

NISSUNA UMANA INVESTIGAZIONE SI PUO DIMANDARE VERA SCIENZA
S'ESSA NON PASSA PER LE MATEMATICHE DIMOSTRAZIONI
LEONARDO DA VINCI

vol. 4

no. 1

2016

MATHEMATICS AND MECHANICS
of
Complex Systems

GÉRARD GAGNEUX AND OLIVIER MILLET
MODELING CAPILLARY HYSTERESIS
IN UNSATURATED POROUS MEDIA



MODELING CAPILLARY HYSTERESIS IN UNSATURATED POROUS MEDIA

GÉRARD GAGNEUX AND OLIVIER MILLET

This paper deals with the modeling of cyclic hysteresis phenomena for flows in unsaturated porous media, using a dynamic regularization process of Sobolev type. The addition of a kinematic regularizing term of third-order partial derivatives, depending on a strictly positive, small real parameter, enables us to capture the missing information of the ill-posed hysteresis phenomena via Rankine–Hugoniot and “entropy” inequalities. When this parameter tends to zero, an oriented hysteresis loop, corresponding to the realistic problem modeled, emerges from the flow of an associated auxiliary ordinary differential equation.

1. Introduction

The modeling of moisture transport in partially saturated porous media is of major importance for civil engineering, soil physics, and pharmaceutical applications. The hysteresis effects, often neglected in the modeling as they are difficult to be taken into account, play a central role in the imbibition and drying process.

In this paper, we propose an original modeling of cyclic hysteresis phenomena in partially saturated porous media, in the simplified case of water–air flows. The approach used is based on the artificial introduction of an unstable spinodal interval and on Sobolev’s method of dynamic regularization, inspired by the works of P. I. Plotnikov [1996; 1994], publicized by L. C. Evans and M. Portilheiro [2004; Evans 2004]. The hysteresis graph is replaced by Cartesian curves and an artificial spinodal interval generating instabilities, with associated attractive–repulsive dynamics.

The additional information to describe the hysteresis effects is introduced on the form of entropy-type inequalities. This way, the asymptotic limit of viscous approximate solutions generates effects of irreversibility and enables us to recover the expected hysteresis loop.

Communicated by Francesco dell’Isola.

MSC2010: primary 35K65; secondary 47J40, 76S05.

Keywords: capillary hysteresis loop, unsaturated porous media.

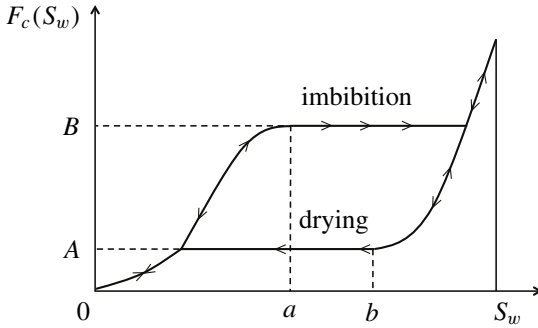


Figure 1. Graph of F_c representing the hysteresis loop during an imbibition-drying process.

2. The physical problem

2.1. Richards equation. The flow of two fluid phases (water–air flow), isothermal and immiscible, in an unsaturated porous medium is considered. To focus on the study of the hysteresis effects, in particular irreversibility, the gravity is neglected and the porous medium is assumed to be homogeneous and isotropic.¹ Moreover, the water vapor in the gas phase is neglected and the air pressure is assumed to be constant and equal to the atmospheric pressure.²

The water saturation S_w is classically governed by a Richards equation,

$$\varphi \frac{\partial S_w}{\partial t} - \Delta \Theta_c(S_w) = 0, \quad (1)$$

where φ denotes the porosity of the porous medium considered. We assume that the residual saturation of each fluid is equal to zero.

Even if the mathematical analysis of this equation is now well stated [Gagneux and Madaune-Tort 1995; Lions 1969], it ignores the hysteresis and dynamic effects that play a major role in the behavior of unsaturated porous media.

2.2. Hysteresis modeling. The capillary hysteresis effects can be modeled with a multivalued operator F_c whose oriented graph F_c of \mathbb{R}^2 is represented in Figure 1. The circulation sense depends both on the values of S_w and on the sign of $\frac{\partial S_w}{\partial t}$. It characterizes the imbibition and drying phases, through the differential inclusion³

$$0 \in \left\{ \varphi \frac{\partial S_w}{\partial t} - \Delta F_c \left(S_w, \text{sign} \left(\frac{\partial S_w}{\partial t} \right) \right) \right\}. \quad (\mathcal{P}_{\text{hyst}})$$

¹The analysis holds also for the anisotropic case.

²It is equivalent to assume that the gas phase moves fast and is connected to outside.

³In the sense of [Aubin and Cellina 1984].

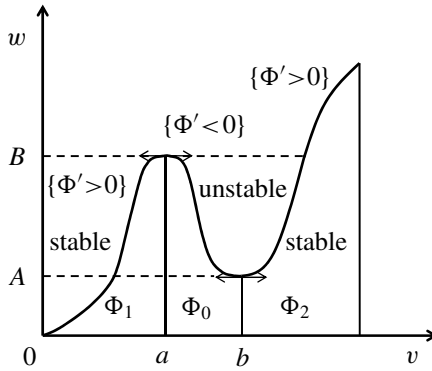


Figure 2. The spinodal interval $[a, b]$ and the unstable part of the graph of Φ .

It is equivalent to search a pair (S_w, q_w) with $q_w \in \{F_c(S_w, \text{sign}(\frac{\partial S_w}{\partial t}))\}$ which satisfies

$$\varphi \frac{\partial S_w}{\partial t} - \Delta q_w = 0 \quad (2)$$

associated to a Cauchy initial condition $S_{w(0)}$ and Neumann homogeneous boundary conditions. To simplify the problem without loss of generality, we will consider in the following a normalized porosity $\varphi = 1$ (this is always possible using a homothetical time scaling, when φ is constant).

In a first step, using a mathematical artifice, we replace the part of the graph F_c representing the loop by a cubic Cartesian curve Φ (Figure 2). Then, to approach problem $(\mathcal{P}_{\text{hyst}})$, a nonlinear monotone diffusion equation with an ad hoc “spinodal” interval $]a, b[$ is introduced. The graph F_c is replaced on $]a, b[$ by a cubic spline function, denoted Φ_0 , whose slope is strictly negative everywhere on $]a, b[$ and assuring a \mathcal{C}^1 continuity at (a, B) and (b, A) with the preserved part.

The graph F_c is decomposed by splitting its domain of definition into three distinct parts. This leads us to introduce three injective functions, Φ_0 , Φ_1 and Φ_2 , defined on $[a, b]$, $[0, a]$ and $[b, 1]$, respectively. We denote by β_0 , β_1 and β_2 their respective inverse functions and by Φ the numerical function of class \mathcal{C}^1 on $[0, 1]$ whose graph is the joining of the graphs of Φ_0 , Φ_1 and Φ_2 .

This substitution enables us to give sense to the initial formal problem in a suitable mathematical functional framework, via the initial–boundary value system

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta \Phi(v) = 0 & \text{in } Q =]0, T[\times \Omega, \\ \frac{\partial \Phi(v)}{\partial n} = 0 & \text{on } \Sigma =]0, T[\times \Gamma, \\ v(0) = S_{w(0)} & \text{in } \Omega, \end{cases} \quad (P_\Phi)$$

where Ω denotes a bounded domain of \mathbb{R}^d , $d \geq 1$, with a Lipschitz boundary Γ and an associated external unit normal vector n . The new forward–backward problem (P_Φ) with variable parabolicity direction is ill posed without any supplementary information, because of the nonmonotonic function Φ . The dynamic regularization process that follows will enable us to regularize the problem. Note that a problem similar to (P_Φ) has been studied in [Smarrazzo and Tesei 2010; 2012; Smarrazzo 2008].

3. Dynamic regularization process of Sobolev type

The classical notations that follow are introduced. Let Ω be a bounded domain of \mathbb{R}^d whose boundary Γ is a Lipschitz manifold of dimension $d - 1$. For $T > 0$, we write $Q =]0, T[\times \Omega$, $\Sigma =]0, T[\times \Gamma$ and let Δ be the Laplacian operator⁴ of \mathbb{R}^d . We denote by $H^s(\Omega)$, $s \in \mathbb{R}$, the classical Hilbert spaces [Lions and Magenes 1968]. For all $\varepsilon > 0$, the embedding of $H^s(\Omega)$ into $H^{s-\varepsilon}(\Omega)$ is compact. Moreover, we identify $L^2(\Omega) = H^0(\Omega)$ to its dual, so that the dual of $H^1(\Omega)$, denoted $H^1(\Omega)'$, can be identified to an superspace of $L^2(\Omega)$ with $H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^1(\Omega)'$, the embeddings being dense and continuous. In addition, an initial state $S_{w(0)} \in L^\infty(\Omega)$ satisfying $0 \leq S_{w(0)} \leq 1$ a.e. in Ω is given.

The dynamical regularization process of Sobolev type used is based on “artificial viscosity”. A parameter $\lambda > 0$ is introduced in the initial ill-posed problem (P_Φ) , which is transformed into the third order boundary problem

$$\begin{cases} \frac{\partial v_\lambda}{\partial t} - \Delta \Phi(v_\lambda) - \lambda \Delta \frac{\partial v_\lambda}{\partial t} = 0 & \text{in } Q =]0, T[\times \Omega, \\ \frac{\partial}{\partial n} \left(\Phi(v_\lambda) + \lambda \frac{\partial v_\lambda}{\partial t} \right) = 0 & \text{on } \Sigma =]0, T[\times \Gamma, \\ v_\lambda(0) = S_{w(0)} & \text{in } \Omega. \end{cases} \quad (P_\Phi)_\lambda$$

We then introduce the auxiliary dynamical unknown w_λ defined by

$$w_\lambda = \Phi(v_\lambda) + \lambda \frac{\partial v_\lambda}{\partial t}, \quad (3)$$

or equivalently

$$\frac{\partial v_\lambda}{\partial t} = \frac{w_\lambda - \Phi(v_\lambda)}{\lambda}, \quad t > 0. \quad (4)$$

Note that the dynamics created by (4) drive the system onto stable parts of the graph, as we will see in the sequel.

⁴In the sense of distributions.

Problem $(P_\Phi)_\lambda$ may be rewritten as

$$\begin{cases} \frac{\partial v_\lambda}{\partial t} - \Delta w_\lambda = 0 & \text{in } Q =]0, T[\times \Omega, \\ \frac{\partial w_\lambda}{\partial n} = 0 & \text{on } \Sigma =]0, T[\times \Gamma, \\ v_\lambda(0) = S_{w(0)} & \text{in } \Omega. \end{cases} \quad (5)$$

Equivalently, for nearly all t , w_λ is a solution of the elliptic problem parametrized in time⁵

$$\begin{cases} w_\lambda - \lambda \Delta w_\lambda = \Phi(v_\lambda) & \text{in } \Omega, t > 0, \\ \frac{\partial w_\lambda}{\partial n} = 0 & \text{on } \Gamma, t > 0. \end{cases} \quad (6)$$

The “ $-$ Laplacian” operator (denoted again $-\Delta$) specifically associated with homogeneous Neumann boundary conditions on Γ is obviously a nonbounded operator from $L^2(\Omega)$ into $L^2(\Omega)$ whose domain is $H^2(\Omega)$. We introduce the Yosida regularization $-\Delta_\lambda := -\Delta(J_\lambda) = (I - J_\lambda)/\lambda$ and its resolvent $J_\lambda = (I - \lambda \Delta)^{-1}$. According to (6), it follows that

$$w_\lambda = J_\lambda \Phi(v_\lambda). \quad (7)$$

Problem $(P_\Phi)_\lambda$ can be formulated again in a well-posed form⁶ for the operator $-\Delta_\lambda \Phi$ in $L^2(\Omega)$:

$$\begin{cases} \frac{\partial v_\lambda}{\partial t} - \Delta_\lambda \Phi(v_\lambda) = 0, & t \in]0, T[, \\ v_\lambda(0) = S_{w(0)} & \text{in } L^2(\Omega) \text{ and a.e. in } \Omega. \end{cases} \quad (P_\lambda)$$

The following proposition summarizes the properties of the solutions of (P_λ) :

Proposition 1. *Let us denote by $g : \mathbb{R} \rightarrow \mathbb{R}$ a Lipschitz nondecreasing function, a so-called “entropy function”, and let*

$$G(r) = \int_0^r g(s) ds, \quad G_\Phi(r) = \int_0^r g(\Phi(s)) ds, \quad g_{1/2}(r) = \int_0^r \sqrt{g'(s)} ds, \quad r \in \mathbb{R}. \quad (8)$$

From the Rademacher theorem [Evans and Gariepy 1992], g' is a bounded Borelian representative of the derivative of g in its class.

For all $\lambda > 0$, the solution v_λ of (P_λ) associated to w_λ has the following properties:

(i) *Estimations using entropy inequations (for each entropy function g):*

$$\frac{\partial}{\partial t} G_\Phi(v_\lambda) \leq \operatorname{div}(g(w_\lambda) \nabla w_\lambda) - g'(w_\lambda) |\nabla w_\lambda|^2 \quad \text{in } Q. \quad (9)$$

⁵According to an observation of [Evans and Portilheiro 2004].

⁶Thanks to the Cauchy–Lipschitz–Picard theorem via a first-order differential equation with given initial condition.

Using the function G defined in (8), the last inequality can be rewritten as

$$\frac{\partial}{\partial t} G_{\Phi}(v_{\lambda}) \leq \Delta(G(w_{\lambda})) - G''(w_{\lambda})|\nabla w_{\lambda}|^2 \quad \text{in } Q.$$

The following inequalities hold:

$$\int_{\Omega} G_{\Phi}(v_{\lambda}(t, x)) dx \leq \int_{\Omega} G_{\Phi}(v_{\lambda}(s, x)) dx \leq \int_{\Omega} G_{\Phi}(S_{w(0)}) dx, \quad t > s > 0,$$

$$\|g_{\frac{1}{2}}(w_{\lambda})\|_{L^2([0, T]; H^1(\Omega))}^2 \stackrel{\text{def}}{=} \int_Q g'(w_{\lambda})|\nabla w_{\lambda}|^2 dx dt \leq C_g, \quad C_g = C(g).$$

(ii) We have the following uniform a priori estimates:

$$\|v_{\lambda}\|_{L^{\infty}(Q)} + \|w_{\lambda}\|_{L^{\infty}(Q)} \leq C_1,$$

$$\|w_{\lambda}\|_{L^2([0, T]; H^1(\Omega))} + \sqrt{\lambda} \left\| \frac{\partial v_{\lambda}}{\partial t} \right\|_{L^2(Q)} \leq C_2,$$

$$\left\| \frac{\partial v_{\lambda}}{\partial t} \right\|_{L^2([0, T]; (H^1(\Omega))')} \leq C_3.$$

The frame constants depend on the extremum values of Φ and $S_{w(0)}$.

Proof. The general principle of the proof of this proposition may be found in [Evans 2004, p. 427]. This classical computation is somewhat akin to an entropy flux calculation for a hyperbolic conservation law, through choices of nondecreasing functions g (see also [Gagneux and Millet 2015] for more details). We note that the inequality (9) is straightforward from the following relation, for any function Φ :

$$\frac{\partial}{\partial t} G_{\Phi}(v_{\lambda}) - \operatorname{div}(g(w_{\lambda})\nabla w_{\lambda}) = -g'(w_{\lambda})|\nabla w_{\lambda}|^2 - (g(w_{\lambda}) - g(\Phi(v_{\lambda}))) \frac{w_{\lambda} - \Phi(v_{\lambda})}{\lambda} \quad (10)$$

stated in [Evans and Portilheiro 2004; Evans 2004; Plotnikov 1996]. \square

4. Study of capillary effects

4.1. Generalized “entropic” solutions. It follows from the uniform estimates of Proposition 1 that we can find subsequences $\{v_{\lambda_k}\}$ and $\{w_{\lambda_k}\}$ and a pair⁷ (v, w) such that, as $\lambda_k \rightarrow 0$,

$$v_{\lambda_k} \rightharpoonup v \quad \text{in } L^{\infty}(Q) \text{ weakly-}^*, \quad (11)$$

$$\frac{\partial v_{\lambda_k}}{\partial t} \rightharpoonup \frac{\partial v}{\partial t} \quad \text{in } L^2([0, T]; (H^1(\Omega))') \text{ weakly,} \quad (12)$$

$$w_{\lambda_k} \rightharpoonup w \quad \text{in } L^{\infty}(Q) \text{ weakly-}^* \text{ and in } L^2([0, T]; H^1(\Omega)) \text{ weakly,} \quad (13)$$

⁷A vanishing viscosity limit.

and

$$w_{\lambda_k} - \Phi(v_{\lambda_k}) \rightarrow 0 \quad \text{in } L^p(Q) \text{ strongly, for any finite } p.$$

Furthermore, we can assume that⁸

$$v_{\lambda_k} \rightarrow v \quad \text{in } C^0([0, T]; H^1(\Omega)') \text{ strongly.} \quad (14)$$

The associated Cauchy condition is given by

$$\begin{cases} v(0, \cdot) = S_{w(0)} & \text{in } H^1(\Omega)', \text{ a.e. in } \Omega, \\ 0 \leq v(t, \cdot) \leq 1 & \text{in } \Omega, t > 0, \\ \int_{\Omega} v(t, x) dx = \int_{\Omega} S_{w(0)}(x) dx, & t > 0. \end{cases}$$

In addition, the pair⁹ (v, w) belongs to the functional frame

$$\begin{cases} v \in L^\infty(Q) \cap C^0([0, T]; H^1(\Omega)'), \\ \frac{\partial v}{\partial t} \in L^2([0, T]; H^1(\Omega)'), \\ w \in L^\infty(Q) \cap L^2([0, T]; H^1(\Omega)), \end{cases}$$

and is a solution of the boundary value problem

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta w = 0 & \text{in } \mathcal{D}'(Q) \text{ and } L^2([0, T]; H^1(\Omega)'), \\ \frac{\partial w}{\partial n} = 0 & \text{on } \Sigma, \\ v(0, \cdot) = S_{w(0)} & \text{a.e. in } \Omega. \end{cases}$$

Because of the nonmonotonicity of Φ , the information (11)–(14) is not sufficient to conclude that $w = \Phi(v)$, as we will see in what follows.

Let us denote by Ξ the complementary of the set of points of \mathcal{L}^{d+1} -approximate continuity of v according to the rigorous definition of the shock wave [Evans and Gariépy 1992; Gagneux and Madaune-Tort 1995]. The set Ξ is Borelian and \mathcal{L}^{d+1} -negligible because v is in $L^\infty(Q)$.

Let us assume that Ξ is a countable union of smooth hypersurfaces Ξ^i of \mathbb{R}^{d+1} which admit a unit normal vector $v^i = (v_1^i, \dots, v_d^i, v_{d+1}^i) = (\tilde{v}^i, v_{d+1}^i)$.

Using the usual notations for jumps in hyperbolic scalar laws and for the Hausdorff measure \mathcal{H}^d , very informally, the pair (v, w) satisfies the Rankine–Hugoniot and entropy conditions for all i , integrating by parts locally in a vicinity of a given transition interface via appropriate smooth functions with compact support:

$$\begin{aligned} v_{d+1}^i[v] &= \tilde{v}^i \cdot [\nabla w] \quad \text{and} \quad [w] = 0 \quad \mathcal{H}^d\text{-a.e. on } \Xi^i, \\ v_{d+1}^i[G_\Phi(v)] - \tilde{v}^i \cdot [\nabla w]g(w) &\leq 0 \quad \mathcal{H}^d\text{-a.e. on } \Xi^i. \end{aligned} \quad (15)$$

⁸From a classical compactness result of J. A. Dubinskii [Lions 1969, pp. 141–142].

⁹The pair (v, w) is called a *generalized solution* of the problem (P_Φ) .

Relation (15) may be written in the form

$$v_{d+1}^i([G_\Phi(v)] - g(w)[v]) \leq 0 \quad \mathcal{H}^d\text{-a.e. on } \Xi^i \quad (16)$$

with the notations (8) for the definition of g (entropy function) and G_Φ . In this form, relation (16) will be very useful to highlight the further developments.

4.2. Associated hysteresis effects. The analysis of hysteresis effects relies on the following proposition:

Proposition 2. *There exist three \mathcal{L}^{d+1} -measurable and bounded functions, Λ_0 , Λ_1 and Λ_2 , that are representative of the respective influence of the three branches of the graph of Φ^{-1} (in the sense of the set theory) through the functions β_0 , β_1 and β_2 . Moreover, we have*

$$0 \leq \Lambda_i \leq 1 \quad \text{and} \quad \sum_{i=0}^2 \Lambda_i = 1 \quad \mathcal{L}^{d+1}\text{-a.e. in } Q.$$

In addition, when λ_k tends to 0^+ ,

$$\mu(v_{\lambda_k}) \text{ converges to } \sum_{i=0}^2 \Lambda_i \mu(\beta_i(w)) \text{ in } L^\infty(Q) \text{ weakly-}^*$$

for any numerical continuous function μ . In addition, we have the following strong convergences:

$$w_{\lambda_k} \text{ and } \Phi(v_{\lambda_k}) \text{ converge to } w \text{ in } L^p(Q) \text{ strongly for any finite } p.$$

Finally, for any Lipschitz nondecreasing function g , we have the entropy relation in the sense of the measures in Q

$$\frac{\partial}{\partial t} \left(\sum_{i=0}^2 \Lambda_i G_\Phi(\beta_i(w)) \right) \leq \operatorname{div}(g(w)\nabla w) - g'(w)|\nabla w|^2 \quad \text{in } Q. \quad (17)$$

Proof. The difficult technical proof of this proposition is not detailed here and can be found in [Evans and Portilheiro 2004; Evans 2004; Plotnikov 1996; 1994] with some adjustments. \square

When Λ_0 is equal to zero everywhere,¹⁰ the method provides a response corresponding to the initial problem, thanks to the information contained in the complementary entropy relation (17). That is the main goal of the following proposition, based on the complementary information on the entropy given by (15)–(16), which enables us to determine the sense of circulation of the hysteresis loop.

¹⁰That corresponds in the final result to the neutralization of the decreasing part of the cubic introduced artificially to create a repulsive region.

Proposition 3. *In the framework of Proposition 2, we assume that*

$$\Lambda_0 = 0 \quad \text{in } Q, \quad \Lambda_1 = 1 \quad \text{in } Q_1 \quad \text{and} \quad \Lambda_2 = 1 \quad \text{in } Q_2,$$

where Q_1 and Q_2 are two open subsets of the cylinder Q , with a Lipschitz interface $\Sigma_{1,2} = \overline{Q_1} \cap \overline{Q_2}$ admitting a unit normal vector $v = (v_1, \dots, v_d, v_{d+1})$ of $\mathbb{R}^d \times \mathbb{R}$, oriented into Q_1 .

Using the notations of Figure 2, the problem can be written more precisely as a problem with free surface:

$$\begin{cases} v = \beta_1(w) \text{ and } \frac{\partial v}{\partial t} - \Delta \Phi_1(v) = 0 & \text{in } Q_1, \\ v = \beta_2(w) \text{ and } \frac{\partial v}{\partial t} - \Delta \Phi_2(v) = 0 & \text{in } Q_2. \end{cases} \quad (18)$$

As a consequence of the information contained in the Rankine–Hugoniot and entropy relations (15)–(16), which are justified here along the shock wave $\Sigma_{1,2}$, the sign of the component v_{d+1} of the normal vector v , i.e., its orientation during the time, is specified by the relations

$$\begin{cases} v_{d+1} = 0 & \text{if } A < w < B, \\ v_{d+1} \geq 0 & \text{if } w = A, \\ v_{d+1} \leq 0 & \text{if } w = B. \end{cases} \quad (19)$$

4.3. Interpretation of the results. According to Proposition 3, the expected hysteresis effect is well described by the pair (v, w) , the generalized solution of (P_Φ) . The change of the expression of the state law, which governs the diffusion process according to the values of the reduced saturation, is given by (18). Moreover, relation (19) reveals that the interface $\Sigma_{1,2}$ evolves only if w takes the value A or B (see Figure 1).

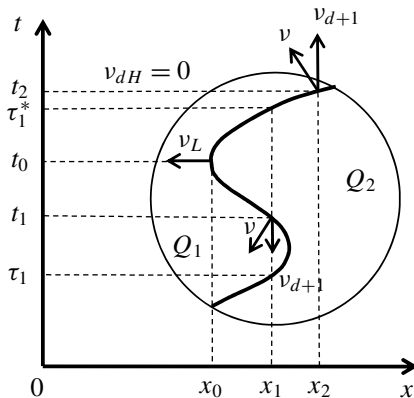


Figure 3. More general case illustrating Proposition 3.

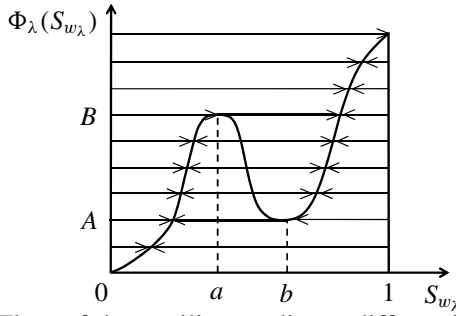


Figure 4. Flow of the auxiliary ordinary differential equation and emergence of the hysteresis effects when $\lambda \rightarrow 0$.

To illustrate the resulting hysteretic behavior, let us consider the generic example of [Figure 3](#) on the previous page, zooming on a time interval representing three possible states of the point x_1 at three different times. We focus on the possible states corresponding to the abscissa x_1 .

At the point (t_1, x_1) , we have a jump from Q_1 to Q_2 , $w = B$, and we are in the imbibition phase (see also [Figure 1](#)). On the contrary, at the point (t_2, x_2) , we have a jump from Q_2 to Q_1 , $w = A$, and we are in the drainage phase.

Therefore, the entropy method linked to the Sobolev regularization leads to a hysteresis loop similar to that obtained for Stefan’s supercooling problem [[Evans 2004](#)]. The flow of the auxiliary ordinary differential equation (3) leads to hysteresis effects when $\lambda \rightarrow 0$ ([Figure 4](#)).

5. Conclusion

The hysteresis phenomena of flows in unsaturated porous media has been modeled with success, using the artificial introduction of an unstable spinodal interval and on a dynamic regularization process of Sobolev type.

References

- [Aubin and Cellina 1984] J.-P. Aubin and A. Cellina, *Differential inclusions: set-valued maps and viability theory*, Grundlehren der Math. Wissenschaften **264**, Springer, Berlin, 1984.
- [Evans 2004] L. C. Evans, “A survey of entropy methods for partial differential equations”, *Bull. Amer. Math. Soc. (N.S.)* **41**:4 (2004), 409–438.
- [Evans and Gariépy 1992] L. C. Evans and R. F. Gariépy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992.
- [Evans and Portilheiro 2004] L. C. Evans and M. Portilheiro, “Irreversibility and hysteresis for a forward–backward diffusion equation”, *Math. Models Methods Appl. Sci.* **14**:11 (2004), 1599–1620.
- [Gagneux and Madaune-Tort 1995] G. Gagneux and M. Madaune-Tort, *Analyse mathématique de modèles non linéaires de l’ingénierie pétrolière*, Mathématiques & Applications **22**, Springer, Berlin, 1995.

- [Gagneux and Millet 2015] G. Gagneux and O. Millet, “A model for capillary hysteresis loops in unsaturated porous media theory”, 2015. Submitted to *Math. Mech. Solids*.
- [Lions 1969] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
- [Lions and Magenes 1968] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications, I*, Travaux et Recherches Mathématiques **17**, Dunod, Paris, 1968.
- [Plotnikov 1994] P. I. Plotnikov, “Passage to the limit with respect to viscosity in an equation with a variable direction of parabolicity”, *Differentsial'nye Uravneniya* **30**:4 (1994), 665–674, 734. In Russian; translated in *Differential Equations* **30**:4 (1994), 614–622.
- [Plotnikov 1996] P. I. Plotnikov, “Forward–backward parabolic equations and hysteresis”, pp. 183–209, 257–258 in *Boundary-value problems of mathematical physics and related problems of function theory, XXVII*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **233**, 1996. Reprinted in *J. Math. Sci.* **93**:5 (1999), 747–766.
- [Smarrazzo 2008] F. Smarrazzo, “On a class of equations with variable parabolicity direction”, *Discrete Contin. Dyn. Syst.* **22**:3 (2008), 729–758.
- [Smarrazzo and Tesei 2010] F. Smarrazzo and A. Tesei, “Long-time behavior of solutions to a class of forward–backward parabolic equations”, *SIAM J. Math. Anal.* **42**:3 (2010), 1046–1093.
- [Smarrazzo and Tesei 2012] F. Smarrazzo and A. Tesei, “Degenerate regularization of forward–backward parabolic equations: the regularized problem”, *Arch. Ration. Mech. Anal.* **204**:1 (2012), 85–139.

Received 10 Sep 2015. Revised 3 Dec 2015. Accepted 15 Jan 2016.

GÉRARD GAGNEUX: gerard.maryse.gagneux@gmail.com

Laboratoire des Sciences de l'Ingénieur pour l'Environnement, UMR-CNRS 7356,
Université de La Rochelle, avenue Michel Crépeau, 17042 La Rochelle Cedex 1, France

OLIVIER MILLET: olivier.millet@univ-lr.fr

Laboratoire des Sciences de l'Ingénieur pour l'Environnement, UMR-CNRS 7356,
Université de La Rochelle, avenue Michel Crépeau, 17042 La Rochelle Cedex 1, France



MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS

msp.org/memocs

EDITORIAL BOARD

ANTONIO CARCATERRA
ERIC A. CARLEN
FRANCESCO DELL'ISOLA
RAFFAELE ESPOSITO
ALBERT FANNJIANG
GILLES A. FRANCFORT
PIERANGELO MARCATI
JEAN-JACQUES MARIGO
PETER A. MARKOWICH
MARTIN OSTOJA-STARZEWSKI
PIERRE SEPPECHER
DAVID J. STEIGMANN
PAUL STEINMANN
PIERRE M. SUQUET

Università di Roma "La Sapienza", Italia
Rutgers University, USA
(CO-CHAIR) Università di Roma "La Sapienza", Italia
(TREASURER) Università dell'Aquila, Italia
University of California at Davis, USA
(CO-CHAIR) Université Paris-Nord, France
Università dell'Aquila, Italy
École Polytechnique, France
DAMTP Cambridge, UK, and University of Vienna, Austria
(CHAIR MANAGING EDITOR) Univ. of Illinois at Urbana-Champaign, USA
Université du Sud Toulon-Var, France
University of California at Berkeley, USA
Universität Erlangen-Nürnberg, Germany
LMA CNRS Marseille, France

MANAGING EDITORS

MICOL AMAR
CORRADO LATTANZIO
ANGELA MADEO
MARTIN OSTOJA-STARZEWSKI

Università di Roma "La Sapienza", Italia
Università dell'Aquila, Italy
Université de Lyon-INSA (Institut National des Sciences Appliquées), France
(CHAIR MANAGING EDITOR) Univ. of Illinois at Urbana-Champaign, USA

ADVISORY BOARD


ADNAN AKAY
HOLM ALTENBACH
MICOL AMAR
HARM ASKES
TEODOR ATANACKOVIĆ
VICTOR BERDICHEVSKY
GUY BOUCHITTÉ
ANDREA BRAIDES
ROBERTO CAMASSA
MAURO CARFORE
ERIC DARVE
FELIX DARVE
ANNA DE MASI
GIANPIETRO DEL PIERO
EMMANUELE DI BENEDETTO
BERNOLD FIEDLER
IRENE M. GAMBA
DAVID Y. GAO
SERGEY GAVRILYUK
TIMOTHY J. HEALEY
DOMINIQUE JEULIN
ROGER E. KHAYAT
CORRADO LATTANZIO
ROBERT P. LIPTON
ANGELO LUONGO
ANGELA MADEO
JUAN J. MANFREDI
CARLO MARCHIORO
GÉRARD A. MAUGIN
ROBERTO NATALINI
PATRIZIO NEFF
ANDREY PIATNITSKI
ERRICO PRESUTTI
MARIO PULVIRENTI
LUCIO RUSSO
MIGUEL A. F. SANJUAN
PATRICK SELVADURAI
ALEXANDER P. SEYRANIAN
MIROSLAV ŠILHAVÝ
GUIDO SWEERS
ANTOINETTE TORDÉSILLAS
LEV TRUSKINOVSKY
JUAN J. L. VELÁZQUEZ
VINCENZO VESPRI
ANGELO VULPIANI

Carnegie Mellon University, USA, and Bilkent University, Turkey
Otto-von-Guericke-Universität Magdeburg, Germany
Università di Roma "La Sapienza", Italia
University of Sheffield, UK
University of Novi Sad, Serbia
Wayne State University, USA
Université du Sud Toulon-Var, France
Università di Roma Tor Vergata, Italia
University of North Carolina at Chapel Hill, USA
Università di Pavia, Italia
Stanford University, USA
Institut Polytechnique de Grenoble, France
Università dell'Aquila, Italia
Università di Ferrara and International Research Center MEMOCS, Italia
Vanderbilt University, USA
Freie Universität Berlin, Germany
University of Texas at Austin, USA
Federation University and Australian National University, Australia
Université Aix-Marseille, France
Cornell University, USA
École des Mines, France
University of Western Ontario, Canada
Università dell'Aquila, Italy
Louisiana State University, USA
Università dell'Aquila, Italia
Université de Lyon-INSA (Institut National des Sciences Appliquées), France
University of Pittsburgh, USA
Università di Roma "La Sapienza", Italia
Université Paris VI, France
Istituto per le Applicazioni del Calcolo "M. Picone", Italy
Universität Duisburg-Essen, Germany
Narvik University College, Norway, Russia
Università di Roma Tor Vergata, Italy
Università di Roma "La Sapienza", Italia
Università di Roma "Tor Vergata", Italia
Universidad Rey Juan Carlos, Madrid, Spain
McGill University, Canada
Moscow State Lomonosov University, Russia
Academy of Sciences of the Czech Republic
Universität zu Köln, Germany
University of Melbourne, Australia
École Polytechnique, France
Bonn University, Germany
Università di Firenze, Italia
Università di Roma La Sapienza, Italia

MEMOCS (ISSN 2325-3444 electronic, 2326-7186 printed) is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

Cover image: "Tangle" by © John Horigan; produced using the *Context Free* program (contextfreetart.org).

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing
<http://msp.org/>

© 2016 Mathematical Sciences Publishers

Gradient materials with internal constraints Albrecht Bertram and Rainer Glüge	1
Unified geometric formulation of material uniformity and evolution Marcelo Epstein and Manuel de León	17
Electromechanics of polarized lipid bilayers David J. Steigmann and Ashutosh Agrawal	31
Orthogonal polynomials and Riesz bases applied to the solution of Love's equation Pierluigi Vellucci and Alberto Maria Bersani	55
Modeling capillary hysteresis in unsaturated porous media G�rard Gagneux and Olivier Millet	67
Discrete double-porosity models for spin systems Andrea Braides, Valeria Chiad� Piat and Margherita Solci	79
Correction to "On the theory of diffusion and swelling in finitely deforming elastomers" Gary J. Templet and David J. Steigmann	103

MEMOCS is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Universit  dell'Aquila, Italy.

