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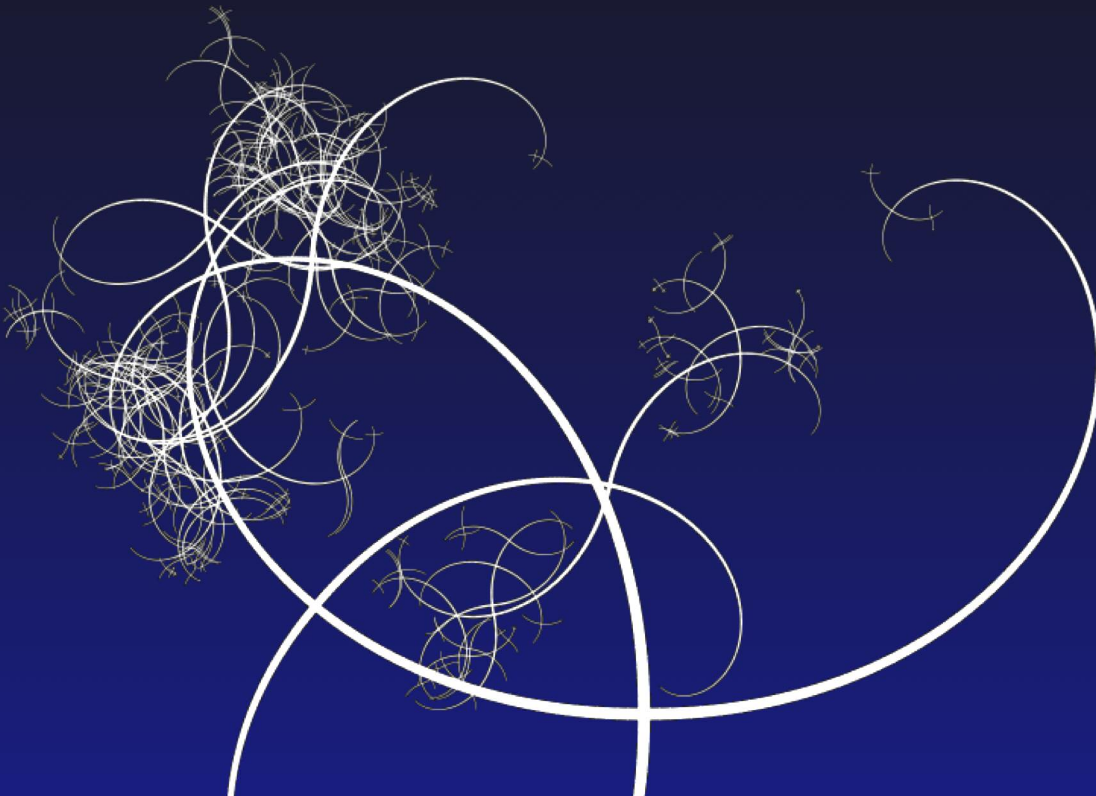
LEONARDO DA VINCI



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"MATHEMATICS" AND "PHYSICS" IN THE SCIENCE OF HARMONICS

STEFANO ISOLA

Some aspects of the role that the science of harmonics has played in the history of science are discussed in light of Russo's investigation of the history of the concepts of "mathematics" and "physics".

1. The rambling route of the ancient scientific method

In several places in Russo's writings on the history of science, one can find enlightening discussions about the meanings of the concepts of "physics" and "mathematics", along with the particular notions of truth involved in them; see, e.g., [58, Chapter 6.6; 60, Chapter 15; 56; 57]. Both terms derive from the Greek: the original meaning of the former was the investigation of *everything that lives, grows or*, more generally, *comes into existence*, whereas the latter referred to *all that is studied*, thus deriving its meaning not from its content but from its method. In the Hellenistic period, the term "physics" continued to be used to indicate that sector of philosophy that addressed nature (the other sectors being ethics and logic), thus corresponding to what came to be called "natural philosophy" in modern times. On the other hand, the term "mathematics" was used to indicate all the disciplines (including geometry, arithmetic, harmonics, astronomy, optics, mechanics, hydrostatics, pneumatics, geodesy and mathematical geography) that shared the same method of investigation, based on the construction of theories by which "theorems" are proved, leaning on explicitly stated initial assumptions. Its meaning thus corresponded to what we call "exact sciences" and refers to a unitary body of scientific disciplines alien to the modern distinction between physical and mathematical sciences.

In antiquity, how the scope of mathematics contrasted with that of physics was a topic of much debate. According to some key testimonials reported and discussed in [56] — in particular that due to Geminus (and reported by Simplicius) in the first

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century B.C.[1] and, much later, that of Thomas Aquinas[2] — the "physicist" would be able to grasp the "substance" of reality using philosophical categories, whereas a characteristic feature of the work of the astronomer, that is, the "mathematician", is its incapability to assert absolute truths, in that he is able to rigorously deduce/construct a number of consequences from previously stated *hypotheses* whose ultimate validity remains however out of control.

To better understand this discrepancy, let's step back again to highlight another important methodological difference between natural philosophy and exact science in antiquity, in that the former operates on a single level of discourse, where data from experience and thoughts are organized so as to produce "directly" a rational account of the perceptions themselves.[3] In particular, natural philosophy starts "from the things which are more knowable and obvious to us and proceeds towards those which are clearer and more knowable by nature" [4, p. 184a], thus revealing the alleged genuine, mind-independent nature of things. This is also reflected in the use of language. As reported by the fifth-century Alexandrian scholar Ammonius, "...Aristotle teaches what the things principally and immediately signified by sounds [e.g., names and verbs] are, and these are thoughts. Through these as means

[1] "The physicist will prove each fact by considerations of essence or substance, of force, of its being better that things should be as they are, or of coming into being and change; the astronomer will prove them by the properties of figures or magnitudes, or by the amount of movement and the time that is appropriate to it. Again, the physicist will in many cases reach the cause by looking to creative force; but the astronomer, when he proves facts from external conditions, is not qualified to judge of the cause... sometimes he invents by way of hypothesis, and states certain expedients by the assumption of which the phenomena will be saved. For example, why do the sun, the moon, and the planets appear to move irregularly? We may answer that, if we assume that their orbits are eccentric circles or that the stars describe an epicycle, their apparent irregularity will be saved; and it will be necessary to go further and examine in how many different ways it is possible for these phenomena to be brought about..." [34, p. 276].

[2] "Reason may be employed in two ways to establish a point: firstly, for the purpose of furnishing sufficient proof of some principle, as in natural science, where sufficient proof can be brought to show that the movement of the heavens is always of uniform velocity. Reason is employed in another way, not as furnishing a sufficient proof of a principle, but as confirming an already established principle, by showing the congruity of its results, as the theory of eccentrics and epicycles is considered as established in astronomy, because thereby the sensible appearances of the heavenly movements can be explained; not, however, as if this proof were sufficient, forasmuch as some other theory might explain them" [1, pp. 63–64].

[3] In criticizing Protagoras's statement that man is the measure of all things, Aristotle says, "We say that knowledge and sense-perception are the measure of things because our recognition of something is due to them" [41, p. 184]. To him, therefore, sense-perception and knowledge are the faculties that furnish all our understanding of things and thus exhausted all possible meanings of the expression "criterion of truth". In Hellenistic practice, however, other meanings of this expression were put forward (of which Protagoras's dictum could be considered a likely precursor) including the Stoics' infallible act of cognition based on *kataleptic* impressions (self-certifying acts of sense-perceptions) as well as the hypothetico-deductive method of exact sciences.

we signify things; and it is not necessary to consider anything else as intermediate between the thought and the thing, as the Stoics do, who assume what they name to be the meaning [*lekton*]" [61, p. 77].

Thus, at variance with the Aristotelian point of view, the early Stoics considered it necessary to distinguish between the pronounced sound and the meaning of what is pronounced as an intermediate link between a thought and a sound. The same kind of epistemological attitude characterized the exact sciences — which flourished in the same period of the early Stoic school — with their specific effort to overcome the illusion of being able to build intellectual schemes based directly on perceptible reality and the elaboration of abstract languages capable of describing not only aspects of the sensible world but also other designable realities; see [58], in particular Chapter 6. The existence of a double level of discourse seems therefore an essential feature of exact sciences, in that their assertions do not directly concern the things of the natural world but rather theoretical entities which are obtained by a procedure of "pruning" which allows one to focus on certain aspects of the *phainomena* — that is, what appears to the senses and calls for an explanation — and to ignore those considered unessential. In brief, the methodological mark of exact sciences consists in the construction of simplified models of aspects of reality which, starting from suitable but "unjustified" hypotheses, operate on their internal entities in a logically rigorous way and then move back to the real world. Note that, by its very nature, every hypothesis is somehow "false", so nothing prevents different models based on different hypotheses of being capable of "saving" the same phenomena. In addition, while the assertions obtained at the theoretical level are "objective" and universally valid, the *correspondence rules* which transform the entities involved in the real world and the claims about them into theoretical entities and theoretical statements are instead historically determined. For example, in Hellenistic scientific theories dealing with phenomena related to the sense of sight, devices such as ruler and compass, designed to assist in the construction of the straight line and the circle, as well as in the measurement of their parts, incorporate the correspondence rules relating theoretical statements of geometry or optics to concrete objects. In theories of acoustic-musical phenomena, this role was played instead by the canon (see below). In both cases, the "concrete objects" — drawings with ruler and compass and pitches produced by a plucked string, respectively — are not rough natural data: rather, they are the result of a somewhat refined human activity, which in turn is rooted in the historical and cultural context.

Although rarely acknowledged, the scientific method, as a cultural product of earlier Hellenistic times, underwent a rapid decline in the context of a more general cultural collapse that occurred during the second century B.C.[4] Notwithstanding

[4]Particularly dramatic were the years 146–145 B.C., with the sharp hardening of the Roman policy in the Mediterranean that had among its consequences the reduction of Macedonia to a Roman

the loss of a major part of ancient knowledge, the memory of Hellenistic science survived thanks to a series of geographically localized revival periods.[5] On the other hand, a peculiar feature of these revivals was the insertion of individual contents, recovered from ancient science or derived from it, into foreign overarching systems of thought which provided their main motivating framework.[6] In particular, "in the Age of Galileo", Russo says, "the exact science preserved the unity that distinguished the Greek models, from which it drew the terminology, but the ancient method was rarely understood. Not that the explanation reported by Simplicus had been forgotten, but few, as Stevin, used the freedom of choice of the hypotheses to build models; much more frequently the relative arbitrariness of the initial assumptions appeared (as it had appeared to Simplicius and Thomas Aquinas) as a particularity (as well as an oddity) of the method of the 'mathematician', which determined its inferiority with respect to philosophers and theologians, who knew how to distinguish 'truth' from 'falsehood'" [56, p. 37]. The idea that the hypothetico-deductive method was mostly a *limit* that prevented approaching the absolute truth peaked with Newton. In the well known General Scholium added to the Principia in 1713, he writes, "But hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses [*hypotheses non fingo*]. For whatever is not deduced from the phenomena, is to be called an hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction" [45, p. 392].

It is worth stressing that the term "phenomena" is used here with a meaning which differs considerably from the ancient one, in that it refers to something which lies beyond our perception.[7] Likewise, the term "hypothesis" was given the new meaning — still in use — of a statement lying at the beginning of our interpretation of the external world but waiting to be corroborated or refuted as soon as the "facts" are known with sufficient detail. Thus, in every genuine search for the

province, the razing of Carthage and Corinth and the heavy political interference in Egypt with persecution and extermination of the Greek intellectual class [59, Chapter 5].

[5]The first of them was the resumption of scientific studies in imperial times, whose main protagonists were Heron, Ptolemy and Galen. The next ones occurred in the sixth-century Byzantine world, then in the medieval Islamic world (eighth to ninth centuries) and finally in Western Europe, from the "twelfth-century Renaissance" until the Renaissance par excellence of early modern times [58, Chapter 11].

[6]We shall discuss below an example which illustrates this fact in connection with Ptolemy's work.

[7]Think of the absolute motions of material bodies with respect to the immovable space which, coexisting with Aristarchan heliocentrism, cannot correspond to any observable datum (see the discussion given in [58, Chapter 11.7]). In a letter of 1698, Newton affirmed, "I am inclined to believe some general laws of the Creator prevailed with respect to the agreeable or unpleasing affections of all our *senses*" [46, Letter XXIX].

truth, hypotheses cannot be anything but a hindrance.[8] As it is well known, Newtonianism was presented in the European continent as the philosophy of progress. The most famous of his supporters was Voltaire, who in the preface of the French translation of the Principia dismissed as "foolish" the followers of vortices formed by the "thin matter" of Descartes and Leibniz and affirmed that only a follower of Newton could be truly called a "physicist". Indeed, according to Russo, the spread of Newtonian mechanics has brought with it the way of reasoning on the basis of which "the exact science got broken into two stumps: 'mathematics' and 'physics'. Both of them inherited from the ancient 'mathematics' the quantitative approach and several technical results, and from the ancient 'physics' (that is from natural philosophy) the idea of producing statements which are absolutely 'true'. The essential difference was lying in the nature of such truth. While the truth of the assumptions of 'mathematics' (called postulates) was considered immediately evident, the assumptions of 'physics' (called principles) were regarded true inasmuch as they are 'proven by the phenomena'... It is plain that these differences were strictly connected to the diverse nature attributed to the entities studied by the two disciplines: the 'mathematical' entities, although usable to describe concrete objects, were considered abstract, whereas the 'physical' entities were considered as concrete as the objects they were referring to" [56, pp. 42–43]. In both cases, the "truth" of a scientific theory (e.g., a theory of the planetary motions) does not lie in its capability to "save the phenomena" (e.g., to determine with some accuracy the observable position of a planet at any time) but becomes something that one can "prove" by means of its own instruments, in the same way in which one can prove a statement on the entities internal to the theory itself. If so, a scientific theory would cease to be a theoretical model, instead becoming a system of statements set to describe the true nature of the real world.[9]

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[8] As d'Alembert wrote, "it is not at all by vague and arbitrary hypotheses that we can hope to know nature; it is by thoughtful study of phenomena, by the comparisons we make among them, by the art of reducing, as much as that may be possible, a large number of phenomena to a single one that can be regarded as their principle" [23, p. 22], and a little further, "let us conclude that the single true method of philosophizing as physical scientists consists either in the application of mathematical analysis to experiments, or in observation alone, enlightened by the spirit of method, aided sometimes by conjectures when they can furnish some insights, but rigidly dissociated from any arbitrary hypotheses" [23, p. 25]. As an aside, this semantic transformation may have played a role in claiming a "historic mission" to human knowledge, from the naivety of the myth towards the final enlightenment, passing through an increasing control of the sources of error which allows one to progressively overcome all "false hypotheses" (that is, "prejudices"): a kind of secularized version of the medieval millenarianism, of which, among others, Newton was an ardent supporter.

[9] Such a prescientific position is proudly maintained by Voltaire in the entry "System" of [65, p. 224], which starts by stating, "We understand by *system* a supposition; for if a system can be proved, it is no longer a system, but a truth. In the meantime, led by habit, we say the celestial system, although we understand by it the real position of the stars". By the way, and not surprisingly,

Following this reshaping of the scientific enterprise, some disciplines have been counted on one side and others on the other; still others have been somehow internally divided or eventually disappeared, as we shall see below in a particular example. Referring to the already cited writings of Russo for a discussion of the splitting between "mathematics" and "physics" in nineteenth and twentieth centuries, let us just remark that the final failure of the efforts towards a methodological reunification of the exact science, of which Poincaré was a prominent exponent, and the prevailing of powerful trends towards specialization and fragmentation of the scientific disciplines, if on one side has led some to wonder what mystery lies behind the "unreasonable effectiveness" of mathematics in providing accurate descriptions of the phenomena [67], on the other side prompted one of the greatest contemporary mathematicians to acknowledge in this trend a severe crisis of science itself: "In the middle of the twentieth century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course, in total ignorance of any other sciences. They first began teaching their ugly scholastic pseudomathematics to their students, then to schoolchildren (forgetting Hardy's warning that ugly mathematics has no permanent place under the Sun)" [7]; see also [57] for a further discussion.

2. Acoustic-musical phenomena

> "Ho detto che la nostra scienza o arte musicale fu dettata dalla matematica. Doveva dire costruita. Essa scienza non nacque dalla natura,... ma ebbe origine ed ha il suo fondamento in quello che è giustamente chiamato seconda natura, ma che altrettanto a torto quanto facilmente e spesso è confuso e scambiato... colla natura medesima, voglio dire nell'assuefazione. Le antiche assuefazioni de' greci... furono l'origine e il fondamento della scienza musicale da' greci determinata, fabbricata e a noi ne' libri e nell'uso tramandata, dalla qual greca scienza vien per comun consenso e confessione la nostra europea" (G. Leopardi, *Zibaldone* [40, 3125–3126]).

Today musical theory is mainly "the study of the structure of music", whereas originally it was part of mathematics. What happened in the meantime? In order to get an idea, it is necessary to go back again to the rambling route of the ancient scientific method through the subsequent history. Resuming what was said in the previous section in a concise albeit vague way, we can say that the general objects of Greek science were not so much the "laws" of the natural world viewed as

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this entry proceeds by strengthening the idea of a necessary progression of the knowledge by denying that Aristarchus introduced heliocentrism.

an entity independent from the man who observes it but rather those indubitable epistemological data provided by the *phainomena* resulting from the interaction between subject and object through active perception. In particular, the models of Hellenistic exact science were primarily suited for that purpose: the creation of theoretical entities as intermediate utterances between the real objects and abstract truths has the effect of making that interaction available to conscious manipulation. This gets a peculiar meaning within the context of music theory which, as such, establishes sound, the material aspect of music, as something which can be knowingly investigated in connection with human experience. Although music — perhaps the most unfathomable expression of psychic activity — might not seem properly suited to scientific analysis, the investigation of acoustic-musical phenomena nonetheless provides an example where the epistemological opposition sketched in the previous section occurred with a striking character within the same domain, as we shall now briefly outline.

It is rather well known that Pythagorean music theory — as a part of their program of liberation of the soul by means of the intellectual perception of proportions in all things — starts from the recognition that the harmonic intervals can be expressed as simple numerical ratios. The following "Pythagorean principle" has been viewed as the first "natural law" expressed in terms of numerical entities (see, e.g., [11]): if two sounding bodies, such as stretched strings or sounding pipes, have lengths which are in simple proportions, and all other aspects are kept fixed, together they will produce musical intervals which are judged by the ear to be in harmonious agreement, or "consonant". Conversely, all intervals that the ear accepts as consonant can be represented as ratios of numbers from the tetrad 1, 2, 3, 4.[10] The harmonic system of Philolaus (see, e.g., [18; 19]), for example, is a structure of intervals externally limited by the octave (*diapason*), whose ratio is $2:1$, and internally articulated by intervals of fifths (*diapente*), with ratio $3:2$, and fourths (*diatessaron*), with ratio $4:3$. If we want to find four quantities — for instance the lengths of the strings of a four-string lyre — that, taken in pairs, reproduce these ratios, then we can choose a unit of measure so that the longest string is 12 units, the intermediate ones 9 and 8 and the shortest 6. It is clear that the system of reciprocal ratios, and therefore the whole harmonic structure, does not change if the strings have lengths 12, 9, 8 and 6 meters, centimeters, stadiums, etc. Finally, observing that $(3:2):(4:3) = 9:8$, the interval of a tone, equivalent to the difference between a fifth and a fourth, is represented by the ratio 9:8. The octave is thus "harmonically" divided into two fourths spaced by a tone.[11]

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[10] The question of which observations lay behind the detection of these ratios and when this happened is hard to answer [13].

[11] Note that $6 \cdot 12 = 8 \cdot 9$, i.e., the four numbers are in geometrical proportion. Moreover, $8 = 2 : (\frac{1}{6} + \frac{1}{12})$ and $9 = (6+12) : 2$; namely, 8 and 9 are the harmonic mean and the arithmetic mean,

Let us point out that in the transition from the Hellenic to the Hellenistic period mathematics becomes an exact science, in the sense specified above, not only by distinguishing theoretical entities from concrete objects but also from pure abstractions in the "platonic" sense. While discussing the subjects for the education of the "Guardians" of the Republic, Plato lets Socrates conceive that "as the eyes are designed to look up at the stars, so are the ears to hear harmonic motions", therefore agreeing that astronomy and music theory are *sister sciences*, as the Pythagoreans said [50, 530d]. On the other hand, those scholars are judged inadequate to reach the "true knowledge" beyond the sensible world in that "their method exactly corresponds to that of the astronomer; for the numbers they seek are those found in these heard concords, but they do not ascend to generalized problems and the consideration which numbers are inherently concordant and which not and why in each case" [50, 531c]. Clearly, the just mentioned "ascension" above experience does not need to be embedded in a theory. Rather, it would rely on "evidences" per se.

In a different direction, music theory, or at least that part of it dealing with tuning systems, was set to become a scientific discipline by putting together the arithmetic theory of proportions and the recognition of the proportionality between the pitch of the sounds and the speed of the vibrations that produce them,[12] a conceptual step that according to some sources had been made in the circle of Archytas in about 400 B.C. [13; 36]. The "experimental device" enabling the establishment of a correspondence between concords and numerical ratios was the canon (*kanon harmonikos*), an instrument that in its simplest form is made of a single string stretched between two bridges fixed on a rigid base and equipped with another movable bridge by which one may divide the string into two parts, yielding sounds of variable pitch. One can further imagine a row fixed at its base on which the positions of the movable bridge corresponding to the notes can be marked. The name of the entire device is then a metonym for the line segment that represents it as a theoretical entity. The theory outlined in the *Sectio canonis*, attributed to Euclid, deals precisely with the harmonic divisions of this segment, i.e., with those divisions corresponding to musical intervals judged to be consonant [25]; see also [26]. In this work, far away from any mystical efflorescence about the music of the cosmos, a scheme of division of the octave by means of the theory of proportions contained in the *Elements* is proposed with the aim of producing patterns of consonant intervals adoptable in practice, e.g., when tuning musical

respectively, of the extremes 6 and 12. The exclusion of the geometric mean in "Pythagorean" music theory is justified by an impossibility result due to Archytas (see below).

[12] The recognition of the nature of sound as vibration of air, with alternation of rarefaction and compression, can already be found in Aristotle's *Problemata* [3] as well as in the Peripatetic *De audibilibus* [5], whereas the idea of a sound wave is attested at least as early as in the Stoa.

instruments. Along this path, the branch of Greek music theory referred to as the *science of harmonics* entered the unitary body of Hellenistic mathematics, along with astronomy, arithmetic, geometry, optics, topography, pneumatics, mechanics and other disciplines [58, Chapter 3].

In the short introduction of the *Sectio canonis*, the author establishes a correspondence between musical intervals and numerical ratios and states the main hypothesis underlying the model: *consonant intervals correspond to multiple or epimoric ratios*.[13] The rationale of this postulate relies on the observation that, as consonant intervals produce a perception of unity or tonal fusion between the notes, they must correspond to numbers which are given a "single name" in relation to one another.[14] On the other hand, this postulate is clearly false not only because it includes among the concords also intervals considered dissonant by the Pythagorean principle stated above, such as the tone $9:8$ or the ratio $5:4$ (natural major third), but also because it counts as dissonant the interval composed by an octave plus a fourth, represented by the ratio $8:3$, unanimously recognized as consonant by the music theorists of antiquity (exactly as an octave plus a fifth, that is, $3:1$). However, this is not a problem in itself, for all hypotheses are somehow "false": what matters is that the theory based on them is consistent and suited to save the phenomena which it aims to model. The introduction is then followed by twenty propositions: the first nine, of pure "number theory", provide a deductive construction of the Philolaus harmonic system sketched above, whereas the remaining ones form the part properly relevant to tuning systems. Of particular interest is the third proposition, which states that neither one nor more mean proportionals can be inserted within an epimoric interval. In particular, it is not possible to divide the octave into equal parts that form a rational relationship with the octave itself.[15]

The consequences of this simple result have been the subject of a controversy which has lasted for over two millennia, at the basis of which there is the distinction

[13] Greek arithmetic classified ratios into three basic types, which reduced to lowest terms corresponding to $n:1$ (*multiple*), $(n+1):n$ (*epimoric* or *superparticular*) or $(n+m):n$, $n>m>1$ (*epimeric* or *superpartient*). Note that the first two are in a one-to-one correspondence: $p:q$ is multiple if and only if $p:(p-q)$ is epimoric so that q is the greatest common divisor of both p and $p-q$. In particular, the octave $2:1$, participating to both consonant classes, is the "consonance of the consonances".

[14] The interpretation of this seemingly arbitrary correspondence is controversial. According to some scholars [11], the "single name" has to be ascribed to the fact that, unlike the epimeric ratios, multiple and epimoric ratios were indicated with a one-word name, like *epitritos*, "third in addition", for $4:3$. According to a different interpretation [26] (based on [54, §I.5]), the "single name" is not a linguistic unity but a numerical one, corresponding to the greatest common "part" which composes the notes in both multiple and epimoric ratios (see footnote 13).

[15] Nor would it be possible to divide in this way the fifth, the fourth or the whole tone. This seems to be the first impossibility result surely ascribable to an author as Boethius [12, §III.11] reports a proof of it given by Archytas.

between natural and tempered tunings [10; 37]. We'll not dwell here on the ways in which the different music theorists have conceived the division of the tonal continuum; see, e.g., [9; 14; 20]. Rather, we shall briefly discuss how in this domain the epistemological opposition between natural philosophy and mathematics manifested itself. To this end, we recall that the first writings of some importance dealing with Greek music theory are those of Aristoxenus and Theophrastus, both students of Aristotle and both harshly critical of the doctrine according to which the pitch can be conceived as a quantitative attribute of sound, representable by numbers. As representatives of the Peripatetic school, they were mainly interested in the "natural qualities" of the object of investigation. For instance, Theophrastus claims that differences of pitch are due to differences in the "shape" of the sounds' movement, not to differences of velocity, frequency of impact or the like, inasmuch as high notes travel in a straight line from the object to the ear while low ones spread more evenly all around the object; see his "*De musica*" excerpt in [52, pp. 61–65] and also [63]. In criticizing the "Pythagorean approach", he maintains that "if every interval were a quantity, and if melody arose from differences between notes, the melody would be as it is because it is a number. But if it were nothing but a number, everything numerable would participate in melody too, to the extent that it does in number". This illustrates in some way the single level of discourse maintained by natural philosophy about which we were talking in the previous section, where there is no space for intermediate entities in between the concrete objects and the abstract thoughts about them. A rather similar position is held by Aristoxenus, the leading musical theorist of antiquity. His *Harmonic elements* opens with the subject of vocal motion within the musical topos in which it moves, which is the continuum whose maximal range and minimal internal intervals are defined solely by what the human voice is capable of doing and by what the human ear can apprehend the moving voice to be doing. In particular, he claims that harmonic properties such as consonance are firstly subjects of experience by a musically trained ear and cannot be traced to numerical ratios. He wrote indeed, "we endeavor to supply proofs that will be in agreement with the phenomena, in this unlike our predecessors. For some of these introduced extraneous reasoning, and rejecting the senses as inaccurate fabricated rational principles, asserting that height and depth of pitch consist in certain numerical ratios and relative rates of vibration — a theory utterly extraneous to the subject and quite at variance with the phenomena" [6, pp. 188–189]. Although one may argue that the polemical target here is mostly the Platonic treatment of Pythagorean music theory, in the rejection of rational arguments based on assumptions external to the musical experience itself, we can see the demand that every assumption must be justified by the phenomena, that is, a substantial disclaiming of the scientific method.[16]

[16] In particular, Aristoxenus and his followers "admitted" that it was possible to divide the tone into two equal parts. This, of course, does not "contradict" the third proposition of the *Sectio*

Be that as it may, with the crisis of Hellenistic civilization and in particular after the dramatic cultural collapse which occurred midway through the second century B.C., the scientific methodology rapidly disappeared together with the very possibility of understanding the need of the *lekta* — the conceptual constructions intermediate between the thoughts and the things — in the devising of meaningful representations of the relationships between human activity and the natural world. As Russo says, "in the imperial age, when the notion of theoretical models had been lost, such entities were conceivable only as real objects: the alternative between 'bodies' and 'incorporeal beings' thus became ineluctable. Some such entities were indeed made corporeal — witness the crystalline celestial spheres which replaced the spheres of Eudoxus of Cnidus and the epicycles of Apollonius of Perga. Likewise, the 'visual rays' of optics reacquired the character of physical objects emitted by the eyes, which was not present in Euclid's theory. ... Other entities, such as those of geometry, were given an incorporeal reality. This placed geometry in the realm of Platonic thought, a position that Hellenistic mathematics had left behind" [58, p. 232].

A similar fate befell the basic entities of the science of harmonics, such as the musical intervals, which lost the character of theoretical entities gained within Hellenistic science to be identified (again) either with corporeal items — such as the discrete movements of a melodic voice — or else with purely ideal abstractions, entities considered as much real as they are not attainable — such as the harmonic ratios composing the Zodiac or the human soul — both deemed to possess quantitative features to which reason can be "directly" applied by assigning them appropriate numbers.

An important example is provided by Ptolemy in the first book of his *Harmonica*, where, no longer being able to grasp the methodological tenets maintained by his Hellenistic sources, he falls back upon epistemological bases close to the Peripatetic ones, without thereby giving up the claim of employing refined mathematical tools inherited from his predecessors (yet conceived in the Platonic sense). For example, with a kind of inversion of the Hellenistic rule that requires a theory to save the *phainomena*, he says, "The purpose of the harmonicist would be to preserve in every way the reasoned hypotheses of the canon which do not in any way at all conflict with the perceptions as most people interpret them, just as the purpose of the astronomer is to preserve the hypotheses

canonis — implying that no (rational) mean proportional can be inserted between 9 and 8 — which characterizes intervals as elements of a theoretical model (hence defined solely by the hypotheses underlying the model itself). Rather, it results from the direct experience of placing one's finger on the string at the point corresponding to the division into equal semitones. A theoretical construction of the equal temperament has been made almost two millennia later by Stevin (see below).

of the heavenly movements concordant with observable paths. Even these hypotheses are themselves assumed from what is clear and roughly apparent, but with the help of reason they discover detail with as much accuracy as is possible. For in every subject it is inherent in observation and knowledge to demonstrate that the works of nature have been crafted with some reason and prearranged cause and completed not at all in random" [54, §I.2, §§I.5.13–21].[17] Like other scholars of his time, Ptolemy's "criterion of truth" is dictated by a strange kind of "concord" between theory and observation, where the model, although highly mathematized, has lost its meaning as a theoretical entity and taken over the former prescientific meaning of direct representation of the known reality. In this regard, he seems to want to frame Hellenistic scientific results within philosophical arguments of the classical period. Although one might regard this as a dialectical strategy to give maximum credibility to the position he wants to hold, in this way he ends up denying the method of his Hellenistic predecessors, deeming legitimate only one theory: that whose assumptions are entirely justified by the phenomena and at the same time reflect the "rationality" of nature's works.[18]

Akin to Galen's craving to strike a balance between the "rationalist" and "empiricist" schools of medicine [30],[19] Ptolemy loudly distinguishes his approach to the study of consonances from that of the "excessively rationalist" Pythagoreans — accused of accepting rationally justifiable statements even when they are contradicted by the senses[20] — and that of the "overly empirical" Aristoxeneans, for whom audible harmonies are not subject to mathematical analysis at all. Since for him the objects of sense-perception and thought are (again) identical (cf. footnote 3), though apprehended in different ways, he feels entitled to set his "hypotheses" in the form of alleged mathematical counterparts of the relevant perceptual impressions. In doing so, Ptolemy assigns to the science of harmonics the task of explaining the audible and inaudible harmonies by reference to the formal, quantitative attributes of the different pitches, as to astronomy that of explaining the movements of the

[17] This goes hand in hand with some passages of the *Almagest*, for example where he says, "Now it is our purpose to demonstrate for the five planets, just as we did for the sun and moon, that all their anomalies can be represented by uniform circular motion, since these are proper to the nature of divine beings, while disorder and nonuniformity are alien [to such beings]" [53, §9.2].

[18] One should perhaps also include evidence that these narrow epistemological bases will actively operate in the development of modern science.

[19] A discussion on Ptolemy's epistemological affinity with his contemporary Galen can be found in [41].

[20] His concern focused in particular on the fact — already mentioned above — that the interval corresponding to the ratio 8 : 3 (*diapason* plus *diatessaron*), although unanimously recognized as consonant, was "deductively" counted as dissonant on the basis of the postulate opening the *Sectio canonis* [54, §I.5, §§I.12.4–8].

observable heavenly bodies by reference to the formal features of the spheres or other bodies on which they are physically carried. In this regard, the use of mathematics would serve mostly as a "rational criterion" to assist the senses in making fine discriminations.

As an aside, in the Alexandrian milieu of the imperial age, where the lingua franca was still Greek but life and thought were dominated by a cohort consisting of astrological fatalism, gnostic dualism and transcendent monotheism, even mathematics was mostly plunged into an atmosphere of irrationalism, with the distancing from the deductive method and the return of numerology. At the same time, among the objects of musical "perception", the prototype was considered the "music of the spheres", an old conception dating back at least to Plato's *Timaeus* [49] and *Republic*, resumed by Nicomachus [47] and subsequently by Ptolemy himself in the third book of his *Harmonica*.[21] Thereafter, the Platonic connection between planetary motion and music became a cornerstone of the *musica speculativa* — which together with *musica poetica* and *musica practica* constituted the quadrivial discipline of *musica*.[22] The regaining of a corporeal nature of the crystalline spheres to which the heavenly bodies were said to be attached goes hand in hand with the resumption of the celestial harmony as a "perceptible" datum, although emanating from an incorporeal and inaudible reality.[23] Both subjects were then transmitted through centuries by sheer copying[24] until they were taken seriously

[21] After comparing the various harmonic functions with several aspects of the human soul [54, §§III.4–7], he proceeds by regarding the zodiac circle as a vibrating string and comparing the principal astrological "aspects" (angles between heavenly bodies that were believed to modify their degree of influence) with musical consonances, thereby explaining their differing "effectiveness" [54, §§III.8–9].

[22] This tripartition of music reflected Aristotle's division of knowledge (*epistēmē*) into *theōrētikē*, *poiētikē* and *praktikē* and was codified by the sixth-century Roman philosopher Boethius [12] as *musica mundana*, *musica humana* and *musica instrumentalis*. More generally, the resumption of a pre-Hellenistic classification of knowledge (in particular that outlined in Plato's *Republic*) becomes glaringly obvious with the reduction of the manifold Hellenistic sciences to the *quadrivium*, consisting of arithmetic, geometry, music and astronomy, which along with the *trivium* (made by grammar, logic and rhetoric) constituted the seven "liberal arts" that articulated the preparatory training for the study of theology in medieval times.

[23] Note that Aristotle, who believed in the existence of the rigid sidereal sphere [2, Chapter 6], refuted the conception of celestial music on the basis of physical arguments [2, Chapter 9]. On the other hand, in the cultural context to which we are referring, it would have seemed vain to refute on physical basis such Platonic mythological representations.

[24] Or else they were transmitted by anthological syntheses of the *prisca sapientia*, such as the commentary on Cicero's *Somnium scipionis* by the fifth-century neoplatonist Macrobius [42], where he drew comprehensively on the whole body of Pythagorean, Orphic and Platonic teachings and cosmology. How deep the decline of science at the end of the ancient world was can be grasped from the fact that, although Macrobius faithfully reports the ratios corresponding to the Pythagorean consonances, he does not even understand that they are *ratios*. For instance, he justifies the fact that the tone 9 : 8 cannot be divided into equal parts (a consequence of the Archytas impossibility result

again in early modern times. For example, Kepler's "estimate" of the thickness of the crystal sphere of fixed stars[25] went together with his tentative attempts to improve Ptolemy's harmonic investigations by searching for musical proportions in various quantities in the Solar system, such as the periods of the (heliocentric) planetary motions.[26] Note however that although the faith in the "harmony of the world" had played an indubitable role in the reappearance of mathematics as the pivotal language of the resurgent sciences in early modern times, when at the time of Newton the terms "physics" and "mathematics" got the new meanings we have discussed above, science had begun to need different images; hence, celestial music became old-fashioned as a scientific subject and eventually became a purely literary metaphor.[27]

Altogether, the early modern resumption of studies on the science of harmonics took different forms, sometimes in open conflict with each other, often revealing with particular vividness the prevailing beliefs on the more general meaning of the scientific enterprise [17; 33; 16]. A well known example is the harsh conflict between Vincenzo Galilei (the father of Galileo) and Zarlino, where, among other things, to the "well ordered" Nature of Zarlino, which whispers to the human ear the true consonances, Galilei opposed the image of a Nature which proceeds "without cognition" (*senza cognitione*), with principles and purposes unrelated to man, and against which man takes advantage of the mechanical arts to an end that nature cannot achieve [31; 69]; see also [66, Chapter 2; 48]. Among the seventeenth-century scholars who took an active role in producing musical theories, like Simon Stevin, Kepler, Isaac Beeckman, Descartes, Mersenne, Francis Bacon, Galileo Galilei, Lord Brouncker, John Wallis, Christian Huygens, Robert Hooke and others, only the first one seems to have retained the option to build a model based on a free choice of hypotheses.[28]

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discussed above) not with the nonexistence of a rational square root of 2 (and hence of 8) but with the fact that 9 is not divisible by 2 [42, Book Two, Chapter I.21–23].

[25]He estimated about two German miles [38, p. 288].

[26]We have to recognize that, unlike the first, the second concern was fruitful, as it is well known that the search for a harmonic correspondence between the periods of revolution and the radii of planets' orbits eventually led to the celebrated "third law" [39].

[27]Nevertheless, Newton himself had imagined recovering the lost *prisca sapientia* in which, among other things, the inverse square law of gravitational attraction between the planets would have been encrypted within Pythagoras' music of the spheres [43].

[28]This is the subject of a short treatise written in Dutch where, among other things, the equal temperament (i.e., the geometrical division of the octave into twelve equal parts, each corresponding to a ratio $1 : \sqrt[12]{2}$) is constructed on the basis of two postulates. The first says that, as one part of a string is to another, so is the coarseness of the sound of the one to that of the other. The second says that natural singing is in the major diatonic scale, and in this scale all whole tones are equal and so are the semitones [62].

Somewhat later, in the age of Lights, and thus after the splitting between "physics" and "mathematics", an interesting confrontation about the science of harmonics took place between Euler and d'Alembert; see, e.g., [8, Chapter 4]. The swiss mathematician — perhaps the last representative of the conception of music as a part of mathematics — at age 24 finished writing his major work on the subject, the *Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae* (1731), whose main goal was to give an answer to the old question of why certain sounds are pleasant and others are not, an answer which would feature not only the perception of single intervals but also sequences of chords or even of a complete musical piece. He pursued this goal by assuming that any pleasure comes from the perception of a "perfection", which in turn is embodied in a notion of *order* that can be measured by an *exponent* calculated only in terms of the arithmetic proportions associated to the pitches of the tones involved.[29] As a consequence, the fact that some people appreciate the use of some chords and others not is explained by saying that the latters' ear is not trained enough to perceive the order hidden in them. As we have pointed out previously about the *Sectio canonis*, the assumptions underlying this construction also cannot exempt themselves from being somehow "false": for instance, the same "exponent", and thus the same degree of pleasantness, is associated to a musical piece regardless if it is played forward or backward, which is in general something far from usual experience. But as we have seen, this is not a problem in itself, at least as long as things remain consistent with the ancient scientific method. It is not clear (to me) whether Euler considered the problem dealt with in his treatise as one of defining the value of something which was not defined before or as a Platonic search for some "true" value. Be that as it may, even only in his ambition to model far more than just a system of tuning, he exposed himself to the criticism that Johann Bernoulli leveled against him in 1731: "... you have derived the rule which establishes how the notes are to be combined, so that an intelligent ear can take delight in them. I think that this is appropriate for a musician who is more concerned about the accuracy of a piece of music than its effect, which satisfies the listener; a person of this kind will undoubtedly find enjoyment and delight, if you have written this down and examine it and find that it is well composed in accordance with the fundamental rules; but as a piece of music is usually played to ears that are devoid of understanding, and are not able to recognize the ratio between the beats of the intervals produced by the strings, and are even less able to count, then I believe that the same ears will appreciate or refuse the same piece

[29] See [27, pp. 197–427], where he starts from a masterful generalization of some previous ideas of Galileo [32] and Mersenne [44] according to which a chord of two sounds is all the more consonant when the "coinciding" blows resulting from the two sounds are in higher proportion in the whole of the produced blows. An example of calculation of this exponent is given below.

of music, depending on whether they are used to this or that kind of music" [28, pp. 146–150].

But a critique of a different tenor was put forward by d'Alembert who, having in mind the example, bad for him, of the "mathematician" Euler, embodied the role of the "physicist", in the sense advocated by his mentor, Voltaire. In particular, he maintained that in music theory there is no place for "demonstration" — insofar as that term is reserved to "mathematics" — and one should adopt an "empirical-deductive" methodology modeled on that of Newton. As he made clear in the preface of his widely read treatise on music theory,[30] issued in 1752, his main purpose in writing the work was "to show how one may deduce from a single experiment the laws of harmony which artists had arrived at only, so to speak, by groping" [22, p. vi]. The single experiment had to do in this case with Rameau's *corps sonore*, that is, any resonating system which, besides the fundamental frequency (sounding pitch), also generates a series of harmonically related overtones, such as the octave, the perfect twelfth (the octave of the perfect fifth), the major seventeenth (the double octave of the major third) and so on. To d'Alembert, the resonance of the sonorous body was the "most probable origin of harmony, and the cause of that pleasure which we receive from it". He thus strove to structure music as a science based on a single "principle" which is somehow "dictated by nature" and from which one should deduce "by an easy operation of reason, the chief and most essential laws of harmony".

The different positions embodied by the two scholars resulted in several controversies, among which the one about the possible solutions of the wave equation is perhaps the best known, although its current reconstructions usually neglect musical motivations and implications. In particular, the last account on the subject written by d'Alembert ends with a polemical stance against the music theory maintained by Euler, in which the very possibility of dealing with a musical phenomenon in terms of theoretical entities seems denied: "It is clear from the preceding formulae that, given an equal tension and thickness, the number of vibrations in the same time is inversely proportional to the length of the strings. As the higher or lower sound of the strings depends on their larger or smaller number of vibrations in the given time, it is undoubtedly for this reason that some very capable modern authors have considered it possible to represent the sounds by means of the logarithms of the ratios between the lengths of the strings. This idea is ingenious, and would appear to be based equally on figures of speech in acoustics and music, when we say that if four strings a, b, c, d are geometrically proportional, the interval formed by sounds a and b will be equal to the interval formed by c and d; hence it was

[30] It is a kind of résumé of the music-theoretic writings of the great composer Jean-Philippe Rameau [55], where he thought to find a paradigm of systematic method and synthetic structure which somehow confirmed his own scientific ideas [22]; see also [8; 15].

considered possible to conclude that the logarithms of the relationships $a : b$ and $c : d$ represented the intervals between the sounds. But undoubtedly this conclusion was not claimed to be anything more than a purely arbitrary supposition; the words interval between sounds, equality and difference of intervals are only abbreviated figures of speech, which should not be given a wider meaning than they really have. Sounds are merely sensations, and consequently they do not in reality have any ratio with one another; sounds cannot be compared, any more than colours can; all that is needed is a little attention to hear this..." [21] (cited in [64]). A further interesting confrontation between Euler and d'Alembert's methodologies concerned the interpretation of the (widely used) dominant seventh chord, namely a chord made out of a root, a major third, a perfect fifth and a minor seventh. Its name comes from the fact that it occurs naturally in the seventh chord built upon the fifth degree — the dominant — of a given major diatonic scale. For example, in the case of the C-major diatonic scale, we get the aggregate G-B-d-f. To d'Alembert, this aggregate was a nice major triad G-B-d to which the dissonant seventh f is added to unambiguously mark the root tone. Differently said, the dissonance G-B-d-f is there just to indicate to the listeners that the piece being played must be in the key of C. Euler discussed this topic in one of the three or four articles that he devoted to music theory in his later years; see "Conjectures sur la raison de quelques dissonances généralement reçues dans la musique" (1766) [27, pp. 508–515].[31] To briefly review Euler's argument, we start by recalling that he worked with "just intonation", i.e., the system of ratios described by Ptolemy and revived in the sixteenth century by Zarlino to account for the intervals used in polyphonic music.[32] A portion of this system, covering a perfect twelfth interval, and reduced to a series of whole numbers, is presented in the following table:

C	D	E	F	G	A	B	c	d	e	f	g
24	27	30	32	36	40	45	48	54	60	64	72

The seventh chord G-B-d-f is then expressed by the ratios $36 : 45 : 54 : 64$, to which Euler assigns the exponent given by their least common multiple, that is, $2^6 \cdot 3^3 \cdot 5 = 8640$. One recognizes that it is the tone f that troubles this chord. Indeed, if we omit this tone, we obtain the much simpler ratios $4 : 5 : 6$, whose exponent is $2^2 \cdot 3 \cdot 5 = 60$. From this, "it seems that the addition of the note f ruins the harmony of this consonance too much for it to have a place in music. However,

[31] We shall use the English translation [29].

[32] Thus, the system replaced the so-called Pythagorean system — constructed with the perfect fifth $3 : 2$ as the only reference interval besides the octave — which remained in use until the late Middle Ages, meeting the needs of the monophonic composition and medieval parallel singing. The subsequent invention of polyphony claimed an increasingly frequent use of intervals of third and sixth, which in the Pythagorean scale are not very consonant [68; 10].

to the ear's judgment, this dissonance is at worst disagreeable and has been used in music with great success. It even seems that musical composition acquires a certain force from it, and without it would bee too smooth and dull. Here we have quite a paradox, where the theory seems to be in contradiction with the practice, to which I will try to give an explanation" [29, §4]. The explanation of Euler is based on the following hypothesis: the organ of hearing is accustomed to taking as simple proportions all proportions that differ very little from it so that the difference is almost imperceptible. For example, in equal temperament, the fifth is expressed by the (irrational) ratio $1 : \sqrt[12]{2}^7$, which hardly differs from the proportion of $2 : 3$, but the ear is not bothered too much by this small discrepancy[33] and in hearing the interval C : G one may safely "think" the ratio $2 : 3$. More generally, if the proportions expressing a combination of tones are too complicated, the ear will "substitute" a close approximation that is simpler. "Thus the heard proportions are different than the true, and it is from them that we must judge the true harmony and not from the actual numbers" [29, §12]. According to this assumption, the effect of listening to the dominant seventh chord, which corresponds to the tones 36, 45, 54, 64, is absolutely the same as listening to the tones 36, 45, 54, 63, which yield the proportion $4 : 5 : 6 : 7$, whose exponent is $2^2 \cdot 3 \cdot 5 \cdot 7 = 420$, about twenty times smaller than the "true" one.

In the perspective of the present work, we can recognize in this construction a way of "saving the phenomena" (the strange acceptance by the ear of a "dissonant" acoustic aggregate) in the same spirit as the ancient exact sciences and therefore the product of the scientific activity of someone who has not yet introjected the division between "mathematics" and "physics" as was vogue in his time. Euler's explanation is often presented as the legacy of an outdated attitude, still attached to a calcified "Pythagorean tradition", whereas d'Alembert would belong to "the right side of History"; see, e.g., [64, p. 289; 8, pp. 139–141]. Indeed, unlike Newtonian mechanics, which, although leaning on outlandish foundations, rather quickly has developed into a true scientific theory, the science of harmonics eventually has fallen apart, in that the subsequent evolution has gradually ousted music theory from the field of direct interest of the majority of scientists. On the one hand, *musica theorica* has been largely absorbed into *musica practica*, written by musicians for musicians and mainly focused on the empirical problems of harmony and counterpoint; on the other hand, the theoretical work on the phenomena regarding musical perception was broken up into several branches, with the result that the acoustic problems that traditionally were part of music theory were detached from

[33] Adopting the tempered fifth amounts to using the convergent $\frac{7}{12}$ of (the continued fraction expansion of) $\log_2(\frac{3}{2})$. The next convergent being $\frac{24}{41}$, the error is smaller than $(12 \cdot 41)^{-1}$, that is, about a hundredth of a tone.

their musical context to become subjects treated separately by the "natural sciences", such as physics[34], physiology[35] and psychology[36]. Nowadays the scenario is rather involved, with the coexistence of several tendencies which mostly ignore each other. On the one side, we witness a significant renewal of interest on mathematical modeling of some aspects of music theory, with the search for structural, and to some extent universal, principles in the formation of musical scales; see, e.g., [37] and the references therein. In other directions, the massive advent of new information technologies in the last decades has created an unprecedented situation in which quantitative methods based on the automatic processing of large masses of data invade all fields, including music. Besides the indubitable enrichment with new sound media and new composing techniques (often directly inspired by mathematical constructions such as probability theory or game theory), as far as the new quantitative treatments of musical-acoustical phenomena are concerned — with the related conceptualizations and cultural trends — we have to say that the aims and the methodologies adopted in this context are often placed quite far from those embodied by the exact sciences. This calls for a critical analysis which is still to come.

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[34] Such treatments mainly revolved around investigations on the "nature of sound", in some aspects analogous to that in which the ancient optics, a science of vision, became the study of the "nature of light".

[35] This includes major achievement on the physiological analysis of the tonotopic structure of the inner ear, starting with the important work of Helmholtz [35] until more recent findings on the band structure of the basilar membrane [51], and the study of the auditory cortex by the so-called "neurosciences". By the way, this research can be viewed as providing a physiological basis both to the ancient principle of consonance and to the Eulerian principle mentioned above.

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STEFANO ISOLA: `stefano.isola@gmail.com`
Scuola di Scienze e Tecnologie, Università di Camerino, I-62032 Camerino, Italy

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