Journal of Software for Algebra and Geometry

The QuiverTools package for SageMath and Julia

Pieter Belmans, Hans Franzen and Gianni Petrella

vol 16 2026

JSAG **16** (2026), 31–44
https://doi.org/10.2140/jsag.2026.16.31

The Journal of Software for
Algebra and Geometry

The `QuiverTools` package for SageMath and Julia

PIETER BELMANS, HANS FRANZEN AND GIANNI PETRELLA

ABSTRACT: We introduce `QuiverTools`, a new software package, available in both a SageMath and Julia version, to study quivers and their moduli spaces of representations. Its key features are the computation of general subdimension vectors, leading to canonical decompositions, and checking the existence of (semi)stable representations, as well as the enumeration of Harder–Narasimhan types and related calculations for Teleman quantization. Computations related to intersection theory on quiver moduli are also implemented.

1. INTRODUCTION. The representation theory of quivers and the geometry of moduli spaces of quiver representations are rich subjects, surveyed in [18; 32; 34]. They admit very explicit calculations: their origin lies close to linear algebra, making an algorithmic approach viable.

Many aspects can be understood from the properties of the infinite root system defined by the *Kac form* in (2), the symmetrization of the Euler bilinear form of the quiver, as evidenced by the program initiated by Kac [20; 21; 22; 23]. For example, the dimension vectors of indecomposable representations of a quiver correspond to the positive roots of the associated root system. Many other properties of quiver representations and their moduli spaces are likewise encoded in properties of the associated root system and Kac–Moody Lie algebra, as surveyed in [34].

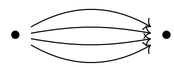
For other results describing properties of quiver representations and their moduli spaces, the essential ingredient is the *Euler form*, defined in (1), itself; see, e.g., its many appearances in [18; 32]. It will be the main tool for the type of results treated in this article, whose goal is to describe some of the features of `QuiverTools`, an implementation of many algorithms pertaining to quivers and their moduli spaces of representations. The software is available both as a library for SageMath [33] and as a Julia package [5]; see [4].

Related software. `QuiverTools` does not treat *individual* quiver representations, for which explicit algorithms exist too. This is functionality provided by `QPA` [30]. Likewise, `QuiverTools` does not deal with enumerative invariants of quivers (with potential). This functionality is provided by `CoulombHiggs` [29] and `msinvar` [27].

MSC2020: 14D20, 16G20.
Keywords: quiver moduli.
`QuiverTools` version 1.0

Structure of the article. In Sections 2 and 4 we recall some basic definitions and results about quivers, their representations, and their moduli spaces. In Sections 3, 5, 6 and 7 we subsequently discuss how important representation-theoretic or algebro-geometric questions can be rephrased in terms of the Euler form of the quiver, and thus allow for an explicit algorithmic approach, implemented in `QuiverTools`.

We illustrate some of its features using the running example of the 4-Kronecker quiver



and dimension vector $(2, 3)$. This setting already exhibits many interesting properties of quiver moduli, whilst it allows for succinct output. In fact, there are still many open questions for this particular example; e.g., the results in [26], or the description in [1], are established for the 3-Kronecker quiver, and are still open for the 4-Kronecker quiver.

Each SageMath example requires `QuiverTools` to be imported. You have to install it once, by running

```
sage --pip install git+https://github.com/QuiverTools/QuiverTools.git@v1.1
```

in your shell, and the import is done by executing

```
from quiver import *
```

Not all existing functionality is discussed. For more features, one is referred to the documentation on https://quiver.tools.

2. Representation theory of quivers. A *quiver* Q is a directed multigraph, given by a finite set of vertices Q_0 and a finite set of arrows Q_1. Each arrow $\alpha \in Q_1$ has a *source* $\mathrm{s}(\alpha)$ and a *target* $\mathrm{t}(\alpha)$, which are vertices in Q_0.

The *Euler form* of a quiver Q is the bilinear form

$$\langle -, - \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z} : (\boldsymbol{d}, \boldsymbol{e}) \mapsto \sum_{i \in Q_0} d_i e_i - \sum_{a \in Q_1} d_{\mathrm{s}(a)} e_{\mathrm{t}(a)}. \tag{1}$$

The motivation behind this definition and terminology will be clear following (9). By symmetrizing the Euler form we obtain the so-called *Kac form* on $\mathbb{Z}^{Q_0}$, given by

$$(\boldsymbol{d}, \boldsymbol{e}) := \langle \boldsymbol{d}, \boldsymbol{e} \rangle + \langle \boldsymbol{e}, \boldsymbol{d} \rangle. \tag{2}$$

We fix an algebraically closed field $\boldsymbol{k}$ of characteristic 0. A *representation* of Q is a collection of $\boldsymbol{k}$-vector spaces V_i for every $i \in Q_0$, and linear maps $V_\alpha : V_{\mathrm{s}(\alpha)} \to V_{\mathrm{t}(\alpha)}$ for every $\alpha \in Q_1$. We say that a representation has *dimension vector* $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$ if for all $i \in Q_0$, $\dim(V_i) = d_i$. In this case we write $\underline{\dim}(V) = \boldsymbol{d}$.

Given two representations V and W of Q, a *morphism* of representations from V to W is a collection of linear maps $\{\phi_i : V_i \to W_i\}_{i \in Q_0}$, such that for every arrow $\alpha \in Q_1$, one has

$$W_\alpha \circ \phi_{\mathrm{s}(\alpha)} = \phi_{\mathrm{t}(\alpha)} \circ V_\alpha. \tag{3}$$

One of the first results on representations of quivers is provided by Kac's theorem [20]. A nonzero representation V is said to be *indecomposable* if it cannot be written as the direct sum of two nonzero representations. On the other hand, Kac considers the symmetric bilinear form (2) of a quiver, equipping $\mathbb{Z}^{Q_0}$ with the structure of a root system, for which the following can be shown.

Theorem 2.1 (Kac). *Let Q be a quiver. The following are equivalent for a dimension vector $\boldsymbol{d}$:*

(1) *There exists an indecomposable representation V for which $\underline{\dim}(V) = \boldsymbol{d}$.*

(2) *$\boldsymbol{d}$ is a root in the root system, i.e., $(\boldsymbol{d}, \boldsymbol{d}) \leq 2$.*

The next two ingredients will form the basis for geometric considerations in what follows. If we fix a dimension vector $\boldsymbol{d}$, a representation corresponds to a point in the *representation space*

$$\mathrm{R}(Q, \boldsymbol{d}) := \prod_{\alpha \in Q_1} \mathrm{Mat}_{d_{\mathrm{t}(\alpha)}, d_{\mathrm{s}(\alpha)}}(\boldsymbol{k}). \tag{4}$$

The group $\mathrm{GL}(\boldsymbol{d}) := \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\boldsymbol{k})$ acts on $\mathrm{R}(Q, \boldsymbol{d})$ by base change; its orbits are, by definition, isomorphism classes of representations of Q.

3. GENERAL REPRESENTATIONS OF QUIVERS. The building block for many algorithms for moduli of quiver representations is the notion of general representations, introduced by Schofield [35].

Definition 3.1. Let $\mathcal{P}$ be a property of a representation of a quiver Q. We say that the *general representation* of dimension vector $\boldsymbol{d}$ satisfies $\mathcal{P}$ if there exists a nonempty Zariski-open subset U of $\mathrm{R}(Q, \boldsymbol{d})$ such that every representation in U satisfies $\mathcal{P}$.

The representations of a quiver Q form an abelian category whose objects have all finite length, and thus a Krull–Schmidt category. Accordingly, each representation can be decomposed uniquely into a finite direct sum of indecomposable ones. The shape of this decomposition for the general representation in $\mathrm{R}(Q, \boldsymbol{d})$ can be described in terms of the quiver Q and the dimension vector $\boldsymbol{d}$. This is closely related to the following class of dimension vectors.

Definition 3.2. A dimension vector $\boldsymbol{e} \leq \boldsymbol{d}$ is called a *general subdimension vector* of $\boldsymbol{d}$ if the general representation in $\mathrm{R}(Q, \boldsymbol{d})$ admits a subrepresentation of dimension vector $\boldsymbol{e}$. We introduce the notation $\boldsymbol{e} \hookrightarrow \boldsymbol{d}$ to indicate that $\boldsymbol{e}$ is a general subdimension vector of $\boldsymbol{d}$.

The subset of $\mathrm{R}(Q, \boldsymbol{d})$ of representations admitting a subrepresentation of fixed subdimension vector can be shown to be Zariski-closed. General subdimension vectors $\boldsymbol{e}$ are thus characterized by the (a priori) stronger requirement that *every* representation of dimension vector $\boldsymbol{d}$ admits a subrepresentation of dimension vector $\boldsymbol{e}$; see [35, Theorem 3.3]. There exists an algorithmic characterization of general subdimension vectors [35, Theorem 5.2].

Theorem 3.3 (Schofield). *Let Q be a quiver and $\boldsymbol{d}$ a dimension vector. A subdimension vector $\boldsymbol{e} \leq \boldsymbol{d}$ is general if and only if for every general subdimension vector $\boldsymbol{e}'$ of $\boldsymbol{e}$, the inequality $\langle \boldsymbol{e}', \boldsymbol{d} - \boldsymbol{e} \rangle \geq 0$ holds.*

This algorithmic result is an ingredient of many other algorithms for quivers and quiver moduli. It can be described and implemented recursively: denoting the set of general subdimension vectors of $\boldsymbol{d}$ by $\boldsymbol{d}$, we have

$$\boldsymbol{d} = \{\boldsymbol{e} \leq \boldsymbol{d} \mid \forall \boldsymbol{e}' \in \boldsymbol{e} : \langle \boldsymbol{e}', \boldsymbol{d} - \boldsymbol{e} \rangle \geq 0\}. \tag{5}$$

The appearance of the Euler form in this recursive characterization is the first of many instances where it plays an essential role.

We will now describe an equivalent characterization of general subdimension vectors, allowing us to introduce concepts which will be useful later.

Definition 3.4. Let Q be a quiver and let $\boldsymbol{d}$ and $\boldsymbol{d}'$ be two dimension vectors. We define the *general hom*, denoted $\hom(\boldsymbol{d}, \boldsymbol{d}')$, and the *general ext*, denoted $\operatorname{ext}(\boldsymbol{d}, \boldsymbol{d}')$, as

$$\begin{aligned} \hom(\boldsymbol{d}, \boldsymbol{d}') &:= \{\dim(\operatorname{Hom}(V, W)) \mid V \in \mathrm{R}(Q, \boldsymbol{d}), W \in \mathrm{R}(Q, \boldsymbol{d}')\}, \\ \operatorname{ext}(\boldsymbol{d}, \boldsymbol{d}') &:= \{\dim(\operatorname{Ext}(V, W)) \mid V \in \mathrm{R}(Q, \boldsymbol{d}), W \in \mathrm{R}(Q, \boldsymbol{d}')\}. \end{aligned} \tag{6}$$

Since the functions

$$\begin{aligned} \dim(\operatorname{Hom}(-,-)) &: \mathrm{R}(Q, \boldsymbol{d}) \times \mathrm{R}(Q, \boldsymbol{d}') \to \mathbb{Z}, \\ \dim(\operatorname{Ext}(-,-)) &: \mathrm{R}(Q, \boldsymbol{d}) \times \mathrm{R}(Q, \boldsymbol{d}') \to \mathbb{Z} \end{aligned} \tag{7}$$

are upper semicontinuous (e.g., by considering the exact sequence in (8)), they reach their minimum on an open subset of their domain. The general representations $V \in \mathrm{R}(Q, \boldsymbol{d})$ and $W \in \mathrm{R}(Q, \boldsymbol{d}')$ thus satisfy $\dim(\operatorname{Hom}(V, W)) = \hom(\boldsymbol{d}, \boldsymbol{d}')$ and $\dim(\operatorname{Ext}(V, W)) = \operatorname{ext}(\boldsymbol{d}, \boldsymbol{d}')$.

The general hom and ext turn out to be closely related to the Euler form. Given two dimension vectors $\boldsymbol{d}$ and $\boldsymbol{d}'$, then, for every $V \in \mathrm{R}(Q, \boldsymbol{d})$ and every $W \in \mathrm{R}(Q, \boldsymbol{d}')$, there exists a 4-term exact sequence

$$0 \to \operatorname{Hom}(V, W) \to \bigoplus_{i \in Q_0} \operatorname{Hom}_{\boldsymbol{k}}(V_i, W_i) \to \bigoplus_{\alpha \in Q_1} \operatorname{Hom}_{\boldsymbol{k}}(V_{\mathrm{s}(\alpha)}, W_{\mathrm{t}(\alpha)}) \to \operatorname{Ext}(V, W) \to 0, \tag{8}$$

where the middle morphism sends $\{\phi_i : V_i \to W_i\}_{i \in Q_0}$ to $\{W_\alpha \circ \phi_{\mathrm{s}(\alpha)} - \phi_{\mathrm{t}(\alpha)} \circ V_\alpha\}_{\alpha \in Q_1}$. This 4-term sequence (8) (and its variations, see, e.g., (21)) are an important tool in the study of quiver representations and the geometry of quiver moduli.

Computing the dimensions in the sequence (8) tells us that for every $V \in \mathrm{R}(Q, \boldsymbol{d})$, $W \in \mathrm{R}(Q, \boldsymbol{d}')$, we have

$$\langle \boldsymbol{d}, \boldsymbol{d}' \rangle = \dim(\operatorname{Hom}(V, W)) - \dim(\operatorname{Ext}(V, W)) = \hom(\boldsymbol{d}, \boldsymbol{d}') - \operatorname{ext}(\boldsymbol{d}, \boldsymbol{d}'). \tag{9}$$

The following central result of Schofield allows one to compute general hom and ext using the Euler form and general subdimension vectors; see [35, Theorem 5.4]. We recall the following notation from [35], dual to Definition 3.2: for a dimension vector $\boldsymbol{f}$ such that the general representation in $\mathrm{R}(Q, \boldsymbol{d})$ admits a quotient representation of dimension vector $\boldsymbol{f}$, we write $\boldsymbol{d} \twoheadrightarrow \boldsymbol{f}$.

Theorem 3.5 (Schofield). *Let $\boldsymbol{d}$ and $\boldsymbol{d}'$ be dimension vectors for the quiver Q. Then,*

$$\mathrm{ext}(\boldsymbol{d}, \boldsymbol{d}') = \max_{\boldsymbol{e} \hookrightarrow \boldsymbol{d}} \{-\langle \boldsymbol{e}, \boldsymbol{d}' \rangle\} = \max_{\boldsymbol{d}' \twoheadrightarrow \boldsymbol{f}} \{-\langle \boldsymbol{d}, \boldsymbol{f} \rangle\}. \tag{10}$$

This number is thus computed by enumerating all general subdimension vectors, using the recursive procedure in Theorem 3.3. In light of Theorem 3.5, Theorem 3.3 can be rephrased in terms of the general extension group.

Corollary 3.6. *A subdimension vector $\boldsymbol{e}$ of $\boldsymbol{d}$ is general if and only if* $\mathrm{ext}(\boldsymbol{e}, \boldsymbol{d} - \boldsymbol{e}) = 0$.

We illustrate these notions with `QuiverTools`:

```
sage: Q = KroneckerQuiver(4); d = (2, 3);
sage: Q.all_general_subdimension_vectors(d)
[(0, 0), (0, 1), (0, 2), (0, 3), (1, 3), (2, 3)]
sage: Q.general_ext((1, 2), d)
4
sage: Q.general_hom((1, 2), d)
0
```

Kac [20] proved that the dimension vectors of the indecomposable summands in the decomposition of the general representation of Q of dimension vector $\boldsymbol{d}$ (which exists because the category is a Krull–Schmidt category) only depend on Q and $\boldsymbol{d}$.

Theorem 3.7 (Kac). *Let Q be a quiver and $\boldsymbol{d}$ be a dimension vector. There exists a unique collection of dimension vectors $\{\boldsymbol{d}^i\}_{i=1}^n$ summing to $\boldsymbol{d}$ and such that for all $i \neq j$,*

$$\mathrm{ext}(\boldsymbol{d}^i, \boldsymbol{d}^j) = 0. \tag{11}$$

Moreover, the general representation of Q of dimension vector $\boldsymbol{d}$ decomposes as a direct sum $\bigoplus_{i=1}^n V^i$, where $\underline{\dim}(V^i) = \boldsymbol{d}^i$, and V^i is indecomposable.

This decomposition is accordingly called the *canonical decomposition* of the dimension vector $\boldsymbol{d}$. In fact, the dimension vectors $\{\boldsymbol{d}^i\}_{i=1}^n$ appearing in the canonical decomposition are not just roots, but they are *Schur roots*: if $\boldsymbol{d}^i$ is one of the dimension vectors appearing in the decomposition, then for a general representation V^i with this dimension vector, one has that $\mathrm{End}(V^i) \cong \boldsymbol{k}$.

An effective characterization of the canonical decomposition follows from Theorem 3.5, which can be used to recursively decompose a dimension vector into general summands, as well as to determine which dimension vectors are indecomposable: by iterating over the general subdimension vectors of $\boldsymbol{d}$, if we find some $\boldsymbol{e} \hookrightarrow \boldsymbol{d}$ for which $\mathrm{ext}(\boldsymbol{e}, \boldsymbol{d} - \boldsymbol{e}) = \mathrm{ext}(\boldsymbol{d} - \boldsymbol{e}, \boldsymbol{e}) = 0$, then the general representation of dimension vector $\boldsymbol{d}$ must split as a direct sum. If on the other hand there is no such $\boldsymbol{e}$, then $\boldsymbol{d}$ must be indecomposable.

We illustrate these notions with QuiverTools, with the 3-vertex example illustrating what happens in [7, Example 11.1.4]:

```
sage: Q1 = KroneckerQuiver(4);
sage: Q1.canonical_decomposition((2, 3))
((2, 3),)
sage: Q2 = ThreeVertexQuiver(1, 1, 1)
sage: Q2.is_root((1, 2, 1))
True
sage: Q2.is_schur_root((1, 2, 1))
False
sage: Q2.canonical_decomposition((1, 2, 1))
((0, 1, 0), (1, 1, 1))
```

4. Moduli spaces of representations of quivers. As introduced in Section 2, the general linear group $\mathrm{GL}(\boldsymbol{d})$ acts on the representation space $\mathrm{R}(Q, \boldsymbol{d})$ via change of basis. One can consider the spectrum of the invariant ring

$$\mathrm{M}(Q, \boldsymbol{d}) := \mathrm{Spec}(\mathcal{O}(\mathrm{R}(Q, \boldsymbol{d}))^{\mathrm{GL}(\boldsymbol{d})}) = \mathrm{R}(Q, \boldsymbol{d}) / \mathrm{GL}(\boldsymbol{d}). \tag{12}$$

It is a moduli space, parametrizing semisimple representations. This invariant ring is generated by the traces of compositions along oriented cycles in the quiver; see [25]. In particular, if Q is acyclic then $\mathrm{M}(Q, \boldsymbol{d})$ is a point.

In order to obtain a better-behaved moduli space, the following notion of stability, involving a choice of *stability parameter* θ, was introduced in [24].

Definition 4.1. Let $\theta \in \mathrm{Hom}(\mathbb{Z}^{Q_0}, \mathbb{Z})$ be such that $\theta(\boldsymbol{d}) = 0$. A representation $V \in \mathrm{R}(Q, \boldsymbol{d})$ is said to be *θ-stable*, respectively *θ-semistable*, if for all of its proper nonzero subrepresentations W, we have $\theta(\underline{\dim}(W)) < 0$, respectively $\theta(\underline{\dim}(W)) \leq 0$.

We denote by $\mathrm{R}^{\theta\text{-st}}(Q, \boldsymbol{d})$ and $\mathrm{R}^{\theta\text{-sst}}(Q, \boldsymbol{d})$ the Zariski-open subsets of $\mathrm{R}(Q, \boldsymbol{d})$ of θ-stable, respectively θ-semistable representations. As it turns out, both admit geometric quotients by the action of $\mathrm{GL}(\boldsymbol{d})$, where the former quotient is always smooth, and the latter is always projective over $\mathrm{R}(Q, \boldsymbol{d}) / \mathrm{GL}(\boldsymbol{d})$. These geometric quotients are the better-behaved moduli spaces sought. We denote them by

$$\begin{aligned} \mathrm{M}^{\theta\text{-st}}(Q, \boldsymbol{d}) &:= \mathrm{R}^{\theta\text{-st}}(Q, \boldsymbol{d}) /\!\!/_{\theta} \mathrm{GL}(\boldsymbol{d}), \\ \mathrm{M}^{\theta\text{-sst}}(Q, \boldsymbol{d}) &:= \mathrm{R}^{\theta\text{-sst}}(Q, \boldsymbol{d}) /\!\!/_{\theta} \mathrm{GL}(\boldsymbol{d}). \end{aligned} \tag{13}$$

For more on their construction and properties, see, e.g., [32]. One standard choice of stability parameter is the *canonical* stability parameter for $\boldsymbol{d}$, defined as $\theta_{\mathrm{can}} := \langle \boldsymbol{d}, - \rangle - \langle -, \boldsymbol{d} \rangle$. For Q acyclic this will often lead to smooth projective Fano varieties [10], making them particularly relevant for algebraic geometers. This is for instance the case in our running example.

Having introduced these moduli spaces, we naturally wish to know for which triples $(Q, \boldsymbol{d}, \theta)$ they are nonempty, i.e., for which such triples (semi)stable representations exist. These questions can be answered by verifying, for the given stability parameter θ, whether the necessary (semi)stability inequalities hold for general subdimension vectors: if every general subdimension vector $\boldsymbol{e} \hookrightarrow \boldsymbol{d}$ satisfies $\theta(\boldsymbol{e}) \leq 0$, then the general representation is semistable, vice versa, if some $\boldsymbol{e} \hookrightarrow \boldsymbol{d}$ does not satisfy said inequality, then no representation can be θ-stable. This makes for an explicit algorithm, building on Section 3; see, e.g., [6, Theorem 3].

We illustrate these notions with `QuiverTools`: in our running example, the canonical stability condition (and its positive rescalings, which do not affect stability) is the only one for which the moduli space is interesting:

```
sage: Q = KroneckerQuiver(4); d = (2, 3);
sage: theta = Q.canonical_stability_parameter(d)
sage: Q.has_stable_representation(d, theta)
True
sage: Q.has_stable_representation(d, -theta)
False
```

As explained in [32], one can compute various geometric invariants of quiver moduli spaces. We compute these using `QuiverTools` for our running example:

```
sage: Q = KroneckerQuiver(4); d = (2, 3);
sage: theta = Q.canonical_stability_parameter(d);
sage: M = QuiverModuliSpace(Q, d, theta);
sage: M.dimension()
12
sage: M.picard_rank()
1
sage: M.index()
4
```

5. Harder–Narasimhan stratification. For the action of a reductive group G on an affine variety X, Hesselink has shown in [16] that the complement of the semistable locus admits a certain stratification into locally closed subsets. This stratification of X is not unique, as it depends on the choice of a norm on a certain maximal torus inside G, and in general it is difficult to compute.

In the case of the action of $\mathrm{GL}(\boldsymbol{d})$ on $\mathrm{R}(Q, \boldsymbol{d})$, this stratification is characterized by an effective description of the *Harder–Narasimhan filtration* of each representation.

For a stability parameter θ and some $\boldsymbol{a} \in \mathbb{Z}_{\geq 1}^{Q_0}$, we define the associated *slope function* as

$$\mu = \mu_{\theta,\boldsymbol{a}} : \mathbb{Z}_{\geq 0}^{Q_0} \setminus \{\boldsymbol{\Gamma}\} \to \mathbb{Q} : \boldsymbol{e} \mapsto \frac{\theta(\boldsymbol{e})}{\boldsymbol{a} \cdot \boldsymbol{e}}. \tag{14}$$

The standard choice is $\boldsymbol{a} = \boldsymbol{\Delta}$, for which $\boldsymbol{a} \cdot \boldsymbol{e} = \sum_{i \in Q_0} e_i$. Different choices of $\boldsymbol{a}$ correspond to different choices of a norm on the maximal torus of $\mathrm{GL}(\boldsymbol{d})$; see [19, Theorem 3.8].

Definition 5.1. Let Q and $\boldsymbol{d}$ be a quiver and a dimension vector, and let μ be a slope function. A representation $V \in \mathrm{R}(Q, \boldsymbol{d})$ is *μ-stable*, respectively *μ-semistable*, if every proper nonzero subrepresentation $W \subset V$ satisfies the inequality $\mu(\underline{\dim}(W)) < \mu(\underline{\dim}(V))$, respectively $\mu(\underline{\dim}(W)) \leq \mu(\underline{\dim}(V))$.

Note that we do not require that $\mu(\underline{\dim}(V)) = 0$.

This more general notion of stability allows us to associate to each representation a unique filtration by subrepresentations, as it is done in [17, Theorem 2.5] and independently in [31, Proposition 2.5].

Proposition 5.2. *Let Q and $\boldsymbol{d}$ be a quiver and a dimension vector, and let μ be a slope function. Every representation $V \in \mathrm{R}(Q, \boldsymbol{d})$ admits a unique filtration*

$$0 = V^0 \subset V^1 \subset \cdots \subset V^\ell = V, \tag{15}$$

which has the property that the successive quotients

$$W^i := V^i / V^{i-1}$$

are μ-semistable of strictly decreasing slope, i.e.,

$$\mu(W^1) > \mu(W^2) > \cdots > \mu(W^\ell). \tag{16}$$

We call this filtration the *Harder–Narasimhan filtration* of V associated to μ. The sequence of dimension vectors $(\underline{\dim}(W^1), \ldots, \underline{\dim}(W^\ell))$ is called the *Harder–Narasimhan type* of V, and we denote it by $\boldsymbol{d}^* = (\boldsymbol{d}^1, \ldots, \boldsymbol{d}^\ell)$. We see that a representation V is μ-semistable if and only if its Harder–Narasimhan type is the trivial one, i.e., $(\underline{\dim}(V))$.

Proposition 5.2 gives us an algorithm to enumerate the Harder–Narasimhan types, by recursively building them: one looks for all subdimension vectors $\boldsymbol{d}^1$ of $\boldsymbol{d}$ of strictly larger slope which admit a semistable representation, and then one builds all possible Harder–Narasimhan types for the remaining $\boldsymbol{d} - \boldsymbol{d}^1$.

One can show, see [19, Proposition 3.4], that the Hesselink stratification of the unstable locus is characterized by Harder–Narasimhan types: two representations in $\mathrm{R}(Q, \boldsymbol{d})$ belong to the same stratum if and only if their Harder–Narasimhan types are equal. To enumerate all the Hesselink strata it suffices then to enumerate all the possible Harder–Narasimhan types; see [2, Section 3] for more details.

We illustrate these notions with QuiverTools:

```
sage: Q = KroneckerQuiver(4); d = (2, 3); theta = (3, -2);
sage: M = QuiverModuliSpace(Q, d, theta)
sage: M.all_harder_narasimhan_types()
(((2, 3),),
 ((1, 1), (1, 2)),
 ((2, 2), (0, 1)),
 ((2, 1), (0, 2)),
 ((1, 0), (1, 3)),
 ((1, 0), (1, 2), (0, 1)),
 ((1, 0), (1, 1), (0, 2)),
 ((2, 0), (0, 3)))
```

Betti numbers of quiver moduli. One direct consequence of this enumeration is the ability to compute Betti numbers for quiver moduli. This is done in [31, Section 6], by translating the method originally introduced by Harder and Narasimhan in [15] to compute the Betti numbers of moduli spaces of vector bundles on curves. We refer to [31] for more details. We compute the Betti numbers of our running example using QuiverTools:

```
sage: Q = KroneckerQuiver(4); d = (2, 3); theta = (3, -2);
sage: M = QuiverModuliSpace(Q, d, theta)
sage: M.betti_numbers()
[1, 0, 1, 0, 3, 0, 4, 0, 7, 0, 8, 0, 10, 0, 8, 0, 7, 0, 4, 0, 3, 0, 1, 0, 1]
```

The computation of Betti numbers makes it possible to experiment and find patterns for families of quiver moduli, as used for instance in [12, Appendix A].

6. Teleman quantization. In Section 5 we introduced the Hesselink stratification of a variety X under the action of a reductive group G. It is used in [14] to obtain *Teleman quantization*, which makes it possible to study the cohomology of coherent sheaves on $X/\!/\mathrm{G}$. For quiver moduli problems, Teleman quantization can be used effectively thanks to computations performed in [2]. The authors already used this tool in [2; 28] to prove several results about rigidity, existence of partial tilting objects and semiorthogonal decompositions.

Setup. We will introduce the necessary notation in the general setting of GIT, and subsequently specialize it to quiver moduli.

In the Hesselink stratification, each stratum S of the unstable locus $X \setminus X^{\mathrm{sst}}$ is identified by a one-parameter subgroup λ_S of G, that is in a certain sense "the most responsible" subgroup for the instability of the points in S. Each such λ_S acts on $\det(\mathrm{N}^\vee_{S/X})|_{S^{\lambda_S}}$, with an integer weight that we denote by η_S.

Let F be a G-linearized coherent sheaf on X so that $F|_{X^{\mathrm{sst}}}$ descends to $\mathcal{F}$ on the quotient $X^{\mathrm{sst}}/\!/\mathrm{G}$. For each stratum S, denote the set of λ_S-weights of $F|_{S^{\lambda_S}}$ by $\mathrm{W}(F, S)$. Teleman quantization then states that, if on each stratum S, all the λ_S-weights of F are strictly smaller than η_S, then there exists an isomorphism

$$\mathrm{H}^\bullet(X^{\mathrm{sst}}, F|_{X^{\mathrm{sst}}})^{\mathrm{G}} \cong \mathrm{H}^\bullet(X, F)^{\mathrm{G}}. \tag{17}$$

Once again, this result is hard to apply effectively in general, because describing the Hesselink stratification is a priori difficult. In the case of quiver moduli however, the latter is characterized by the Harder–Narasimhan types (see [2, Theorem 3.11]), so it can be described effectively using the previous discussion.

For quiver moduli. The variety of interest is $\mathrm{R}(Q, \boldsymbol{d})$, with the action of $\mathrm{GL}(\boldsymbol{d})$, and we index all the terms introduced above by their Harder–Narasimhan types $\boldsymbol{d}^*$. For each stratum $S_{\boldsymbol{d}^*}$, the corresponding one-parameter subgroup of $\mathrm{GL}(\boldsymbol{d})$ is denoted by $\lambda_{\boldsymbol{d}^*}$. For every $\mathrm{GL}(\boldsymbol{d})$-equivariant coherent sheaf F on $\mathrm{R}(Q, \boldsymbol{d})$, we denote the set of $\lambda_{\boldsymbol{d}^*}$-weights of $F|_{S_{\boldsymbol{d}^*}{}^{\lambda_{\boldsymbol{d}^*}}}$ by $\mathrm{W}(F, \boldsymbol{d}^*)$. The weight of the natural action of $\lambda_{\boldsymbol{d}^*}$ on $\det(\mathrm{N}^\vee_{S_{\boldsymbol{d}^*}/\mathrm{R}(Q,\boldsymbol{d}^*)})|_{S_{\boldsymbol{d}^*}{}^{\lambda_{\boldsymbol{d}^*}}}$ is denoted by $\eta_{\boldsymbol{d}^*}$.

In the context of quiver moduli problems, Teleman quantization is stated as follows, using that the action of $\mathrm{GL}(\boldsymbol{d})$ on $\mathrm{R}^{\theta\text{-sst}}(Q, \boldsymbol{d})$ is free, for the last isomorphism.

Theorem 6.1. *With the previous notation, let F be a* $\mathrm{GL}(\boldsymbol{d})$*-equivariant coherent sheaf that descends to* $\mathcal{F}$. *If on every stratum* $S_{\boldsymbol{d}^*}$ *we have*

$$\max \mathrm{W}(F, \boldsymbol{d}^*) < \eta_{\boldsymbol{d}^*}, \tag{18}$$

then for every $k \in \mathbb{Z}$ *we have*

$$\mathrm{H}^k(\mathrm{R}(Q, \boldsymbol{d}), F)^{\mathrm{GL}(\boldsymbol{d})} \cong \mathrm{H}^k(\mathrm{R}^{\theta\text{-sst}}(Q, \boldsymbol{d}), F|_{\mathrm{R}^{\theta\text{-sst}}(Q,\boldsymbol{d})})^{\mathrm{GL}(\boldsymbol{d})} \cong \mathrm{H}^k(\mathrm{M}^{\theta\text{-sst}}(Q, \boldsymbol{d}), \mathcal{F}). \tag{19}$$

In particular, since $\mathrm{R}(Q, \boldsymbol{d})$ is affine, we have vanishing of higher cohomology.

Corollary 6.2. *Under the previous assumptions, we have for all* $k \geq 1$ *the vanishing*

$$\mathrm{H}^k(\mathrm{M}^{\theta\text{-sst}}(Q, \boldsymbol{d}), \mathcal{F}) = 0. \tag{20}$$

The effective computations of the numbers $\eta_{\boldsymbol{d}^*}$ are performed in [2, Corollary 3.18], where these are shown to be expressed in terms of the Euler form of Q. `QuiverTools` implements these computations, making it possible to verify (18). We illustrate this in `QuiverTools` using our running example:

```
sage: Q = KroneckerQuiver(4); d = (2, 3); theta = (3, -2);
sage: M = QuiverModuliSpace(Q, d, theta);
sage: hn_types = M.all_harder_narasimhan_types(proper=True)
sage: table(
....:     [["Harder-Narasimhan type hn", "eta_hn"]] + \
....:     [[hn, M.teleman_bound(hn)] for hn in hn_types]
....: )
  Harder-Narasimhan type hn          eta_hn
  ((1, 1), (1, 2))                   25
  ((2, 2), (0, 1))                   30
  ((2, 1), (0, 2))                   70
  ((1, 0), (1, 3))                   55
  ((1, 0), (1, 2), (0, 1))           140
  ((1, 0), (1, 1), (0, 2))           125
  ((2, 0), (0, 3))                   120
```

Universal families on quiver moduli. An application of the Teleman quantization theorem are vanishing results for the cohomology of interesting vector bundles on quiver moduli.

Let Q be an acyclic quiver and $\boldsymbol{d}$ be a coprime dimension vector for which stable representations exist. For a sufficiently general choice of stability parameter θ, the moduli space $\mathrm{M}^{\theta\text{-sst}}(Q, \boldsymbol{d})$ is a smooth projective variety admitting a universal family $\mathcal{U} = \bigoplus_{i \in Q_0} \mathcal{U}_i$. This universal family has the property that for a closed point in the moduli space, the fiber of the bundle $\mathcal{U}$ is the representation of Q corresponding to that point of the moduli space. Its summands appear in the four-term exact sequence

$$0 \to \mathcal{O}_{\mathrm{M}^{\theta\text{-sst}}(Q,\boldsymbol{d})} \to \bigoplus_{i \in Q_0} \mathcal{U}_i^\vee \otimes \mathcal{U}_i \to \bigoplus_{\alpha \in Q_1} \mathcal{U}_{\mathrm{s}(\alpha)}^\vee \otimes \mathcal{U}_{\mathrm{t}(\alpha)} \to \mathrm{T}_{\mathrm{M}^{\theta\text{-sst}}(Q,\boldsymbol{d})} \to 0. \tag{21}$$

This is the globalization of the exact sequence in (8), see [1, Lemma 4.2] for its construction. The Teleman weights of $\mathcal{U}_i$, and more importantly for our applications $\mathcal{U}_i^\vee \otimes \mathcal{U}_j$, are computed in [2, Lemma 3.19 and Proposition 3.20].

QuiverTools implements the computation of these weights [2, Proposition 3.20], and, under mild conditions, of the canonical line bundle $\omega_{\mathrm{M}^{\theta\text{-sst}}(Q,\boldsymbol{d})}$, which is described in [28, Proposition 4.7]. We illustrate this in QuiverTools using our running example:

```
sage: Q = KroneckerQuiver(4); d = (2, 3); theta = (3, -2);
sage: M = QuiverModuliSpace(Q, d, theta);
sage: U0 = M.weights_universal_bundle(0)
sage: U1 = M.weights_universal_bundle(1)
sage: omega = M.weights_canonical()
sage: hn_types = M.all_harder_narasimhan_types(proper=True)
sage: table(
....:     [["Harder-Narasimhan type hn", "W(U0, hn)", \
....:     "W(U1, hn)", "W(omega, hn)"]] + \
....:     [[hn, U0[hn], U1[hn], omega[hn]] for hn in hn_types]
....: )
  Harder-Narasimhan type hn              W(U0, hn)   W(U1, hn)      W(omega, hn)
  ((1, 1), (1, 2))                       [5, 0]      [5, 0, 0]      [20]
  ((2, 2), (0, 1))                       [5, 5]      [5, 5, 0]      [40]
  ((2, 1), (0, 2))                       [10, 10]    [10, 5, 5]     [80]
  ((1, 0), (1, 3))                       [10, 5]     [5, 5, 5]      [60]
  ((1, 0), (1, 2), (0, 1))               [25, 15]    [15, 15, 10]   [160]
  ((1, 0), (1, 1), (0, 2))               [20, 15]    [15, 10, 10]   [140]
  ((2, 0), (0, 3))                       [15, 15]    [10, 10, 10]   [120]
```

7. INTERSECTION THEORY. The final feature we will discuss concerns computations in intersection theory [8; 11]. In this subject, the goal is to construct a (graded) ring which encodes how subvarieties intersect, with respect to an equivalence relation, so that we can move subvarieties around. In computer algebra, this equivalence relation is usually numerical equivalence. This choice makes the Chow ring a finite-dimensional graded algebra, whose multiplication encodes intersections of subvarieties. Via the Chern character, coherent sheaves are also encoded by this Chow ring.

For certain varieties, like Grassmannians, there exist very explicit tools to study these Chow rings, which can be implemented, and put to great use, see [13]. Similarly, when a moduli space of quiver representations is a smooth projective variety admitting a universal family, there exists

- an effective presentation of its Chow ring $\mathrm{CH}^\bullet(\mathrm{M}^{\theta\text{-st}}(Q,\boldsymbol{d}))$ [9], together with
- an explicit expression for its Todd class $\mathrm{td}_{\mathrm{M}^{\theta\text{-st}}(Q,\boldsymbol{d})}$ and point class [pt]; see [1].

This already goes a long way in the direction of Schubert calculus, as developed for Grassmannians. The presentation of the Chow ring of quiver moduli has the Chern classes of the summands of the universal family $\bigoplus_{i\in Q_0}\mathcal{U}_i$ as generators. This description makes it particularly tractable to compute Euler characteristics of sheaves, provided one knows how to express them in the Chern classes of the $\mathcal{U}_i$, or compute degrees of line bundles, as in [1; 26].

Euler characteristic. For the Euler characteristic of a sheaf, one uses the *Hirzebruch–Riemann–Roch theorem*, which states that

$$\chi(\mathrm{M}^{\theta\text{-st}}(Q,\boldsymbol{d}),\mathcal{E}) := \sum_{i=0}^{\dim \mathrm{M}^{\theta\text{-st}}(Q,\boldsymbol{d})} (-1)^i \dim_{\boldsymbol{k}} \mathrm{H}^i(\mathrm{M}^{\theta\text{-st}}(Q,\boldsymbol{d}),\mathcal{E}) = \int_{\mathrm{M}^{\theta\text{-st}}(Q,\boldsymbol{d})} \mathrm{ch}(\mathcal{E})\,\mathrm{td}_{\mathrm{M}^{\theta\text{-st}}(Q,\boldsymbol{d})} \tag{22}$$

where the integral notation refers to taking the coefficient of the integrand in front of the point class $[\mathrm{pt}] \in \mathrm{CH}^{\dim}(\mathrm{M}^{\theta\text{-st}}(Q,\boldsymbol{d}))$. These Euler characteristics can be used together with the vanishing results described in Section 6, to know the number of global sections of certain vector bundles; see also [3] for canonical identifications of global sections of $\mathcal{U}_i^\vee \otimes \mathcal{U}_j$.

Intersection numbers. Another important type of computations are intersection numbers. For instance, the top self-intersection of an anticanonical divisor, or *canonical volume*, i.e.,

$$\int_{\mathrm{M}^{\theta\text{-st}}(Q,\boldsymbol{d})} \mathrm{c}_1(-\mathrm{K}_X)^{\dim \mathrm{M}^{\theta\text{-st}}(Q,\boldsymbol{d})}, \tag{23}$$

or degree of the anticanonical line bundle is an important invariant in birational geometry, and the classification of varieties.

We illustrate these features in QuiverTools using our running example, by computing the Hilbert series of the generator of the Picard group, and the degree of the moduli space. Note that our running example is a Fano variety [10], so we are computing the degree of the projective embedding provided by the (very) ample anticanonical line bundle. These are important numerical invariants of varieties, which make it possible to anticipate identifications, as in [1]:

```
sage: Q = KroneckerQuiver(4); d = (2, 3);
sage: theta = Q.canonical_stability_parameter(d);
sage: M = QuiverModuliSpace(Q, d, theta);
sage: chi = (-1, 1);
sage: CH = M.chow_ring(chi=chi)
sage: CH.gens()
(x1_1bar, x0_2bar, x1_1bar, x1_2bar, x1_3bar)
sage: # CH.gens() lists the generators of the polynomial ring in the presentation,
sage: # after the quotient by the relations.
sage: # Here, x0_1bar is equivalent to x1_1bar, so it appears twice.
sage: pt = M.point_class(chi=chi);
sage: # Computing the Hilbert series of M
sage: H = M.chern_character_line_bundle(vector(theta) / M.index())
sage: [M.integral(H ** i) for i in range(M.dimension())]
[1,
 126,
 4032,
```

```
 59268,
 531839,
 3395882,
 16907632,
 69626910,
 246885947,
 775675824,
 2205490144,
 5766791394]
sage: # Computing the degree of M, which is the integral of -K_M^dim(M)
sage: M.degree(theta)
1996824248320
```

ACKNOWLEDGEMENTS. Belmans was partially supported by the Luxembourg National Research Fund (FNR–17113194). Franzen was partially supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) SFB-TRR 358/1 2023 "Integral Structures in Geometry and Representation Theory" (491392403). Petrella was supported by the Luxembourg National Research Fund (FNR–17953441). We thank the referee for their helpful comments.

SUPPLEMENT. The online supplement contains version 1.0 of QuiverTools.

REFERENCES.

[1] P. Belmans and H. Franzen, "On Chow rings of quiver moduli", *Int. Math. Res. Not.* **2024**:13 (2024), 10255–10272. MR

[2] P. Belmans, A.-M. Brecan, H. Franzen, G. Petrella, and M. Reineke, "Rigidity and Schofield's partial tilting conjecture for quiver moduli", *J. Éc. polytech. Math.* **12** (2025), 1345–1379. MR

[3] P. Belmans, A.-M. Brecan, H. Franzen, and M. Reineke, "Vector fields and admissible embeddings for quiver moduli", *Moduli* **2** (2025), art. id. e3. MR

[4] P. Belmans, H. Franzen, and G. Petrella, "QuiverTools", 2026, available at https://github.com/QuiverTools/QuiverTools.jl.

[5] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, "Julia: a fresh approach to numerical computing", *SIAM Rev.* **59**:1 (2017), 65–98. MR

[6] H. Derksen and J. Weyman, "Semi-invariants of quivers and saturation for Littlewood–Richardson coefficients", *J. Amer. Math. Soc.* **13**:3 (2000), 467–479. MR

[7] H. Derksen and J. Weyman, *An introduction to quiver representations*, Graduate Studies in Mathematics **184**, Amer. Math. Soc., Providence, RI, 2017. MR

[8] D. Eisenbud and J. Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge Univ. Press, 2016. MR

[9] H. Franzen, "Chow rings of fine quiver moduli are tautologically presented", *Math. Z.* **279**:3-4 (2015), 1197–1223. MR

[10] H. Franzen, M. Reineke, and S. Sabatini, "Fano quiver moduli", *Canad. Math. Bull.* **64**:4 (2021), 984–1000. MR

[11] W. Fulton, *Intersection theory*, Ergebnisse der Math. (3) **2**, Springer, 1984. MR

[12] S. Galkin, N. C. Leung, C. Li, and R. Xiong, "A-D-E diagrams, Hodge–Tate hyperplane sections and semisimple quantum cohomology", preprint, 2025. arXiv 2509.01101

[13] D. R. Grayson, M. E. Stillman, S. A. Strømme, D. Eisenbud, and C. Crissman, "Schubert2: computation in intersection theory", 2026, available at https://macaulay2.com. A *Macaulay*2 package.

[14] D. Halpern-Leistner, "The derived category of a GIT quotient", *J. Amer. Math. Soc.* **28**:3 (2015), 871–912. MR

[15] G. Harder and M. S. Narasimhan, "On the cohomology groups of moduli spaces of vector bundles on curves", *Math. Ann.* **212** (1974/75), 215–248. MR

[16] W. H. Hesselink, "Uniform instability in reductive groups", *J. Reine Angew. Math.* **303/304** (1978), 74–96. MR

[17] L. Hille and J. A. de la Peña, "Stable representations of quivers", *J. Pure Appl. Algebra* **172**:2-3 (2002), 205–224. MR

[18] V. Hoskins, "Parallels between moduli of quiver representations and vector bundles over curves", *Symmetry Integrability Geom. Methods Appl.* **14** (2018), art. id. 127. MR

[19] V. Hoskins, "Stratifications for moduli of sheaves and moduli of quiver representations", *Algebr. Geom.* **5**:6 (2018), 650–685. MR

[20] V. G. Kac, "Infinite root systems, representations of graphs and invariant theory", *Invent. Math.* **56**:1 (1980), 57–92. MR

[21] V. G. Kac, "Some remarks on representations of quivers and infinite root systems", pp. 311–327 in *Representation theory, II* (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), edited by V. Dlab and P. Gabriel, Lecture Notes in Math. **832**, Springer, 1980. MR

[22] V. G. Kac, "Infinite root systems, representations of graphs and invariant theory, II", *J. Algebra* **78**:1 (1982), 141–162. MR

[23] V. G. Kac, "Root systems, representations of quivers and invariant theory", pp. 74–108 in *Invariant theory* (Montecatini, 1982), edited by F. Gherardelli, Lecture Notes in Math. **996**, Springer, 1983. MR

[24] A. D. King, "Moduli of representations of finite-dimensional algebras", *Quart. J. Math. Oxford Ser.* (2) **45**:180 (1994), 515–530. MR

[25] L. Le Bruyn and C. Procesi, "Semisimple representations of quivers", *Trans. Amer. Math. Soc.* **317**:2 (1990), 585–598. MR

[26] S. Makarova and J. Meng, "Full exceptional sequence for a fine quiver moduli space", preprint, 2024. arXiv 2412.15390v1

[27] S. Mozgovoy, "msinvar", 2024, available at https://github.com/smzg/msinvar.

[28] G. Petrella, "Partial semiorthogonal decompositions for quiver moduli", *J. Symbolic Comput.* **131** (2025), art. id. 102448. MR

[29] B. Pioline, "CoulombHiggs", 2023, available at https://github.com/bpioline/CoulombHiggs.

[30] QPA-team, "QPA – Quivers, path algebras and representations – a GAP package", 2022, available at https://folk.ntnu.no/oyvinso/QPA/.

[31] M. Reineke, "The Harder–Narasimhan system in quantum groups and cohomology of quiver moduli", *Invent. Math.* **152**:2 (2003), 349–368. MR

[32] M. Reineke, "Moduli of representations of quivers", pp. 589–637 in *Trends in representation theory of algebras and related topics*, edited by A. Skowroński, Eur. Math. Soc., Zürich, 2008. MR

[33] *SageMath, version 10.7*, 2025, available at https://www.sagemath.org.

[34] O. G. Schiffmann, "Kac polynomials and Lie algebras associated to quivers and curves", pp. 1393–1424 in *Proceedings of the International Congress of Mathematicians*, *II*: *Invited lectures* (Rio de Janeiro 2018), edited by B. Sirakov et al., World Sci. Publ., Hackensack, NJ, 2018. MR

[35] A. Schofield, "General representations of quivers", *Proc. London Math. Soc.* (3) **65**:1 (1992), 46–64. MR

RECEIVED: 24 Jun 2025 REVISED: 22 Dec 2025 ACCEPTED: 10 Feb 2026

PIETER BELMANS:

p.belmans@uu.nl
Mathematical Institute, Utrecht University, Utrecht, Netherlands

HANS FRANZEN:

hans.franzen.math@gmail.com
Stuttgart, Germany

GIANNI PETRELLA:

gianni.petrella@uni.lu
Department of Mathematics, Université de Luxembourg, Luxembourg