NISSUNA UMANA INVESTIGAZIONE SI PUO DIMANDARE VERA SCIENZIA
S'ESSA NON PASSA PER LE MATEMATICHE DIMOSTRAZIONI
LEONARDO DA VINCI

vol. 2 no. 2 2014

Mathematics and Mechanics *of* Complex Systems

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MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS
Vol. 2, No. 2, 2014
dx.doi.org/10.2140/memocs.2014.2.109

A MIXED BOUNDARY VALUE PROBLEM IN POTENTIAL THEORY FOR A BIMATERIAL POROUS REGION: AN APPLICATION IN THE ENVIRONMENTAL GEOSCIENCES

A. P. S. SELVADURAI

This paper develops an exact closed-form solution to a mixed boundary value problem in potential theory for an elliptical opening located at an impervious interface separating two dissimilar, nondeformable porous media. The resulting solution provides a convenient result for estimating the leakage rate of an incompressible fluid retained in the system at a hydraulic potential difference. The result for the elliptical opening is also used to provide a Pólya–Szegö-type estimate for leakage rates from openings of arbitrary shape located at the impermeable interface. The extension of the study to include leakage into a transversely isotropic porous medium with the plane of isotropy inclined to the impervious boundary is also discussed.

1. Introduction

In environmental geosciences there are a number of instances where porous media are separated by relatively impervious barriers. Such situations can be found where periodic geologic deposition results in laminated regions where a fluid transmissivity contrast can be created by sedimentation of fine-grained material that will form a nearly impervious barrier. Examples of these include varved clays and other stratified geological media, where the scale of the impervious layers can range from a few millimeters to meters [Tschebotarioff 1951; Bear and Verruijt 1987; Phillips 1991; Selvadurai and Carnaffan 1997; Selvadurai et al. 2005]. An example that has direct relevance to the application discussed in this paper is the problem of geosynthetic liners, made of polymeric materials and used quite extensively as engineered barriers to prevent the migration of fluids containing hazardous and toxic chemicals from reaching potable groundwater regimes. Nearly all current-day waste management endeavors, ranging from sanitary landfills to storage reservoirs from resource extraction, use a variation of the concept of a geosynthetic liner to

Communicated by Felix Darve.
MSC2010: primary 31A10; secondary 31A25.
Keywords: potential theory, mixed boundary value problem, bimaterial region, Darcy flow, leakage from barrier, bounds for leakage rates, transverse isotropic permeability, mathematical geosciences.

contain the hazardous components of toxic fluids. The implicit assumption in these endeavors is that the polymeric material will maintain its integrity as a barrier, in perpetuity. This is a naive expectation for any man-made material, particularly a geosynthetic material that is prone to degradation upon exposure to leachates and other chemicals and ultraviolet light. Recent experimental and theoretical investigations [Yu and Selvadurai 2005; 2007; Selvadurai and Yu 2006a; 2006b] indicate that the flexible polymeric material can be rendered brittle as a result of the loss of plasticizer, induced by the leaching action of relatively common chemicals such as acetone and ethanol. Since the plasticizer contributes to the flexibility of the geosynthetic material, the leaching process can lead to its embrittlement, which can serve as a location for the development of cracks through which the retained contaminant fluids can be released. The assessment of the leakage rate from the retained fluid is therefore of interest to estimate the environmental hazard associated with the long term use of geosynthetics.

This paper examines the problem of the leakage through an elliptical opening or defect that is located at the impermeable interface between two isotropic porous geological media (Figure 1). The formulation of the problem takes into consideration the Darcy flow properties of the porous media adjacent to the impervious barrier containing the elliptical defect. The paper examines the mixed boundary value in potential theory that can be applied to the fluid migration from an elliptical defect at an impervious boundary. It is shown that an exact solution to this problem can be obtained using the results originally presented by Lamb [1927] and recently applied by Selvadurai [2010], who developed a result for the intake shape factor for a *circular fluid intake* terminating at the interface between a hydraulically transversely isotropic porous medium and an impervious stratum with the principal planes of permeability inclined to the impervious interface. This analytical approach is used to develop an exact analytical result for the leakage from an elliptical defect in an otherwise impervious geosynthetic liner. The analytical procedure yields an exact closed-form solution, which takes into consideration the permeability characteristics of porous regions on either side of the impervious barrier. It is shown that the analytical result for the elliptical defect can also be used to develop a set of bounds for estimating the leakage rate from a defect of arbitrary shape. In the particular instance when the barrier with the elliptical defect is in contact with a single transversely isotropic porous medium with the plane of transverse isotropy inclined to the impervious interface, a closed-form analytical result can be developed for the leakage rate from the elliptical defect.

2. The mixed boundary value problem

We restrict attention to a bimaterial porous region that contains a plane impermeable barrier at the interface of the two porous media and fluid leakage occurs

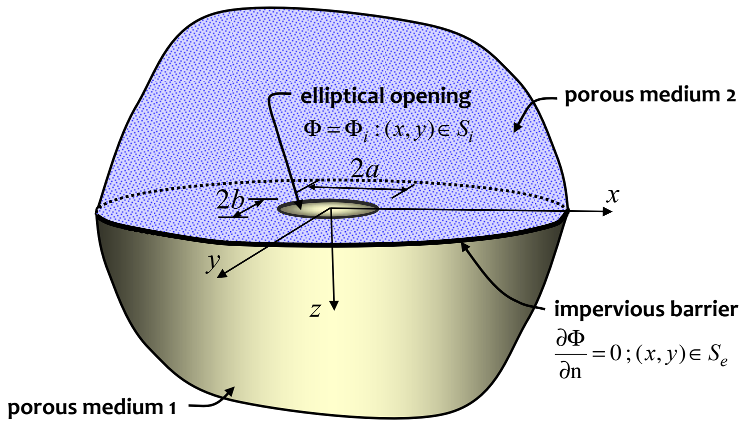

Figure 1. Elliptical opening at an impermeable interface separating two dissimilar porous media.

through an *elliptical defect* in the impervious barrier (Figure 1). The fluid flow characteristics are governed by Darcy's law and the permeabilities are defined by K_1 and K_2 for the respective regions. The reduced Bernoulli potential associated with Darcy flow is defined by $\Phi(\boldsymbol{x})$ and this neglects the velocity potential. The datum is taken as the plane of the interface and we assume that the regions 1 and 2 are subjected, respectively, to far-field reduced Bernoulli potentials Φ_1 and Φ_2, with $\Phi_1 > \Phi_2$. For purposes of model development, we assume that the nondeformable dissimilar porous region shown in Figure 1 contains fluids with similar properties, although it should be noted that the viscosity and other properties of the contaminating leachates can be different from those of groundwater. In order to develop a convenient analytical result that can be used to estimate the *steady fluid leakage* through the crack, we shall adopt this assumption. It can be shown that for isochoric Darcy flow in an isotropic nondeformable porous medium the mass conservation law gives $\nabla.\boldsymbol{v} = 0$, where $\boldsymbol{v}(\boldsymbol{x})$ is the velocity vector and $\boldsymbol{x}$ is the spatial coordinates. For an isotropic porous medium, Darcy's law can be written as

$$\boldsymbol{v}(\boldsymbol{x}) = -\frac{K\gamma_w}{\mu}\nabla\Phi, \tag{1}$$

where K is the permeability, γ_w is the unit weight of the fluid, and μ is its dynamic viscosity. Combining Darcy's law and the fluid mass conservation principle, we obtain the partial differential equation governing $\Phi(\boldsymbol{x})$ as

$$\nabla^2\Phi(\boldsymbol{x}) = 0, \tag{2}$$

where ∇^2 is Laplace's operator. We consider the mixed boundary value problem in potential theory referred to a halfspace region, the boundary of which is subject to the relevant Dirichlet and Neumann boundary conditions applicable to an impermeable region with an elliptical opening. The planar region S_i corresponding to the elliptical defect is defined by

$$S_i: \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1, \tag{3a}$$

while the region S_e exterior to the defect is defined by

$$S_e: \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 > 1, \tag{3b}$$

where a and b are, respectively, the semimajor and the semiminor axes of the elliptical region. We consider the mixed boundary value problem in potential theory referred to the elliptical aperture such that

$$(\Phi)_{z=0} = \Phi_i = \text{constant}, \quad (x, y) \in S_i, \tag{4}$$

and

$$\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} = 0, \quad (x, y) \in S_e, \tag{5}$$

where Φ_i is the constant Bernoulli potential over the elliptical interior region, which is dictated by the constant pressure potential over this region, and the datum is taken as the plane of the defect. Since the problem examined has a three-dimensional configuration, the regularity conditions applicable to a semiinfinite domain should also be satisfied. In this case the far-field potential in region 1 is Φ_1. For the solution of the mixed boundary value problem posed by (4) and (5) we assume, however, that the potential $\Phi(\boldsymbol{x})$ decays uniformly to zero as $\boldsymbol{x} \to \infty$. Since the Bernoulli potential is indeterminate to an arbitrary constant, the far-field value can be added to satisfy the value of the constant potential regularity condition as $\boldsymbol{x} \to \infty$. The solution to the mixed boundary value problem in potential theory, governed by the partial differential equation (2) and mixed boundary conditions (4) and (5), can be developed in a variety of ways, the most widely accepted being the formulation that employs a generalized ellipsoidal coordinate system and by developing the solution to the opening with an elliptical plan form as a limiting case of an ellipsoid. This approach was used by Lamb [1927] to develop a solution for the motion of a perfect fluid through an elliptical aperture. The result can also be developed using the formal developments in potential theory given by Morse and Feshbach [1953, Section 10.3]. Similar developments have been used in [Green and Sneddon 1950; Kassir and Sih 1968; Selvadurai 1982; Walpole 1991] in developing canonical results for elliptical cracks and elliptical inclusions embedded in isotropic and transversely isotropic elastic solids. The solution can be most conveniently formulated in relation to a set of ellipsoidal coordinates (ξ, η, ζ) of the point (x, y, z), which are the roots of the cubic equation in θ defined by

$$\frac{x^2}{(a^2+\theta)} + \frac{y^2}{(b^2+\theta)} + \frac{z^2}{\theta} - 1 = 0. \tag{6}$$

The ellipsoidal coordinate system (ξ, η, ζ) chosen ensures that the interior Dirichlet region S_i corresponds to the ellipse $\xi = 0$ and the exterior Neumann region S_e corresponds to a hyperboloid of one sheet $\eta = 0$. The mixed boundary conditions (4) and (5) can be explicitly satisfied by the harmonic function

$$\Phi(x, y, z) = \frac{a\Phi_i}{\mathrm{K}(\sigma)} \int_{\xi}^{\infty} \frac{ds}{\sqrt{s(a^2+s)(b^2+s)}}, \tag{7}$$

where

$$\xi = a^2(\mathrm{sn}^{-2}\, u - 1) \tag{8}$$

and $\mathrm{sn}\, u$ represents the Jacobian elliptic function defined by

$$\int_0^{\mathrm{sn}(u,\sigma)} \frac{dt}{\sqrt{(1-t^2)(1-\sigma^2 t^2)}} = (u, \sigma). \tag{9}$$

In a numerical evaluation of $\mathrm{sn}\, u$, it is convenient to express the function in the series form:

$$\begin{aligned}\mathrm{sn}(u, \sigma) = u - (1+\sigma^2)\frac{u^3}{3!} + (1 + 14\sigma^2 + \sigma^4)\frac{u^5}{5!} \\ - (1 + 135\sigma^2 + 135\sigma^4 + \sigma^6)\frac{u^7}{7!} + \cdots .\end{aligned} \tag{10}$$

The complete elliptic integral of the first kind $\mathrm{K}(\sigma)$ is defined by

$$\mathrm{K}(\sigma) = \int_0^{\pi/2} \frac{d\varsigma}{\sqrt{1-\sigma^2 \sin^2\varsigma}}, \quad \sigma = \left(\frac{a^2-b^2}{a^2}\right)^{1/2}. \tag{11}$$

We can generalize the result (7) to account for the effect of the far-field potential Φ_1 ($> \Phi_i$). This involves simply changing the potential Φ_i in (7) to $(\Phi_1 - \Phi_i)$. The fluid velocity at the elliptical aperture associated with region 1 is now given by

$$\begin{aligned}v_z^{(1)}(x, y, 0) &= -\frac{K_1\gamma_w}{\mu}\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} \\ &= \frac{K_1\gamma_w(\Phi_1-\Phi_i)}{b\mu\mathrm{K}(\sigma)} \frac{1}{\sqrt{1-x^2/a^2-y^2/b^2}}, \quad (x, y) \in S_i,\end{aligned} \tag{12}$$

where K_1 is the permeability of the porous region 1. The flow rate out of the elliptical aperture is given by

$$Q = \frac{K_1 \gamma_w (\Phi_1 - \Phi_i)}{\mu b \mathrm{K}(\sigma)} \iint_{S_i} \frac{dx\, dy}{\sqrt{1 - x^2/a^2 - y^2/b^2}} = \frac{2\pi a (\Phi_1 - \Phi_i) \gamma_w K_1}{\mu \mathrm{K}(\sigma)}. \quad (13)$$

A similar result can be developed for the potential flow problem where flow takes place from the porous halfspace region 2 at a far-field potential Φ_2 ($< \Phi_i$), which gives the velocity field in the interface approached from region 2 as

$$\begin{aligned} v_z^{(2)}(x, y, 0) &= -\frac{K_1 \gamma_w}{\mu} \left(\frac{\partial \Phi}{\partial z} \right)_{z=0} \\ &= \frac{K_1 \gamma_w (\Phi_i - \Phi_2)}{b \mu \mathrm{K}(\sigma)} \frac{1}{\sqrt{1 - x^2/a^2 - y^2/b^2}}, \quad (x, y) \in S_i, \end{aligned} \quad (14)$$

and the flow rate into region 2 is given by

$$Q = \frac{2\pi a (\Phi_i - \Phi_2) \gamma_w K_2}{\mu \mathrm{K}(\sigma)}. \quad (15)$$

The value of the interface potential Φ_i can be obtained, ensuring continuity of the velocity field at the interface: that is,

$$v_z^{(1)}(x, y, 0) = v_z^{(2)}(x, y, 0), \quad (16)$$

which gives $\Phi_i = (K_1 \Phi_1 + K_2 \Phi_2)/(K_1 + K_2)$. The leakage rate through the elliptical aperture can now be obtained by eliminating Φ_i in either (13) or (15), which gives

$$Q = \frac{2\pi a (\Phi_1 - \Phi_2) \gamma_w K_1 K_2}{\mu (K_1 + K_2) \mathrm{K}(\sigma)}. \quad (17)$$

It is important to note that the result (17) is the *exact closed-form solution* for the steady leakage of an incompressible fluid through an elliptical cavity located at the impermeable interface separating isotropic nondeformable porous regions of dissimilar permeability. The potential problem that is solved satisfies the governing equations of potential flow, the mixed boundary conditions applicable to the potential problem and, continuity of *both* the *potential* and the *flow velocity* at the interface where Dirichlet conditions are prescribed. From the *uniqueness theorem* applicable to mixed boundary value problems in potential theory, this solution is unique [Zauderer 1989, Section 6.8; Selvadurai 2000a, Section 5.7; 2000b, Section 9.5]. It should be noted that the solution is identical even if a *continuity of total flux boundary condition* is imposed on the interface rather than a *continuity of flow velocity*. It is noted from (12) and (14) that although the velocity at the boundary of the elliptical entry region is singular, the volume flow rate to the elliptical cavity region is finite. In the special case when the permeability value of one region becomes large (for example, $K_2 \to \infty$), (17) reduces to

$$Q = \frac{2\pi a(\Phi_1 - \Phi_2)\gamma_w K_1}{\mu \mathrm{K}(\sigma)}. \tag{18}$$

For the special case when the elliptical opening has the shape of a circular region of radius a, $\mathrm{K}(0) \to \pi/2$ and (18) reduces to the classical result that can be obtained by solving the associated mixed boundary value problem in potential theory for the circular opening at an impervious interface, by appeal to the theory of dual integral equations [Sneddon 1966]. The solution presented for the elliptical defect is valid for all aspect ratios of the defect, which permits the evaluation of leakage rates from narrow cracks. Comparisons of the analytical estimates with computational results are given in [Selvadurai 2012] and the analytical solution provides a benchmark for calibration of computational modeling of the potential problem. It is also worth noting that in instances where the separate porous regions display spatial heterogeneity with a *log normal* variation in the permeability, which can be characterized by an effective permeability such as the *geometric mean* [Selvadurai and Selvadurai 2010], the result (17) can be used to estimate the leakage from the elliptical opening.

3. Fluid leakage from a defect with an arbitrary plan form

The result (17) can also be used to estimate or develop bounds for the leakage from a *damaged region* of arbitrary area A_D, where the bounds for the effective permeability K_D are obtained by considering equivalent elliptical regions that either *inscribe* or *circumscribe* the region A_D (Figure 2), that is,

$$\bar{Q}_I \le \frac{Q\mu}{2\pi \gamma_w (\Phi_1 - \Phi_2)\sqrt{A_D K_1 K_2}} \le \bar{Q}_C. \tag{19}$$

In (19), $\bar{Q}_n$ $(n = I, C)$ denote the nondimensional flow rates, which refer to the elliptical regions that either *inscribe* (I) or *circumscribe* (C) the region A_D and are given by

$$\bar{Q}_n = \frac{a_n\sqrt{K_1 K_2}}{\sqrt{A_D}(K_1 + K_2)\mathrm{K}(\sigma_n)} \quad (n = I, C). \tag{20}$$

The nondimensionalization is accomplished by ensuring that the bounds correspond to nondimensional versions derived from (20). Here, a_n and b_n $(n = I, C)$ are, respectively, the *semimajor* and *semiminor* axes of the elliptical region associated with the inscribed and circumscribed ellipses that contain the region of area A_D, and

$$\mathrm{K}(\sigma_n) = \int_0^{\pi/2} \frac{d\varsigma}{\sqrt{1 - \sigma_n^2 \sin^2 \varsigma}}, \quad \sigma_n = \left(\frac{a_n^2 - b_n^2}{a_n^2}\right)^{1/2}. \tag{21}$$

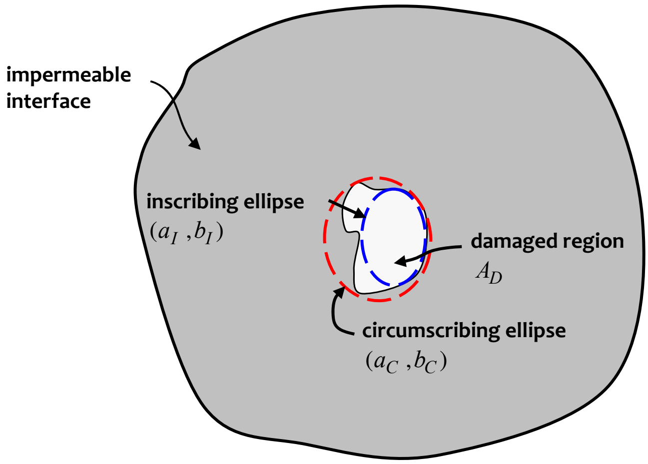

Figure 2. Defect with an arbitrary shape located at an impermeable interface between dissimilar nondeformable porous media.

This approach for bounding the result follows procedures that were proposed by Maxwell [1892] for problems in electrostatics and applied by Pólya and Szegö [1945; 1951] (see also [Protter and Weinberger 1984]) for a variety of problems in potential theory and elastostatics to obtain bounds to problems that are usually regarded as analytically intractable [Galin 1961; Selvadurai 1983].

4. Fluid leakage from an elliptical defect into a hydraulically transversely isotropic porous medium

We now consider the problem of a transversely isotropic or stratified porous medium of semiinfinite extent where the stratifications are inclined at an angle α to the surface of the halfspace (Figure 3). The surface of the halfspace is impervious except over an elliptical opening through which fluids can either enter or exit the transversely isotropic porous medium. The orientation of the elliptical opening is such that its minor axis is aligned with the y-axis of the spatial coordinate system (x, y, z). To an extent, this is a simplification; otherwise the potential problem would only be amenable to a complicated ellipsoidal harmonic function formulation.

The permeability matrix for the hydraulically transversely isotropic material, $[\boldsymbol{K}^{\mathrm{P}}]$, referred to the principal directions aligned along the normal (n) and tangential (t) directions, is given by

$$[\boldsymbol{K}^{\mathrm{P}}] = \begin{bmatrix} K_t & 0 & 0 \\ 0 & K_t & 0 \\ 0 & 0 & K_n \end{bmatrix}, \tag{22}$$

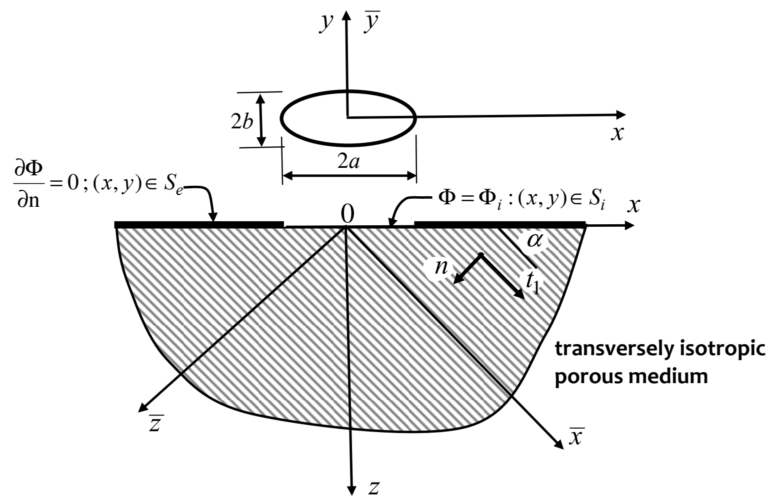

Figure 3. A mixed boundary value problem for flow from an elliptical opening located at the surface of a transversely isotropic porous halfspace.

and the permeability matrix $[\boldsymbol{K}^{\mathrm{RC}}]$ referred to the rectangular Cartesian coordinate system (x, y, z) is given by

$$[\boldsymbol{K}^{\mathrm{RC}}] = [\boldsymbol{H}]^{\mathrm{T}}[\boldsymbol{K}^{\mathrm{P}}][\boldsymbol{H}], \tag{23}$$

where

$$[\boldsymbol{H}] = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}. \tag{24}$$

The mixed boundary value problem governing flow into the transversely isotropic elastic halfspace is of the type given by the boundary conditions (4) and (5) indicated previously. It can be shown that this mixed boundary value problem, if formulated in the conventional way, will give rise, even for a circular opening, to a series of dual integral equations of unmanageable complexity. The approach adopted here follows from the transformation technique proposed by Selvadurai [2010], which leads to a manageable problem. Referring to Figure 3, where the principal planes of hydraulic transverse isotropy are aligned with the coordinate system $(\bar{x}, \bar{y}, \bar{z})$, the fluid flow in the transversely isotropic porous medium is given by

$$K_t\left(\frac{\partial^2\Phi}{\partial\bar{x}^2} + \frac{\partial^2\Phi}{\partial\bar{y}^2}\right) + K_n\frac{\partial^2\Phi}{\partial\bar{z}^2} = 0. \tag{25}$$

At the elliptical region on which the Dirichlet condition is prescribed,

$$\bar{x} = a\cos\alpha, \quad \bar{y} = b, \quad \bar{z} = a\sin\alpha. \tag{26}$$

We now make the conventional transformation that converts (25) from a pseudo-Laplacian to a Laplacian form, using the following transformations:

$$\bar{x} = X, \quad \bar{y} = Y, \quad \bar{z} = Z\sqrt{\frac{K_n}{K_t}}, \tag{27}$$

which reduces (25) to the Laplacian form

$$\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Phi}{\partial Y^2} + \frac{\partial^2\Phi}{\partial Z^2} = 0. \tag{28}$$

The coordinate directions X, Y, and Z are aligned with the axes $\bar{x}$, $\bar{y}$, and $\bar{z}$, respectively; the new orientation of the halfspace region is defined in relation to the axes $(\tilde{x}, \tilde{y}, \tilde{z})$ and the inclination of the halfspace is defined by β as shown in Figure 4. The transformations (27), however, transform the original elliptical cavity with semimajor axis a and semiminor axis b to an ellipse with semimajor axis c and semiminor axis b, and the inclination β of the plane of the elliptical Dirichlet region at the interface (Figure 4) gives

$$X = c\cos\beta, \quad Y = b, \quad Z = c\sin\beta. \tag{29}$$

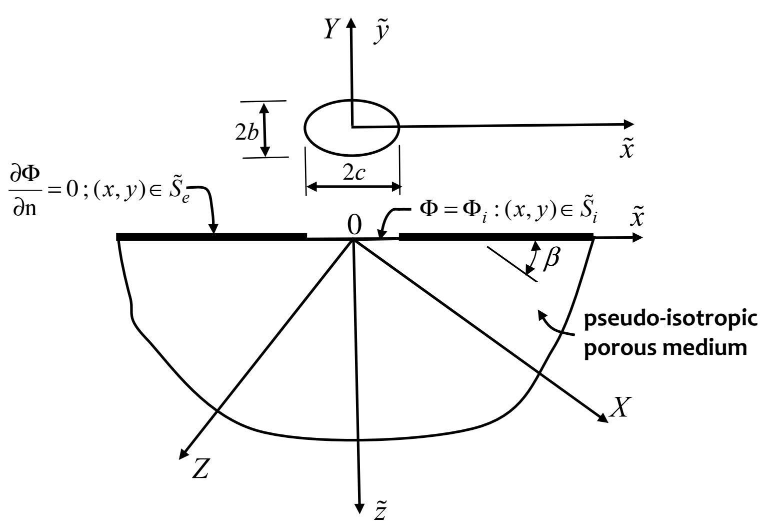

Figure 4. A mixed boundary value problem for flow from an elliptical opening located at the surface of a pseudoisotropic porous medium.

Using (27) and (29), we obtain

$$c^2 = a^2\left\{\cos^2\alpha + \frac{K_t}{K_n}\sin^2\alpha\right\}. \tag{30}$$

Since the representation of the potential problem in the (X, Y, Z) coordinate system is isotropic, the choice of reference coordinate system to analyze the mixed boundary value problem referred to the Dirichlet boundary conditions prescribed on the transformed elliptical region is arbitrary and it is convenient to select the coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ as shown in Figure 4. Referring to this coordinate system, the mixed boundary value problem deals with Dirichlet boundary conditions prescribed within the elliptical opening $\tilde{S}_i$ and null Neumann boundary conditions prescribed in the exterior region $\tilde{S}_e$, where

$$\tilde{S}_i\colon \left(\frac{\tilde{x}}{c}\right)^2 + \left(\frac{\tilde{y}}{b}\right)^2 \leq 1, \quad \text{and} \quad \tilde{S}_e\colon \left(\frac{\tilde{x}}{c}\right)^2 + \left(\frac{\tilde{y}}{b}\right)^2 > 1, \tag{31}$$

where c will be the major axis if $K_t/K_n > 1$ and a minor axis if $K_t/K_n < 1$. Following procedures similar to those outlined in Section 2 it can be shown that the relevant solution for satisfying the mixed boundary conditions

$$(\Phi)_{\tilde{z}=0} = \Phi_i = \text{constant}, \quad (\tilde{x}, \tilde{y}) \in \tilde{S}_i, \tag{32}$$

$$\left(\frac{\partial\Phi}{\partial\tilde{z}}\right)_{\tilde{z}=0} = 0, \quad (\tilde{x}, \tilde{y}) \in \tilde{S}_e, \tag{33}$$

is given by

$$\Phi(\tilde{x}, \tilde{y}, z) = \frac{c\Phi_i}{\mathrm{K}(\rho)}\int_\xi^\infty \frac{ds}{\sqrt{s(c^2+s)(b^2+s)}}, \tag{34}$$

where

$$\rho = \sqrt{\frac{K_n\cos^2\alpha + K_t\sin^2\alpha - (b/a)^2}{K_n\cos^2\alpha + K_t\sin^2\alpha}}. \tag{35}$$

The flow rate at the elliptical opening is given by

$$Q = \frac{2\pi a\gamma_w\Phi_i\sqrt{K_nK_t}}{\mu\mathrm{K}(\rho)}\sqrt{\cos^2\alpha + \frac{K_t}{K_n}\sin^2\alpha}. \tag{36}$$

When $K_t = K_n = K_0$, (36) is independent of the orientation α and reduces to a result similar to that given by (18).

The mixed boundary value problem involving dissimilar transversely isotropic porous halfspace regions with planes of transverse isotropy that are arbitrary (see Figure 5) cannot be solved in an exact fashion similar to that outlined previously. This limitation arises from the fact that when both the hydraulic transverse isotropies of the two regions and the inclinations of the principal directions are arbitrary, in the

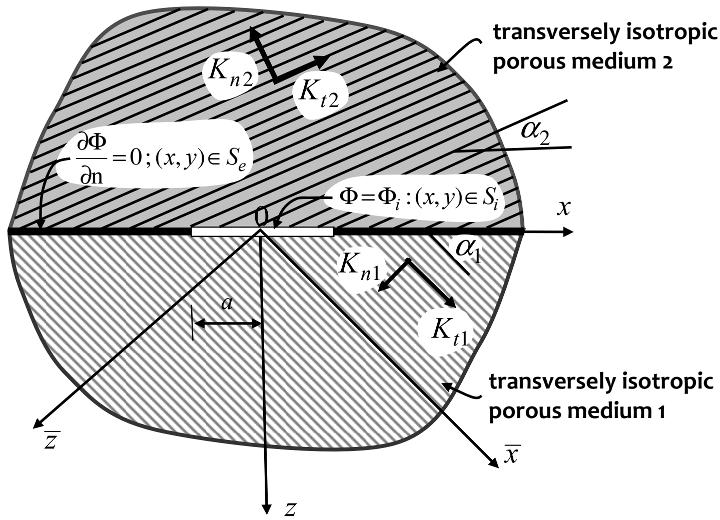

Figure 5. Elliptical defect at an impermeable interface between two dissimilar transversely isotropic porous media.

transformed configuration the dimensions of the ellipses will be different (by virtue of (30)) and will be unequal even if the orientations of the transverse isotropies are made to coincide along one principal direction. An approximate solution to the leakage through an elliptical opening separating dissimilar transversely isotropic regions under far-field potentials Φ_1 and Φ_2 (with $\Phi_1 > \Phi_2$) can be evaluated by assuming that the *total flow rate* at the elliptical defect is the same for both transversely isotropic regions irrespective of the misalignment in their principal axes.

The flow rate can be estimated from the result

$$Q \simeq \frac{2\pi a \gamma_w (\Phi_1 - \Phi_2)}{\mu \displaystyle\sum_{i=1,2} \frac{\mathrm{K}(\rho_i)}{\sqrt{K_{ni} K_{ti} \{\cos^2 \alpha_i + (K_{ti}/K_{ni}) \sin^2 \alpha_i\}}}}, \tag{37}$$

where

$$\rho_i = \sqrt{\frac{K_{ni} \cos^2 \alpha_i + K_{ti} \sin^2 \alpha_i - (b/a)^2}{(K_{ni} \cos^2 \alpha_i + K_{ti} \sin^2 \alpha_i)}}. \tag{38}$$

In the instance when $K_{n1} = K_{t1} = K_1$ and $K_{n2} = K_{t2} = K_2$, the solution will be independent of α_i $(i = 1, 2)$ and $\rho_i = \sigma$, which is defined by (11). In this case the result (37) reduces to the exact solution given by (17). Similarly as $K_{n2} = K_{t2} \to \infty$, and $K_{n1} = K_n$, $K_{t1} = K_t$, (37) reduces to the result (36) for a single halfspace.

5. Concluding remarks

The mixed boundary value problem in potential theory for a halfspace region, where Dirichlet boundary conditions are prescribed over an elliptical region and null Neumann boundary conditions are prescribed exterior to the elliptical domain, can be used to develop solutions to porous media flow problems of interest to environmental geosciences. The exact closed-form solutions developed for leakage through an elliptical defect located at an impermeable interface between nondeformable dissimilar porous media can be extended to develop exact closed-form results that can be used to benchmark the accuracy of computational schemes for Darcy flow where singular behavior in the velocity field at the boundary is rarely incorporated in the solution scheme.

Acknowledgements

The work described in this paper was supported by the 2003 Max Planck Research Award in the Engineering Sciences from the Max-Planck-Gesellschaft, Germany, and through a Discovery Grant awarded by the Natural Sciences and Engineering Research Council of Canada. The author is grateful to the reviewers for providing valuable comments that led to improvements in the paper.

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Received 7 Aug 2012. Revised 13 Mar 2013. Accepted 20 Apr 2013.

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MEMOCS (ISSN 2325-3444 electronic, 2326-7186 printed) is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

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MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS
vol. 2 no. 2 2014

MEMOCS is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

2326-7186(2014)2:2;1-F