points of intersection of the lines of \mathcal{A} , depicted as



The underlying matroid M of \mathcal{A} , on the ground set $\{0, 1, 2, 3\}$, is realized by the matrix A by labeling the columns with $0, 1, 2, 3$, respectively. The lattice of flats $\mathcal{L}(M)$ of M is represented by



There are four rank 1 flats, corresponding to the lines L_0, L_1, L_2, L_3 , and four rank 2 flats, corresponding to the points P_1, P_2, P_3, P_4 . Let $\mathcal{G} = \mathcal{L}(M) \setminus \{\emptyset\}$ be the maximal building set of $\mathcal{L}(M)$. Then, from Proposition 4.1, the wonderful compactification \bar{Y} of the complement $Y = \mathbb{P}^2 \setminus \cup \mathcal{A}$ with respect to \mathcal{G} is the blow-up of \mathbb{P}^2 at the points P_1, P_2, P_3, P_4 . In particular, \bar{Y} is a smooth projective surface, all Weil divisors are Cartier [17, Proposition II.6.11] and the class group is isomorphic to the Picard group [17, Corollary II.6.16]. From [17, Proposition V.3.2], the Picard group of \bar{Y} has a basis given by

$$\text{Pic}(\bar{Y}) = \langle [H], [E_1], \dots, [E_t] \rangle, \tag{1}$$

where $[H]$ is the class of the strict transform H of a general line in \mathbb{P}^2 , and $[E_i]$ is the class of the exceptional divisor E_i of the blow-up at P_i .

The Bergman fan $\Sigma \subseteq \mathbb{R}^4/\mathbb{R}\mathbf{1}$ of M with respect to \mathcal{G} has eight rays, denoted $\{\rho_i : 0 \leq i \leq 7\}$. Their primitive ray vectors are given by the columns of the matrix

$$\begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 & \rho_6 & \rho_7 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

where $\rho_0, \rho_1, \rho_2, \rho_3$ correspond to the rank 1 flats in \mathcal{G} , which in turn correspond to the lines L_0, L_1, L_2, L_3 , respectively, and $\rho_4, \rho_5, \rho_6, \rho_7$ correspond to the rank 2 flats in \mathcal{G} , that correspond to the points P_1, P_2, P_3, P_4 , respectively. Since \mathcal{G} is the maximal building set, the maximal cones of Σ are just the maximal chains of the lattice of flats $\mathcal{L}(M)$.

By using the isomorphism in [Theorem 4.2](#), let $[Y_{\rho_i}]$ denote the class in $A^*(\bar{Y})$ isomorphic to the class of the torus invariant divisor of X_Σ associated to the ray ρ_i . Expressing these divisors in the Picard basis (1), we have

$$\begin{aligned} [Y_{\rho_0}] &= [H] - [E_1] - [E_2], & [Y_{\rho_4}] &= [E_1], \\ [Y_{\rho_1}] &= [H] - [E_1] - [E_3], & [Y_{\rho_5}] &= [E_2], \\ [Y_{\rho_2}] &= [H] - [E_2] - [E_3] - [E_4], & [Y_{\rho_6}] &= [E_3], \\ [Y_{\rho_3}] &= [H] - [E_1] - [E_4], & [Y_{\rho_7}] &= [E_4]. \end{aligned} \tag{2}$$

Let $\mathbb{C}[y_0^{\pm 1}, y_1^{\pm 1}, y_2^{\pm 1}]$ be the Laurent ring of the torus

$$T^3 = \{(1 : y_0 : y_1 : y_2) : y_0, y_1, y_2 \in \mathbb{C}^*\} \subseteq \mathbb{P}^3.$$

The embedding $Y \hookrightarrow T^3$ is given by $(x_0 : x_1 : x_2) \mapsto (x_0 : x_1 : x_2 : x_0 + x_1)$, and the Laurent ideal of Y inside T^3 is $I = (-1 - y_0 + y_2)$.

Now, let C be the conic in \mathbb{P}^2 passing through the points P_1, P_2 and P_3 given by the equation $x_0x_1 + x_0x_2 + x_1x_2$. The ideal of C in T^3 is $(y_0 + y_1 + y_0y_1) + I$. We expect the class of its strict transform in \bar{Y} to be $[2H - E_1 - E_2 - E_3]$. We now verify this with our package, using the function `classWonderfulCompactification`.

```
i2 : R = QQ[y_0,y_1,y_2];
i3 : I = ideal(-1-y_0+y_2);
i4 : f = y_0+y_1+y_0*y_1;
i5 : raysList = {{-1,-1,-1},{1,0,0},{0,1,0},
                {0,0,1},{0,-1,0},{-1,0,-1},
                {1,1,0},{0,1,1}};
i6 : conesList = {{4,0},{4,1},{4,3},{5,0},{5,2},
                 {6,1},{6,2},{7,2},{7,3}};
i7 : X = normalToricVariety (raysList, conesList);
i8 : D = classWonderfulCompactification(X,I,f)
o8 = X_{0} + X_{4} + X_{1}
o8 : ToricCycle on X
```

To check that this is the result we expect, compare with (2). Note that we have (tropically) dehomogenized the rays of X_Σ with respect to the first coordinate in order to be consistent with our choice of coordinates of T^3 .

4B. The moduli space $\overline{M}_{0,n}$. The Deligne–Mumford compactification of the moduli space $M_{0,n}$ can be realized as a wonderful compactification; see, for instance, [27, Example 6.7.16]. Therefore, we can apply to $\overline{M}_{0,n}$ the machinery described in the previous section. As an application, we compute one of the 15 Keel–Vermeire divisors of $\overline{M}_{0,6}$, using one of the equations listed in [16, Table 2]. These divisors, found independently by Keel and Vermeire [35], were the first example of an effective divisor of $\overline{M}_{0,n}$ whose class lies outside the cone generated by the classes of the boundary divisors, answering in the negative a conjecture of Fulton; see [25].

```
i2 : R = QQ[x_0..x_8];
i3 : I = ideal {-x_0+x_3+x_4, -x_1+x_3+x_5, -x_2+x_3+x_6,
              -x_0+x_2+x_7, -x_1+x_2+x_8, -x_0+x_1+1};
i4 : X = normalToricVariety fan tropicalVariety I;
i5 : f = x_0*x_1-x_2*x_3;
i6 : D = classWonderfulCompactification(X,I,f);
i7 : D = toricDivisorFromCycle(D)
o7 = X_2 - X_5 - 2*X_6 + X_7 + 2*X_9 + 2*X_10 - X_11 + 2*X_13 + 2*X_14 - X_17
o7 : ToricDivisor on X
```

Now fix the Picard basis of X_Σ given by the boundary divisors associated to the rays of Σ with primitive ray vectors not equal to the standard vectors e_i . The complement of this Picard basis is indexed by the list $l = \{0, 1, 2, 4, 5, 7, 11, 13, 21\}$. The function `makeTransverse` computes a divisor linearly equivalent to a given divisor D , with support disjoint from a given list l . We use this function to compute a representation of the class of the Keel–Vermeire divisor computed above, in the Picard basis we fixed:

```
i8 : l = {0,1,2,4,5,7,11,13,21};
i9 : D = makeTransverse(D,l)
o9 = X_3 - X_6 - X_8 + X_9 + 2*X_10 - X_14 - X_16 + X_17 + 2*X_18 + 2*X_19
      + 2*X_20 - X_22 - X_24
o9 : ToricDivisor on X
```

Finally, we verify that the obtained divisor is outside the cone generated by the classes of boundary divisors. In order to do so, we interface with `Polymake` [13] by using the function `polymakeConeContains`:

```
i10 : D = apply(#rays X, i->D#i);
i11 : Bdivisors = apply(#rays X, i-> makeTransverse(X_i,l));
i12 : Bdivisors = apply(Bdivisors, B-> apply(#rays X, i->B#i));
i13 : polymakeConeContains(D,Bdivisors)
o13 = false
```

In [18] it was proved, by using computational methods, that the boundary divisors and the Keel–Vermeire divisors generate the effective cone of $\overline{M}_{0,6}$. In [5] it was proved that the effective cone of $\overline{M}_{0,n}$

for $n \geq 10$ is not polyhedral. The problem of determining the effective cone of $\bar{M}_{0,n}$ for $n \in \{7, 8, 9\}$ is still open. Some examples of extremal effective divisors on $\bar{M}_{0,7}$ were found in [4; 8; 28]. We performed computations similar to those displayed above on $\bar{M}_{0,7}$ and found the mentioned examples with a brute-force approach. More recently, in [31], several thousands of extremal effective divisors on $\bar{M}_{0,7}$ were found.

4C. Characteristic polynomials. Our last application is an explicit verification of a theorem proved by Huh and Katz [20] about characteristic polynomials of realizable matroids.

Let \mathcal{A} be an arrangement of $n + 1$ hyperplanes on \mathbb{P}^d , let M be its underlying matroid of rank $d + 1$, and let $\mathcal{L}(M)$ be the lattice of flats of M . The *characteristic polynomial* of M is

$$\chi_M(q) = \sum_{F \in \mathcal{L}(M)} \mu(\emptyset, F) q^{d+1-r(F)},$$

where μ is the Möbius function of $\mathcal{L}(M)$; see [33, Section 3.7]. The *reduced characteristic polynomial* of M is $\bar{\chi}_M(q) = \chi_M(q)/(q - 1)$.

Now we embed the complement $Y = \mathbb{P}^d \setminus \cup \mathcal{A}$ in $T^n \subseteq \mathbb{P}^n$, as described in Section 4A, and consider the Cremona map

$$\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n, \quad (x_0, \dots, x_n) \mapsto (x_0^{-1}, \dots, x_n^{-1}).$$

Finally, let \bar{Z} be the closure in $\mathbb{P}^n \times \mathbb{P}^n$ of the graph Z of the restriction $\varphi|_Y$.

Theorem 4.4 (Huh and Katz [20]). *Define the integers $a_i \in \mathbb{Z}$ by the formula*

$$\bar{\chi}_M(q) = \sum_{i=0}^d (-1)^i a_i q^{d-i}.$$

Then

$$[\bar{Z}] = \sum_{i=0}^d a_i [\mathbb{P}^{r-i} \times \mathbb{P}^i] \in A_d(\mathbb{P}^n \times \mathbb{P}^n).$$

Example 4.5. Let G be the graph

$$G = \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \quad A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Let M be the rank 3 graphic matroid of G , realized by the matrix A above. The characteristic polynomial of M coincide with the chromatic polynomial of G . Let $\mathbb{C}[x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ be the Laurent ring of the torus

$$T^4 = \{(1 : x_0 : x_1 : x_2 : x_3) : x_0, x_1, x_2, x_3 \in \mathbb{C}^*\} \subseteq \mathbb{P}^4.$$

Let \mathcal{A} be the hyperplane arrangement realizing M . More explicitly, the normal vectors of its hyperplanes are the columns of the matrix A . The Laurent ideal of the hyperplane arrangement complement $Y = \mathbb{P}^2 \setminus \cup \mathcal{A}$ embedded in $T^4 \subseteq \mathbb{P}^4$ is given by $I = (-1 + x_0 + x_2, -1 + x_1 + x_3)$.

Now consider a copy of T^4 with Laurent ring $\mathbb{C}[x_4^{\pm 1}, x_5^{\pm 1}, x_6^{\pm 1}, x_7^{\pm 1}]$. Let $Z \subseteq T^4 \times T^4$ be the graph of the Cremona map $\varphi: \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ restricted to Y . The ideal of Z in the Laurent ring $\mathbb{C}[x_0^{\pm 1}, \dots, x_7^{\pm 1}]$ of $T^4 \times T^4$ is generated by I and the polynomials $x_i x_{i+4} - 1$ for $i \in \{0, 1, 2, 3\}$.

```
i2 : R = QQ[x_0..x_7];
i3 : I = ideal(-1+x_0+x_2,-1+x_1+x_3,
              x_0*x_4-1,x_1*x_5-1,x_2*x_6-1,x_3*x_7-1);
o3 : Ideal of R
i4 : P4 = toricProjectiveSpace 4;
i5 : X = NormalToricVarieties$cartesianProduct(P4,P4);
i6 : D = classFromTropical(X,I)
o6 = 4*X_{0,1,2,3,5,6} + 4*X_{0,1,2,5,6,7} + X_{0,1,5,6,7,8}
o6 : ToricCycle on X
```

We obtained $[\bar{Z}] = [\mathbb{P}^2 \times \mathbb{P}^0] + 4[\mathbb{P}^1 \times \mathbb{P}^1] + 4[\mathbb{P}^0 \times \mathbb{P}^2]$. We now verify that the coefficients of this class are the same, up to sign, to those of the (reduced) chromatic polynomial of G :

```
i7 : needsPackage "Graphs";
i8 : G = graph({{0,1},{1,2},{2,3},{3,0},{0,2}});
i9 : p = chromaticPolynomial G
      4      3      2
o9 = x  - 5x  + 8x  - 4x
o9 : ZZ[x]
i10 : x = (ring p)_0;
i11 : p/(x-1)
      3      2
o11 = x  - 4x  + 4x
o11 : frac(ZZ[x])
```

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SUPPLEMENT. The [online supplement](#) contains version 1.0 of TropicalToric.

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