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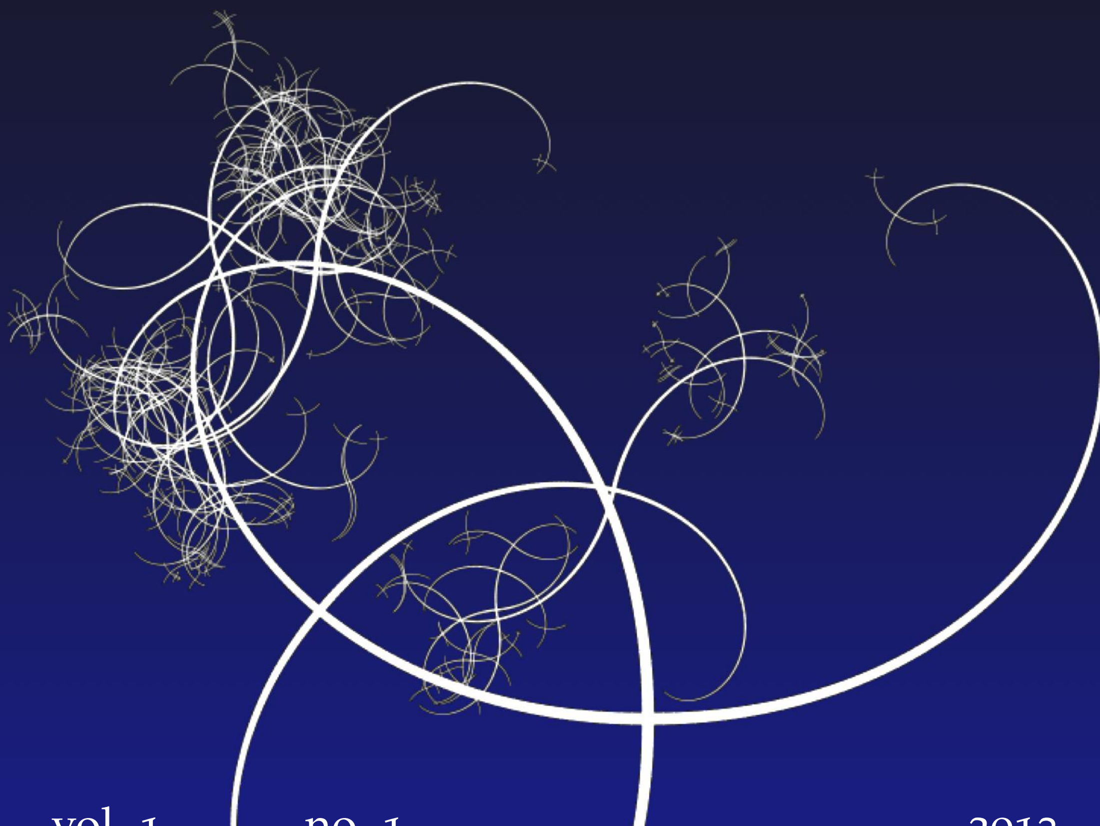

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LOCALIZATION OF POINT VORTICES UNDER CURVATURE PERTURBATIONS

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We discuss the effect of curvature on the dynamics of a two-dimensional inviscid incompressible fluid with initial vorticity concentrated in N small disjoint regions, that is, the classical point vortex system. We recall some results about point vortex dynamics on simply connected surfaces with constant curvature K, that is, plane, spherical, and hyperbolic surfaces. We show that the effect of curvature can be treated as a smooth perturbation to the Green's function of the equation related to the stream function in the planar case. Then we obtain as a main result that the localization property of point vortices, already proved for the plane, is preserved also under the effect of curvature perturbation.

1. Introduction

Vortex dynamics is a fundamental topic in fluid mechanics. In the framework of ideal incompressible fluid it is described by the Euler equation. A classical approximation made in order to study vortex dynamics analytically in two dimensions is to treat singular vorticity distributions. This means replacing a partial differential equation with infinite degrees of freedom with a system of ordinary differential equations with N degrees of freedom. This point vortex model was first introduced by Helmholtz in 1858 and Kirchhoff in 1876; it was also treated in classic textbooks like [Batchelor 1967] (for a detailed historical review see [Llewellyn Smith 2011]). The study of point vortex dynamics is still an important topic in mathematical physics, a "classical mathematics playground" as stated in [Aref 2007]. It finds its physical roots in the analysis of the dynamics of a two-dimensional inviscid incompressible fluid with initial vorticity sharply concentrated in N small disjoint regions. There are many papers devoted to the mathematical analysis of this model in the framework of dynamical systems (see, for example, [Newton 2001]) and mathematical fluid mechanics (see, for example, [Marchioro and Pulvirenti 1994]). However a critical point of this model is the divergence of the velocity computed in the point where the single vortex is localized. This (infinite) term could be skipped in a heuristic way from a physical point of view because it is a self-interaction term.

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But from a mathematical point of view a rigorous connection between the Euler equation and the point vortex model is given by the proof that, if

$$\omega_0(x)\,dx \to \sum_{i=1}^{N} a_i \delta_{x_i}(dx),$$

then

$$\omega_t(x)\,dx \to \sum_{i=1}^{N} a_i \delta_{x_i(t)}(dx).$$

The proof of this fact was firstly given in [Marchioro and Pulvirenti 1993]. In more detail, it was proved there (and in [Marchioro 1998] that the time evolution of a system of vortices initially concentrated in N small disjoint regions of diameter ϵ remains concentrated in N disjoint vortices with diameter $d(\epsilon) \to 0$ as $\epsilon \to 0$. This property of the point vortex model is called *localization*.

Here we analyze the effect of curvature on the dynamics of N sharply concentrated vortices. There are a number of papers [Hally 1980; Kimura 1999] devoted to the analysis of vortex motion on surfaces with constant curvature, for example, on spheres [Crowdy 2006] and hyperbolic surfaces [Hwang and Kim 2009]. Moreover, Boatto [2008] treated the perturbative effect of curvature on the stability of a ring of vortices.

The main aim of this paper is to prove that the property of localization of the dynamics of point vortex motion is preserved under the effect of curvature perturbation. Actually, we show that the effect of the constant curvature K of the surface on the dynamics can be treated by means of a smooth perturbation to the Green's function of the plane case. Then we include this perturbation in an external mean field and we show that the localization of the vortices under the effect of curvature is essentially a corollary to the theorem of localization stated in [Marchioro and Pulvirenti 1993].

This result is really interesting from a physical point of view because it states that strong concentrated vortices remain concentrated under the effect of curvature. For example, in the spherical case we can apply it to the dynamics of vortices over the Earth's surface. Moreover, we can generalize this result to any regular surface that can be locally approximated with a Riemannian manifold with constant curvature K.

The plan of the paper is as follows: in Section 2 we introduce the constitutive equations of the point vortex model, in Section 3 we recall some useful results about point vortex dynamics on surfaces with constant curvature, and in Section 4 we discuss the main result, recalling the localization theorem in the planar case and proving that it also works taking into account the effect of curvature.

2. Point vortex motion in fluid mechanics

Here we introduce the constitutive equations of point vortex motion in the whole plane $\mathbb{R}^2$. Consider the Euler equation about a two-dimensional inviscid incompressible fluid with unitary density:

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \quad \nabla \cdot u = 0, \quad \omega = \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1, \tag{2-1}$$

with boundary condition $u \to 0$ as $|x| \to \infty$. Here $u \equiv (u_1, u_2)$ denotes the velocity field.

Then we define the stream function $\psi(x, t)$ such that $u(x, t) = \nabla^\perp \psi(x, t)$, with $\nabla^\perp \equiv (\partial_2, -\partial_1)$. It is immediate to see that

$$\omega(x, t) = -\nabla^2 \psi(x, t), \tag{2-2}$$

that is, a Poisson-type equation with ω as a source term. We notice that formally the stream function plays the role of a Hamiltonian; this explains the great interest in the point vortex system in the field of dynamical systems.

By using the definition of a stream function, we find the explicit form of the velocity field by means of the Green's function of (2-2):

$$u(x, t) = \nabla^\perp \psi = \int \nabla^\perp G(r, r')\omega(r')\,dr', \tag{2-3}$$

depending on the initial conditions on the vorticity and the domain. If the initial vorticity field is generated by N disjoint point vortices, we use an initial condition given by a measure

$$\omega(x, 0)\,dx = \sum_{i=1}^{N} a_i \delta_{x_i}(dx), \tag{2-4}$$

where a_i is the vortex intensity of the i-vortex situated at x_i. This is the so-called point vortex system. The dynamics of the N point vortex system is defined by the Green's function of (2-2). It clearly depends on the domain. For example, in the whole plane $\mathbb{R}^2$, the evolution equations for a system of point vortices is given by

$$\frac{dx_i(t)}{dt} = -\nabla^\perp \sum_{j=1;\, j\neq i}^{N} a_j G(x_i(t), x_j(t)), \quad x_i(t=0) = x_i, \tag{2-5}$$

where $G(x_i, x_j) = 1/(2\pi)\ln|x_i(t) - x_j(t)|$ is the Green's function of (2-2) in the plane. It appears as a discrete solution of the Euler equation.

Starting from this mathematical formulation, there are a great number of possible investigations about the point vortex system. In the framework of mathematical fluid mechanics, a wide discussion of the properties of such systems and the rigorous relation with the Euler equation can be found in [Marchioro and Pulvirenti

1994]. In the framework of dynamical systems there are a great number of papers devoted to the analysis of integrability, relative equilibria, and applications; we refer to [Newton 2001].

In the next section we recall the explicit form of the Green's function in the planar, spherical, and hyperbolic cases.

3. Dynamics of point vortices on surfaces of constant curvature

Here we recall the main results about the Green's function of the Poisson equation over surfaces with constant curvature K. First of all, we recall that the three surfaces with constant curvature, a sphere ($K > 0$), a Euclidean plane ($K = 0$), and a hyperbolic plane ($K < 0$), can be considered as three different situations inside a family of Riemannian manifolds with the curvature K as a parameter. We refer to [Kimura 1999] for an unified geometrical setting of this problem. In this work the fundamental solution of the Poisson equation over a spherical surface is given as a function of the geodesic distance r from the north pole of the sphere, that is, $r = \theta R = \theta/\sqrt{K}$, with θ its colatitude and R the radius of the sphere. Then Kimura found, in a direct way, the Green's function for the hyperbolic case as a function of the same variables. We can prove that the Green's function G_K only depends upon the geodesic distance r and is given by

$$2\pi G_{K>0} = -\ln\sin\frac{\sqrt{K}\,r}{2} \qquad \text{for a spherical surface,} \qquad r \in \left(0, \frac{\pi}{\sqrt{K}}\right), \tag{3-1}$$

$$2\pi G_0 = -\ln r \qquad \text{for a plane,} \qquad r \in (0, \infty), \tag{3-2}$$

$$2\pi G_{K<0} = -\ln\tanh\frac{\sqrt{|K|}\,r}{2} \qquad \text{for a hyperbolic surface,} \qquad r \in (0, \infty). \tag{3-3}$$

Then if we take the difference $\Delta_{K>0}$ between (3-1) and (3-2), we obtain

$$\Delta_{K>0}(r) = -\ln\sin\frac{\sqrt{K}\,r}{2} + \ln r = \ln\frac{r}{\sin\left(\frac{1}{2}\sqrt{K}r\right)}. \tag{3-4}$$

This is a continuous function, with a bounded first derivative for $r \in (0, \pi/\sqrt{K})$ where the Green's function is defined, that is, it is also a Lipschitz function. This last statement has a central role in the following discussion. Actually, we can treat the effect of curvature as a Lipschitz perturbation to the Green's function of the planar case. The same reasoning can be applied in the hyperbolic case. In this case we find a Lipschitz function $\Delta_{K<0}$ for $r \in (0, \infty)$.

Moreover, it is simple to check by Taylor expansion that the planar case can be recovered in the limit $K \to 0$. In more detail, when considering the limit $K \to 0$, we obtain:

$$2\pi G_{K>0} = -\ln[\sin(\sqrt{K}r/2)] + \ln(\sqrt{K}/2) \sim -\ln(r) - \frac{K}{24}r^2 + \dots, \tag{3-5}$$

$$2\pi G_{K<0} = -\ln[\tanh(\sqrt{|K|}r/2)] + \ln(\sqrt{|K|}/2) \sim -\ln(r) + \frac{|K|}{12}r^2 + \dots. \tag{3-6}$$

Then it's clear that the effect of the curvature on the dynamics can be parametrized as a smooth perturbation to the Green's function on the plane.

Finally we can write the Green's function of the Poisson equation over a surface with constant curvature as:

$$G(r) = G_0(r) + \Delta_K(r), \tag{3-7}$$

where $G_0 = -1/(2\pi)\ln(r)$ is the Green's function on the plane and $\Delta_K(r)$ is a Lipschitz perturbation dependent on the curvature K as previously defined.

This means that, from (2-3), the velocity field of the fluid over a surface with constant curvature is given by

$$u(x,t) = \nabla^{\perp}\psi(x,t) = u_0(x,t) + u_K(x,t), \tag{3-8}$$

where

$$u_0(x,t) = \int \nabla^{\perp} G_0(r,r')\omega(r')\,dr',$$
$$u_K(x,t) = \int \nabla^{\perp} \Delta_K(r,r')\omega(r')\,dr'.$$

As already discussed, we can treat the contribution u_K due to the curvature effect as a Lipschitz field. Then from a Lagrangian point of view the fluid particle satisfies the following equation:

$$\frac{dx(t)}{dt} = u_0(x,t) + u_K(x,t). \tag{3-9}$$

In the following we will use directly $u(x,t)$ for the velocity field of the planar case.

4. Localization of the vortices under curvature perturbation

In the planar case, we call "localization" the following property of the dynamics of a system of point vortices: the time evolution of N concentrated vortices, according to the Euler equation in the two-dimensional case, remains concentrated in N small disjoint regions of diameter $d(\epsilon) \to 0$ as $\epsilon \to 0$ [Marchioro and Pulvirenti 1993; Marchioro 1998]. This result provides a rigorous connection between the Euler equation and the point vortex model, giving a complete justification for skipping the divergent self-interaction term in the point vortex dynamics (for a full discussion of this point see [Marchioro and Pulvirenti 1994]).

In more detail, we recall the following localization theorem:

Theorem 4.1 [Marchioro 1998]. *Consider an initial datum*

$$\omega_\epsilon(x,0)=\sum_{i=1}^{N}\omega_{\epsilon;i}(x,0) \tag{4-1}$$

where $\omega_{\epsilon;i}(x,0)$ is a function with a definite sign supported in a region $\Lambda_{\epsilon;i}$ such that

$$\Lambda_{\epsilon;i}=\operatorname{supp}\omega_{\epsilon;i}(x,0)\subset\Sigma(z_i|\epsilon),\qquad \Sigma(z_i|\epsilon)\cap\Sigma(z_j|\epsilon)=0 \text{ if } i\neq j, \tag{4-2}$$

for ϵ small enough. Here $\Sigma(z|r)$ denotes the circle of center z and radius r. The intensity of any single vortex is

$$\int dx\,\omega_{\epsilon;i}(x,0)\equiv a_i\in\mathbb{R}, \tag{4-3}$$

independent of ϵ and we assume

$$|\omega_{\epsilon;i}(x,0)|\leq M\epsilon^{-\gamma},\qquad M>0,\qquad \gamma>0. \tag{4-4}$$

Denote by $\omega_\epsilon(x,t)$ the time evolution of (4-1) *according to the Euler equation with boundary condition $u\to 0$ as $|x|\to\infty$. Then, for any fixed time T, for any $\alpha\in\left[0,\frac{1}{3}\right)$ and $0\leq t\leq T$, we have*:

- *For all $d>0$, there exists $\epsilon_0(d,T)$ such that, if $\epsilon<\epsilon_0$, then $\omega_\epsilon(x,t)=\sum_{i=1}^{N}\omega_{\epsilon,i}(x,t)$. Moreover,* $\operatorname{supp}\omega_{\epsilon;i}(x,t)\subset\Sigma(z_i(t)|d)$, *where $d\to 0$ as $\epsilon\to 0$ and $z_i(t)$ is the solution of the differential system*

$$\dot{z}_i(t)=\sum_{j=1;i\neq j}^{N}a_i\nabla^{\perp}G(|z_i-z_j|),\qquad \nabla^{\perp}=(\partial_2,-\partial_1),\qquad z_i(0)=z_i, \tag{4-5}$$

  where $G(\cdot)$ is the Green's function of the Poisson equation in the planar case with vanishing boundary condition at infinity.

- *For any continuous bounded function $f(x)$*

$$\lim_{\epsilon\to 0}\int\omega_\epsilon(x,t)f(t)=\sum_{i}^{N}a_i f(z_i(t)). \tag{4-6}$$

The value of $T>0$ must be such that there are no collapses for any $t<T$; a complete discussion of the existence of such a T is given in [Marchioro and Pulvirenti 1994].

Note that this formulation is an improvement of the previous result stated in [Marchioro and Pulvirenti 1993], giving a much better estimate of the support $d(\epsilon)$ of the vortices.

The main step for the proof of this theorem is to study the localization of a single vortex, simulating the effect of the other $N-1$ vortices with a Lipschitz external field $F(x,t)$. In this case the motion of the vortex is described by the Euler equation in the weak form:

$$\frac{d}{dt}\omega(f) = \omega[(u+F)\cdot\nabla f], \tag{4-7}$$

where $\omega(f(x)) = \int dx\,\omega(x,t)f(x)$ and $f(x)$ is a bounded smooth function. From a Lagrangian point of view, we have

$$\frac{dx}{dt} = u(x,t) + F(x,t). \tag{4-8}$$

Then, defining the center of vorticity as

$$B_\epsilon(t) \equiv \int x\omega_\epsilon(x,t)\,dt, \tag{4-9}$$

we state the following theorem about the localization of a single blob:

Theorem 4.2. *Suppose that*

$$\mathrm{supp}|\omega_\epsilon(x,0)| \subset \Sigma(x^*|\epsilon) \tag{4-10}$$

and

$$|\omega_\epsilon(x,0)| \le M\epsilon^{-\gamma}, \quad M>0, \quad \gamma>0, \quad \int dx\,\omega_\epsilon(x,0) = 1. \tag{4-11}$$

Then, there exists $C(\beta,T)>0$, *with* $\beta>0$, *such that for* $0\le t\le T$

$$\mathrm{supp}|\omega_\epsilon(x,t)| \subset \Sigma(B(t)|d) \tag{4-12}$$

where

$$d = C(\beta,T)\epsilon^\beta, \tag{4-13}$$

and $B(t)$ *is the solution of the ordinary differential equation*

$$\frac{dB(t)}{dt} = F(B(t),t), \tag{4-14}$$

$$B(0) = x^*. \tag{4-15}$$

We refer to [Marchioro 1998] for the complete proof of this theorem, and in the Appendix we sketch the proof for the utility of the reader. Here we again remark that one of the central assumptions is about the Lipschitz continuity of the simulating external field.

Starting from these results, we can finally state our main result: the localization property of point vortices is preserved in surfaces with constant curvature.

Theorem 4.3. *Consider an initial datum*

$$\omega_\epsilon(x,0)=\sum_{i=1}^{N}\omega_{\epsilon;i}(x,0), \tag{4-16}$$

where $\omega_{\epsilon;i}(x,0)$ is a function with a definite sign supported in a region $\Lambda_{\epsilon;i}$ such that

$$\Lambda_{\epsilon;i}=\operatorname{supp}\omega_{\epsilon;i}(x,0)\subset\Sigma(z_i|\epsilon),\quad \Sigma(z_i|\epsilon)\cap\Sigma(z_j|\epsilon)=0 \ \textit{if}\ i\neq j, \tag{4-17}$$

for ϵ small enough. The intensity of any single vortex is

$$\int dx\,\omega_{\epsilon;i}(x,0)\equiv a_i\in\mathbb{R}, \tag{4-18}$$

independent of ϵ and we assume

$$|\omega_{\epsilon;i}(x,0)|\le M\epsilon^{-\gamma},\quad M>0,\quad \gamma>0. \tag{4-19}$$

Denote by $\omega_\epsilon(x,t)$ the time evolution of (4-1) *on a surface of constant curvature K according to the Euler equation*, *then Theorem 4.1 holds.*

The proof of this theorem is similar to that of the planar case, considering first of all the localization of a single vortex. We have shown that the effect of curvature on the Green's function can be treated as a Lipschitz perturbation to the Green's function of the planar case. Then the localization of a single vortex is a corollary of Theorem 4.2

Corollary 4.4. *Consider a single point vortex such that*

$$\begin{gathered}\operatorname{supp}|\omega_\epsilon(x,0)|\subset\Sigma(x^*|\epsilon),\quad |\omega_\epsilon(x,0)|\le M\epsilon^{-\gamma},\\ M>0,\quad \gamma>0,\quad \int dx\,\omega_\epsilon(x,0)=1.\end{gathered} \tag{4-20}$$

Denote by $\omega_\epsilon(x,t)$ the time evolution on a surface with constant curvature K, according to the Euler equation. Then, there exists $C(\beta,T)>0$, with $\beta>0$, such that for $0\le t\le T$

$$\operatorname{supp}|\omega_\epsilon(x,t)|\subset\Sigma(B(t)|d), \tag{4-21}$$

where

$$d=C(\beta,T)\epsilon^{\beta}, \tag{4-22}$$

and $B(t)$ is the solution of the ordinary differential equation

$$\frac{dB(t)}{dt}=F_K(B(t),t),\quad B(0)=x^*. \tag{4-23}$$

where $F_K(x,t)$ is a Lipschitz field including in a single term the effect of the curvature (depending on K) and the effect of the other $N-1$ vortices on the dynamics of the single vortex.

The main improvement was proved in Section 3: the velocity term linked to the curvature effect is a Lipschitz function. Then we include the effect of curvature on the motion in a single Lipschitz term in (4-7). It is simple then to come back to the general case of the N vortices and to prove the main result.

We conclude that the localization theorem for point vortices moving on surfaces with constant curvature is a consequence of the analysis given in Section 3 about the effect of the curvature on vortex dynamics. Then its proof is exactly the same as that of the planar case discussed in [Marchioro 1998].

This result is valid for any regular surface. Actually, it is always possible to approximate locally these surfaces with manifolds with constant curvature K. Then we can develop exactly the same reasoning, including the curvature effect in an external Lipschitz continuous field. Moreover, the localization holds also in the presence of internal frontiers such as continents on the Earth's surface. Again, the physical meaning of this rigorous result is that it permits one to skip the singular part of the self-interacting term in the point vortex model, previously neglected in the basis of heuristic physical reasoning.

Appendix: Proof of Theorem 4.2

Here we give a synthetic idea of the rather technical proof of the localization theorem stated in [Marchioro 1998], recalling the fundamental steps. The main difficulty is due to the singularity of the kernel in the velocity expression

$$u(x,t) = \int K(x-y)\omega_\epsilon(y,t)\,dy, \tag{A.1}$$

where

$$K(x-y) = \nabla^\perp G(x-y) = \frac{\nabla^\perp \ln|x-y|}{2\pi}$$

in the planar case without boundaries.

First we introduce the moment of inertia I_ϵ with respect to the center of vorticity defined in (4-9):

$$I_\epsilon = \int \omega_\epsilon(x,t)(x-B_\epsilon(t))^2\,dx. \tag{A.2}$$

We want to show that the main part of the vorticity is concentrated around the center of vorticity. It is simple to prove that if $F=0$ then B_ϵ and I_ϵ are constant along the motion, bringing us to (4-14).

If $F \neq 0$ then

$$\frac{dI}{dt} = 2\int (x - B_\epsilon(t))F(x,t)\omega_\epsilon(x,t)\,dx. \tag{A.3}$$

Then, by using the Lipschitz condition on $F(x,t)$ we find

$$\left|\frac{dI}{dt}\right| \leq 2L\int (x - B_\epsilon(t))^2\omega_\epsilon(x,t)\,dx = 2LI_\epsilon(t), \tag{A.4}$$

and integrating we obtain

$$\left|\frac{dI}{dt}\right| \leq I_0 e^{2Lt}, \tag{A.5}$$

so that

$$\lim_{\epsilon\to 0} I_\epsilon(t) = 0 \text{ at least as } \epsilon^2. \tag{A.6}$$

Hence we find that the main part of the vorticity remains concentrated around the center of vorticity. However we have to give an estimate of the mass and velocity of the filaments of vorticity generated by fluid particles near the boundaries and spreading out from the initially concentrated field. With this purpose we prove that the mass of vorticity near the boundary of the support is very small when $\epsilon \to 0$. Here the main technical complication is due to the singularity of the kernel in (A.1). First we introduce a nonnegative function $W_R \in C^\infty(\mathbb{R}^2)$ satisfying the following conditions, for a fixed $C_1 > 0$:

$$W_R(r) = \begin{cases} 1 & \text{if } |r| < R, \\ 0 & \text{if } |r| > 2R, \end{cases} \tag{A.7}$$

$$|\nabla W_R(r)| < \frac{C_1}{R}, \tag{A.8}$$

$$|\nabla W_R(r) - \nabla W_R(r')| < \frac{C_1}{R^2}|r - r'|. \tag{A.9}$$

Then we define a regularized measure of the mass of vorticity outside $\Sigma(B_\epsilon(t)|r)$:

$$\mu_t(R) = 1 - \int dx\; W_R(x - B_\epsilon(t))\omega_\epsilon(x,t), \tag{A.10}$$

such that if $\operatorname{supp}\omega_\epsilon(x,t) \subset \Sigma(B_\epsilon(t)|r)$ then $\mu_t(R) = 0$. Hence it gives a direct measure of the localization of the vorticity field.

We evaluate the growth in time of such a measure:

$$\frac{d\mu_t}{dt} = -\int dx\; \nabla W_R(x - B_\epsilon(t))\Big(u(x,t) + F(x,t) - \frac{dB}{dt}\Big)\omega_\epsilon(x,t). \tag{A.11}$$

Using (4-14) and (A.1), we obtain

$$\frac{d\mu_t}{dt} = -\int dx\, \omega_\epsilon(x,t)\nabla W_R(x-B_\epsilon(t)) \int dy\, K(x-y)\omega_\epsilon(y,t)$$
$$-\int dx\, \omega_\epsilon(x,t)\nabla W_R(x-B_\epsilon(t)) \int dy\, \omega_\epsilon(y,t)(F(x,t)-F(y,t)). \quad \text{(A.12)}$$

To give an estimate to the first term of (A.12), we split the integration domain into many different rings, defined by the following sets:

- if $h<n$, $T_h \equiv \{(x,y) | x \notin \Sigma(B_\epsilon(t)|R),\ y \in \Sigma(B_\epsilon(t)|a_h) - \Sigma(B_\epsilon(t)|a_{h-1})\}$,
- if $h=n$, $T_n \equiv \{(x,y) | x \notin \Sigma(B_\epsilon(t)|R),\ y \notin \Sigma(B_\epsilon(t)|a_{n-1})\}$,
- if $h<n$, $S_h \equiv \{(x,y) | y \notin \Sigma(B_\epsilon(t)|R),\ x \in \Sigma(B_\epsilon(t)|a_h) - \Sigma(B_\epsilon(t)|a_{h-1})\}$,
- if $h=n$, $S_n \equiv \{(x,y) | y \notin \Sigma(B_\epsilon(t)|R),\ x \notin \Sigma(B_\epsilon(t)|a_{n-1})\}$,

where $a_0=0$, $a_1=1$, and $a_k=2a_{k-1}$.

Starting from the set T_h, where $\nabla W_R(y)=0$, we obtain

$$\left| \int_D dy\omega_\epsilon(x,t)\omega_\epsilon(y,t)\nabla W_R(x-B_\epsilon(t))K(x-B_\epsilon(t)) \right.$$
$$\left. + \int_D dy\omega_\epsilon(x,t)\omega_\epsilon(y,t)\nabla W_R(x-B_\epsilon(t))\big[K(x-y)-K(x-B_\epsilon(t))\big] \right|, \quad \text{(A.13)}$$

where $D \equiv \Sigma(B_\epsilon(t)|a_h) - \Sigma(B_\epsilon(t)|a_{h-1})$.

In (A.13) the first term is null because $\nabla W_R(x)\cdot K(x)=0$. Moreover, we observe that

$$|K(x-y)-K(x)| < \text{const.}\, \frac{\rho}{|x|(|x|-\rho)} \quad \text{if} \quad |y|<\rho<|x|, \quad \text{(A.14)}$$

so the contribution to (A.13) due to T_h is bounded by

$$\text{(A.13)} \le \text{const.}\, \frac{m_t(R)}{R}\left\{\frac{\epsilon}{R^2} + \sum_{h=2}^{n-1} \frac{a_h\epsilon^2}{R(R-a_h)a_{h-1}^2}\right\} \le \frac{\epsilon}{R^3} m_t(R), \quad \text{(A.15)}$$

where $m_t(R) = 1-\int \omega_\epsilon(y,t)\,dy$ is the vorticity mass outside $\Sigma(B_\epsilon(t)|R)$.

The contribution due to T_n is simply bounded by using the fact that $\nabla W_R(r)$ removes the singularity of the kernel, because

$$|(\nabla W_R(x)-\nabla W_R(y))K(x-y)| \le \frac{1}{R^2} \quad \text{where} \quad R=|x-B_\epsilon(t)|. \quad \text{(A.16)}$$

Then it is also simple to give an estimate to the second term of (A.12) by using the Lipschitz continuity of $F(x,t)$. Recollecting all the terms we have

$$\left|\frac{d\mu(R)}{dt}\right| \le A(R,\epsilon)m_t(R), \quad \text{(A.17)}$$

where the explicit expression of $A(R,\epsilon)$ comes directly from the previous reasoning by simple calculation. We observe, from the definition of regularized mass, that

$$m_t(R) \le \mu_t\Big(\frac{R}{2}\Big); \tag{A.18}$$

using this inequality in (A.17) and integrating we obtain

$$\mu_t(R) \le \mu_0(R) + A(R)\int_0^t \mu_t\Big(\frac{R}{2}\Big)dt. \tag{A.19}$$

Then we can use an iterative procedure:

$$\begin{aligned}\mu_t(R) &\le \mu_0(R) + A(R)\int_0^t \mu_t\Big(\frac{R}{2}\Big)dt\\ &\le \mu_0(R) + \mu_0\Big(\frac{R}{2}\Big)A(R)\int dt + A(R)A\Big(\frac{R}{2}\Big)\int_0^t dt_1\int_0^{t_1} dt\mu_t\Big(\frac{R}{4}\Big),\end{aligned} \tag{A.20}$$

choosing the number n of iterations so that $n\to\infty$ as $\epsilon\to 0$ and $\mu_0(R2^{-n})=0$. We finally have that

$$m_t(R) \le \frac{(\text{const.})^n}{n!} \to 0 \text{ as } \epsilon\to 0 \text{ faster than any power of } \epsilon. \tag{A.21}$$

This means that the vorticity mass becomes very small near the boundary if we take strong concentrations, that is, $\epsilon\to 0$. Then it is also simple to prove that the velocity field generated by the fluid particles near the boundary vanishes for strong concentrations. Finally the main theorem is achieved essentially from these results.

Here we have just recalled the main steps of the proof, leaving to the reader the most part of calculations. As just seen, the property of the solution about the Lipschitz continuity of the field generated by the other $N-1$ vortices is central for the proof of the localization.

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