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A DIRECT APPROACH TO NONLINEAR SHELLS WITH APPLICATION TO SURFACE-SUBSTRATE INTERACTIONS

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The paper develops a direct, intrinsic approach to the equilibrium equations of bodies coated by a thin film with a nonlinear shell like structure. The forms of the equations in the reference and actual configurations are considered. The equations are shown to coincide with those obtained by using coordinate systems on the film or on the thin shell.

1. Introduction

This paper presents equilibrium equations of the system consisting of a bulk solid and attached boundary film. The film is assumed to exhibit resistance to flexural deformations in that its energy is a nonlinear function of the boundary first-order deformation gradient and of a second-order tensor that represents a suitable version of its curvature in the deformed state. Such a theory was developed by Steigmann and Ogden [1997a; 1997b; 1999] in dimensions $n = 2$ (plane deformations) and $n = 3$ (full three-dimensional deformations). The case $n = 2$ has also been treated by Fried and Todres [Fried and Todres 2005]. The cited works by Steigmann and Ogden generalize the situation in [Gurtin and Murdoch 1975; Podio-Guidugli 1988; Podio-Guidugli and Vergara-Caffarelli 1990; Steigmann and Li 1995; Steinmann 2008], where the film is modeled as a nonlinear membrane, i.e., its energy is assumed to depend only on the first surface deformation gradient.

Steigmann and Ogden used a variational principle to derive the equilibrium equations (among other things) and to show that they coincide with those of thin nonlinear shells; see [Sanders 1963; Cohen and De Silva 1966; Naghdi 1971; Pietraszkiewicz 1989].

The purpose of the present paper is to derive a direct, index-free, form of the balance equations. This approach allows a more unified understanding of the underlying mechanics than the coordinate-based approach, where one is typically forced to cover the manifold with coordinate patches.

The formalism I adopt is different from the intrinsic approaches in [Delfour and Zolésio 1997] and [Favata and Podio-Guidugli 2011]. The basic feature of

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the present work is the treatment of the surface quantities as tensors in the three-dimensional space and not just on the tangent space to the shell at the given point. This allows for a fully tensorial form of the equilibrium equations. Both the referential and the actual configurations are considered. The main results are the intrinsic form of the first variation of the surface energy Proposition 5.2 and the associated equilibrium equations (15), the fully intrinsic form (17) of the effective second-order stress tensor, the spatial intrinsic form of the equilibrium equations Proposition 6.1, and the tangential and normal components of the equilibrium equations Proposition 6.2. Up to the last mentioned item, no coordinate system is invoked to derive the results. However, for reasons of comparison with the existing coordinate approaches, in Section 7 I give the coordinate form of the main results and show that they coincide with those obtained by different methods.

As for the intrinsic tensor calculus, only tensors in euclidean space will be employed, of orders 0, 1, 2, and 3. Tensors of orders 0 and 1 are scalars and vectors from R^n ($n \geq 2$; typically $n = 2$ or $n = 3$). Second-order tensors are either R-valued bilinear forms on $\mathrm{R}^n \times \mathrm{R}^n$ or linear transformations from R^n to R^n; we do not distinguish these two interpretations graphically. The set of all second-order tensors is denoted by Lin. The third-order tensors are mostly interpreted as R^n-valued bilinear forms on $\mathrm{R}^n \times \mathrm{R}^n$. The set of all third-order tensors is denoted by T.

2. Geometry of deformation of a coated body

We identify the material points of the body with their positions $\mathbf{x}$ in a reference configuration $\Omega \subset \mathrm{R}^n$, where $n \geq 2$ is arbitrary but in applications $n = 2$ or $n = 3$. We assume that Ω is a bounded open set with sufficiently smooth boundary $\partial\Omega$ with the unit outer normal $\mathbb{n}$. We consider the bulk solid to be coated with an elastic surface $\mathcal{S} \subset \partial\Omega$. We assume that $\mathcal{S}$ is a relatively open subset of $\partial\Omega$ with a smooth boundary $\partial\mathcal{S}$ without corners.

The deformation of the coated body is described by a sufficiently smooth map from the closure $\operatorname{cl}\Omega$ of Ω so that the deformation $\mathbb{y}$ of the coating, i.e., the restriction of $\mathbf{y}$ to the closure $\operatorname{cl}\mathcal{S}$ of $\mathcal{S}$, is well defined and sufficiently smooth.

The deformation $\mathbf{y}$ of the bulk body is described by the bulk deformation gradient

$$\mathbf{F} = \nabla\mathbf{y},$$

which is assumed to exist, be continuous, and of positive determinant, at every point of $\operatorname{cl}\Omega$. Here ∇ indicates the gradient with respect to the position in the reference configuration and at a given point of $\operatorname{cl}\Omega$, $\mathbf{F}$ is interpreted as a linear transformation from R^n to R^n.

The surface deformation $\mathbb{y}$ of the coating is described by the surface deformation gradient and by the referential version of the curvature tensor of the coating in the deformed configuration to be defined below. For the referential surface $\mathcal{S}$ of the coating we introduce the curvature tensor $\mathbb{L}$ (also called the Weingarten tensor), defined by

$$\mathbb{L} = \nabla \mathbb{m}, \tag{1}$$

where ∇ denotes the surface gradient. We here adopt the following convention for the surface differentiation of maps with values is a finite-dimensional vector space V defined on a manifold $\mathcal{M}$ of dimension k in R^n: if $f : \mathcal{M} \to V$ is a smooth map then for every $\mathbf{x} \in \mathcal{M}$ the surface gradient $\nabla f(\mathbf{x})$ of f at $\mathbf{x}$ is a linear map from the whole space R^n to V which satisfies $\nabla f(\mathbf{x})\mathbb{P}(\mathbf{x}) = \nabla f(\mathbf{x})$, where $\mathbb{P}(\mathbf{x})$ is a projection from R^n onto the tangent space $\operatorname{Tan}(\mathcal{M}, \mathbf{x})$ of $\mathcal{M}$ at $\mathbf{x}$, and

$$\lim_{\substack{\bar{\mathbf{x}} \to \mathbf{x} \\ \bar{\mathbf{x}} \in \mathcal{M}, \bar{\mathbf{x}} \neq \mathbf{x}}} \frac{|f(\bar{\mathbf{x}}) - f(\mathbf{x}) - \nabla f(\mathbf{x})(\bar{\mathbf{x}} - \mathbf{x})|}{|\bar{\mathbf{x}} - \mathbf{x}|} = 0.$$

This convention differs from the alternative view [Federer 1969, Subsection 3.1.22; Gurtin and Murdoch 1975; Gurtin 2000], where the surface gradient at the given point is interpreted as a linear transformation from $\operatorname{Tan}(\mathcal{M}, \mathbf{x})$ to V. The latter is just the restriction of our $\nabla f(\mathbf{x})$ to $\operatorname{Tan}(\mathcal{M}, \mathbf{x})$. Below we apply the same convention to the derivatives of the response functions for the surface energy with respect to the surface deformation gradient and curvature. Our convention has the advantage that the surface gradient at different points of $\mathcal{M}$ is an element of the same linear space and one can thus iterate the procedure to define the second surface gradient $\nabla^2 f(\mathbf{x})$ of f at $x \in \mathcal{M}$ as the surface gradient of the surface gradient. Thus $\nabla^2 f(\mathbf{x}) = \nabla(\nabla f)(\mathbf{x})$ and we interpret $\nabla^2 f(\mathbf{x})$ as a bilinear transformation from $\mathrm{R}^n \times \mathrm{R}^n$ to V, defined by

$$\nabla^2 f(\mathbf{x})(\mathbf{a}, \mathbf{b}) = \nabla(\nabla f \mathbf{b})\mathbf{a}$$

for every $\mathbf{a}$, $\mathbf{b} \in \mathrm{R}^n$. A comparison with [Murdoch and Cohen 1979/80] shows that the second gradient as defined there is similarly the present second gradient restricted to $\operatorname{Tan}(\mathcal{M}, \mathbf{x}) \times \operatorname{Tan}(\mathcal{M}, \mathbf{x})$. In [Steigmann and Ogden 1999] the notion of the second gradient is employed in the special case of the second surface gradient. We shall see below that this notion of the second surface gradient coincides with the present one also as far as the definition domain is concerned. We note that the bilinear map $\nabla^2 f(\mathbf{x})$ is generally nonsymmetric, but its restriction to $\operatorname{Tan}(\mathcal{M}, \mathbf{x}) \times \operatorname{Tan}(\mathcal{M}, \mathbf{x})$ is symmetric. If $V = \mathrm{R}^n$, i.e., if f is a scalar function, we identify the linear transformation $\nabla f(\mathbf{x})$ from R^n to R with an equally denoted element of R^n via the identification $\nabla f(\mathbf{x})\mathbf{a} = \nabla f(\mathbf{x}) \cdot \mathbf{a}$ for each $\mathbf{a} \in \mathrm{R}^n$. The equation $\nabla f(\mathbf{x})\mathbb{P}(\mathbf{x}) = \nabla f(\mathbf{x})$ then reduces to $\mathbb{P}(\mathbf{x})\nabla f(\mathbf{x}) = \nabla f(\mathbf{x})$, i.e., $\nabla f(\mathbf{x})$ is an element of the tangent space to $\mathcal{M}$ at $\mathbf{x}$. Similarly, the second gradient $\nabla^2 f(\mathbf{x})$ is identified with a second-order tensor in Lin via $\nabla^2 f(\mathbf{x})(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \nabla^2 f(\mathbf{x})\mathbf{b}$ for each $\mathbf{a}, \mathbf{b} \in \mathrm{R}^n$.

We refer to [Šilhavý 2011, Appendix A and B] for more details on the present conventions on the derivatives and gradients.

We define the surface deformation gradient $\mathbb{F}$ of $\mathbb{y}$ by

$$\mathbb{F} = \nabla \mathbb{y}.$$

At the given point $\mathbf{x}$ of $\mathcal{S}$, $\mathbb{F}$ is a second-order tensor on R^n which is assumed to map $\mathrm{Tan}(\mathcal{S}, \mathbf{x})$ onto $\mathrm{Tan}(\bar{\mathcal{S}}, \mathbb{y}(\mathbf{x}))$, where $\bar{\mathcal{S}} = \mathbb{y}(\mathcal{S})$ is the actual configuration of the coating. We denote by $\mathbb{P}$ the orthogonal projection onto the tangent space to $\mathcal{S}$ and by $\overline{\mathbb{P}}$ the orthogonal projection onto the tangent space to $\bar{\mathcal{S}}$. Then we have

$$\mathbb{F}\mathbb{P} = \mathbb{F}, \quad \overline{\mathbb{P}}\mathbb{F} = \mathbb{F}.$$

The tensor $\mathbb{F}$ is always noninvertible. However, we denote by $\mathbb{F}^{-1}$ the pseudoinverse, which at a given point of $\mathcal{S}$ is a linear transformation from R^n to R^n satisfying

$$\mathbb{F}^{-1}\mathbb{F} = \mathbb{P}, \quad \mathbb{F}\mathbb{F}^{-1} = \overline{\mathbb{P}}.$$

Then $\mathbb{F}^{-1}$ always exists and is determined uniquely. If, for a given point of $\mathcal{S}$, $\mathbf{F}$ is the bulk deformation gradient at that point, then

$$\mathbb{F} = \mathbf{F}\mathbb{P}, \quad \mathbb{F}^{-1} = \mathbf{F}^{-1}\overline{\mathbb{P}},$$

where $\mathbf{F}^{-1}$ is the inverse of $\mathbf{F}$ in the standard sense.

We assume that the response of the coating depends on the first and second deformation gradients, but on the second deformation gradient $\nabla^2\mathbb{y}$ only through a combination that can be regarded as the curvature tensor of the deformed configuration $\bar{\mathcal{S}}$ viewed from the reference configuration. That is, we introduce a bilinear form $\mathbb{K}$ which is identified with an equally denoted second-order tensor by

$$\mathbb{K}(\mathbf{a}, \mathbf{b}) = \bar{\mathbb{n}} \cdot \nabla^2\mathbb{y}(\mathbb{P}\mathbf{a}, \mathbb{P}\mathbf{b})$$

for every $\mathbf{a}, \mathbf{b} \in \mathrm{R}^n$, where

$$\bar{\mathbb{n}} = \frac{\mathrm{cof}\,\mathbb{F}\,\mathbb{n}}{|\mathrm{cof}\,\mathbb{F}\,\mathbb{n}|}$$

is the unit outer normal to $\bar{\mathcal{S}}$. Here $\mathrm{cof}\,\mathbb{F}$ is the cofactor tensor of $\mathbb{F}$. If for any map B on $\mathrm{R}^n \times \mathrm{R}^n$ we introduce the symbol $\mathrm{B} \circ (\mathbb{P}, \mathbb{P})$ to denote the map on $\mathrm{R}^n \times \mathrm{R}^n$ given by

$$\mathrm{B} \circ (\mathbb{P}, \mathbb{P})(\mathbf{a}, \mathbf{b}) = \mathrm{B}(\mathbb{P}\mathbf{a}, \mathbb{P}\mathbf{b})$$

for any $\mathbf{a}, \mathbf{b} \in \mathrm{R}^n$, then we have

$$\mathbb{K} = \bar{\mathbb{n}} \cdot \nabla^2\mathbb{y} \circ (\mathbb{P}, \mathbb{P})$$

and

$$\mathbb{K} = \mathbb{K} \circ (\mathbb{P}, \mathbb{P}).$$

Also, when viewed as a second-order tensor, $\mathbb{K}$ satisfies

$$\mathbb{P}\mathbb{K}\mathbb{P} = \mathbb{K}.$$

It is useful to note that the curvature $\bar{\mathbb{L}} = \overline{\nabla}\bar{\mathbb{n}}$ of the surface $\bar{\mathcal{S}}$ is

$$\bar{\mathbb{L}} = -\mathbb{F}^{-\mathrm{T}}\mathbb{K}\mathbb{F}^{-1}.$$

Here $\overline{\nabla}$ is the surface gradient on $\bar{\mathcal{S}}$, i.e., the surface gradient as defined above, but for maps on $\mathcal{M} = \bar{\mathcal{S}}$, and $\mathbb{F}^{-\mathrm{T}} := (\mathbb{F}^{-1})^{\mathrm{T}}$ where T denotes the transposition.

If $\mathcal{M} \subset \mathrm{R}^n$ is a smooth manifold of dimension k and if

$$\mathbb{Q} : \mathcal{M} \to \mathrm{Lin}(\mathrm{R}^n, V) \tag{2}$$

is a map on $\mathcal{M}$ with values in the space $\mathrm{Lin}(\mathrm{R}^n, V)$ of linear transformations from R^n to a finite-dimensional inner product space V, we define the surface divergence $\mathbb{div}\,\mathbb{Q} : \mathcal{M} \to V$ by

$$\mathbb{a} \cdot \mathbb{div}\,\mathbb{Q} = \mathrm{tr}\big(\nabla(\mathbb{Q}^{\mathrm{T}}\mathbb{a})\big) \tag{3}$$

for each $\mathbb{a} \in V$ where the transpose

$$\mathbb{Q}^{\mathrm{T}} : \mathcal{M} \to \mathrm{Lin}(V, \mathrm{R}^n)$$

is defined by $\mathbf{b} \cdot \mathbb{Q}^{\mathrm{T}}\mathbb{a} = \mathbb{Q}\mathbf{b} \cdot \mathbb{a}$ for each $\mathbf{b} \in \mathrm{R}^n$ and $\mathbb{a} \in V$. It follows directly from the definition that if $\mathbb{a} : \mathcal{M} \to V$ then

$$\mathbb{div}(\mathbb{Q}^{\mathrm{T}}\mathbb{a}) = \mathbb{a} \cdot \mathbb{div}\,\mathbb{Q} + \mathbb{Q} \cdot \nabla\mathbb{a}, \tag{4}$$

where

$$\mathbb{Q} \cdot \nabla\mathbb{a} := \mathrm{tr}\big(\mathbb{Q}^{\mathrm{T}}\nabla\mathbb{a}\big).$$

If $V = \mathrm{R}$, we identify $\mathbb{Q} : \mathcal{M} \to \mathrm{Lin}(\mathrm{R}^n, \mathrm{R})$ with a vector field $\mathbb{q} : \mathcal{M} \to \mathrm{R}^n$ by $\mathbb{Q}\mathbf{a} = \mathbb{q} \cdot \mathbf{a}$ for each $\mathbf{a} \in \mathrm{R}^n$ and define the divergence of $\mathbb{q}$ to be the divergence of $\mathbb{Q}$; thus $\mathbb{div}\,\mathbb{q}$ is a scalar field defined by

$$\mathbb{div}\,\mathbb{q} = \mathrm{tr}(\nabla\mathbb{q})$$

and (3) can be rewritten as $\mathbb{a} \cdot \mathbb{div}\,\mathbb{Q} = \mathbb{div}(\mathbb{Q}^{\mathrm{T}}\mathbb{a})$ for each $\mathbb{a} \in V$.

We say that $\mathbb{Q}$ as in (2) is superficial if $\mathbb{Q} = \mathbb{Q}\mathbb{P}$. In particular if $\mathbb{Q} : \mathcal{M} \to \mathrm{Lin}(\mathrm{R}^n, \mathrm{R})$ and $\mathbb{q} : \mathcal{M} \to \mathrm{R}^n$ are related as above, $\mathbb{Q}$ is superficial if and only if $\mathbb{q}$ is tangential, i.e., $\mathbb{q}$ is an element of the tangent space at every point: $\mathbb{P}\mathbb{q} = \mathbb{q}$.

We define the relative boundary $\partial\mathcal{M}$ of $\mathcal{M}$ by $\partial\mathcal{M} = \mathrm{cl}\,\mathcal{M} \setminus \mathcal{M}$ where $\mathrm{cl}\,\mathcal{M}$ is the closure of $\mathcal{M}$ in R^n. We assume that $\partial\mathcal{M}$ is sufficiently smooth. We denote by $\mathbb{m}$ the relative normal to $\partial\mathcal{M}$. This is a map which, at a given point of $\partial\mathcal{M}$, is an element to the tangent space to $\mathcal{M}$ defined, e.g., by $\mathbb{m} = \nabla\varphi/|\nabla\varphi|$ where φ is a function defined locally on $\mathcal{M}$ such that the equation $\varphi = 0$ expresses locally $\partial\mathcal{M}$. We here assume that $\mathcal{M}$ can be extended to a smooth manifold of dimension

k in R^n which contains $\mathrm{cl}\,\mathcal{M}$; this makes the tangent space to $\mathcal{M}$ defined also at the points of $\partial\mathcal{M}$ and the equation $\varphi = 0$ makes sense. The surface divergence theorem asserts that if $\mathbb{Q}$ as in (2) is superficial then

$$\int_{\mathcal{M}} \mathbb{div}\,\mathbb{Q}\,d\mathcal{H}^k = \int_{\partial\mathcal{M}} \mathbb{Q}\mathbb{m}\,d\mathcal{H}^{k-1}.$$

Here $\mathcal{H}^k$ is the k-dimensional area measure on $\mathcal{M}$ and $\mathcal{H}^{k-1}$ the $(k-1)$-dimensional area measure on $\partial\mathcal{M}$.

Furthermore, the surface Piola transformation asserts that if $\mathbb{Q}$ is as in (2) is superficial, if $\mathbf{\Phi} : \mathcal{M} \to \overline{\mathcal{M}} := \mathbf{\Phi}(\mathcal{M})$ is a diffeomorphism and if $\overline{\mathbb{Q}}$ is a field defined on $\overline{\mathcal{M}}$ by

$$\overline{\mathbb{Q}} = \mathbb{j}^{-1}\mathbb{Q}\mathbb{F}^{\mathrm{T}},$$

where $\mathbb{j}$ is the jacobian of $\mathbf{\Phi}$, $\mathbb{F} := \nabla\mathbf{\Phi}$, then $\overline{\mathbb{Q}}$ is superficial and

$$\overline{\mathbb{div}}\,\overline{\mathbb{Q}} = \mathbb{j}^{-1}\,\mathbb{div}\,\mathbb{Q} \tag{5}$$

where $\overline{\mathbb{div}}$ is the surface divergence on $\overline{\mathcal{M}}$, i.e., the surface divergence as defined above, but for fields defined on $\overline{\mathcal{M}}$. Below we need the cases $\mathcal{M} = \mathcal{S}$ and $\mathcal{M} = \partial\mathcal{S}$. Moreover, we shall employ $V = \mathrm{R}$, $V = \mathrm{R}^n$ and $V = \mathrm{Lin}$, i.e., $\mathbb{Q}$ will be a vector field, second-order tensor field and third-order tensor field. We refer to [Marsden and Hughes 1983, Chapter 1] for abstract formulations of Stokes' theorem and surface Piola transformation on manifolds from which the present euclidean cases follow.

3. Constitutive assumptions

We assume that the bulk body is made of a nonlinear hyperelastic material with the bulk stored energy $\tilde{f} : \mathrm{Lin}_+ \to \mathrm{R}$, where Lin_+ is the set of all second-order tensors with positive determinant. For a given deformation $\mathbf{y} : \mathrm{cl}\,\Omega \to \mathrm{R}^n$ the stored energy field is given by the constitutive equation

$$f(\mathbf{x}) = \tilde{f}(\mathbf{F}(\mathbf{x})), \quad \mathbf{x} \in \mathrm{cl}\,\Omega, \tag{6}$$

where $\mathbf{F}$ is the bulk deformation gradient. For the coating $\mathcal{S}$ we assume that for each $\mathbf{x} \in \mathcal{S}$ we have a surface stored energy function $\tilde{\mathbb{f}}_{\mathbf{x}} : D_{\mathbf{x}} \to \mathrm{R}^n$ where $D_{\mathbf{x}}$ is the set of all pairs $(\mathbb{F}, \mathbb{K}) \in \mathrm{Lin} \times \mathrm{Lin}$ such that $\mathbb{F}\mathbb{P}(\mathbf{x}) = \mathbb{F}$ and $\mathbb{K}$ is symmetric and satisfies $\mathbb{P}(\mathbf{x})\mathbb{K}\mathbb{P}(\mathbf{x}) = \mathbb{K}$. For a given deformation $\mathbb{y} : \mathcal{S} \to \mathrm{R}^n$ the field of superficial stored energy $\mathbb{f}$ on $\mathcal{S}$ is given by the constitutive equation

$$\mathbb{f}(\mathbf{x}) = \tilde{\mathbb{f}}_{\mathbf{x}}(\mathbb{F}(\mathbf{x}), \mathbb{K}(\mathbf{x})), \quad \mathbf{x} \in \mathcal{S}, \tag{7}$$

where $\mathbb{F}$ and $\mathbb{K}$ are defined in Section 2. We note that the response function for the superficial stored energy depends on $\mathbf{x}$ since we are forced to assume that the domain $D_{\mathbf{x}}$ is different for different $\mathbf{x} \in \mathcal{S}$. The same applies for the derivatives of $\tilde{\mathrm{f}}_{\mathbf{x}}$. However, below we simplify the notation and suppress the dependence of $\tilde{\mathrm{f}}_{\mathbf{x}}$ on $\mathbf{x}$ and write simply $\tilde{\mathrm{f}}$ in place of $\tilde{\mathrm{f}}_{\mathbf{x}}$. The same convention applies for the derivatives of $\tilde{\mathrm{f}}$. The constitutive assumption (7) is employed in [Steigmann and Ogden 1999] in a coordinate form, who refer to [Hilgers and Pipkin 1992; Cohen and De Silva 1966] for earlier employments of the same hypothesis. In $n = 2$ (plane deformations of the bulk body), equivalent hypotheses have been made in [Steigmann and Ogden 1997a; 1997b; Fried and Todres 2005]. See Section 7 (below) for the coordinate version of this assumption for $n = 3$.

Following [Steigmann and Ogden 1999], we also treat the coating $\mathcal{S}$ as a general second grade material, i.e., we treat the superficial stored energy as a function of the first and second surface gradients. More precisely, we introduce a third-order tensor $\mathbb{G}$ interpreted as an R^n-valued bilinear form on $\mathrm{R}^n \times \mathrm{R}^n$ given by

$$\mathbb{G}(\mathbf{a}, \mathbf{b}) = \mathbb{\nabla}^2 \mathbb{y}(\mathbb{P}\mathbf{a}, \mathbb{P}\mathbf{b})$$

for all $\mathbf{a}, \mathbf{b} \in \mathrm{R}^n$, note that

$$\mathbb{K}(\mathbf{a}, \mathbf{b}) = \bar{\mathbb{n}} \cdot \mathbb{G}(\mathbf{a}, \mathbf{b}), \tag{8}$$

and introduce a response function $\hat{\mathrm{f}} \equiv \hat{\mathrm{f}}_{\mathbf{x}} : E_{\mathbf{x}} \to \mathrm{R}$ related to $\tilde{\mathrm{f}}$ by

$$\hat{\mathrm{f}}(\mathbb{F}, \mathbb{G}) = \tilde{\mathrm{f}}(\mathbb{F}, \mathbb{K})$$

where $\mathbb{G}$ and $\mathbb{K}$ are related by (8). The domain $E_{\mathbf{x}}$ of $\hat{\mathrm{f}}$ consists of all pairs $(\mathbb{F}, \mathbb{G}) \in \mathrm{Lin} \times T$, where T is the space of all R^n-valued bilinear forms on $\mathrm{R}^n \times \mathrm{R}^n$, satisfying

$$\mathbb{F} = \mathbb{F}\mathbb{P}, \quad \mathbb{G} = \mathbb{G} \circ (\mathbb{P}, \mathbb{P}). \tag{9}$$

The field f is then given by the constitutive equation

$$\mathrm{f}(\mathbf{x}) = \hat{\mathrm{f}}(\mathbb{F}(\mathbf{x}), \mathbb{G}(\mathbf{x})), \quad \mathbf{x} \in \mathcal{S}.$$

4. The total energy

The total energy F of the bulk body plus the coating is assumed in the form

$$\mathsf{F}(\mathbf{y}) = \mathsf{E}_{\mathrm{b}}(\mathbf{y}) + \mathsf{E}_{\mathrm{c}}(\mathbf{y}) + \mathsf{W}(\mathbf{y})$$

for each deformation $\mathbf{y} : \mathrm{cl}\,\Omega \to \mathrm{R}^n$, where $\mathsf{E}_{\mathrm{b}}(\mathbf{y})$ is the internal energy of the bulk body, $\mathsf{E}_{\mathrm{c}}(\mathbf{y})$ is the internal energy of the coating, and $\mathsf{W}(\mathbf{y})$ is the potential energy of the loads. Here

$$\mathsf{E}_{\mathrm{b}}(\mathbf{y}) = \int_{\Omega} f \, d\mathcal{L}^n,$$

where f is given by the constitutive equation (6) and $\mathcal{L}^n$ is the n-dimensional volume in R^n,

$$\mathsf{E_c}(\mathbf{y}) \equiv \mathsf{E_c}(\mathbb{y}) = \int_{\mathcal{S}} \mathbb{f}\, d\mathcal{H}^{n-1}$$

where $\mathbb{f}$ is given by the constitutive equation (7) and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional area on $\partial\Omega$, and

$$\mathsf{W}(\mathbf{y}) = -\int_{\Omega} \mathbf{y}\cdot\mathbf{b}\, d\mathcal{L}^n - \int_{\partial\Omega} \mathbf{y}\cdot\mathbf{s}\, d\mathcal{H}^{n-1};$$

here $\mathbf{b} : \Omega \to \mathrm{R}^n$ is a prescribed body force and $\mathbf{s} : \partial\Omega \to \mathrm{R}^n$ is a prescribed surface traction on the boundary of the body.

We assume that the response functions for the bulk and surface energies are sufficiently smooth and define the first variation $\delta\mathsf{F}(\mathbf{y}, \mathbf{v})$ of the total energy corresponding to the variation $\mathbf{v} : \mathrm{cl}\,\Omega \to \mathrm{R}^n$ by

$$\delta\mathsf{F}(\mathbf{y}, \mathbf{v}) = \frac{d\mathsf{F}(\mathbf{y} + t\mathbf{v})}{dt}\Big|_{t=0}.$$

We define the first variations $\delta\mathsf{E_b}(\mathbf{y}, \mathbf{v})$ and $\delta\mathsf{E_c}(\mathbf{y}, \mathbf{v})$ of the internal energies and the first variation $\delta\mathsf{W}(\mathbf{y}, \mathbf{v})$ of the potential energy of loads analogously.

5. The first variation of total energy and the Euler Lagrange equations

Proposition 5.1. *For every deformation* $\mathbf{y} : \mathrm{cl}\,\Omega \to \mathrm{R}^n$ *and every variation of deformation* $\mathbf{v} : \mathrm{cl}\,\Omega \to \mathrm{R}^n$, *we have*

$$\delta\mathsf{E_b}(\mathbf{y}, \mathbf{v}) = -\int_{\Omega} \mathbf{v}\cdot \mathrm{div}\,\mathbf{S}\, d\mathcal{L}^n + \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{S}\mathbb{n}\, d\mathcal{H}^{n-1}$$

where $\mathbf{S}$ *is the bulk referential stress given by the constitutive equation*

$$\mathbf{S}(\mathbf{x}) = \tilde{\mathbf{S}}(\mathbf{F}(\mathbf{x})), \quad \textit{with } \mathbf{x} \in \mathrm{cl}\,\Omega,\ \tilde{\mathbf{S}} = \partial_{\mathbf{F}} \tilde{f}.$$

Furthermore,

$$\delta\mathsf{W}(\mathbf{y}, \mathbf{v}) = -\int_{\Omega} \mathbf{v}\cdot\mathbf{b}\, d\mathcal{L}^n - \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{s}\, d\mathcal{H}^{n-1}.$$

This is standard.

Proposition 5.2. *We have*

$$\delta\mathsf{E_c}(\mathbb{y}, \mathbb{v}) = -\int_{\mathcal{S}} \mathbb{v}\cdot\mathbb{div}\,\mathbb{T}\, d\mathcal{H}^{n-1} + \int_{\partial\mathcal{S}} \big(\mathbb{A}^{\perp}\cdot\nabla^{\perp}\mathbb{v} + (\mathbb{T}\mathbb{m} - \mathbb{div}^{\parallel}\mathbb{A}^{\parallel})\cdot\mathbb{v}\big)\, d\mathcal{H}^{n-2} \quad (10)$$

for each deformation $\mathbb{y} : \mathrm{cl}\,\mathcal{S} \to \mathrm{R}^n$ *and for each variation* $\mathbb{v} : \mathrm{cl}\,\mathcal{S} \to \mathrm{R}^n$ *of deformation, where* $\mathcal{H}^{n-2}$ *is the* $(n-2)$*-dimensional area measure on* $\partial\mathcal{S}$,

$$\mathbb{T} = \mathbb{S} - (\mathbb{div}\,\mathbb{A})\mathbb{P} \quad (11)$$

with $\mathbb{S}$ and $\mathbb{A}$ the referential surface stress and the referential surface couple stress given by the constitutive equations

$$\mathbb{S}(\mathbf{x}) = \hat{\mathbb{S}}(\mathbb{F}(\mathbf{x}), \mathbb{G}(\mathbf{x})), \quad \mathbb{A}(\mathbf{x}) = \hat{\mathbb{A}}(\mathbb{F}(\mathbf{x}), \mathbb{G}(\mathbf{x})), \quad \mathbf{x} \in \operatorname{cl} \mathcal{S},$$

with

$$\hat{\mathbb{S}} = \partial_{\mathbb{F}} \hat{\mathbf{f}}\,, \quad \hat{\mathbb{A}} = \partial_{\mathbb{G}} \hat{\mathbf{f}}\,, \tag{12}$$

$\mathbb{div}^{\parallel}$ *denotes the divergence on* $\partial\mathcal{S}$, $\nabla^{\perp}\mathbb{v} := \nabla_{\mathbb{m}}\mathbb{v}$ *is the directional surface gradient of* $\mathbb{v}$ *in the direction of the normal* $\mathbb{m}$, *and* $\mathbb{A}^{\perp}$ *and* $\mathbb{A}^{\parallel}$ *are fields on* $\partial\mathcal{S}$ *given by*

$$\mathbb{A}^{\perp} = \mathbb{A}(\mathbb{m}, \mathbb{m}), \quad \mathbb{A}^{\parallel}\mathbf{a} = \mathbb{A}(\mathbb{P}^{\parallel}\mathbf{a}, \mathbb{m})$$

for each $\mathbf{a} \in \mathrm{R}^n$, *where* $\mathbb{P}^{\parallel}$ *is the orthogonal projection from* R^n *onto the tangent space of* $\partial\mathcal{S}$ *at the given point.*

We here recall that the response function $\hat{\mathbf{f}}$ is defined on the domain $E_{\mathbf{x}}$ which consists of all pairs $(\mathbb{F}, \mathbb{G}) \in \mathrm{Lin} \times T$ such that (9) hold. Thus for a given point of $\mathcal{S}$, the domain of $\hat{\mathbf{f}}$ is a linear subspace of the product $\mathrm{Lin} \times T$. The partial derivatives of $\hat{\mathbf{f}}$ in (12) follow our convention about derivatives on submanifolds of an euclidean space and interpret the total derivative (differential) $\mathrm{D}\,\hat{\mathbf{f}}$ of $\hat{\mathbf{f}}$ as an element of the space $\mathrm{Lin} \times T$, which satisfies

$$\Pi\, \mathrm{D}\,\hat{\mathbf{f}} = \hat{\mathbf{f}} \tag{13}$$

where Π is the orthogonal projection from $\mathrm{Lin} \times T$ onto $E_{\mathbf{x}}$. The value $\mathrm{D}\,\hat{\mathbf{f}}$ is a pair in $\mathrm{Lin} \times T$ and we write $\mathrm{D}\,\hat{\mathbf{f}} = (\partial_{\mathbb{F}}\hat{\mathbf{f}}\,, \partial_{\mathbb{K}}\hat{\mathbf{f}}\,)$ with $\partial_{\mathbb{F}}\hat{\mathbf{f}} \in \mathrm{Lin}$, $\partial_{\mathbb{K}}\hat{\mathbf{f}} \in T$ for the "components" of $\mathrm{D}\,\hat{\mathbf{f}}$. Equation (13) and the definition of $E_{\mathbf{x}}$ gives

$$\partial_{\mathbb{F}}\hat{\mathbf{f}}\,\mathbb{P} = \partial_{\mathbb{F}}\hat{\mathbf{f}}\,, \quad \partial_{\mathbb{K}}\hat{\mathbf{f}} \circ (\mathbb{P}, \mathbb{P}) = \partial_{\mathbb{K}}\hat{\mathbf{f}}\,.$$

Proof. The definition of the variation gives

$$\delta \mathsf{E}_{\mathrm{c}}(\mathbb{y}, \mathbb{v}) = \int_{\mathcal{S}} \big(\mathbb{S} \cdot \nabla\mathbb{v} + \mathbb{A} \cdot (\nabla^2\mathbb{v} \circ (\mathbb{P}, \mathbb{P}))\big)\, d\mathcal{H}^{n-1}.$$

Since $\mathbb{A} \cdot \big(\nabla^2\mathbb{v} \circ (\mathbb{P}, \mathbb{P})\big) = \mathbb{A} \circ (\mathbb{P}, \mathbb{P}) \cdot \nabla^2\mathbb{v}$ and $\mathbb{A} = \mathbb{A} \circ (\mathbb{P}, \mathbb{P})$, this may be rewritten as

$$\delta \mathsf{E}_{\mathrm{c}}(\mathbb{y}, \mathbb{v}) = \int_{\mathcal{S}} (\mathbb{S} \cdot \nabla\mathbb{v} + \mathbb{A} \cdot \nabla^2\mathbb{v})\, d\mathcal{H}^{n-1}.$$

We use the formula $\mathbb{div}(\mathbb{A} \cdot \nabla\mathbb{v}) = \mathbb{div}\,\mathbb{A} \cdot \nabla\mathbb{v} + \mathbb{A} \cdot \nabla^2\mathbb{v} \equiv (\mathbb{div}\,\mathbb{A})\mathbb{P} \cdot \nabla\mathbb{v} + \mathbb{A} \cdot \nabla^2\mathbb{v}$ [see (4)] and employ the surface divergence theorem noting that $\mathbb{A}$ is superficial to obtain

$$\delta \mathsf{E}_{\mathrm{c}}(\mathbb{y}, \mathbb{v}) = \int_{\mathcal{S}} \mathbb{T} \cdot \nabla\mathbb{v}\, d\mathcal{H}^{n-1} + \int_{\partial\mathcal{S}} \mathbb{div}\,\mathbb{A}\mathbb{m} \cdot \mathbb{v}\, d\mathcal{H}^{n-2}.$$

Next we use the formula $\mathbb{div}(\mathbb{T}^{\mathrm{T}}\mathbb{v}) = (\mathbb{div}\,\mathbb{T})\cdot\mathbb{v} + \mathbb{T}\cdot\nabla\mathbb{v}$ and employ the surface divergence theorem to obtain

$$\delta\mathsf{E}_{\mathrm{c}}(\mathbb{y},\mathbb{v}) = -\int_{\mathcal{S}} \mathbb{v}\cdot\mathbb{div}\,\mathbb{T}\,d\mathcal{H}^{n-1} + \int_{\partial\mathcal{S}} \big(\mathbb{A}\mathbb{m}\cdot\nabla\mathbb{v} + \mathbb{T}\mathbb{m}\cdot\mathbb{v}\big)\,d\mathcal{H}^{n-2}. \tag{14}$$

The second integral on the right hand side is further transformed as follows. We write $\nabla\mathbb{v} = \nabla^{\perp}\mathbb{v} + \nabla^{\parallel}\mathbb{v}$ where $\nabla^{\parallel}$ denotes the surface gradient relative to $\partial\mathcal{S}$ to obtain

$$\int_{\partial\mathcal{S}} \mathbb{A}\mathbb{m}\cdot\nabla\mathbb{v}\,d\mathcal{H}^{n-2} = \int_{\partial\mathcal{S}} \big(\mathbb{A}^{\perp}\cdot\nabla^{\perp}\mathbb{v} + \mathbb{A}^{\parallel}\cdot\nabla^{\parallel}\mathbb{v}\big)\,d\mathcal{H}^{n-2}.$$

Recalling that $\partial^2\mathcal{S} = \varnothing$, we use the surface divergence theorem to obtain

$$\int_{\partial\mathcal{S}} \mathbb{div}^{\parallel}(\mathbb{A}^{\parallel\mathrm{T}}\cdot\mathbb{v})\,d\mathcal{H}^{n-2} = 0.$$

Next we invoke the identity $\mathbb{A}^{\parallel}\cdot\nabla^{\parallel}\mathbb{v} = \mathbb{div}^{\parallel}(\mathbb{A}^{\parallel\mathrm{T}}\cdot\mathbb{v}) - \mathbb{v}\cdot\mathbb{div}^{\parallel}\,\mathbb{A}^{\parallel}$. The last two relations provide

$$\int_{\partial\mathcal{S}} \mathbb{A}\mathbb{m}\cdot\nabla\mathbb{v}\,d\mathcal{H}^{n-2} = \int_{\partial\mathcal{S}} \big(\mathbb{A}^{\perp}\cdot\nabla^{\perp}\mathbb{v} - \mathbb{v}\cdot\mathbb{div}^{\parallel}\,\mathbb{A}^{\parallel}\big)\,d\mathcal{H}^{n-2}$$

and this reduces (14) to (10). □

Proposition 5.3. *If* $\delta\mathsf{F}(\mathbf{y},\mathbf{v}) = 0$ *for a given deformation* $\mathbf{y} : \mathrm{cl}\,\Omega \to \mathrm{R}^n$ *and all variations of deformation* $\mathbf{v} : \mathrm{cl}\,\Omega \to \mathrm{R}^n$, *we have the equations*

$$\left.\begin{array}{ll} \operatorname{div}\mathbf{S} + \mathbf{b} = \mathbf{0} & \text{on } \Omega, \\ \mathbf{S}\mathbb{n} = \mathbf{s} & \text{on } \partial\Omega\setminus\mathcal{S}, \\ \mathbb{div}\,\mathbb{T} + \mathbb{p} = \mathbf{0} & \text{on } \mathcal{S} \\ \mathbb{A}^{\perp} = 0, \quad \mathbb{T}\mathbb{m} - \mathbb{div}^{\parallel}\mathbb{A}^{\parallel} = \mathbf{0} & \text{on } \partial\mathcal{S}, \end{array}\right\} \tag{15}$$

where

$$\mathbb{p} = \mathbf{s} - \mathbf{S}\mathbb{n}.$$

If $n = 3$ *and* $\mathbb{t}$ *is the counterclockwise unit tangent vector to* $\partial\mathcal{S}$ *then*

$$\mathbb{A}^{\parallel} = \mathbb{A}(\mathbb{t},\mathbb{m})\otimes\mathbb{t}, \qquad \mathbb{div}^{\parallel}\,\mathbb{A}^{\parallel} = (\mathbb{A}(\mathbb{t},\mathbb{m}))'$$

where $'$ *denotes the derivative with respect to the arc length parameter on* $\partial\mathcal{S}$.

Proof. Collecting the expressions in Propositions 5.1 and 5.2, we obtain

$$\begin{aligned} \delta\mathsf{F}(\mathbf{y},\mathbf{v}) = &-\textstyle\int_{\Omega}\mathbf{v}\cdot(\operatorname{div}\mathbf{S} + \mathbf{b})\,d\mathcal{L}^n + \int_{\partial\Omega\setminus\mathcal{S}}\mathbf{v}\cdot(\mathbf{S}\mathbb{n} - \mathbf{s})\,d\mathcal{H}^{n-1} \\ &-\textstyle\int_{\mathcal{S}}\mathbb{v}\cdot(\mathbb{div}\,\mathbb{T} - \mathbf{S}\mathbb{n} + \mathbf{s})\,d\mathcal{H}^{n-1} \\ &+\textstyle\int_{\partial\mathcal{S}}\big(\mathbb{A}^{\perp}\cdot\nabla^{\perp}\mathbb{v} + (\mathbb{T}\mathbb{m} - \mathbb{div}^{\parallel}\mathbb{A}^{\parallel})\cdot\mathbb{v}\big)\,d\mathcal{H}^{n-2}. \end{aligned} \tag{16}$$

We first consider all variations $\mathbf{v}$ with compact support contained in the open set Ω, then the integrals over $\partial\Omega \setminus \mathcal{S}$, over $\mathcal{S}$ and over $\partial\mathcal{S}$ vanish and the arbitrariness of $\mathbf{v}$ gives $(15)_1$. With this knowledge the volume integral in (16) disappears. We then consider variations $\mathbf{v}$ such that $\mathbf{v} = 0$ on $\operatorname{cl}\mathcal{S}$; then the integrals over $\mathcal{S}$ and $\partial\mathcal{S}$ vanish and the arbitrariness of the values of $\mathbf{v}$ on $\partial\Omega \setminus \mathcal{S}$ gives $(15)_2$. With this knowledge, also the integral over $\Omega \setminus \mathcal{S}$ disappears from (16). Then we consider variations $\mathbb{v} = \mathbf{v}|\operatorname{cl}\mathcal{S}$ such that $\mathbb{v}$ has compact support in the relatively open set $\mathcal{S}$. This gives $(15)_3$ and the integral over $\mathcal{S}$ in (16) disappears. We are thus left with only the integral over $\partial\mathcal{S}$ in (16). Since the variations $\nabla^{\perp}\mathbb{v}$ and $\mathbb{v}|\partial\mathcal{S}$ can be chosen independently, we obtain $(15)_4$. □

We recall that our basic response function for the surface energy was $\tilde{\mathbf{f}}$, expressing the surface stored energy as a function of $\mathbb{F}$ and $\mathbb{K}$. The next proposition expresses the tensor $\mathbb{T}$ occurring in (10) and $(15)_3$ in terms of the derivatives of $\tilde{\mathbf{f}}$.

Proposition 5.4. *We have the following relation for the tensor* $\mathbb{T}$ *in* (11):

$$\mathbb{T} = \partial_{\mathbb{F}}\tilde{\mathbf{f}} - \mathbb{div}\big(\bar{\mathbb{n}} \otimes \mathbb{F}\,\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)\mathbb{F}^{-\mathrm{T}}. \tag{17}$$

Proof. Let us show that the partial derivatives of the functions $\hat{\mathbf{f}}$ and $\tilde{\mathbf{f}}$ are related by

$$\partial_{\mathbb{F}}\hat{\mathbf{f}} = \partial_{\mathbb{F}}\tilde{\mathbf{f}} - \bar{\mathbb{n}} \otimes \big(\mathbb{F}^{-1}\mathbb{G} \cdot \partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big), \qquad \partial_{\mathbb{G}}\hat{\mathbf{f}} = \bar{\mathbb{n}} \otimes \partial_{\mathbb{K}}\tilde{\mathbf{f}} \tag{18}$$

at the corresponding arguments, where $\mathbb{F}^{-1}\mathbb{G} \cdot \partial_{\mathbb{K}}\tilde{\mathbf{f}}$ is a vector satisfying

$$\mathbf{a} \cdot (\mathbb{F}^{-1}\mathbb{G} \cdot \partial_{\mathbb{K}}\tilde{\mathbf{f}}\,) = \mathbb{R} \cdot \partial_{\mathbb{K}}\tilde{\mathbf{f}}$$

for all $\mathbf{a} \in \mathrm{R}^n$, where $\mathbb{R}$ is a second-order tensor satisfying

$$\mathbb{R}(\mathbf{p}, \mathbf{q}) = \mathbf{a} \cdot \mathbb{F}^{-1}\mathbb{G}(\mathbf{p}, \mathbf{q})$$

for all $\mathbf{p}, \mathbf{q} \in \mathrm{R}^n$. Indeed, we interpret $\bar{\mathbb{n}}$ as a function of $\mathbb{F}$ determined locally uniquely by the equations $\mathbb{F}^{\mathrm{T}}\bar{\mathbb{n}} = \mathbf{0}$, $|\bar{\mathbb{n}}| = 1$. This functional interpretation of $\bar{\mathbb{n}}$ makes $\mathbb{K}$ a function of $\mathbb{G}$ and $\bar{\mathbb{n}}$. Differentiating the relation

$$\hat{\mathbf{f}}\,(\mathbb{F}, \mathbb{G}) = \tilde{\mathbf{f}}\,(\mathbb{F}, \bar{\mathbb{n}}(\mathbb{F}) \cdot \mathbb{G}) \tag{19}$$

with respect to $\mathbb{G}$ we obtain $(18)_2$. To obtain $(18)_1$, we first note that interpreting $\bar{\mathbb{n}}$ as a function of $\mathbb{F}$, we have the relation

$$\partial_{\mathbb{F}}\bar{\mathbb{n}}\,\mathbf{A} = -\mathbb{F}^{-\mathrm{T}}\mathbf{A}^{\mathrm{T}}\bar{\mathbb{n}}$$

for each $\mathbf{A} \in \mathrm{Lin}$, where we interpret $\partial_{\mathbb{F}}\bar{\mathbb{n}}$ as a linear transformation from Lin to R^n. Differentiating (19) with respect to $\mathbb{F}$ in the direction of $\mathbf{A} \in \mathrm{Lin}$ and using the above relation for the derivative on $\bar{\mathbb{n}}$, we obtain

$$\partial_{\mathbb{F}}\hat{\mathbf{f}}\, \cdot \mathbf{A} = -(\mathbb{F}^{-\mathrm{T}}\mathbf{A}^{\mathrm{T}}\bar{\mathbb{n}} \cdot \mathbb{G}) \cdot \partial_{\mathbb{K}}\tilde{\mathbf{f}} + \partial_{\mathbb{F}}\tilde{\mathbf{f}}\, \cdot \mathbf{A}, \tag{20}$$

where for any vector $\mathbf{a}$ the symbol $\mathbf{a}\cdot\mathbb{G}$ denotes a second-order tensor defined by

$$(\mathbf{a}\cdot\mathbb{G})(\mathbf{p},\mathbf{q})=\mathbf{a}\cdot\mathbb{G}(\mathbf{p},\mathbf{q})$$

for any $\mathbf{p},\mathbf{q}\in\mathrm{R}^n$. We have the following rearrangements

$$\begin{aligned}(\mathbb{F}^{-\mathrm{T}}\mathbf{A}^{\mathrm{T}}\bar{\mathbb{n}}\cdot\mathbb{G})\cdot\partial_{\mathbb{K}}\tilde{\mathbf{f}}&=(\mathbb{F}^{-\mathrm{T}}\mathbf{A}^{\mathrm{T}}\bar{\mathbb{n}})\cdot\big(\mathbb{G}\cdot\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)\\&=(\mathbf{A}^{\mathrm{T}}\bar{\mathbb{n}})\cdot\big(\mathbb{F}^{-1}\mathbb{G}\cdot\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)\\&=\big(\bar{\mathbb{n}}\otimes\big(\mathbb{F}^{-1}\mathbb{G}\cdot\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)\big)\cdot\mathbf{A}\end{aligned}$$

which reduces (20) to

$$\partial_{\mathbb{F}}\hat{\mathbf{f}}\,\cdot\mathbf{A}=-\big(\bar{\mathbb{n}}\otimes\big(\mathbb{F}^{-1}\mathbb{G}\cdot\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)\big)\cdot\mathbf{A}+\partial_{\mathbb{F}}\tilde{\mathbf{f}}\,\cdot\mathbf{A}$$

and the arbitrariness of $\mathbf{A}$ gives $(18)_1$.

Using relations (18) one finds from the definition (11) of $\mathbb{T}$ that

$$\mathbb{T}=\mathbb{S}-(\mathbb{div}\,\mathbb{A})\mathbb{P}=\partial_{\mathbb{F}}\tilde{\mathbf{f}}\,-\bar{\mathbb{n}}\otimes\big(\mathbb{F}^{-1}\mathbb{G}\cdot\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)-\mathbb{div}\big(\bar{\mathbb{n}}\otimes\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)\mathbb{P}$$

and the proof of (17) is completed by noting the following easily provable identity

$$\mathbb{div}\big(\bar{\mathbb{n}}\otimes\mathbb{F}\,\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)\mathbb{F}^{-\mathrm{T}}=\bar{\mathbb{n}}\otimes\big(\mathbb{F}^{-1}\mathbb{G}\cdot\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)+\mathbb{div}\big(\bar{\mathbb{n}}\otimes\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)\mathbb{P}.\qquad\square$$

6. The spatial form of equilibrium equations

The spatial form of the equilibrium equations (i.e., that on the deformed configuration of the film) to be derived below admits a splitting into the tangential and normal components with the tangential component given by a second-order equation and the normal component a fourth-order equation with the iterated surface divergence.

Proposition 6.1. *Assume that the stored energy $\tilde{\mathbf{f}}$ is objective in the sense that*

$$\tilde{\mathbf{f}}\,(\mathbf{Q}\mathbb{F},\mathbb{K})=\tilde{\mathbf{f}}\,(\mathbb{F},\mathbb{K})$$

for all orthogonal tensors $\mathbf{Q}$ and all arguments $\mathbb{F}$ and $\mathbb{K}$ from the domain of $\tilde{\mathbf{f}}$. Then $(15)_3$ is equivalent to

$$\overline{\mathbb{div}}\,\overline{\mathbb{T}}+\mathbb{j}^{-1}\mathbb{p}=\mathbf{0},\tag{21}$$

where $\mathbb{j}=|\mathrm{cof}\,\mathbb{F}|$ is the jacobian of the transformation $\mathbb{y}:\mathcal{S}\to\bar{\mathcal{S}}:=\mathbb{y}(\mathcal{S})$, $\overline{\mathbb{div}}$ is the divergence on the actual configuration $\bar{\mathcal{S}}$, and

$$\overline{\mathbb{T}}=\mathbb{j}^{-1}\mathbb{T}\mathbb{F}^{\mathrm{T}}\equiv\overline{\mathbb{N}}-\overline{\mathbb{L}}\overline{\mathbb{M}}-\bar{\mathbb{n}}\otimes(\overline{\mathbb{P}}\,\overline{\mathbb{div}}\,\overline{\mathbb{M}}),\tag{22}$$

where

$$\overline{\mathbb{N}}=\mathbb{j}^{-1}\,\partial_{\mathbb{F}}\tilde{\mathbf{f}}\,\mathbb{F}^{\mathrm{T}},\quad\overline{\mathbb{M}}=\mathbb{j}^{-1}\mathbb{F}\,\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\mathbb{F}^{\mathrm{T}}\tag{23}$$

and $\bar{\mathbb{L}} = \bar{\nabla}\bar{\mathbb{n}}$ *is the curvature of the deformed configuration of the film. Equations* (15)$_4$ *are equivalent to*

$$\bar{\mathbb{M}}(\bar{\mathbb{m}}, \bar{\mathbb{m}}) = 0, \quad \bar{\mathbb{T}}\bar{\mathbb{m}} - \overline{\mathbb{div}}^{\|}(\bar{\mathbb{n}} \otimes \bar{\mathbb{M}}\bar{\mathbb{m}}) = \mathbf{0}, \tag{24}$$

where $\bar{\mathbb{m}}$ *is the unit normal to* $\partial\bar{\mathcal{S}}$ *in the tangent space to* $\bar{\mathcal{S}}$ *and* $\overline{\mathbb{div}}^{\|}$ *is the divergence relative to* $\partial\bar{\mathcal{S}}$. *If* $n = 3$, *then* (24)$_2$ *reads*

$$\bar{\mathbb{T}}\bar{\mathbb{m}} - \big(\bar{\mathbb{n}}\bar{\mathbb{M}}(\bar{\mathbb{t}}, \bar{\mathbb{m}})\big)' = \mathbf{0},$$

where $\bar{\mathbb{t}}$ *is the counterclockwise unit vector tangent to* $\partial\bar{\mathcal{S}}$ *and the superscript* $'$ *denotes the derivative with respect to the arc length parameter on* $\partial\bar{\mathcal{S}}$.

Here and below in this section we distinguish the objects related to the deformed configuration by a superimposed bar. Here $\bar{\mathbb{N}}$ is the normal stress and $\bar{\mathbb{M}}$ the couple stress in the film.

Proof. Note first that a standard argument based on the objectivity implies that $\bar{\mathbb{N}}$, given by (23)$_1$, is a symmetric tensor. Furthermore, the definition (23)$_1$ immediately implies that $\bar{\mathbb{N}}\bar{\mathbb{n}} = \mathbf{0}$ and $\bar{\mathbb{N}}\bar{\mathbb{P}} = \bar{\mathbb{N}}$, where $\bar{\mathbb{P}}$ is the orthogonal projection from R^n onto the tangent space to $\bar{\mathcal{S}}$, which by the symmetry implies that $\bar{\mathbb{P}}\bar{\mathbb{N}} = \bar{\mathbb{N}}$.

We note further that if $\bar{\mathbb{T}}$ is given by (22)$_1$ then (21) is equivalent to (15)$_3$ by the surface version of the Piola transformation (5). To obtain the equivalent form (22)$_2$, we note that by (17) we have

$$\bar{\mathbb{T}} = \bar{\mathbb{N}} - \mathbb{j}^{-1}\,\mathbb{div}\big(\bar{\mathbb{n}} \otimes \mathbb{F}\,\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)\bar{\mathbb{P}}. \tag{25}$$

Employing the surface version of the Piola transformation once more and invoking the formula for the divergence of a tensor product, we find that

$$\mathbb{j}^{-1}\,\mathbb{div}\big(\bar{\mathbb{n}} \otimes \mathbb{F}\,\partial_{\mathbb{K}}\tilde{\mathbf{f}}\,\big)\bar{\mathbb{P}} = \big(\overline{\mathbb{div}}(\bar{\mathbb{n}} \otimes \bar{\mathbb{M}})\big)\bar{\mathbb{P}} = \bar{\mathbb{L}}\bar{\mathbb{M}} + \bar{\mathbb{n}} \otimes \bar{\mathbb{P}}\,\overline{\mathbb{div}}\,\bar{\mathbb{M}}.$$

The insertion into (25) yields (22)$_2$.

Equation (24)$_1$ is clearly equivalent to the first equation in (15)$_4$. To obtain the equivalence of (24)$_2$ and the second equation in (15)$_4$, we employ the Piola transformation to the passage from $\partial\mathcal{S}$ to $\partial\bar{\mathcal{S}}$. We note that the jacobian of this transformation is

$$\dot{\mathbb{j}} = \mathbb{j}|\mathbb{F}^{-\mathrm{T}}\mathbb{m}|$$

and the unit normal $\bar{\mathbb{m}}$ to $\partial\bar{\mathcal{S}}$ is given by

$$\bar{\mathbb{m}} = \mathbb{F}^{-\mathrm{T}}\mathbb{m}/|\mathbb{F}^{-\mathrm{T}}\mathbb{m}|.$$

The basic relation (5) of the Piola transformation is then

$$\overline{\mathbb{div}}^{\|}(\mathbb{j}^{-1}|\mathbb{F}^{-\mathrm{T}}\mathbb{m}|^{-1}\mathbb{A}^{\|}\mathbb{F}^{\mathrm{T}}) = \mathbb{j}^{-1}|\mathbb{F}^{-\mathrm{T}}\mathbb{m}|^{-1}\,\mathbb{div}^{\|}\,\mathbb{A}^{\|}$$

which reduces the second equation in (15)$_4$ to

$$\overline{\mathbb{T}}\overline{\mathbb{m}} + \overline{\mathbb{div}}^{\parallel}(\mathbb{j}^{-1}|\mathbb{F}^{-\mathrm{T}}\mathbb{m}|^{-1}\mathbb{A}^{\parallel}\mathbb{F}^{\mathrm{T}}) = \mathbf{0}.$$

Recalling (18)$_2$, we note that

$$\mathbb{A} = \bar{\mathbb{n}} \otimes \partial_{\mathbb{K}} \tilde{\mathbb{f}}$$

and we use this in the following computation:

$$\begin{aligned}
\mathbb{j}^{-1}|\mathbb{F}^{-\mathrm{T}}\mathbb{m}|^{-1}\mathbb{A}^{\parallel}\mathbb{F}^{\mathrm{T}}\mathbf{a} &= \mathbb{j}^{-1}|\mathbb{F}^{-\mathrm{T}}\mathbb{m}|^{-1}\mathbb{A}(\mathbb{P}^{\parallel}\mathbb{F}^{\mathrm{T}}\mathbf{a}, \mathbb{m})\\
&= \mathbb{j}^{-1}|\mathbb{A}(\mathbb{P}^{\parallel}\mathbb{F}^{\mathrm{T}}\mathbf{a}, \mathbb{F}^{\mathrm{T}}\overline{\mathbb{m}})\\
&= \mathbb{j}^{-1}|\mathbb{A}(\mathbb{F}^{\mathrm{T}}\overline{\mathbb{P}}^{\parallel}\mathbf{a}, \mathbb{F}^{\mathrm{T}}\overline{\mathbb{m}})\\
&= \overline{\mathbb{M}}(\overline{\mathbb{P}}^{\parallel}\mathbf{a}, \overline{\mathbb{m}})\bar{\mathbb{n}}\\
&= \overline{\mathbb{M}}\big((\overline{\mathbb{P}} - \overline{\mathbb{m}} \otimes \overline{\mathbb{m}})\mathbf{a}, \overline{\mathbb{m}}\big)\bar{\mathbb{n}}\\
&= \overline{\mathbb{M}}(\mathbf{a}, \overline{\mathbb{m}})\bar{\mathbb{n}} - \overline{\mathbb{M}}(\overline{\mathbb{m}}, \overline{\mathbb{m}})(\overline{\mathbb{m}} \cdot \mathbf{a})\bar{\mathbb{n}}\\
&= \overline{\mathbb{M}}(\mathbf{a}, \overline{\mathbb{m}})\bar{\mathbb{n}},
\end{aligned}$$

where we have used (24)$_1$. Thus we conclude that

$$\mathbb{j}^{-1}|\mathbb{F}^{-\mathrm{T}}\mathbb{m}|^{-1}\mathbb{A}^{\parallel}\mathbb{F}^{\mathrm{T}} = \bar{\mathbb{n}} \otimes \overline{\mathbb{M}}\overline{\mathbb{m}}$$

and the above computation also shows that $\overline{\mathbb{M}}\overline{\mathbb{m}}$ is a tangential vector on $\partial\bar{\mathcal{S}}$, i.e., $\overline{\mathbb{P}}^{\parallel}\overline{\mathbb{M}}\overline{\mathbb{m}} = \overline{\mathbb{M}}\overline{\mathbb{m}}$. Thus we have (24)$_2$. □

Proposition 6.2. *The tangential and normal components of* (21) *read*

$$\left.\begin{aligned}
\overline{\mathbb{P}}\,\overline{\mathbb{div}}(\overline{\mathbb{N}} - \overline{\mathbb{L}}\overline{\mathbb{M}}) - \overline{\mathbb{L}}\,\overline{\mathbb{div}}\,\overline{\mathbb{M}} + \mathbb{j}^{-1}\mathbb{p}^{\parallel} = \mathbf{0},\\
\overline{\mathbb{div}}(\overline{\mathbb{P}}\,\overline{\mathbb{div}}\,\overline{\mathbb{M}}) + \overline{\mathbb{L}} \cdot \overline{\mathbb{N}} - \overline{\mathbb{L}}^2 \cdot \overline{\mathbb{M}} - \mathbb{j}^{-1}\mathbb{p}^{\perp} = 0,
\end{aligned}\right\} \tag{26}$$

where

$$\mathbb{p}^{\parallel} = \overline{\mathbb{P}}\mathbb{p}, \quad \mathbb{p}^{\perp} = \bar{\mathbb{n}} \cdot \mathbb{p}.$$

The tangential and normal components of (24)$_2$ *read*

$$\left.\begin{aligned}
(\overline{\mathbb{N}} - 2\overline{\mathbb{L}}\overline{\mathbb{M}})\overline{\mathbb{m}} = \mathbf{0},\\
\overline{\mathbb{m}} \cdot \overline{\mathbb{div}}\,\overline{\mathbb{M}} + \overline{\mathbb{div}}^{\parallel}(\overline{\mathbb{P}}^{\parallel}\overline{\mathbb{M}}\overline{\mathbb{m}}) = 0
\end{aligned}\right\} \tag{27}$$

on $\partial\bar{\mathcal{S}}$. *If* $n = 3$ *then* (27)$_2$ *reads*

$$\overline{\mathbb{m}} \cdot \overline{\mathbb{div}}\,\overline{\mathbb{M}} + \big(\overline{\mathbb{M}}(\bar{\mathbb{t}}, \overline{\mathbb{m}})\big)' = 0. \tag{28}$$

We recall that $\overline{\mathbb{N}}$ and $\overline{\mathbb{M}}$ depend on the first and second gradients of the surface deformation, which gives that (26)$_1$ is of the second order in the deformation and (26)$_2$ of the fourth order.

Proof. To obtain the tangential component of (21), we use the identity

$$\overline{\mathbb{P}}\,\overline{\mathrm{div}}\,\overline{\mathbb{T}} = \overline{\mathbb{P}}^2\,\overline{\mathrm{div}}\,\overline{\mathbb{T}} = \overline{\mathbb{P}}\,\overline{\mathrm{div}}(\overline{\mathbb{P}}\overline{\mathbb{T}}) - \overline{\mathbb{P}}\overline{\nabla}\overline{\mathbb{P}}\overline{\mathbb{T}}, \tag{29}$$

where we note that employing the formula $\overline{\mathbb{P}}\overline{\mathbb{N}} = \overline{\mathbb{N}}$ we obtain

$$\overline{\mathbb{P}}\overline{\mathbb{T}} = \overline{\mathbb{N}} - \overline{\mathbb{L}}\overline{\mathbb{M}}. \tag{30}$$

Furthermore, differentiating $\overline{\mathbb{P}} = \mathbf{1} - \overline{\mathbb{n}} \otimes \overline{\mathbb{n}}$ one finds that the directional gradient in $\bar{\mathcal{S}}$ of $\overline{\mathbb{P}}$ in the direction $\mathbf{a}$ satisfies

$$\overline{\mathbb{P}}\overline{\nabla}_{\mathbf{a}}\overline{\mathbb{P}}\mathbf{b} = -\overline{\mathbb{L}}\mathbf{a}(\overline{\mathbb{n}} \cdot \mathbf{b})$$

for each $\mathbf{a}, \mathbf{b} \in \mathrm{R}^n$; it follows that

$$\overline{\mathbb{P}}\overline{\nabla}\overline{\mathbb{P}}\overline{\mathbb{T}} = -\overline{\mathbb{L}}\overline{\mathbb{T}}^{\mathrm{T}}\overline{\mathbb{n}} = \overline{\mathbb{L}}\,\overline{\mathrm{div}}\,\overline{\mathbb{M}}. \tag{31}$$

Inserting (30) and (31) into (29), one obtains $(26)_1$.

To obtain the normal component of (21), we employ the identity

$$\overline{\mathbb{n}} \cdot \overline{\mathrm{div}}\,\overline{\mathbb{T}} = \overline{\mathrm{div}}(\overline{\mathbb{T}}^{\mathrm{T}}\overline{\mathbb{n}}) - \overline{\mathbb{T}} \cdot \overline{\mathbb{L}}. \tag{32}$$

Since $\overline{\mathbb{N}}$ is symmetric, we have $\overline{\mathbb{N}}^{\mathrm{T}}\overline{\mathbb{n}} = \overline{\mathbb{N}}\overline{\mathbb{n}} = \mathbf{0}$; combining with $\overline{\mathbb{L}}\overline{\mathbb{n}} = \mathbf{0}$, we find

$$\overline{\mathbb{T}}^{\mathrm{T}}\overline{\mathbb{n}} = -\overline{\mathbb{P}}\,\overline{\mathrm{div}}\,\overline{\mathbb{M}}. \tag{33}$$

Also

$$\overline{\mathbb{T}} \cdot \overline{\mathbb{L}} = \overline{\mathbb{N}} \cdot \overline{\mathbb{L}} - \overline{\mathbb{L}}\overline{\mathbb{M}} \cdot \overline{\mathbb{L}}. \tag{34}$$

Inserting (33) and (34) into (32), we obtain $(26)_2$.

To obtain (27), we invoke the identity

$$\begin{aligned}\overline{\mathrm{div}}^{\parallel}(\overline{\mathbb{n}} \otimes \overline{\mathbb{M}}\overline{\mathbb{m}}) &= \overline{\nabla}^{\parallel}\overline{\mathbb{n}}\overline{\mathbb{M}}\overline{\mathbb{m}} + \overline{\mathbb{n}}\,\overline{\mathrm{div}}^{\parallel}(\overline{\mathbb{M}}\overline{\mathbb{m}}) \\ &= \overline{\nabla}\overline{\mathbb{n}}\overline{\mathbb{P}}^{\parallel}\overline{\mathbb{M}}\overline{\mathbb{m}} + \overline{\mathbb{n}}\,\overline{\mathrm{div}}^{\parallel}(\overline{\mathbb{P}}^{\parallel}\overline{\mathbb{M}}\overline{\mathbb{m}}) \\ &= \overline{\mathbb{L}}\overline{\mathbb{M}}\overline{\mathbb{m}} + \overline{\mathbb{n}}\,\overline{\mathrm{div}}^{\parallel}(\overline{\mathbb{P}}^{\parallel}\overline{\mathbb{M}}\overline{\mathbb{m}})\end{aligned}$$

and combining with the second expression in (22) we conclude that $(24)_2$ is equivalent to

$$\overline{\mathbb{N}}\overline{\mathbb{m}} - 2\overline{\mathbb{L}}\overline{\mathbb{M}}\overline{\mathbb{m}} - \overline{\mathbb{n}}\Big(\big((\overline{\mathbb{P}}\,\overline{\mathrm{div}}\,\overline{\mathbb{M}}) \cdot \overline{\mathbb{m}}\big) + \overline{\mathrm{div}}^{\parallel}(\overline{\mathbb{P}}^{\parallel}\overline{\mathbb{M}}\overline{\mathbb{m}})\Big) = \mathbf{0}.$$

Taking the tangential and normal components, we obtain (27). □

7. Coordinate expressions

For the purpose of comparison with the existing literature, we now establish the component expressions of the main formulas.

Throughout this section we use the convention that the Greek indices α, β, γ run from 1 to $n-1$ while the Latin indexes i, j, k run from 1 to n. We use the Einstein summation convention for repeated indices.

We assume that $\mathcal{S}$ is parametrized by a map $\mathbf{\Phi} : D \to \mathcal{S}$, where $D \subset \mathrm{R}^{n-1}$ and note that we can express the general maps m defined on $\mathcal{S}$ as functions $\tilde{m}$ of the variables $(\theta^1, \dots, \theta^{n-1}) \in D$ of the parametrization $\mathbf{\Phi}(\theta^1, \dots, \theta^{n-1})$. The maps m and $\tilde{m}$ are related by $\tilde{m} = m \circ \mathbf{\Phi}$ but below we denote both $\tilde{m}$ and m by the same letter m. A Greek subscript following a comma denotes the partial differentiation with respect to the corresponding variable in the collection $(\theta^1, \dots, \theta^{n-1})$.

We denote by $\mathbf{e}_i \equiv \mathbf{e}^i$ the canonical basis in R^n and introduce the coordinate vectors $\mathbb{e}_\alpha$ in the tangent space of $\mathcal{S}$ by

$$\mathbb{e}_\alpha = \mathbf{\Phi}_{,\alpha}.$$

We denote by $\mathbb{e}^\beta$ the dual basis in the tangent space, satisfying

$$\mathbb{e}_\alpha \cdot \mathbb{e}^\beta = \delta_\alpha^\beta$$

and the Christoffel symbols $\Gamma_{\alpha\beta}^\gamma$ defined by

$$\Gamma_{\alpha\beta}^\gamma = \mathbb{e}^\gamma \cdot \mathbb{e}_{\alpha,\beta}.$$

We then have

$$\mathbb{e}_{\alpha,\beta} = \Gamma_{\alpha\beta}^\gamma \mathbb{e}_\gamma - \mathbb{L}_{\alpha\beta}\mathbb{n}, \quad \mathbb{e}^\gamma_{,\beta} = -\Gamma_{\alpha\beta}^\gamma \mathbb{e}^\alpha - \mathbb{L}_\beta^\gamma \mathbb{n},$$

where $\mathbb{L}_{\alpha\beta} = \mathbb{L}\mathbb{e}_\alpha \cdot \mathbb{e}_\beta$ and $\mathbb{L}_\beta^\gamma = \mathbb{L}\mathbb{e}^\gamma \cdot \mathbb{e}_\beta$, and where $\mathbb{L}$ is the curvature tensor of the reference configuration of the film, see (1). (We note in passing that the second fundamental form of $\mathcal{S}$ is given by $b_{\alpha\beta} = -\mathbb{L}_{\alpha\beta}$.) If V is a finite-dimensional vector space and $f : \mathcal{S} \to V$ a class 2 mapping then

$$\nabla f = f_{,\alpha} \otimes \mathbb{e}^\alpha \tag{35}$$

and consequently

$$\nabla^2 f = (f_{,\alpha} \otimes \mathbb{e}^\alpha)_{,\beta} \otimes \mathbb{e}^\beta. \tag{36}$$

Differentiating the above product by the product rule and employing the formula for $\mathbb{e}^\alpha_{,\beta}$ given above we obtain

$$\nabla^2 f = f_{,\alpha\beta} \otimes \mathbb{e}^\alpha \otimes \mathbb{e}^\beta - f_{,\alpha} \otimes \Gamma_{\gamma\beta}^\alpha \mathbb{e}^\gamma \otimes \mathbb{e}^\beta - \mathbb{L}_\beta^\alpha f_{,\alpha} \otimes \mathbb{n} \otimes \mathbb{e}^\beta.$$

It follows that

$$\nabla^2 f \circ (\mathbb{P}, \mathbb{P}) = f_{,\alpha\beta} \otimes \mathbb{e}^\alpha \otimes \mathbb{e}^\beta - f_{,\alpha} \otimes \Gamma^\alpha_{\gamma\beta} \mathbb{e}^\gamma \otimes \mathbb{e}^\beta.$$

If V is a finite-dimensional vector space and $\mathbb{Q} : \mathcal{S} \to \mathrm{Lin}(\mathrm{R}^n, V)$ a superficial map then we have the formula

$$\mathbb{div}\, \mathbb{Q} = \mathbb{J}^{-1}(\mathbb{J}\mathbb{Q}^\alpha)_{,\alpha}, \tag{37}$$

where $\mathbb{Q}^\alpha = \mathbb{Q}\mathbb{e}^\alpha$ and $\mathbb{J} = (\det \nabla\mathbf{\Phi}^{\mathrm{T}} \nabla\mathbf{\Phi})^{1/2}$ is the jacobian of $\mathbf{\Phi}$. This formula coincides with the well known expression for the divergence based on covariant derivatives of tangential vector fields. However, with the divergence defined in (3), Formula (37) holds for an arbitrary superficial field $\mathbb{Q} : \mathcal{S} \to \mathrm{Lin}(\mathrm{R}^n, V)$, where V is arbitrary finite-dimensional vector space with inner product, in particular also for second and third-order tensor fields, whereas (37) does not hold for divergence based on the covariant derivative of tensor fields of order ≥ 2. See also (43) (below).

By (35) and (36), the first and second surface deformation gradients are determined by the components of the deformation function $\mathbb{y}^i := \mathbb{y} \cdot \mathbf{e}^i$ as follows:

$$\mathbb{F} = \nabla\mathbb{y} = \mathbb{F}^i_\alpha \mathbf{e}_i \otimes \mathbb{e}^\alpha,$$

$$\nabla^2\mathbb{y} = \mathbb{F}^i_{\alpha,\beta} \mathbf{e}_i \otimes \mathbb{e}^\alpha \otimes \mathbb{e}^\beta - \mathbb{F}^i_{,\alpha}(\Gamma^\alpha_{\gamma\beta} \mathbf{e}_i \otimes \mathbb{e}^\gamma \otimes \mathbb{e}^\beta + \mathbb{L}^\alpha_\beta \mathbf{e}_i \otimes \mathbb{n} \otimes \mathbb{e}^\beta),$$

where

$$\mathbb{F}^i_\alpha = \mathbb{y}^i_{,\alpha}.$$

It follows that

$$\mathbb{G} = \nabla^2\mathbb{y} \circ (\mathbb{P}, \mathbb{P}) = (\mathbb{F}^i_{\alpha,\beta} - \mathbb{F}^i_{,\gamma}\Gamma^\gamma_{\alpha\beta})\mathbf{e}_i \otimes \mathbb{e}^\alpha \otimes \mathbb{e}^\beta. \tag{38}$$

Let us express the energy as a function of $\mathbb{F}^i_\alpha$ and $\mathbb{F}^i_{\alpha,\beta}$, viz.,

$$\hat{\mathbb{f}}(\mathbb{F}, \mathbb{G}) = \bar{\mathbb{f}}(\mathbb{F}^i_\alpha, \mathbb{F}^i_{\alpha,\beta}),$$

where $\mathbb{F}$, $\mathbb{G}$ and $\mathbb{F}^i_\alpha$, $\mathbb{F}^i_{\alpha,\beta}$ are related by the formulas established above. This is the assumption employed in [Steigmann and Ogden 1999].

Proposition 7.1. *We have*

$$\delta\mathsf{E}_\mathrm{c}(\mathbb{y}, \mathbb{v}) = -\int_{\mathcal{S}} \mathbb{J}^{-1}(\mathbb{J}\mathbb{T}^\alpha_i)_{,\alpha}\mathbb{v}^i \, d\mathcal{H}^{n-1} + \int_{\partial\mathcal{S}} (\mathbb{M}^{\alpha\beta}_i \mathbb{v}^i_{,\alpha}\mathbb{m}_\beta + \mathbb{T}^\alpha_i \mathbb{m}_\alpha \mathbb{v}^i)\, d\mathcal{H}^{n-2} \tag{39}$$

for each $\mathbf{v} \in C^2(\mathrm{cl}\,\mathcal{S}, \mathrm{R}^n)$, *where*

$$\mathbb{T}^\alpha_i = \partial_{\mathbb{F}^i_\alpha}\bar{\mathbb{f}} - \mathbb{J}^{-1}(\mathbb{J}\mathbb{M}^{\alpha,\beta}_i)_{,\beta}, \quad \mathbb{M}^{\alpha\beta}_i = \partial_{\mathbb{F}^i_{\alpha,\beta}}\bar{\mathbb{f}}; \tag{40}$$

furthermore $\mathbb{T}^\alpha_i$ *and* $\mathbb{M}^{\alpha\beta}_i$ *are the components of the tensors* $\mathbb{T}$ *and* $\mathbb{M}$, *satisfying*

$$\mathbb{T} = \mathbb{T}^\alpha_i \mathbf{e}^i \otimes \mathbb{e}_\alpha, \quad \mathbb{M} = \mathbb{M}^{\alpha\beta}_i \mathbf{e}^i \otimes \mathbb{e}_\alpha \otimes \mathbb{e}_\beta.$$

Up to a change of notation, (39) coincides with Equation (4.8) of [Steigmann and Ogden 1999].

Proof. If $\mathbb{T}_i^\alpha$ and $\mathbb{M}_i^{\alpha\beta}$ are the components of $\mathbb{T}$ and $\mathbb{M}$, we see that (10) reduces to (39). Thus it remains to prove (40). Let the components $\mathbb{S}_i^\alpha$ of $\mathbb{S}$ be defined by

$$\mathbb{S} = \mathbb{S}_i^\alpha \mathbf{e}^i \otimes \mathbb{e}_\alpha.$$

The components of the partial derivatives $\hat{\mathbf{f}}$ and the partial derivatives of $\bar{\mathbf{f}}$ are related as follows:

$$\mathbb{S}_i^\alpha \equiv \partial_{\mathbb{F}} \hat{\mathbf{f}}\,|_i^\alpha = \partial_{\mathbb{F}^i_\alpha} \bar{\mathbf{f}} + \partial_{\mathbb{F}^i_{\gamma,\beta}} \bar{\mathbf{f}}\,\Gamma^\alpha_{\gamma\beta}, \quad \mathbb{M}_i^{\alpha\beta} \equiv \partial_{\mathbb{G}} \hat{\mathbf{f}}\,|_i^{\alpha\beta} = \partial_{\mathbb{F}^i_{\alpha,\beta}} \bar{\mathbf{f}}, \tag{41}$$

where we have used (38). Equation $(40)_2$ then follows immediately from $(41)_2$. To prove $(40)_1$, we note that by (37), we have

$$\begin{aligned}(\mathbb{div}\,\mathbb{M})_i &= \mathbb{J}^{-1}(\mathbb{J}\,\mathbb{M}_i^{\alpha\beta}\mathbb{e}_\alpha)_{,\beta}\\ &= \mathbb{J}^{-1}(\mathbb{J}\,\mathbb{M}_i^{\alpha\beta})_{,\beta}\mathbb{e}_\alpha + \mathbb{M}_i^{\alpha\beta}\mathbb{e}_{\alpha,\beta}\\ &= \mathbb{J}^{-1}(\mathbb{J}\,\mathbb{M}_i^{\alpha\beta})_{,\beta}\mathbb{e}_\alpha + \mathbb{M}_i^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta}\mathbb{e}_\gamma - \mathbb{L}_{\alpha\beta}\mathbb{n}),\end{aligned}$$

where $(\mathbb{div}\,\mathbb{M})_i = (\mathbb{div}\,\mathbb{M})^{\mathrm{T}}\mathbf{e}_i$. Thus

$$((\mathbb{div}\,\mathbb{M})\mathbb{P})_i = \mathbb{J}^{-1}(\mathbb{J}\,\mathbb{M}_i^{\alpha\beta})_{,\beta}\mathbb{e}_\alpha + \mathbb{M}_i^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\mathbb{e}_\gamma, \tag{42}$$

where $((\mathbb{div}\,\mathbb{M})\mathbb{P})_i = ((\mathbb{div}\,\mathbb{M})\mathbb{P})^{\mathrm{T}}\mathbf{e}_i$. Combining (42) with $(41)_1$, we obtain $(40)_1$. □

Next, let us express the energy as a function $\bar{\bar{\mathbf{f}}}$ of $\mathbb{F}^i_\alpha$ and $\mathbb{K}_{\alpha\beta} := \mathbb{K}(\mathbb{e}_\alpha, \mathbb{e}_\beta)$. We have

$$\mathbb{K}_{\alpha\beta} = \bar{\mathbb{n}}_i \mathbb{G}^i_{\alpha\beta},$$

where

$$\mathbb{G}^i_{\alpha\beta} = \mathbf{e}^i \cdot \mathbb{G}(\mathbb{e}_\alpha, \mathbb{e}_\beta) = \mathbb{F}^i_{\alpha,\beta} - \mathbb{F}^i_{,\gamma}\Gamma^\gamma_{\alpha\beta},$$

where we have used (38). Using the relation $\bar{\mathbb{n}}_i \mathbb{F}^i_{,\gamma} = 0$ we obtain

$$\mathbb{K}_{\alpha\beta} = \bar{\mathbb{n}}_i \mathbb{F}^i_{\alpha,\beta}.$$

The function $\bar{\bar{\mathbf{f}}}$ satisfies the relation

$$\bar{\bar{\mathbf{f}}}(\mathbb{F}^i_\alpha, \mathbb{K}_{\alpha\beta}) = \bar{\mathbf{f}}(\mathbb{F}^i_\alpha, \mathbb{F}^i_{\alpha,\beta}).$$

Proposition 7.2. *In terms of the partial derivatives of* $\bar{\bar{\mathbf{f}}}$ *we have*

$$\mathbb{T}_i^\alpha = \partial_{\mathbb{F}^i_\alpha}\bar{\bar{\mathbf{f}}} - \mathbb{J}^{-1}(\mathbb{F}^{-1})^\alpha_j\big(\mathbb{J}\,\bar{\mathbb{n}}_i\,\mathbb{F}^j_\beta\,\partial_{\mathbb{K}_{\beta\gamma}}\bar{\bar{\mathbf{f}}}\big)_{,\gamma},$$

where the components $(\mathbb{F}^{-1})^{\alpha}_{j}$ *are defined via the identification*

$$\mathbb{F}^{-1} = (\mathbb{F}^{-1})^{\alpha}_{j} \mathbb{e}_{\alpha} \otimes \mathbf{e}^{j},$$

where $\mathbb{F}^{-1}$ *is the pseudoinverse of* $\mathbb{F}$.

Proof. This follows immediately from Proposition 5.4. □

We note that the parametrization $\mathbf{\Phi}$ of the referential surface $\mathcal{S}$ which introduces a coordinate system $\theta^1, \dots, \theta^{n-1}$ on $\mathcal{S}$ gives, via the composition with the deformation $\mathbb{y}$ a parametrization $\bar{\mathbf{\Phi}} := \mathbb{y} \circ \mathbf{\Phi}$ of the deformed surface $\bar{\mathcal{S}}$ which introduces the coordinate system $\theta^1, \dots, \theta^{n-1}$ on $\bar{\mathcal{S}}$. If m is a function defined on $\bar{\mathcal{S}}$ with values in a finite-dimensional vector space, we use the subscript comma followed by the index α to denote the derivative of $m \circ \bar{\mathbf{\Phi}}$ with respect to θ^{α}. The coordinate vectors $\bar{\mathbb{e}}^{\alpha}$ corresponding to the coordinate system $\theta^1, \dots, \theta^{n-1}$ on $\bar{\mathcal{S}}$ are given by $\bar{\mathbb{e}}^{\alpha} = \mathbb{F}\mathbb{e}^{\alpha}$ and the dual vectors are $\bar{\mathbb{e}}_{\alpha} = \mathbb{F}^{-\mathrm{T}} \mathbb{e}_{\alpha}$. We denote by $\bar{\Gamma}^{\gamma}_{\alpha\beta}$ the Christoffel symbols corresponding to the coordinate system $\theta^1, \dots, \theta^{n-1}$ on $\bar{\mathcal{S}}$, given by

$$\bar{\Gamma}^{\gamma}_{\alpha\beta} = \bar{\mathbb{e}}^{\gamma} \cdot \bar{\mathbb{e}}_{\alpha,\beta}.$$

We denote by a vertical bar followed by an index α the covariant differentiation on $\bar{\mathcal{S}}$ using the Christoffel symbols $\bar{\Gamma}^{\gamma}_{\alpha\beta}$, i.e., if $\bar{\mathbb{v}} = \bar{\mathbb{v}}^{\alpha} \bar{\mathbb{e}}_{\alpha}$ and $\overline{\mathbb{A}} = \overline{\mathbb{A}}^{\alpha\beta} \bar{\mathbb{e}}_{\alpha} \otimes \bar{\mathbb{e}}_{\beta}$ is a tangential vector and superficial tensor defined on $\bar{\mathcal{S}}$ then

$$\bar{\mathbb{v}}^{\alpha}|_{\beta} = \bar{\mathbb{v}}^{\alpha}_{,\beta} - \bar{\Gamma}^{\alpha}_{\beta\gamma} \bar{\mathbb{v}}^{\gamma},$$

$$\overline{\mathbb{A}}^{\alpha\beta}|_{\gamma} = \overline{\mathbb{A}}^{\alpha\beta}_{,\gamma} - \bar{\Gamma}^{\alpha}_{\gamma\delta} \overline{\mathbb{A}}^{\delta\beta} - \bar{\Gamma}^{\beta}_{\gamma\delta} \overline{\mathbb{A}}^{\alpha\delta}.$$

We shall also employ the divergences based on the covariant differentiation, i.e., the objects $\bar{\mathbb{v}}^{\alpha}|_{\alpha}$ and $\overline{\mathbb{A}}^{\alpha\beta}|_{\beta}$. It is easy to see that the superficial derivative is related to the just mentioned divergences by

$$\mathbb{div}\, \overline{\mathbb{A}} = \bar{\mathbb{v}}^{\alpha}|_{\alpha}, \quad \overline{\mathbb{P}}\, \mathbb{div}\, \overline{\mathbb{A}} = \overline{\mathbb{A}}^{\alpha\beta}|_{\beta}\, \bar{\mathbb{e}}_{\alpha}. \tag{43}$$

For the subsequent discussion we define the superficial right Cauchy–Green tensor components $\mathbb{C}_{\alpha\beta} = \mathbb{F}^{i}_{\alpha} \mathbb{F}^{i}_{\beta}$. Furthermore, assume that the stored energy $\tilde{\mathbb{f}}$ is objective and let us express the energy as a function $\hat{\hat{\mathbb{f}}}$ of $\mathbb{C}_{\alpha\beta}$ and $\mathbb{K}_{\alpha\beta}$, i.e.,

$$\hat{\hat{\mathbb{f}}}(\mathbb{C}_{\alpha\beta}, \mathbb{K}_{\alpha\beta}) = \tilde{\mathbb{f}}(\mathbb{F}, \mathbb{K}) = \bar{\bar{\mathbb{f}}}(\mathbb{F}^{i}_{\alpha}, \mathbb{K}_{\alpha\beta}).$$

We note that this is possible by the objectivity.

Proposition 7.3. *Assume that the stored energy* $\tilde{\mathbb{f}}$ *is objective and denote by* $\overline{\mathbb{N}}^{\alpha\beta}$ *and* $\overline{\mathbb{M}}^{\alpha\beta}$ *the components of* $\overline{\mathbb{N}}$ *and* $\overline{\mathbb{M}}$ *identified by*

$$\overline{\mathbb{N}} = \overline{\mathbb{N}}^{\alpha\beta} \bar{\mathbb{e}}_{\alpha} \otimes \bar{\mathbb{e}}_{\beta}, \quad \overline{\mathbb{M}} = \overline{\mathbb{M}}^{\alpha\beta} \bar{\mathbb{e}}_{\alpha} \otimes \bar{\mathbb{e}}_{\beta}.$$

In terms of these components, Equations (26) read

$$\left.\begin{aligned}(\overline{\mathbb{N}}^{\alpha\beta}-\overline{\mathbb{L}}^{\alpha}_{\gamma}\overline{\mathbb{M}}^{\gamma\beta})|_{\beta}-\overline{\mathbb{L}}^{\alpha}_{\gamma}\overline{\mathbb{M}}^{\gamma\beta}|_{\beta}+\mathbb{j}^{-1}\mathbb{p}^{\alpha}=0,\\ \overline{\mathbb{M}}^{\alpha\beta}|_{\alpha\beta}+\overline{\mathbb{L}}_{\alpha\beta}\overline{\mathbb{N}}^{\alpha\beta}-\overline{\mathbb{L}}_{\alpha\gamma}\overline{\mathbb{L}}^{\gamma}_{\beta}\cdot\overline{\mathbb{M}}^{\alpha\beta}-\mathbb{j}^{-1}\mathbb{p}^{\perp}=0,\end{aligned}\right\}\tag{44}$$

where

$$\mathbb{p}^{\parallel}=\mathbb{p}^{\alpha}\bar{\mathbb{e}}_{\alpha}.$$

If $n=3$, the system of boundary conditions (24)$_1$, (27)$_1$, and (28) is equivalent to

$$\left.\begin{aligned}\overline{\mathbb{M}}^{\alpha\beta}\overline{\mathbb{m}}_{\alpha}\overline{\mathbb{m}}_{\beta}=0,\\ (\overline{\mathbb{N}}^{\alpha\beta}-2\overline{\mathbb{L}}^{\alpha}_{\gamma}\overline{\mathbb{M}}^{\gamma\beta})\overline{\mathbb{m}}_{\beta}=0,\\ \overline{\mathbb{M}}^{\alpha\beta}|_{\beta}\,\overline{\mathbb{m}}_{\alpha}+(\overline{\mathbb{M}}^{\alpha\beta}\bar{\mathbb{t}}_{\alpha}\overline{\mathbb{m}}_{\beta})'=0,\end{aligned}\right\}\tag{45}$$

where $\overline{\mathbb{m}}_{\alpha}$ and $\bar{\mathbb{t}}_{\alpha}$ are the components of the unit normal and tangent to $\partial\bar{S}$ given by

$$\overline{\mathbb{m}}=\overline{\mathbb{m}}_{\alpha}\bar{\mathbb{e}}^{\alpha},\quad \bar{\mathbb{t}}=\bar{\mathbb{t}}_{\alpha}\bar{\mathbb{e}}^{\alpha},$$

and the superscript $'$ denotes the derivative with respect to the arc length parameter on $\partial\bar{S}$. One has

$$\overline{\mathbb{N}}^{\alpha\beta}=\mathbb{j}^{-1}\,\partial_{\mathbb{C}_{\alpha\beta}}\hat{\hat{\mathbf{f}}},\quad \overline{\mathbb{M}}^{\alpha\beta}=\mathbb{j}^{-1}\,\partial_{\mathbb{K}_{\alpha\beta}}\hat{\hat{\mathbf{f}}}.\tag{46}$$

Apart from differences in notation, Equations (44) coincide with Equations (4.37) of [Steigmann and Ogden 1999]. They also coincide with the first and second of equations (9.47) of [Naghdi 1971] when the latter are specialized to the case of equilibrium of a shell, and with the equations in Theorem 7.1-3 of [Ciarlet 2000].

Proof. Equations (44) follow from (26) and the identities (43).

To prove (46), we note that differentiating the relation

$$\tilde{\mathbf{f}}(\mathbb{F}^{i}_{\alpha}\mathbf{e}_{i}\otimes\mathbb{e}^{\alpha},\mathbb{K}_{\alpha\beta}\mathbb{e}^{\alpha}\otimes\mathbb{e}^{\beta})=\bar{\bar{\mathbf{f}}}(\mathbb{F}^{i}_{\alpha},\mathbb{K}_{\alpha\beta})$$

one obtains

$$\partial_{\mathbb{F}}\tilde{\mathbf{f}}\;=\;\partial_{\mathbb{F}^{i}_{\alpha}}\bar{\bar{\mathbf{f}}}\,\mathbf{e}^{i}\otimes\mathbb{e}_{\alpha},\quad \partial_{\mathbb{K}}\tilde{\mathbf{f}}\;=\;\partial_{\mathbb{K}_{\alpha\beta}}\bar{\bar{\mathbf{f}}}\,\mathbb{e}_{\alpha}\otimes\mathbb{e}_{\beta}.\tag{47}$$

From (47)$_1$ follows that

$$\overline{\mathbb{N}}_{ij}=\mathbb{j}^{-1}\,\partial_{\mathbb{F}^{i}_{\alpha}}\bar{\bar{\mathbf{f}}}\,\mathbb{F}^{j}_{\alpha},$$

where $\overline{\mathbb{N}}_{ij}$ are the components of $\overline{\mathbb{N}}$ is the orthonormal basis $\mathbf{e}^{i}\equiv\mathbf{e}_{i}$. The components of $\overline{\mathbb{N}}$ in the basis $\bar{\mathbb{e}}_{\alpha}=\mathbb{F}^{i}_{\alpha}\mathbf{e}_{i}$ are then related by $\overline{\mathbb{N}}_{ij}=\overline{\mathbb{N}}^{\alpha\beta}\mathbb{F}^{i}_{\alpha}\mathbb{F}^{j}_{\beta}$, which gives

$$\overline{\mathbb{N}}^{\alpha\beta}=\overline{\mathbb{N}}_{ij}(\mathbb{F}^{-1})^{\alpha}_{i}(\mathbb{F}^{-1})^{\beta}_{j}=\mathbb{j}^{-1}(\mathbb{F}^{-1})^{\alpha}_{i}\,\partial_{\mathbb{F}^{i}_{\beta}}\bar{\bar{\mathbf{f}}}.\tag{48}$$

Likewise, from (47)$_2$ follows that

$$\overline{\mathbb{M}}_{ij}=\mathbb{j}^{-1}\mathbb{F}^{i}_{\alpha}\mathbb{F}^{j}_{\beta}\,\partial_{\mathbb{K}_{\alpha\beta}}\bar{\bar{\mathbf{f}}},$$

where $\overline{\mathbb{M}}_{ij}$ are the components of $\overline{\mathbb{M}}$ in the basis $\mathbf{e}^i \equiv \mathbf{e}_i$. It follows as above that the components $\overline{\mathbb{M}}^{\alpha\beta}$ in the basis $\bar{\mathbb{e}}_\alpha$ are

$$\overline{\mathbb{M}}^{\alpha\beta} = \overline{\mathbb{M}}_{ij}(\mathbb{F}^{-1})^{\alpha}_{i}(\mathbb{F}^{-1})^{\beta}_{j} = \mathbb{j}^{-1}\,\partial_{\mathbb{K}_{\alpha\beta}}\bar{\bar{\mathbb{f}}}\,. \tag{49}$$

Differentiating the relation

$$\hat{\hat{\mathbb{f}}}(\mathbb{F}^i_\alpha \mathbb{F}^i_\beta, \mathbb{K}_{\alpha\beta}) = \bar{\bar{\mathbb{f}}}(\mathbb{F}^i_\alpha, \mathbb{K}_{\alpha\beta})$$

we obtain

$$\partial_{\mathbb{C}_{\alpha\beta}}\hat{\hat{\mathbb{f}}}\,\mathbb{F}^i_\beta = \partial_{\mathbb{F}^i_\alpha}\bar{\bar{\mathbb{f}}}\,, \qquad \partial_{\mathbb{K}_{\alpha\beta}}\hat{\hat{\mathbb{f}}} = \partial_{\mathbb{K}_{\alpha\beta}}\bar{\bar{\mathbb{f}}}\,. \tag{50}$$

A combination of (48) with $(50)_1$ provides $(46)_1$ and a combination of (49) with $(50)_2$ provides $(46)_2$.

The equivalence of the system $(24)_1$, $(27)_1$, and (28) with (45) is proved similarly. □

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