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ABSTRACT: Computer aided design and motion control lead to algebraic systems. Resultants of multivariate polynomials are useful to solve such problems. Following Gelfand, Kapranov and Zelevinsky, we calculate them via the Cayley Formula as determinant of a complex formed by global sections of sheaves. These arise from the Koszul complex generated by the polynomials, which we twist by a reflexive rank one bundle corresponding to the shift of Newton polytopes by a rational vector, introduced by Canny and Emiris. Again, inspired by these authors, we apply tight mixed subdivisions of the polytopes to obtain regular minors of the differentials required to evaluate the Cayley formula. Besides the assumption that the Minkowski sum of all Newton polytopes in the system should be full dimensional, there are no further constraints on the set of exponents defining the input polynomials with indeterminate coefficients. Consequently, our resultant coincide with the definition of D'Andrea and Sombra. This complements the package `SparseResultant` implemented by Staglianò (2021) which requires stricter assumptions, including that each individual Newton polytope must be full-dimensional.

1. INTRODUCTION.

Resultants are used in computer graphics and control theory. They determine the implicit representation of hypersurfaces and allow, with some restrictions, to calculate the roots of algebraic systems. In such problems, the coefficients in the system are initially undefined. Only later are they assigned various floating point numbers. A polynomial with indeterminate coefficients is specified by the exponent vectors $m \in \mathbb{Z}^n$ of its terms. These form a subset $\mathcal{A}_j \subset \mathbb{Z}^n$ of the n-dimensional integer lattice, also referred to as support set. Its convex hull $Q_j := \operatorname{conv}(\mathcal{A}_j)$ is called the Newton polytope. Resultants depend on $n+1$ such supports, which in turn specify a system of polynomials with n unknowns $t_i \in \mathbb{C}^*$ in the algebraic torus

$$f_j(t_1, \ldots, t_n) = \sum_{m \in \mathcal{A}_i} c_{j,m} \boldsymbol{t}^m, \qquad j = 0, \ldots, n. \tag{1}$$

Let us assume the Minkowski sum Q of system's Newton polytopes is n-dimensional, so that its normal fan Σ determines the complete normal toric variety $\boldsymbol{X}$ as described in [12]. The polynomials (1) specify a Koszul complex of sheaves on this variety. It was shown in [20] that after twisting with multiples of any line bundle, its global sections form a complex whose determinant agrees with the resultant. Since the powers are not specified, we use instead a reflexive rank-1-bundle that corresponds to the displacement of Newton polytopes introduced by [6]. In addition, they used tight coherent mixed subdivisions of polytopes

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`ResultantComplexes` version 1.0

to obtain a regular minor of the multivariate Sylvester-matrix which is a multiple of the resultant; see also [26]. We apply this idea to all differentials in the complex. Its determinant turns out to agree with the redefined resultant of [15] which is valid for any family of supports and can be a multiple of an irreducible polynomial. Toric varieties are implemented in package `NormalToricVarieties` based on *Macaulay*2 [21].

1.1. *The complex associated with resultants.* The vertices of Newton polytopes Q_j determine Cartier divisors $D[Q_j]$ on the toric variety $\boldsymbol{X}$, their divisor classes are referred to as $\alpha_j \in \mathrm{Cl}(\boldsymbol{X})$. Accordingly, we assign degree $\alpha_\nu := \sum_{j\in\nu} \alpha_j$ to a sum of polytopes, and α denotes the degree of the Minkowski sum Q of all Newton polytopes. Since we consider the zero locus of polynomials, their coefficients are unique up to scaling and do not vanish simultaneously. So they are in the projectivization of a subspace $V_j \subset \Gamma(\boldsymbol{X}, \mathcal{O}_{\boldsymbol{X}}(\alpha_j))$ of the space of global sections of line bundle $\mathcal{O}_{\boldsymbol{X}}(\alpha_j)$ given by a Newton polytope. This subspace is spanned by monomials which have exponents in the support $\mathcal{A}_j$. All together they add up to an element $\boldsymbol{c} := (c_{j,a})$ in the Cartesian product $\boldsymbol{P} = \mathbb{P}(V_0) \times \cdots \times \mathbb{P}(V_n)$ of projective spaces which we refer to as variety of system's coefficients; its total coordinate ring is denoted by $\boldsymbol{A} = \mathbb{C}[c_{j,a}]$.

The rays $\rho_i \in \Sigma(1)$ in the normal fan of Minkowski sum Q are assigned facet variables x_i. Using the generators $u_i \in \mathbb{Z}^n$ of rays ρ_i and the weights a_i in the hyperplane representation of Q, homogeneous forms F_j are assigned to the polynomials (1) by substituting $\boldsymbol{t}^m$ with $\boldsymbol{x}^p$ with nonnegative exponents $p_i = u_i \cdot m + a_i$. They are elements of the ring $\boldsymbol{R} = \boldsymbol{A}[x_1, \dots, x_s]$, which in turn is graded by divisor classes $\mathrm{Cl}(\boldsymbol{X})$ of the toric variety as in [10]. The forms $F_j \in \boldsymbol{R}$ have degree α_j, they generate the homogeneous ideal $I = (F_0, \dots, F_n)$ in ring $\boldsymbol{R}$ and also determine a Koszul complex $K_\bullet(F_0, \dots, F_n)$ of graded $\boldsymbol{A}$-modules with differential ∂_i:

$$0 \to \bigwedge^{n+1} \boldsymbol{R}^{n+1} \to \cdots \to \bigwedge^{1} \boldsymbol{R}^{n+1} \xrightarrow{\partial_1} \boldsymbol{R} \to \boldsymbol{R}/I \to 0. \tag{2}$$

Resultant complexes $\boldsymbol{C}_\bullet$ are homogeneous components of this graded complex and thus contain finite $\boldsymbol{A}$-modules. Their degree is $\varrho = \alpha + \alpha_\delta$, where the additional shift $\alpha_\delta \in \mathrm{Cl}(\boldsymbol{X})$ is required to obtain exact complexes $\boldsymbol{C}_\bullet \otimes_{\boldsymbol{A}} K(\boldsymbol{A})$ when tensoring with the field $K(\boldsymbol{A})$ of fractions, while the ϱ-component $(\boldsymbol{R}/I)_\varrho$ of the quotient goes to zero. The differentials $d_i : \boldsymbol{C}_i \to \boldsymbol{C}_{i-1}$ are inherited from the Koszul complex, the first of which d_1 is also referred to as multivariate Sylvester matrix. Such complexes are determined in function `resultantComplexes`.

1.2. *Resultants on toric varieties.* All the Cartier divisors $D[Q_j]$ are base point free, since they arise from the Newton polytopes, whose Minkowski sum specifies the complete toric variety $\boldsymbol{X}$. Since each support set $\mathcal{A}_j$ contains the Cartier data of the divisor, the subspace $V_j \subset \Gamma(\boldsymbol{X}, \mathcal{O}_{\boldsymbol{X}}(\alpha_j))$ has no base points either, which implies a morphism $\varphi_j : \boldsymbol{X} \to \mathbb{P}(V_j^*)$ to the projectivization of V_j^*. Here, the individual Newton polytopes Q_j need not be n-dimensional, so their divisors are generally not ample, as described in Chapter 6 of [12]. Any point $\varphi_j(x)$ represents a codimension 1 subspace in $\mathbb{P}(V_j)$ of coefficients at which the polynomial vanishes: $F_j(\boldsymbol{c}, x) = 0$.

The incidence variety $\boldsymbol{W} \subset \boldsymbol{X} \times \boldsymbol{P}$ generated by the ideal I of system (1), is a bundle of irreducible fibers over the normal toric variety $\boldsymbol{X}$ which is likewise irreducible. Hence, the incidence variety is irreducible as well, which also applies to its projection $\pi_p : \boldsymbol{X} \times \boldsymbol{P} \to \boldsymbol{P}$ to the second factor as in [26]. This image $\boldsymbol{Z} = \pi_p(\boldsymbol{W})$ consists of all coefficients $\boldsymbol{c} = (c_{j,a})$ for which the polynomials (1) have a common root within the toric variety $\boldsymbol{X}$, it is already closed since we assumed the Minkowski sum Q to be n-dimensional [12]. If $\boldsymbol{Z}$ yields a hypersurface, the irreducible polynomial defining it, is called the eliminant $\mathrm{Elim}_{\mathcal{A}} \in \boldsymbol{A}$ of the family of support sets. In case of higher codimensions, the eliminant is set to value 1.

By definition, the map $\pi_p : \boldsymbol{W} \to \boldsymbol{Z}$ is dominating. In addition, whenever $\boldsymbol{Z}$ is a hypersphere, the dimensions of both varieties coincide, so that the function field $\mathbb{C}(\boldsymbol{W})$ is a finite field extension of $\mathbb{C}(\boldsymbol{Z})$. Its degree is referred to as the degree $\deg \pi_p = [\mathbb{C}(\boldsymbol{W}) : \mathbb{C}(\boldsymbol{Z})]$ of the mapping π_p. If the dimension of the resultant variety drops, this degree is set to zero. The resultant of sparse systems was redefined [15], it now includes a multiplicity

$$\mathrm{Res}_{\mathcal{A}} = (\mathrm{Elim}_{\mathcal{A}})^{\deg \pi_p}. \tag{3}$$

1.3. ***Mixed subdivisions of support sets.*** Further, we consider coherent mixed subdivisions Δ_ω of the family of support sets $\mathcal{A}_0, \ldots, \mathcal{A}_n$. Such subdivisions are described in [20; 26], they are generated by functions $\omega_j : \mathcal{A}_j \to \mathbb{Q}$, which extend the given support sets $\mathcal{A}_j$ by an additional coordinate $\hat{\mathcal{A}}_j := \{(m, \omega_j(m)) : m \in \mathcal{A}_j\}$. The projection of the lower facets of the convex hull of $\sum_{j=0}^{n} \hat{\mathcal{A}}_j$ back to the first n coordinates yields the cell domain $P_k := \sum_{j=0}^{n} P_{k|j}$, where $P_{k|j}$ is face of Q_j. Each cell of the subdivision is a family $\mathfrak{C}_k = \{C_{k|0}, \ldots, C_{k|n}\}$ of subsets $C_{k|j} \subset \mathcal{A}_j$, such that its convex hulls form the faces $P_{k|j} = \mathrm{conv}(C_{k|j})$. Furthermore, let σ_k be the set of indices of all vertices in a cell; it labels the zero-dimensional faces: $\dim P_{k|j} = 0$ for $j \in \sigma_k$. In the case of tight coherent mixed subdivisions (TCMD), the sum of the dimensions of the individual faces corresponds to that of the ambient space: $\sum_{j=0}^{n} \dim P_{k|j} = n$. Here, each cell has at least one vertex, so that σ_k is never empty. Function `mixedSubdivision` determines such subdivisions using a triangulation of the Cayley embedding of the support as described in [26].

1.4. ***Previous work.*** Inspired by ideas of Cayley, Gelfand et al. [20] associated resultants with the determinant of a complex. In their seminal approach, they assumed the line bundles $O_X(\alpha_j)$ to be very ample, so that each support set $\mathcal{A}_j$ in the system affinely generates the lattice $\mathbb{Z}^n$. With these conditions, the resultant (3) has multiplicity one. Appendix A of [20] describes determinants of exact complexes of vector spaces. To calculate them, certain columns with indices I_i are chosen from the matrices of differentials, such that all submatrices $d_i(\bar{I}_{i-1}, I_i)$ with rows in $\bar{I}_i := \{1, \ldots, r_i\} \setminus I_i$ are regular and square. This way, the determinants can be calculated as in (4) which is implemented in function `CayleyFormula`

$$\det \boldsymbol{C}_\bullet = \prod_{i=1}^{n} \det d_i(\bar{I}_{i-1}, I_i)^{(-1)^{i-1}}. \tag{4}$$

Canny and Emiris [7] introduced the algorithm to determine columns I_1 of the multivariate Sylvester matrix d_1 to obtain the first minor of this product. They assume the family $\boldsymbol{\mathcal{A}}$ of supports to be essential [26] and that their sum affinely generates the lattice $\mathbb{Z}^n$ as in [20]. Here, again, resultant and eliminant coincide. Using coherent mixed subdivisions of the support sets $\mathcal{A}_0 \cdots \mathcal{A}_n$ and a displacement of polytopes by a vector $\delta \in \mathbb{Q}^n$, they determine columns of the multivariate Sylvester matrix, to obtain a submatrix M which is regular, when tensored with the field of fractions $K(\boldsymbol{A})$. Its determinant is a multiple of the resultant: $\det(M) = p(\boldsymbol{c}_1, \dots, \boldsymbol{c}_n)\,\mathrm{Res}_{\mathcal{A}}$, where the extra factor does not depend on the coefficients $\boldsymbol{c}_0$ of the system's first polynomial [26]. By rearranging the polynomials, there are $n+1$ such minors whose greatest common divisor yields the resultant [6; 18].

Sturmels [26] determined the codimension of the resultant variety $\boldsymbol{Z}$ defined by the support sets and showed that it is a hypersphere, if and only if exactly one essential subfamily is included. In case the whole family is essential, it has been shown that the multihomogeneous degrees of the resultant agree with mixed volumes of Newton polytopes:

$$\deg_j(\mathrm{Res}_{\mathcal{A}}) = MV(Q_0, \dots, Q_{j-1}, Q_{j+1}, \dots, Q_n). \tag{5}$$

D'Andrea and Sombra [15] redefined the resultant which is now valid for any family of support sets. Its multihomogeneous degrees are consequently given by (5) without any further assumptions. Multiplicities occur, if there is a proper essential subcollection of support sets $\mathcal{A}_j$ or if the family $\mathcal{A}_0, \dots, \mathcal{A}_n$ does not affinely generate the integer grid $\mathbb{Z}^n$, see [26] for the definition of essentiality.

Cattani et al. [8] consider systems which have scaled Newton polytopes. Initially, they assume generic coefficients $\boldsymbol{c}$ where the system (1) has no zeros in the toric variety $\boldsymbol{X}$ with total coordinate ring $\boldsymbol{S}$. The homogeneous polynomials F_j now generate the ideal $\boldsymbol{I_c} \subset \boldsymbol{S}$. The degrees in Koszul complex (2) are shifted by the canonical class $[K] \in \mathrm{Cl}(\boldsymbol{X})$, so that the ϱ-component of residue class ring $\boldsymbol{S}/\boldsymbol{I_c}$ becomes a one-dimensional vector space. The higher differential to the first term of the complex multiply its elements with the toric Jacobian. Therefore, the first differential of the Weyman complex, which combine these differentials of various orders, contains a column of Bézoutian-like entries which are components of the toric Jacobian. The determinant includes multiplicities as well.

Cox and Dickenstein [11] investigate this approach for more general families $\boldsymbol{\mathcal{A}}$ of supports. For this they determine the dimension of the ϱ-component of residue class ring $\boldsymbol{S}/\boldsymbol{I_c}$ when generic coefficients are inserted. Here, resultant complexes are Weyman complexes which contain global sections as well as a higher cohomology of the sheaf $\mathcal{O}_{\boldsymbol{X}}(K)$ of the canonical divisor.

Furthermore, D'Andrea and Dickenstein [14] generalized the classical Macaulay formula for the resultants of systems of homogeneous polynomials [24]. For this purpose, they combined a complex of Sylvester-like matrices and its dual with a differential in the center, which also contains Bézoutian-like entries. Beyond that, D'Andrea [13] also determined Macaulay-style formulas for arbitrary sparse systems. On this basis, Groh [22] specified a composite tight subdivision generated by various lifting functions. In case of unmixed support sets, Emiris and Konaxis [19] determined a single coherent subdivision. D'Andrea et al. [16] introduced the new concept of admissible subdivisions. Bender et al. [3] investigate systems with determinantial representations for resultants, which are particularly useful.

The Canny–Emiris matrices present the ϱ-component of quotient $\boldsymbol{R}/I$ and their rank drop if we insert special coefficients $\boldsymbol{c} \in \boldsymbol{Z}$ on the resultant variety. Hence they shares some properties of elimination matrices which are described by [5] for projective spaces and generalized for toric varieties in [4]. To verify equivalence, the ϱ-components $I_\varrho = I_\varrho^{\mathrm{sat}}$ of ideals should coincide where $I = (F_0, \ldots, F_n)$ and I^{sat} refers to its saturation with the irrelevant ideal $\boldsymbol{B}(\Sigma)$ of the toric variety $\boldsymbol{X}$, extended with the coefficient ring.

1.5. ***Main result and open questions.*** In package `ResultantComplexes` we determine resultant complexes $\boldsymbol{C}_\bullet$ of systems (1) provided the Minkowski sum of all Newton polytopes is n-dimensional. Beyond this, there are no other restrictions on the family of support sets. Admissible sets of row and column indices are specified so that its determinant $\det \boldsymbol{C}_\bullet$ can be calculated by means of the Cayley formula (4). To prove that it agrees with the resultant, redefined in [15], we investigate certain properties of this complex: It becomes exact when tensored with the quotient field $K(\boldsymbol{A})$ of the coefficient ring. Moreover, having been localized at an irreducible element $\pi \in \boldsymbol{A}$ of this ring, all its higher homology groups vanish, except the first. We conclude that its determinant is equal to the greatest common divisor of all maximal minors of the multivariate Sylvester matrix, Theorem 34, Appendix A in [20].

Moreover, since the ring $\boldsymbol{A}$ of coefficients is a unique factorization domain, the multiplicities of each prime factor π of the determinant equals the length of homology group $H_0(\boldsymbol{C}_\bullet)_\mathfrak{p}$ localized at the principal ideal $\mathfrak{p} = (\pi)$, by Theorem 30 in [20]. Except for the homology localized at the eliminant, all others vanish. Thus, both determinant and resultant are multiples of the eliminant $\mathrm{Elim}_\mathcal{A}$. As their multihomogeneous degrees (5) in $\boldsymbol{A}$ are identical, they coincide:

$$\mathrm{Res}_\mathcal{A} = \det \boldsymbol{C}_\bullet. \tag{6}$$

If experimentally the canonical class of toric variety $\boldsymbol{X}$ is chosen as additional degree, then higher cohomologies of sheaves in the resultant complex have to be considered as well, as in [8; 11]. In addition, function `calcResultant` would be more convenient with the automatic construction of lifting functions and Canny–Emiris shift.

2. TORIC RESULTANT-COMPLEXES. Toric resultant complexes are determined by the support sets $\mathcal{A}_i$ of the polynomials (1) with Newton polytopes Q_i and a degree $\alpha_\delta \in \mathrm{Cl}(\boldsymbol{X})$ to shift the α-homogeneous components in Koszul complex (2). This additional degree is specified by the rational vector $\delta \in \mathbb{Q}^n$, introduced in [7] to displace the cell domains P_k obtained in the tight coherent mixed subdivision of the Newton polytopes, so that their boundaries do not contain a lattice point. The functions `basisComplex` and `CannyEmirisCoef` check these conditions.

To obtain the terms C_i of resultant complexes, we consider all combinations ν of i elements from 0 to n; it will be arranged in ascending order: $\nu_1 < \cdots < \nu_i$. Its complement $\bar{\nu} := \{0, \ldots, n\} \setminus \nu$ is a combination of $n + 1 - i$ elements. It is convenient to use the symbols for set operations also for combinations. Moreover, we denote the unit elements of module $\boldsymbol{R}^{n+1}$ by $\boldsymbol{e}_j$ and their wedge products generating $\bigwedge^i \boldsymbol{R}^{n+1}$ with $\boldsymbol{e}_\nu = \boldsymbol{e}_{\nu_1} \wedge \cdots \wedge \boldsymbol{e}_{\nu_i}$. The homogeneous components of Koszul complex (2) of degree $\varrho = \alpha + \alpha_\delta$ yield a resultant complex $\boldsymbol{C}_\bullet$. As with [11], its terms C_i are formed by direct sums of $\boldsymbol{A}$-modules arising from the global sections of sheaves $\mathcal{O}_{\boldsymbol{X}}(\alpha_{\bar{\nu}} + \alpha_\delta)$, but whose scalars have been extended here by the ring $\boldsymbol{A}$ of coefficients

$$\boldsymbol{C}_i = \bigoplus_{\nu_1 < \cdots < \nu_i} \boldsymbol{A} \otimes \Gamma(\boldsymbol{X}, \mathcal{O}_{\boldsymbol{X}}(\alpha_{\bar{\nu}} + \alpha_\delta))\boldsymbol{e}_\nu. \tag{7}$$

The additional degree α_δ is chosen so that the polytope determined by $\alpha_{\bar{\nu}} + \alpha_\delta$ contains the lattice points $m \in \mathbb{Z}^n \cap (Q_{\bar{\nu}} + \delta)$ in the polytope $Q_{\bar{\nu}}$ of $\alpha_{\bar{\nu}}$ shifted by the Canny Emiris vector $\delta \in \mathbb{Q}^n$. Further, the Minkowski sum $Q_{\bar{\nu}} = \sum_{j \in \bar{\nu}} Q_j$ of Newton polytopes is referred to as partial sum.

2.1. *Divisor equivalent to Canny–Emiris shift.* To obtain the still undetermined degree $\alpha_\delta \in \mathrm{Cl}(\boldsymbol{X})$, we consider the $\mathbb{Q}$-Weil divisor $D[\delta]$ with coordinates $D[\delta]_\rho = u_\rho \delta$ given by the Canny–Emiris vector $\delta \in \mathbb{Q}^n$. Furthermore, L_ρ denotes the saturated facet lattice supported by generator u_ρ of ray $\rho \in \Sigma(1)$ in the fan of the toric variety $\boldsymbol{X}$. Since ray generators are minimal, the rounded-down coordinate $\lfloor D[\delta]_\rho \rfloor$ counts how many affine sublattices parallel to L_ρ are crossed while moving straight from zero point to δ. Each partial sum $Q_{\bar{\nu}}$ specifies a Cartier divisor, and its sum with the Canny–Emiris divisor is a rounded-down $\mathbb{Q}$-Weil divisor as well:

$$D[Q_{\bar{\nu}}] + \lfloor D[\delta] \rfloor = \lfloor D[Q_{\bar{\nu}} + \delta] \rfloor. \tag{8}$$

If we multiply the shifted Minkowski sum $Q_{\bar{\nu}} + \delta$ by the least common multiple of all the denominators in the coordinates of the vector δ, we obtain an integral polytope. Hence, each $D[Q_{\bar{\nu}} + \delta]$ is a $\mathbb{Q}$-Cartier and nef divisor. So, all higher sheaf cohomologies arising from the twisted Koszul complex (2) are zero, according to Theorem 9.3.5 in [12] which generalizes the vanishing theorem of [2] to $\mathbb{Q}$-divisors.

Example 1. As running example we consider a polynomial system given in [26]:

$$\begin{aligned} \mathcal{A}_0 &= \{(0,0), (2,2), (1,3)\}, \\ \mathcal{A}_1 &= \{(0,0), (2,0), (1,2)\}, \\ \mathcal{A}_2 &= \{(2,0), (0,1)\}. \end{aligned} \tag{9}$$

The Minkowski sum Q of their Newton polytopes gives a normal fan with 8 rays, they are shown in Figure 1. Here the Minkowski sum is shifted vertically by a Canny Emiris vector $\delta = \left(0, \frac{2}{3}\right)$, its equivalent divisor is $D[\delta] = -2D_1 - D_2 - D_3 + D_6$.

2.2. *Subdivide the resultant complex.* Using tight mixed coherent subdivision of the family $\boldsymbol{\mathcal{A}}$ of supports, Canny and Emiris [7] determine a maximum minor of the multivariate Sylvester matrix. We transfer their approach to the other differentials $d_i : \boldsymbol{C}_i \to \boldsymbol{C}_{i-1}$ of the resultant complex and consider subdivisions of subfamilies $\boldsymbol{\mathcal{A}}_{\bar{\nu}} := (\mathcal{A}_{\bar{\nu}_1}, \ldots, \mathcal{A}_{\bar{\nu}_s})$ as well.

Definition 2. Families $\mathfrak{C}_{k,\bar{\nu}} = (C_{k,\bar{\nu}_1}, \ldots, C_{k,\bar{\nu}_s})$ of subsets $C_{k,j} \subset \mathcal{A}_j$ with indices j in the complementary combination $\bar{\nu}$ of $s = n + 1 - i$ are referred to as reduced cells.

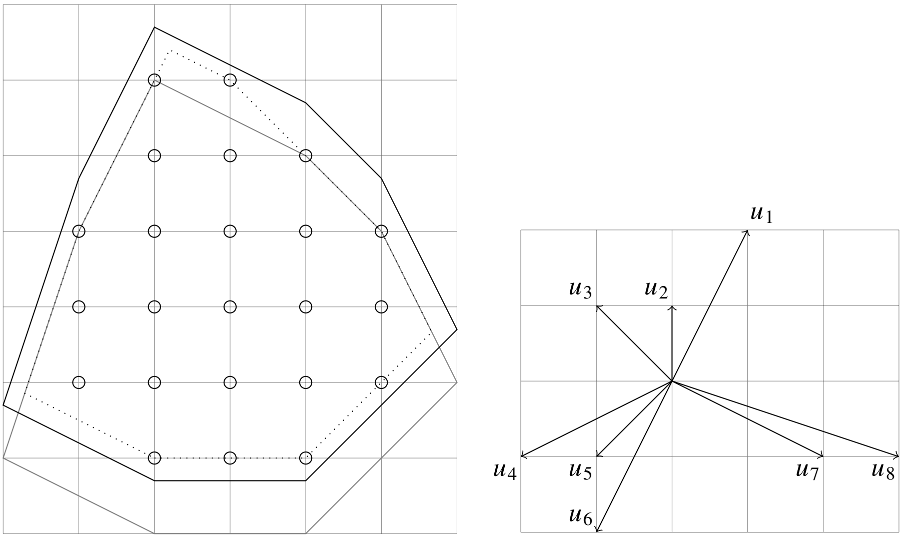

Figure 1. Minkowski sum Q of Newton polytopes outlined with gray lines, its normal fan yields the toric variety $\boldsymbol{X}$. The polytope in black lines shifted by the vector $\delta = \left(0, \frac{2}{3}\right)$ contains 27 grid points. Dashed lines show the polytope of the equivalent divisor $D[\delta] = -2D_1 - D_2 - D_3 + D_6$, its coordinates are the number of facet grids traversed.

We shall see that certain reduced cells form a TCMD of the subfamily $\mathcal{A}_{\bar{\nu}}$. It will be induced by the lifting functions $\omega_j : \mathcal{A}_j \to \mathbb{Q}$ with $j \in \bar{\nu}$. Here, we assume the partial sum $Q_{\bar{\nu}}$ to be n-dimensional. Otherwise, the sheaf $\mathcal{O}_{\boldsymbol{X}}(\alpha_{\bar{\nu}} + \alpha_\delta)$ contains no global sections and therefore would not contribute to the term $\boldsymbol{C}_i$ in accordance with equation (7).

Lemma 3. *All reduced cells $\mathfrak{C}_{k,\bar{\nu}}$ with an index list $\nu \subset \sigma_k$ contained in the vertex set σ_k of the complete cell $\mathfrak{C}_k$ form a subdivision of the family $\mathcal{A}_{\bar{\nu}}$, provided its polytope $Q_{\bar{\nu}}$ is n-dimensional.*

Proof. Given is a tight coherent mixed subdivision Δ_ω of the complete Minkowski sum Q. The facets on the lower hull of the lifted polytope $\hat{Q}_{\bar{\nu}}$ determine a subdivision of the partial sum $Q_{\bar{\nu}}$. Let us consider one of these facets with the inward normal vector $\eta \in \mathbb{Q}^{n+1}$. Assigning faces is additive, which implies the Minkowski sum $\operatorname{face}_\eta \hat{Q}_{\bar{\nu}} + \operatorname{face}_\eta \hat{Q}_\nu = \operatorname{face}_\eta \hat{Q}$ is a facet on the complete polytope's $\hat{Q}$ lower envelope. So, it defines a cell of Δ_ω, say C_k. Since we assume tight subdivisions here, $\operatorname{face}_\eta \hat{Q}_\nu$ must be zero-dimensional. Therefore, the index list ν contains only vertices $\nu \subset \sigma_k$, so that $\operatorname{face}_\eta \hat{Q}_{\bar{\nu}}$ yields a reduced cell $C_{k|\bar{\nu}}$. On the other hand, if $\nu \subset \sigma_k$ is valid, which also means $\dim(\operatorname{face}_\eta \hat{Q}_\nu) = 0$, then each facet of the polytope $\hat{Q}_{\bar{\nu}}$ appears again as a facet of the lifted partial sum $\hat{Q}_{\bar{\nu}}$. This, in turn, is a consequence of the operation: $\hat{Q} \mapsto \operatorname{face}_\eta \hat{Q}$, being linear. Hence, all the reduced cells subdivide the partial sum; and since they originate from C_k, they also form a TCMD. □

Subsequently, the cell domain $P_{k|\bar{\nu}_1} + \cdots + P_{k|\bar{\nu}_s}$ of $\mathfrak{C}_{k,\bar{\nu}}$ is denoted by $P_{k|\bar{\nu}}$. Further, according to the definition of degree α_δ via Canny–Emiris displacement, the characters χ^m of torus $(\mathbb{C}^*)^n$ with exponents $m \in \mathbb{Z}^n \cap (Q_{\bar{\nu}} + \delta)$ inside the shifted partial sum generate the $\mathbb{C}$-vector space of global sections of sheaf

k	$\vartheta_{k\|1}, \vartheta_{k\|2}, \vartheta_{k\|3}$	σ_k
1	[1,2,3], [2], [2]	[1,2]
2	[2,3], [2,3], [1]	2
3	[1,3], [1,2], [2]	2
4	[1,3], [2], [1,2]	1
5	[1], [1,2], [1,2]	0
6	[3], [1,2,3], [2]	[0,2]
7	[3], [2,3], [1,2]	0

k	ν	$\vartheta_{k\|\bar{\nu}_1}, \vartheta_{k\|\bar{\nu}_2}$
5	0	[1,2], [1,2]
6		[1,2,3], [2]
7		[2,3], [1,2]
1	1	[1,2,3], [2]
4		[1,3], [1,2]
1	2	[1,2,3], [2]
2		[2,3], [2,3]
3		[1,3], [1,2]
6		[3], [1,2,3]

k	ν	$\vartheta_{k\|\bar{\nu}_1}$
-	[0,1]	-
6	[0,2]	[1,2,3]
1	[1,2]	[1,2,3]

Table 1. The cells of the subdivision of all Minkowski sums $Q_{\bar{\nu}}$, given by the specified lifting vector.

$\mathcal{O}_X(\alpha_{\bar{\nu}} + \alpha_\delta)$ and hence the $\boldsymbol{A}$-modules in the direct sum (7). Lemma 3 specifies a division of the partial sums $Q_{\bar{\nu}}$ into the nonoverlapping domains $P_{k|\bar{\nu}}$ of reduced cells. Thus, the subdivision Δ_ω can be used to decompose all terms $\boldsymbol{C}_i$ into a direct sum of even finer submodules $\boldsymbol{C}^{\nu}_{i,k}$ which are generated by the i-forms $\chi^m \boldsymbol{e}_\nu$ with exponent vectors m in the shifted domain $P_{k|\bar{\nu}} + \delta$, where the combination $\nu \subset \sigma_k$ consists of i vertices of cell $\mathfrak{C}_k$. Hence, only cells that have at least i vertices contribute to the i-th term of the complex

$$\boldsymbol{A} \otimes \Gamma(\boldsymbol{X}, \mathcal{O}_{\boldsymbol{X}}(\alpha_{\bar{\nu}} + \alpha_\delta))\boldsymbol{e}_\nu = \bigoplus_{k,\nu\subset\sigma_k} \boldsymbol{C}^{\nu}_{i,k}, \quad \text{with } \#\nu = i. \tag{10}$$

The rank r_i of the free $\boldsymbol{A}$-modules $\boldsymbol{C}_i$ depend on the displacement vector δ but not on the specific subdivision. It is determined by the number of lattice points inside the shifted polytopes $Q_{\bar{\nu}} + \delta$ for all the combinations ν of i elements. Each of these points m together with ν represents the basis element $\chi^m \boldsymbol{e}_\nu \in \boldsymbol{C}_i$. Although the resultant complexes $\boldsymbol{C}_\bullet$ depend on the rational Canny–Emiris vector δ, the greatest common divisor of all maximal minors of the multivariate Sylvester matrix d_1 is not affected by this, it will turn out to be the resultant.

Example 4. We continue with the family (9). Figure 2 shows how its partial Minkowski sums are composed of the reduced cells given in Table 1, which were obtained according to Definition 2. This way equation (7) for sparse resultant complexes shall be clarified. The lifting functions given by vector $\omega = -(1, 0, 0; 7, 13, 0; 0, 0)$ generate the subdivision Δ_ω in Table 1. To specify the cell components $C_{k,\bar{\nu}_i}$, here, we number the exponent vectors $m_{i,k}$ in each support set (9), so they are given via index sets $\vartheta_{k,\bar{\nu}_i} \subset \mathbb{N}$ such that $C_{k,\bar{\nu}_i} = \{m_{i,k} \in \mathcal{A}_i : k \in \vartheta_{k,\bar{\nu}_i}\}$.

Bold numbers mark cells assigned to index sets $\bar{I}_k$ described in Section 3.1. Further, the points within the cells in Figure 2 represent the exponents vectors $m \in \mathbb{Z}^2 \cap Q_{\bar{\nu}}$ which form the base elements $\chi^m \boldsymbol{e}_\nu$ of the three terms $\boldsymbol{C}_0$, $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$. With the displacement vector $\delta = (0, 2/3)$, their dimensions are 23, 27 and 4. Function `resultantComplex` returns these exponents in list `mon`, and the flags in list `choice` are true if they belong to index set I_k.

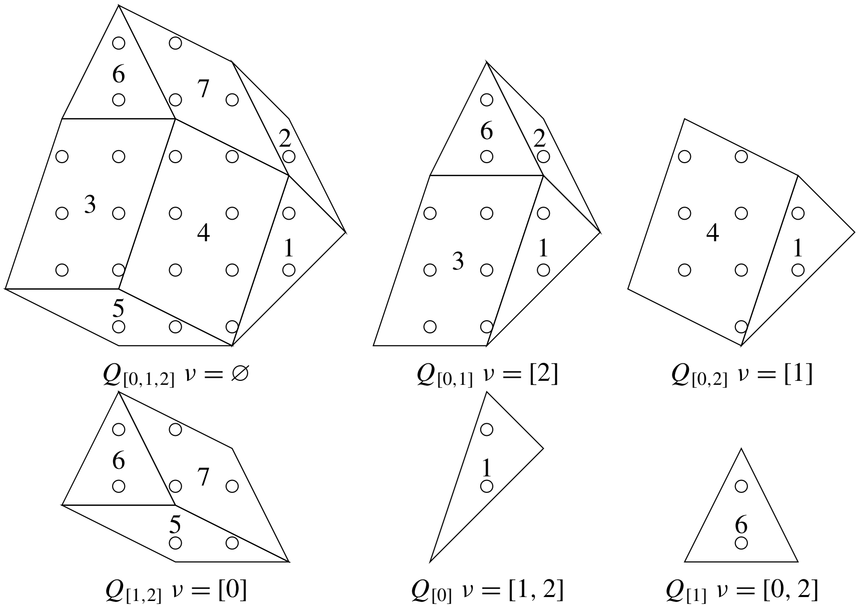

Figure 2. The Minkowski sums $Q_{\bar{\nu}}$ of the given Newton polytopes define the sparse resultant complex.

3. Determinant of the resultant complex. Complexes $\boldsymbol{C}_\bullet$ are called *generically exact* if their tensor product $\boldsymbol{C}_\bullet \otimes_A K(\boldsymbol{A})$ with the field of fractions of their base ring $\boldsymbol{A}$ is exact. The terms of the resultant complex are free modules $\boldsymbol{A}^{r_i}$ with finite rank r_i. To prove that its determinant $\det \boldsymbol{C}_\bullet$ agrees with the resultant, we consider the greatest common divisor (GCD) of all maximal minors of the multivariate Sylvester matrix d_1 at the beginning of the chain. If the assumptions of Theorem 5 are fulfilled, it coincides with the determinant of the complex; see Theorem 34 in Appendix A in the book of Gelfand et al. [20].

Theorem 5. *The determinant of generic exact complexes* $\boldsymbol{C}_\bullet = \boldsymbol{0} \to \boldsymbol{A}^{r_n} \cdots \boldsymbol{A}^{r_1} \xrightarrow{d_1} \boldsymbol{A}^{r_0} \to \boldsymbol{0}$ *of free modules over the ring* $\boldsymbol{A}$ *equals up to a factor in* $\boldsymbol{A}^*$ *the greatest common divisor of all maximal minors of the multivariate Sylvester matrix* d_1*, provided that the homologies* $H_i(\boldsymbol{C}_\bullet)$ *localized at* (π) *vanish for all indices* $i \geq 1$ *and irreducible elements* $\pi \in \boldsymbol{A}$*.* □

The following sections show that resultant complexes $\boldsymbol{C}_\bullet$ fulfill the assumption of Theorem 5. As a consequence, resultants can be calculated by means of the Cayley-formula (4). In the first Section 3.1 we describe how the columns I_i and rows $\bar{I}_{i-1}$ can be chosen to obtain the square submatrices $M_i := d_i(\bar{I}_{i-1}, I_i)$ thus determining the required regular minors $\mathfrak{D}_i := \det C_i$. Since the choice of I_1 and the properties of the first matrix M_1 have been already considered in [6; 26], in this section we investigate the remaining matrices of the chain. They turn out to have a block triangular form with a nonzero main diagonal. To confirm their regularity, the proof for the first matrix M_1 presented [6; 26] is adapted in Section 3.2. In the last Section 3.4, we combine the results previously obtained to prove that the complex $\boldsymbol{C}_\bullet$ is generically exact, and when localized at an irreducible element $\pi \in \boldsymbol{A}$, all homology groups $H_i(\boldsymbol{C}_\bullet \otimes_A \boldsymbol{A}_{(\pi)})$ except the first one vanish.

3.1. ***Admissible subsets and modules.*** To calculate the determinant of a resultant complex via the Cayley formula (4), regular submatrices $d_i(\bar{I}_{i-1}, I_i)$ of the differentials need to be specified. Their column indices are referred to as admissible set I_i, while their rows are given by the complement $\bar{I}_{i-1} \setminus \{1, \ldots, r_i\}$ of the preceding index set, where I_0 shall be the empty set. This corresponds to the Definition 12 in Appendix A of [20]. Both index sets are assigned base elements in each term $\boldsymbol{C}_i$, this way splitting it into a direct sum the admissible module $\boldsymbol{C}_{i|I}$ and its complement $\bar{\boldsymbol{C}}_{i|I}$, respectively. The function `resultantComplex` outputs these admissible sets as return value `choice`:

$$\boldsymbol{C}_i = \boldsymbol{C}_{i|I} \oplus \bar{\boldsymbol{C}}_{i|I}. \tag{11}$$

To specify them, we consider the sets $\sigma_k \subset \{0, \ldots, n\}$, which number the vertices, i.e., the zero-dimensional faces, of the cells $\mathfrak{C}_k$. Moreover, $\xi_k := \min(\sigma_k)$ denote their minimal elements.

Definition 6. The admissible module $\boldsymbol{C}_{i|I} \subset \boldsymbol{C}_i$ is the direct sum of submodules $\boldsymbol{C}^{\nu}_{i,k}$ of i-forms, defined in (10), where ν contains the minimal element of the vertex set: $\xi_k \in \nu$.

Lemma 7. *The admissible module $\boldsymbol{C}_{i|I}$ is isomorphic to the complement $\bar{\boldsymbol{C}}_{i-1|I}$ of its predecessor.*

Proof. Each term in the complex is direct sum of modules $\boldsymbol{C}^{\nu}_{i,k}$, each of which is assigned to a reduced cell $\mathfrak{C}_{k,\bar{\nu}}$ of the subdivision. Accordingly, we decompose the admissible module $\boldsymbol{C}_{i|I}$ into such cell modules $\boldsymbol{C}^{\nu}_{i,k}$ of i-forms $\#\nu = i$ with index $\nu \subset \sigma_k$ contained in the cell's set of vertices; by Definition 6, it contains the minimal element $\xi_k \in \nu$. Let $\chi^m \boldsymbol{e}_\nu \in \boldsymbol{C}^{\nu}_{i,k}$ be an element, its exponent vector $m \in P_{k|\bar{\nu}} + \delta$ is inside the shifted cell domain (10). We consider the function

$$\phi_{k,\nu} : \chi^m \boldsymbol{e}_\nu \mapsto \chi^{(m+P_{k|\xi_k})} \boldsymbol{e}_{\nu\setminus\xi_k}. \tag{12}$$

The exponent vector $m + P_{k|\xi_k}$ of the image lies inside the domain of reduced cell $\mathfrak{C}_{k,[\xi_k]\cup\bar{\nu}}$ because the two cell domains $P_{k|\bar{\nu}}$ and $P_{k|\bar{\nu}\cup[\xi_k]}$ differ by the vertex $P_{k|\xi_k}$

$$P_{k|\bar{\nu}\cup[\xi_k]} = (P_{k|\bar{\nu}_1} + \cdots + P_{k|\bar{\nu}_{n+1-i}}) + P_{k|\xi_k} = P_{k|\bar{\nu}} + P_{k|\xi_k}. \tag{13}$$

Thus, the assignment given in (12) define mappings $\phi_{k,\nu} : \boldsymbol{C}^{\nu}_{i,k} \to \boldsymbol{C}^{\nu\setminus\xi_k}_{i-1,k}$, the direct sum of their images forming the complementary submodule of the preceding term. Altogether, they compose a function $\phi_i : \boldsymbol{C}_{i|I} \to \bar{\boldsymbol{C}}_{i-1|I}$. To specify its inverse ϕ_i^{-1}, we denote with $\chi^{\bar{m}} \boldsymbol{e}_\mu$ the $(i-1)$-forms in the image. Each of them is element of a submodule $\boldsymbol{C}^{\mu}_{i-1,k}$ of some cell, while function (12) relates their index and exponents via: $\nu = [\xi_k, \mu]$ and $\bar{m} = m + P_{k|\xi_k}$, respectively. Accordingly, the inverse splits into mappings $\phi^{-1}_{k,\nu} : \boldsymbol{C}^{\mu}_{i-1,k} \to \boldsymbol{C}^{\nu}_{i,k}$, given by the following assignments (14). By Definition 6, index ξ_k is smaller than any element of the combination μ:

$$\phi^{-1}_{k,\nu} : \chi^{\bar{m}} \boldsymbol{e}_\mu \mapsto \chi^{(\bar{m}-P_{k|\xi_k})} \boldsymbol{e}_{\xi_k} \wedge \boldsymbol{e}_\mu. \tag{14}$$

Associating basis elements in this way, we obtain an isomorphism of modules $\boldsymbol{C}_{i|I}$ and $\bar{\boldsymbol{C}}_{i-1|I}$. Here, the first term of the complex is identical to its complementary submodule, corresponding to $I_0 = \varnothing$, while the last term agrees with the admissible submodule: $\boldsymbol{C}_0 = \bar{\boldsymbol{C}}_{0|I}$ and $\boldsymbol{C}_n = \boldsymbol{C}_{n|I}$. Moreover, this isomorphism implies an one-to-one relation between the admissible sets I_i and the complement of their predecessors $\bar{I}_{i-1}$. Therefore, both sets are of equal cardinality and the chosen submatrices $d_i(\bar{I}_{i-1}, I_i)$ are square indeed. □

The first admissible subset I_1 of columns and thus the matrix $d_1(\bar{I}_0, I_1)$ is equivalent to the Newton matrix given in [7], with the difference that they are considering the maximal elements of the vertex sets. The isomorphisms φ_i of admissible and complementary submodules produce a splitting of the terms of resultant complexes as in [9]. Further, for a system of two equations, the differential d_1 is equal to the Sylvester matrix.

Proposition 8. *The determinants $\mathfrak{D}_i = \det d_i(\bar{I}_{i-1}, I_i)$ do not depend on the coefficients of the last $i-1$ polynomials*: $f_{n+2-i}, \ldots, f_n$.

Proof. If we arrange columns and rows appropriately, $d_i(\bar{I}_{i-1}, I_i)$ will turns out to have a blockwise triangular form with square submatrices on the main diagonal. By Definition 6, this matrix represents a map $\boldsymbol{C}_{i|I} \to \bar{\boldsymbol{C}}_{i-1|I}$ of the admissible submodule to the complement of its predecessor. First, the image is decomposed into a direct sum of cell modules $\boldsymbol{C}^{\mu}_{i,k}$, then the submodules with the same index μ are combined to form

$$\bar{\boldsymbol{C}}^{\mu}_{I} := \oplus_k \boldsymbol{C}^{\mu}_{i-1,k}, \quad \text{with } \mu \subset \sigma_k \setminus \xi_k, \tag{15}$$

and lexicographically order these sums, according to the combinations μ, which index them. The bijections $\phi_{k,\nu}$ specified in (12) determine the inverse map $\phi_i^{-1} : \bar{\boldsymbol{C}}_{i-1|I} \to \boldsymbol{C}_{i|I}$ and therefore produce a decomposition of the admissible module $\boldsymbol{C}^{\mu}_{I}$ into the submodules

$$\boldsymbol{C}^{\mu}_{I} := \phi_i^{-1}(\bar{\boldsymbol{C}}^{\mu}_{I}) = \oplus_k \boldsymbol{C}^{\nu}_{i,k}, \quad \text{with } \boldsymbol{C}^{\nu}_{i,k} = \phi^{-1}_{k,\nu}(\boldsymbol{C}^{\mu}_{i-1,k}). \tag{16}$$

This way, the matrix $d_i(\bar{I}_{i-1}, I_i)$ can be divided into smaller blocks $d_{i|\tilde{\mu},\mu}$, according to submodules $\bar{\boldsymbol{C}}^{\mu}_{I}$ and $\boldsymbol{C}^{\tilde{\mu}}_{I}$. Since ϕ_i is isomorphic, the submatrices $d_{i|\mu,\mu}$ on the main diagonal are square. Let us consider the action of the differential d_i on an i-form $\chi^m \boldsymbol{e}_\nu \in \boldsymbol{C}^{\nu}_{i,k} \subset \boldsymbol{C}^{\mu}_{I}$ within the admissible submodule, with index $\nu = [\xi_k, \mu]$. In this case, the identity $\boldsymbol{e}_\nu = \boldsymbol{e}_{\xi_k} \wedge \boldsymbol{e}_\mu$ is valid. Moreover, each homogeneous polynomial (1) is assigned a sum of characters $f_j = \sum_{m \in \mathcal{A}_i} c_{j,m} \chi^m$:

$$d_i : \chi^m \boldsymbol{e}_\nu \mapsto \chi^m d_i(\boldsymbol{e}_{\xi_k} \wedge \boldsymbol{e}_\mu) = \chi^m f_{\xi_k} \boldsymbol{e}_\mu - \sum_{j \in \mu} \chi^m f_j (\boldsymbol{e}_{\xi_k} \wedge \boldsymbol{e}_{\mu \setminus j}). \tag{17}$$

Forms like $\chi^m f_{\xi_k} \boldsymbol{e}_\mu$ on right-hand side of equation (17) are contained in $\bar{\boldsymbol{C}}^{\mu}_{I}$, and thus determine the main diagonal block $d_{i|\mu,\mu}$. Since ξ_k is smaller than any element of μ, for all $j \in \mu$ the index $\mu'' := [\xi_k, \mu \setminus j]$ of $\boldsymbol{e}_{\xi_k} \wedge \boldsymbol{e}_{\mu \setminus j}$ is lower than μ in the lexicographic order: $\mu'' \prec_{\text{lex}} \mu$. Therefore, all terms of $\chi^m f_j \boldsymbol{e}_{\mu''}$ contribute only to blocks of the upper triangular matrix. Consequently, the blocks $d_{i|\mu',\mu}$ with $\mu \prec_{\text{lex}} \mu'$ vanish. So the matrix $d_i(\bar{I}_{i-1}, I_i)$ has the desired block triangular form, when the submodules $\bar{\boldsymbol{C}}_{\mu|I}$ and $\boldsymbol{C}_{\mu|I}$ are sorted lexicographically. To obtain the minor $\mathfrak{D}_i = \det d_i(\bar{I}_{i-1}, I_i)$, the determinants of all main diagonal blocks can be multiplied:

$$\mathfrak{D}_i = \prod_{\mu} \det d_{i|\mu,\mu}, \quad \text{with } \mu \subset \sigma_k \setminus \xi_k. \tag{18}$$

For the first differential $i = 1$ in the complex, μ is the empty set and there is only a single block in (18) that depends on all polynomial coefficients. Let us consider the case: $i > 1$. According to equation (17) the submatrices $d_{i|\mu,\mu}$ contain the coefficients of the polynomial f_{ξ_k}, where the minimal vertex ξ_k must be less than all components μ_j of the combination. Because it has $i-1$ elements the inequality $\xi_k < n+2-i$ holds. Consequently, the minor $\mathfrak{D}_i$ does not depend on the last $i-1$ polynomials. □

Corollary 9. *All minors $\mathfrak{D}_i = \det d_i(\bar{I}_{i-1}, I_i)$ are a separately homogeneous function with respect to each of the polynomials in the system.*

Proof. The columns of the square main diagonal blocks $d_{i|\mu,\mu}$ contain only coefficients of a specific equation. Thus the determinants $\det d_{i|\mu,\mu}$ are separately homogeneous functions and therefore also the minor $\mathfrak{D}_i$ which is their product. □

3.2. ***Canny Emiris matrices are regular.*** Apart from containing only a subset of the system's polynomials, the square submatrices $d_{i|\mu,\mu}$ on the principal diagonals of the restricted differentials $d_i : \boldsymbol{C}_{i|I} \to \bar{\boldsymbol{C}}_{i-1|I}$ agree with the Newton matrix $M = d_1(\bar{I}_0, I_1)$ introduced by [7]. They proved its determinant $\det M$ does not vanish identically. Their reasoning relies on the convexity of Newton polytopes and can be adapted to all other minors $\mathfrak{D}_i$ of differentials in the complex. Thus, we vary the coefficients of system (1) with an additional parameter: $\lambda \mapsto c_{j,a}\lambda^{\omega_j(a)}$. The functions $\omega_j : \mathcal{A}_j \to \mathbb{Q}$ therein have been used before to lift the support sets $\hat{\mathcal{A}}_j = \{(a, \omega_j(a)) : a \in \mathcal{A}_j\}$ and their convex hulls $\hat{Q}_j = \operatorname{conv}(\hat{\mathcal{A}}_j)$ to generate a TCMD of the family of supports. Canny and Emiris [7] used affine lift functions ω_j this way obtaining coarse mixed subdivisions of the supports, which they determine efficiently via linear programs. Sturmfels [26] considered generic lifting functions to generate any tight coherent mixed subdivision Δ_ω. We also assume this and further refer to the main diagonal blocks $M_\mu := d_{i|\mu,\mu}$ as Canny–Emiris matrices, Table 2.

Proposition 10. *Canny–Emiris matrices $M_\mu : \boldsymbol{C}_I^\mu \to \bar{\boldsymbol{C}}_I^\mu$ are regular for each combination μ.*

Proof. Canny–Emiris matrices M_μ map the admissible modules $\boldsymbol{C}_I^\mu$ to their isomorphic complements $\bar{\boldsymbol{C}}_I^\mu = \phi_i(\boldsymbol{C}_I^\mu)$ in the preceding term. Both are composed of finer modules $\boldsymbol{C}_{i,k}^\nu$ and $\boldsymbol{C}_{i,k}^\mu$, specified in equation (16) and (15), which have basis elements $\chi^p \boldsymbol{e}_\nu$ and $\chi^q \boldsymbol{e}_\mu$, respectively. We index them with exponent vectors p and q. If identity $\phi_{k,\nu}(\chi^p \boldsymbol{e}_\nu) = \chi^q \boldsymbol{e}_\mu$ holds with function (12), let p and q have the same index. In this cases, $p + P_{k|\xi_k} = q$ is valid, while $\nu = [\xi_k, \mu]$ applies by definition. Further, we assign to each column index p in the domain $P_{k|\bar{\nu}}$ of the reduced cell $\mathfrak{C}_{k,\bar{\nu}}$ the number of the minimum vertex of this cell: $p \mapsto j[p] := \xi_k$. For the first Canny–Emiris matrix of the complex, where $\mu = \varnothing$ holds, this function corresponds to the row content (RC) of [7], except that here columns are referred to. As with [26], we insert the modified coefficients $\hat{c}_{j,a} = c_{j,a}\lambda^{\omega_j(a)}$ into the system and obtain polynomials

$\hat{f}_j(\lambda)$ which now depend on a parameter

$$M_\mu : \chi^p \boldsymbol{e}_{j[p]} \wedge \boldsymbol{e}_\mu \mapsto \hat{f}_{j[p]}(\lambda)\chi^p \boldsymbol{e}_\mu = \sum_{a \in \mathcal{A}_{j[p]}} c_{j[p],a} \lambda^{\omega_{j[m]}(a)} \chi^{a+p} \boldsymbol{e}_\mu. \tag{19}$$

Here, the wedge product yields $\boldsymbol{e}_{j[p]} \wedge \boldsymbol{e}_\mu = \boldsymbol{e}_\nu$. For the matrix components $M_{\mu|q,p}$ with differences $a = q - p$ in the support set $\mathcal{A}_{j[p]}$ assigned to the column p, we get the following expression (20); whereas any other entry is zero:

$$M_{\mu|q,p}(\lambda) = c_{j[p],a} \lambda^{\omega_{j[p]}(a)}, \quad \text{with } a = q - p \in \mathcal{A}_{j[p]}. \tag{20}$$

These components agree with the Newton matrix in [7], except that here not all polynomials of the system are included. Again, the product of all main diagonal entries gives the coefficient of determinant $\det M_\mu(\lambda)$ with the smallest λ-exponent. To make this evident, level functions $h_{\bar{\nu}}$ are introduced which assign points within the shifted partial sums $Q_{\bar{\nu}} + \delta$ the height of the lower envelope of their extensions $\hat{Q}_{\bar{\nu}} + (\delta, 0)$

$$h_{\bar{\nu}} : Q_{\bar{\nu}} \to \mathbb{R}, \quad p \mapsto \min(h \in \mathbb{R} : (p, h) \in \hat{Q}_{\bar{\nu}}). \tag{21}$$

The Canny Emiris matrix has nonzero entries only if the two vectors q and p indexing its rows and columns differ by an element of the support set $\mathcal{A}_{j[p]}$. For such pairs of vectors, the convexity of Newton polytopes implies inequality (22) relating the level functions $h_{\bar{\nu}}$ and $h_{\bar{\mu}}$ with $\bar{\mu} = j[p] \cup \bar{\nu}$. Herein, $\omega_j : \mathcal{A}_{j[p]} \to \mathbb{Q}$ denote the lifting functions that generate the TCMD of the support sets

$$h_{\bar{\nu}}(p) + \omega_{j[p]}(a) \geq h_{\bar{\mu}}(q), \quad \text{with } a = q - p \in \mathcal{A}_{j[p]}. \tag{22}$$

Moreover, the geometric Lemma 4.5 in [7] states, that both sides of (22) are equal if and only if the difference $a = q - p$ is identical to the vertex $P_{k|j[p]}$. These are just the vectors indexing the main diagonal components of the Canny–Emiris matrix. To verify its determinant does not vanish, we scale its columns by $\lambda^{h_{\bar{\nu}}(p)}$ and the rows by $\lambda^{-h_{\bar{\mu}}(q)}$; this defines transformed matrix elements

$$\tilde{M}_{\mu|q,p} = \lambda^{-h_{\bar{\mu}}(q)} M_{\mu|q,p} \lambda^{h_{\bar{\nu}}(p)} = c_{j[p],a} \lambda^{\omega_{j[p]}(a) + h_{\bar{\nu}}(p) - h_{\bar{\mu}}(q)}. \tag{23}$$

For main diagonal components the inequality (22) becomes an equation: These entries are constant, while for all nondiagonal components $\tilde{M}_{\mu|q,p}$ the exponent of λ in (23) is always positive. Hence, the product of all main diagonal entries yields the term of the determinant which has the smallest degree. It does not vanish (20), so that $\det \tilde{M}_\mu$ is not the zero-polynomial. Since the two matrices $\tilde{M}_\mu$ and M_μ are equivalent, this statement also holds for the minor $\det M_\mu$. □

3.3. *Determinant of the complex.* A complex $\boldsymbol{C}_\bullet$ of modules over the base ring $\boldsymbol{A}$ is called *generically exact* if its tensor product with the base ring's field of fractions $K(\boldsymbol{A})$ is exact. This product $\boldsymbol{C}_\bullet \otimes_{\boldsymbol{A}} K(\boldsymbol{A})$ yields a complex of vector spaces over $K(\boldsymbol{A})$. We refer to it here as resultant complex $V_\bullet$ of vector spaces V_i with dimensions: $\dim V_i = r_i$ with differentials $d_{i|V}$ inherited from the complex of modules.

Lemma 11. *The toric resultant complex $V_\bullet = \boldsymbol{C}_\bullet \otimes_{\boldsymbol{A}} K(\boldsymbol{A})$ of vector spaces is exact.*

Proof. By Proposition 10 the dimension of each image space $\mathrm{Im}(d_{i+1|V})$ is bounded below by the cardinality of the index set I_{i+1}. Furthermore, because the image $\mathrm{Im}(d_{i+1|V})$ is included in the kernel of $d_{i|V}$ the inequality $\dim \mathrm{Im}(d_{i+1|V}) \leq \dim \mathrm{Ker}(d_{i|V})$ holds. The last part of (24) follows from the estimation $\#I_i \leq \dim \mathrm{Im}(d_{i|V})$, the definition of complementary sets $\bar{I}_i = \{1,\ldots,r_i\} \setminus I_i$ and the identity $\#\bar{I}_i = \#I_{i+1}$, which was stated in Lemma 7:

$$\#I_{i+1} \leq \dim \mathrm{Im}(d_{i+1|V}) \leq \dim \mathrm{Ker}(d_{i|V}) = \dim V_i - \dim \mathrm{Im}(d_{i|V}) \leq \dim V_i - \#I_i = \#I_{i+1}. \tag{24}$$

Since the image $\mathrm{Im}(d_{i+1|V})$ is a subset of the kernel $\mathrm{Ker}(d_{i|V})$ and their dimensions agree, both vector spaces must be identical: Thus, the complex $V_\bullet$ of vector spaces is exact. □

To determine the localized homologies $H_i(\boldsymbol{C}_\bullet)_{(\pi)}$ for each irreducible element $\pi \in \boldsymbol{A}$ as required in Theorem 5, we consider the local ring $\boldsymbol{A}_\mathfrak{p}$ at principal ideals $\mathfrak{p} = (\pi)$ and the complex $\boldsymbol{C}_{\bullet|\mathfrak{p}} := \boldsymbol{C}_\bullet \otimes_{\boldsymbol{A}} \boldsymbol{A}_\mathfrak{p}$ of modules over $\boldsymbol{A}_\mathfrak{p}$. Besides, the differentials d_i in $\boldsymbol{C}_\bullet$ induce differentials $d_{i|\mathfrak{p}} : \boldsymbol{A}_\mathfrak{p}^{d_i} \to \boldsymbol{A}_\mathfrak{p}^{d_{i-1}}$ in the localized complex and, moreover, $H_i(\boldsymbol{C}_\bullet)_{(\pi)} = H_i(\boldsymbol{C}_{\bullet|(\pi)})$ holds.

Lemma 12. *The higher homologies of the localized resultant complex vanish:* $H_i(\boldsymbol{C}_{\bullet|(\pi)}) = 0$ *for index* $i \geq 1$, *at any irreducible element* $\pi \in \boldsymbol{A}$ *of the coefficient ring.*

Proof. The statement means that the two $\boldsymbol{A}_\mathfrak{p}$-modules $\mathrm{Ker}(d_{i|\mathfrak{p}})$ and $\mathrm{Im}(d_{i+1|\mathfrak{p}}) \subset \boldsymbol{A}_\mathfrak{p}^{r_i}$ are equal for all indices $i \geq 1$ and principal ideals $\mathfrak{p} = (\pi)$ generated by irreducible elements $\pi \in \boldsymbol{A}$. This identity will be proved by Nakayama's lemma [1]. In doing so, we consider a complex $\boldsymbol{W}_\bullet$ of the quotient spaces $W_i := \boldsymbol{A}_\mathfrak{p}^{r_i}/\mathfrak{m}_\mathfrak{p}\boldsymbol{A}_\mathfrak{p}^{r_i}$. These are vector spaces over the residue field $\boldsymbol{A}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$ were $\mathfrak{m}_\mathfrak{p}$ is the maximal ideal. Further, $d_{i|W} : W_i \to W_{i-1}$ denote the linear mappings induced by the differentials $d_{i|\mathfrak{p}}$ in the localized complex $\boldsymbol{C}_{\bullet|\mathfrak{p}}$.

Because the units in ring $\boldsymbol{A}$ are the constants, the irreducible polynomial π must depend on some coefficient, say $c_{j,m}$. The equations in the polynomial system can be permuted arbitrary. Thus, Proposition 8 implies that for each index $i \geq 1$ the original matrices d_{i+1} have a minor of order $\#I_{i+1}$, which does not depend on coefficient $c_{j,m}$. This minor $\mathfrak{D}_{i+1}$ cannot be divisible by π in the coefficient field $\boldsymbol{A}$, so that it is not zero in residue field $\boldsymbol{A}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$. Consequently, the inequality $\#I_{i+1} \leq \mathrm{rank}(d_{i+1|W})$ holds for any irreducible element $\pi \in \boldsymbol{A}$, hence (24) is valid and the subspaces $\mathrm{Im}(d_{i+1|W})$ and $\mathrm{Ker}(d_{i|W})$ coincide. Besides, they are also isomorphic to the quotient $\mathrm{Ker}(d_{i|\mathfrak{p}})/\mathfrak{m}_\mathfrak{p}\,\mathrm{Ker}(d_{i|\mathfrak{p}})$. To verify this statement, let us consider

$$\mathrm{Ker}(d_{i|W}) = (\mathrm{Ker}(d_{i|\mathfrak{p}}) + \mathfrak{m}_\mathfrak{p}\boldsymbol{A}_\mathfrak{p}^{r_i})/\mathfrak{m}_\mathfrak{p}\boldsymbol{A}_\mathfrak{p}^{r_i} \cong \mathrm{Ker}(d_{i|\mathfrak{p}})/(\mathrm{Ker}(d_{i|\mathfrak{p}}) \cap \mathfrak{m}_\mathfrak{p}\boldsymbol{A}_\mathfrak{p}^{r_i}). \tag{25}$$

Assume w lies in the intersection $\mathrm{Ker}(d_{i|\mathfrak{p}}) \cap \mathfrak{m}_\mathfrak{p}\boldsymbol{A}_\mathfrak{p}^{r_i}$. This means that $w = \pi v$ is valid, for some element $v \in \boldsymbol{A}_\mathfrak{p}^{r_i}$. Because $\boldsymbol{A}_\mathfrak{p}$ is a principal ideal domain, submodules of $\boldsymbol{A}_\mathfrak{p}^{r_i}$ are free and contain no torsion. Therefore $\pi d_{i|\mathfrak{p}}(v) = 0$ implies $v \in \mathrm{Ker}(d_{i|\mathfrak{p}})$, such that $w \in \mathfrak{m}_\mathfrak{p}\,\mathrm{Ker}(d_{i|\mathfrak{p}})$ holds. As the reverse inclusion is immediately valid, the identity $\mathrm{Ker}(d_{i|\mathfrak{p}}) \cap \mathfrak{m}_\mathfrak{p}\boldsymbol{A}_\mathfrak{p}^{r_i} = \mathfrak{m}_\mathfrak{p}\,\mathrm{Ker}(d_{i|\mathfrak{p}})$ results. Nakayama's lemma applied to quotient (25) of modules implies that the columns of matrix $d_{i+1|\mathfrak{p}}$ generate as well the kernel $\mathrm{Ker}(d_{i|\mathfrak{p}})$ as the image $\mathrm{Im}(d_{i+1|\mathfrak{p}})$. Therefore, both submodules are identical. □

Proposition 13. *The determinant* $\det \boldsymbol{C}_\bullet$ *of complex* $\boldsymbol{C}_\bullet$ *agrees with the greatest common divisor of all maximal minors of the multivariate Sylvester matrix* $d_1 : \boldsymbol{C}_1 \to \boldsymbol{C}_0$.

Proof. According to Lemma 11 the resultant complex $\boldsymbol{C}_\bullet$ is generically exact. Its localized higher homologies $H_i(\boldsymbol{C}_\bullet \otimes_A \boldsymbol{A}_{(\pi)})$ vanish for all indices $i \geq 1$ and irreducible elements $\pi \in \boldsymbol{A}$ of the coefficient ring, by Lemma 12. Hence, $\boldsymbol{C}_\bullet$ fulfills the assumptions of Theorem 5, so that its determinant is equal to the greatest common divisor (GCD) of all maximal minors of the multivariate Sylvester matrix d_1 at the beginning of the chain. □

The admissible submodules given in Definition 6 and their complements determine regular minors of the differentials: $\mathfrak{D}_i = \det d_i(\bar{I}_{i-1}, I_i)$. Therefore, the determinant of complex $\boldsymbol{C}_\bullet$ can be calculated by means of the Cayley formula (4). This way, its degrees of homogeneity with respect to the system's polynomials can be specified. They agree with those of the resultant (5).

Proposition 14. *The determinant of a resultant complex is multihomogeneous in the ring* $\boldsymbol{A}$ *of coefficients, its degrees are given by the mixed volume* (MV) *of the system's Newton polytopes*

$$\boldsymbol{deg}_j(\det \boldsymbol{C}_\bullet) = MV(Q_0, \dots, Q_{j-1}, Q_{j+1}, \dots, Q_n). \tag{26}$$

Proof. According to Corollary 9, all minors in the Cayley formula (4) are multihomogeneous and so is the determinant. Moreover, because of Proposition 8, only the first factor $\mathfrak{D}_1$ depends on the last polynomial in the system, which implies $\deg_n(\det \boldsymbol{C}_\bullet) = \deg_n \mathfrak{D}_1$. This homogeneity degree agrees with the number of columns assigned to the last polynomial in the Canny–Emiris matrix. As a consequence of Definition 6, specifying this matrix, it is obtained by summing up the volumes of all cells C_k which have exactly two vertices, namely $P_{k|j}$ with an index $j < n$ and a vertex $P_{k|n+1}$ of the additional rational polytope. This sum yields the mixed volume (MV) of the first n polytopes Q_j, in agreement with Theorem 7.4 of [7] and Theorem 2.4 of [23]. Neither theorem requires the essentiality of the family of support sets. Since the polynomials $f_0, \dots, f_n$ can be arranged in any order without changing the determinant of the complex, the proposition is valid for all its degrees of homogeneity. □

3.4. *Resultant and determinant of the complex agree.* As before, we consider a principal ideal $\mathfrak{p} = (\pi)$ generated by an irreducible element $\pi \in \boldsymbol{A}$ of the coefficient ring, it is the uniformizing parameter of discrete valuation ring $\boldsymbol{A}_\mathfrak{p}$. Theorem 30 in [20] specifies the multiplicity $\mathrm{ord}_\pi(\det \boldsymbol{C}_\bullet)$ of this element π in the prime factorization of $\det \boldsymbol{C}_\bullet \in \boldsymbol{A}$ as alternating sum of the lengths of all localized homologies $H_i(\boldsymbol{C}_\bullet)_\mathfrak{p}$. According to Lemma 12 all higher homologies with $i \geq 1$ of the resultant complexes vanish, so that only the initial homology $H_0(\boldsymbol{C}_\bullet)_\mathfrak{p}$ have to considered. So, their length, if finite, yields the order of the irreducible element π in the prime factorization

$$\mathrm{ord}_\pi(\det \boldsymbol{C}_\bullet) = \mathrm{length}_{\boldsymbol{A}_\mathfrak{p}}(H_0(\boldsymbol{C}_\bullet)_\mathfrak{p}). \tag{27}$$

The length of the localized homology on the right-hand side is finite when annihilated by some power of the maximal ideal $\mathfrak{m}_\mathfrak{p}$ of local ring $\boldsymbol{A}_\mathfrak{p}$, Lemma 10.52.8 in [25]. To verify this assumption, we consider the homogeneous ideal $I = (F_0, \dots, F_n) \subset \boldsymbol{R}$ generated by the polynomials (1) in the total coordinate ring. It inherits the grading by equivalence classes $\mathrm{Cl}(\boldsymbol{X})$ of von Weil divisors on the toric variety and its ϱ-component agrees with the image of the multivariate Sylvester map: $I_\varrho = \mathrm{Im}(d_1)$, according to the definition of resultant complexes. In addition, the cokernel yields the initial homology $H_0(\boldsymbol{C}_\bullet) = \boldsymbol{R}_\varrho / I_\varrho$, which also coincides with the homogeneous component $(\boldsymbol{R}/I)_\varrho$ of the quotient ring in the Koszul complex (2).

Theorem 15. *The determinant of complex $\boldsymbol{C}_\bullet$ agrees with the toric system's resultant* $\mathrm{Res}_{\mathcal{A}}$.

Proof. The eliminant $\mathrm{Elim}_{\mathcal{A}} \in \boldsymbol{A}$ vanishes on the incidence variety $\boldsymbol{W} = \boldsymbol{V}(I)$ generated by the homogeneous ideal $I = (F_0, \dots, F_n)$ of system (1). It also vanishes on the affine variety $\boldsymbol{V}_a(I)$ in the affine space $\mathbb{C}^{\#\mathcal{A}_0+\cdots+\#\mathcal{A}_n+s}$ of ring $\boldsymbol{R}$. Therefore, according to Hilbert's Nullstellensatz some multiple of the eliminant lies in this ideal: $\mathrm{Elim}^k_{\mathcal{A}} \in I$. First, let $\pi \neq \mathrm{Elim}_{\mathcal{A}}$ be any element of the coefficient ring $\boldsymbol{A}$ that do not agree with the eliminant. Here, the power $\mathrm{Elim}^k_{\mathcal{A}}$ is a unit in the local ring $\boldsymbol{A}_{(\pi)}$ annihilating the $\boldsymbol{A}$-module $\boldsymbol{R}/I$ such that its localization at prime ideal $\mathfrak{p} = (\pi)$ goes to zero: $(\boldsymbol{R}/I)_\mathfrak{p} = 0$; see [17]. Since the degree here is associated with the facet variables x_i of $\boldsymbol{X}$, the order of localization and forming homogeneous components can be reversed in equation (28). Hence, the order (27) of all irreducible elements $\pi \in \boldsymbol{A}$ that do not agree with the eliminant is zero:

$$H_0(\boldsymbol{C}_\bullet)_\mathfrak{p} = ((\boldsymbol{R}/I)_\varrho \otimes_A \boldsymbol{A}_\mathfrak{p}) = ((\boldsymbol{R}/I)_\mathfrak{p})_\varrho. \tag{28}$$

On the other hand, if $\pi = \mathrm{Elim}_{\mathcal{A}}$ is the eliminant, then a multiple of the maximal ideal $\mathfrak{m}^k_\mathfrak{p} \subset \boldsymbol{A}_\mathfrak{p}$ annihilates the localized quotient $(\boldsymbol{R}/I)_\mathfrak{p}$ which therefore has a finite length. Thus, in accordance with equation (27), the determinant of the resultant complex is a multiple of the eliminant, as is the resultant by definition (3). In fact, they agree since their degrees of homogeneity coincide by Proposition 14 and equation (5), respectively: $\det \boldsymbol{C}_\bullet = \mathrm{Res}_{\mathcal{A}}$. Moreover, the length of the localized homology (28) is the degree of the extension of function fields $\mathbb{C}(\boldsymbol{W})$ over $\mathbb{C}(\boldsymbol{Z})$. □

In conclusion, let us consider the extreme case, when some support sets are points. First we assume that exactly one support is a point, say $\mathcal{A}_j = \{m\}$. Here the only cells C_k that contain exactly one vertex are those with $\sigma_k = \mathcal{A}_j$. According to equation (26) and Theorem 2.4 in [23], the sum of their volumes yields the degree of homogeneity g_j with which the coefficient $c_{j,m}$ of the single monomial is included in the resultant. Equation (26) further implies that the homogeneity-degrees of all other polynomials vanish, so that the resultant must have the form: $\mathrm{Res}_{\mathcal{A}} = c^{g_j}_{j,m}$. If the family of support sets contains several points, there is no cell with only one vertex. By equation (26), here, the resultant is constant.

4. EXAMPLE. In the first Section 4.1 we describe a sparse system which has a proper essential subfamily, such that its resultant includes a multiplicity. The terms of the associated complex $\boldsymbol{C}_\bullet$ have high ranks. Although its differentials thus yield extensive matrices, the determinant can be evaluated via Cayley's formula (4) using Maple. This outcome can be verified, because the resultant of the essential subfamily is the same as in the Example 4. After a transformation in the lattice of the exponent vectors, the multiplicity can also be calculated. The second Section 4.2 illustrates the blockwise triangular form of the matrices $d_i(\bar{I}_{i-1}, I_i)$ with $i > 1$, as described in the proof of Proposition 8. Their determinants $\mathfrak{D}_i$ are multihomogeneous elements of the coefficient ring $\boldsymbol{A}$, independent of the last $i-1$ polynomials in the sparse system.

4.1. *Determinant with multiplicity.* Here we consider the 5-dimensional problem (29) where the first 3 support sets are the only essential subfamily. Thus, we can determine its resultant in two different ways: On the one hand directly, as determinant of the resultant complex formed by all the 6 support sets $\mathcal{A}_i$ and on the other hand as multiple of the resultant of the essential subfamily. To calculate this multiplicity we use Proposition 3.13 in [15]:

$$\begin{aligned}
\mathcal{A}_0 &= \{(0,0,0,3,0), (8,4,0,1,4), (10,6,2,0,4)\},\\
\mathcal{A}_1 &= \{(0,0,2,2,0), (2,0,0,2,2), (7,4,3,0,3)\},\\
\mathcal{A}_2 &= \{(0,0,0,1,1), (1,2,3,0,0)\},\\
\mathcal{A}_3 &= \{(3,0,0,0,0), (0,1,3,2,1), (4,2,3,0,1), (0,0,1,2,1)\},\\
\mathcal{A}_4 &= \{(4,3,3,0,1), (0,0,0,3,0), (5,3,2,0,2)\},\\
\mathcal{A}_5 &= \{(3,0,2,0,0), (0,0,3,2,1), (2,0,2,1,0), (6,1,0,0,2)\}.
\end{aligned} \tag{29}$$

The lifting vector (30) subdivides this family $\boldsymbol{\mathcal{A}}$ into 210 cells. Choosing $\delta = \left(-\frac{1}{8}\,\frac{1}{3}\,\frac{1}{5}\,\frac{1}{7}\,\frac{2}{9}\right)$ as Canny Emiris shift vector, yields a four-term complex of free $\boldsymbol{A}$-modules with ranks 1384, 2575, 1378, 187, 0, and 0. Despite this high ranks, the determinant of this resultant complex can be calculated symbolically using the Cayley formula

$$\omega = (4,8,8;\,7,1,7;\,1,7;\,0,6,8,7;\,3,5,6;\,1,4,4,0). \tag{30}$$

The first three polynomials form the essential subfamily of the system. Multiplying all exponents vectors in (29) on the left by the inverse of $\boldsymbol{T}$ defined in (31), we obtain integer support sets $\mathcal{A}'_i$, where the first three reproduce the supports of Example 4, padded with zeros on the right:

$$\boldsymbol{T} = \begin{pmatrix} 1 & 0 & -1 & 0 & 1\\ 3 & 2 & 1 & -1 & 1\\ 0 & 1 & 2 & 0 & 0\\ -1 & 0 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}. \tag{31}$$

Thus, the last two components of elements in the support set $\mathcal{A}'_i$ determine its image $[\mathcal{A}'_i]$ in the quotient of $\mathbb{Z}^5$ with the sublattice affinely generated by the essential family $\boldsymbol{\mathcal{A}}' := (\mathcal{A}'_0, \mathcal{A}'_1, \mathcal{A}'_2)$. The mixed volume of the other supports' Newton polytopes yields: $MV([Q'_3],[Q'_4],[Q'_5]) = 10$. Furthermore, the determinant of matrix (31) is $\det \boldsymbol{T} = 2$, it scales the transformed lattice. So, according to Property 3.13. in [15], we obtain the multiplicity 20 and consequently

$$\mathrm{Res}_{\mathcal{A}} = \mathrm{Res}_{\mathcal{A}'}^{20}. \tag{32}$$

c_{12}	0	c_{11}	0	0	$-c_{33}$	0	0	0	0	0	$-c_{43}$	0	$-c_{43}$	0
0	c_{12}	0	0	0	0	0	0	0	0	$-c_{43}$	0	0	0	0
0	0	c_{13}	0	0	0	$-c_{33}$	0	0	0	0	0	$-c_{42}$	0	0
			c_{21}	0	0	0	0	0	0	0	0	0	0	0
			0	c_{21}	0	0	0	0	0	0	0	0	0	0
			0	0	c_{21}	0	0	0	c_{11}	0	0	0	0	0
			0	0	0	c_{22}	0	c_{13}	0	0	0	0	0	0
			0	0	0	0	c_{22}	c_{14}	0	0	0	0	0	0
			0	0	0	0	0	c_{12}	0	0	0	0	0	$-c_{43}$
			0	0	c_{22}	0	0	c_{11}	c_{12}	0	0	0	0	$-c_{41}$
										c_{21}	0	0	0	0
										0	c_{21}	0	0	0
										0	0	c_{22}	0	0
										0	0	c_{21}	c_{22}	c_{33}
										0	c_{22}	0	c_{21}	c_{32}

Table 2. Matrix $d_2(\bar{I}_1, I_2)$ in blockwise triangular form, its subblocks correspond to $\mu =$ [2], [3], [4]. The square Canny–Emiris matrices M_μ on the main diagonal, are further divided: The first three columns of $M_{[2]}$ refer to $\nu = [\xi_\alpha, \mu] = [1, 2]$, so they contain only coefficients $c_{1,k}$ of the first polynomial. Columns 4 to 8 of $M_{[3]}$ refer to $\mu = [2, 3]$, etc.

The resultant of the essential subfamily $\mathcal{A}'$ is given in [26]. Substituting it into the right-hand side of equation (32), we obtained the determinant det $\boldsymbol{C}_\bullet$ of the resultant complex of all 6 support sets, calculated via the Cayley formula (4). The admissible index sets I_k necessary here were determined according to Definition 6 from the TCMD generated by vector (30).

4.2. ***Blockwise triangular submatrices.*** We consider a system with the support sets given in (33). They generate a resultant complex of lower rank, so that the second submatrix $d_2(\bar{I}_1, I_2)$ in the chain, which is blockwise triangular, can be given as Table 2. Again, it is possible to calculate the determinant symbolically:

$$\begin{aligned} \mathcal{A}_0 &= \{(0, 1, 1), (1, 0, 1), (1, 1, 1), (3, 0, 0)\}, \\ \mathcal{A}_1 &= \{(0, 2, 1), (1, 1, 1)\}, \\ \mathcal{A}_2 &= \{(0, 0, 2), (0, 2, 1), (1, 1, 1)\}, \\ \mathcal{A}_3 &= \{(0, 1, 1), (0, 2, 1), (1, 0, 1), (2, 0, 0)\}. \end{aligned} \tag{33}$$

The polynomials defined by these support sets are given by (1). The lifting vector given in (34) subdivides the Minkowski sum Q into 26 cells. With point $Q_4 = \left(\frac{1}{2}\ \frac{2}{9}\ \frac{1}{9}\right)$ as additional polytope, we obtain a four term complex of the dimensions 24, 39, 17 and 2:

$$\omega = (24, 0, 14, 30; 28, 22; 21, 19, 28; 29, 21, 26, 7). \tag{34}$$

In Proposition 8 the basis elements of each term $\boldsymbol{C}_k$ have been arranged so that the submatrices $d_k(\bar{I}_{k-1}, I_k)$ except the first become blockwise triangular as in Table 2, which shows this ordering for the second matrix. It is subdivided into smaller blocks $d_{2|\mu',\mu}$.

The columns of the square Canny–Emiris matrices $M_\mu = d_{2|\mu,\mu}$ on the main diagonal fall into groups according to the indexes $\nu = [\xi_\alpha, \mu]$ which define the elements $x^p \boldsymbol{e}_\nu$ of the column-space. In such a rectangular subblock of M_μ appear only coefficients c_{ik} with $i = \xi_\alpha$ of the polynomial f_i. In this example, the minor can be evaluated symbolically

$$\mathfrak{D}_2 = c_{12}^3 c_{13} c_{21}^4 c_{22}^3 (c_{22}c_{32} - c_{21}c_{33})(c_{12}c_{21} - c_{11}c_{22}). \tag{35}$$

The third minor equals $\mathfrak{D}_3 = c_{12}c_{21}$, whereby M_3 is a two-dimensional upper triangular matrix. According to the Cayley formula (4), the fraction $\mathfrak{D}_2 {\mathfrak{D}_3}^{-1}$ cancels the extraneous factor in $\mathfrak{D}_1$.

SUPPLEMENT. The online supplement contains version 1.0 of `ResultantComplexes`.

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