Nissuna umana investigazione si puo dimandare vera scienzia
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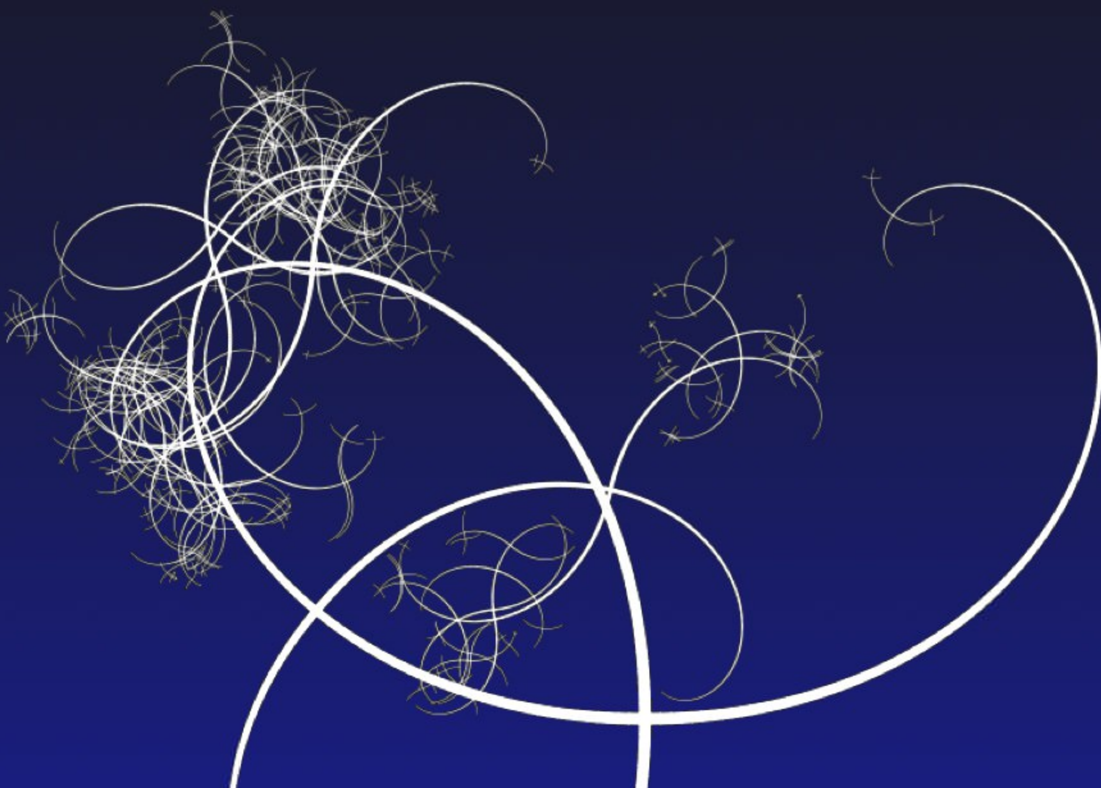
Leonardo da Vinci



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A REMARK ON EIGENVALUE PERTURBATION THEORY AT VANISHING ISOLATION DISTANCE

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Let T be a self-adjoint operator in a separable Hilbert space X, admitting compact resolvent and simple eigenvalues with possibly vanishing isolation distance, and let V be symmetric and bounded. Consider the self-adjoint operator family $T(g) : g \in \mathbb{R}$ in X defined by $T + gV$ on $D(T)$. A simple criterion is formulated ensuring, for any eigenvalue of $T(g)$, the existence to all orders of its perturbation expansion and its asymptotic nature near $g = 0$, with estimates independent of the eigenvalue index. An application to a class of Schrödinger operators is described.

1. Introduction and formulation of the result

The standard Rellich–Kato regular perturbation theory [Kato 1976] applies to *isolated* eigenvalues of finite multiplicity of a densely defined, closed operator T in a Banach space X. We consider here only the particular case in which

- X is a separable Hilbert space,
- T is a self-adjoint operator in X with compact resolvent and simple spectrum, and
- the perturbation is symmetric, regular and linear on the perturbation parameter.

Let the operator $V : D(V) \to X$ be symmetric and T-bounded with relative bound b; i.e., let $D(T) \subset D(V)$, and let there exist $a > 0$ and $b > 0$ such that

$$\|Vu\| \le b\|Tu\| + a\|u\| \quad \text{for all } u \in D(T). \tag{1-1}$$

With $g \in \mathbb{C}$, consider the operator family in X defined as

$$g \mapsto T(g) := Tu + gVu, \qquad D(T(g)) = D(T). \tag{1-2}$$

Then $T(g)$ is closed with nonempty resolvent set for $|g| < 1/b$ and $T(g)^* = T(\bar{g})$ so that $T(g) = T(g)^*$ if $g \in \mathbb{R}$. Let $\lambda \in \mathbb{R}$ be an isolated eigenvalue of T (simple by the above assumption), with isolation distance $d(\lambda) > 0$. Here, let us recall that

$$d(\lambda) := \operatorname{dist}(\operatorname{Spec}(T) \setminus \{\lambda\}, \lambda) > 0. \tag{1-3}$$

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Then for $|g|$ suitably small, $T(g)$ has one and only one simple eigenvalue $\lambda(g)$ such that $\lim_{g\to 0}\lambda(g)=\lambda$ (see, e.g., [Kato 1976, §§VII.2–3] or [Reed and Simon 1978, §XII.1]). The function $g\mapsto\lambda(g)$ is holomorphic in a disk centered at the origin because its Taylor expansion at $g=0$

$$\lambda(g)=\lambda+\sum_{\ell=1}^{\infty}\lambda_\ell g^\ell \tag{1-4}$$

exists and converges for $|g|<r_d(\lambda)$, with $r_d(\lambda)>0$. The coefficients λ_ℓ are generated by (Rayleigh–Schrödinger) perturbation theory. This existence and convergence result depends in a critical way on the positivity of $d(\lambda)$ and therefore does not apply to nonisolated eigenvalues.

To the best of our knowledge, a simple, explicit criterion ensuring existence, let alone convergence, of (Rayleigh–Schrödinger) perturbation theory when $r_d(\lambda)\to 0$ is still missing, even under much stronger assumptions such as $r_d(\lambda)\to 0$ only if $\lambda\to\infty$ and boundedness of V. (For related questions involving the behavior of $r_d(\lambda)$ as $\lambda\to\infty$, we refer the reader to [Reed and Simon 1978, §XIII.5] and to [Brownell and Clark 1961; McLeod 1961; Tamura 1974]). Within this last class of Hilbert space operators, we formulate and prove here a similar criterion, working out the necessary estimates on the behavior of λ_ℓ uniform with respect to the eigenvalue index $\underline{n}$. Under more restrictive assumptions on the vanishing of $r_d(\lambda)$, the explicit dependence on $\underline{n}$ of the above estimates is actually determined.

Our hypotheses are formulated as follows.

(A1) T is a nonnegative self-adjoint operator in the separable Hilbert space X, with compact resolvent and simple spectrum. Its eigenvalues are denoted by $\{\lambda_{\underline{n}}:\underline{n}\in\mathbb{N}^s\}$, $s\geq 1$, and the corresponding (normalized) eigenvectors by $\{\psi_{\underline{n}}:\underline{n}\in\mathbb{N}^s\}$.

(A2) $d(\underline{m},\underline{n}):=|\lambda_{\underline{m}}-\lambda_{\underline{n}}|\to 0$ if and only if $|\underline{m}-\underline{n}|\to\infty$. Here $|n|:=n_1+\cdots+n_s$, $\underline{n}\in\mathbb{N}^s$.

(A3) There are $\Lambda>0$ and $\gamma>s-1$ such that

$$|\lambda_{\underline{m}}-\lambda_{\underline{n}}|^{-1}\leq\Lambda|\underline{m}-\underline{n}|^\gamma,\quad \underline{m}\neq\underline{n}. \tag{1-5}$$

Here $|\underline{x}|^\gamma:=x_1^\gamma+\cdots+x_s^\gamma$.

(A4) $V:X\to X$ is symmetric and bounded (hence self-adjoint). Moreover, there exist $A>\alpha$ and $\alpha>0$ such that

$$|\langle\psi_{\underline{m}},V\psi_{\underline{n}}\rangle|\leq Ae^{-\alpha|\underline{m}-\underline{n}|},\quad |\underline{m}-\underline{n}|\to\infty. \tag{1-6}$$

Since V is bounded, it is a fortiori T-bounded with T-bound 0. Thus, the operator family $g\mapsto T(g)=T+gV$ with $D(T(g))=D(T)$ is type-A real-holomorphic in the sense of Kato [1976, §VII.2] for all $g\in\mathbb{C}$.

Consider now the Rayleigh–Schrödinger perturbation expansion near any eigenvalue $\lambda_{\underline{n}}(g)$ of $H(g)$. The initial point of $\lambda_{\underline{n}}(g)$ is the eigenvalue $\lambda_{\underline{n}}$ of T, $\underline{n} \in \mathbb{Z}^s$. Thus,

$$\lambda_{\underline{n}}(g) = \lambda_{\underline{n}} + \sum_{\ell=1}^{\infty} B_\ell(\underline{n}) g^\ell. \tag{1-7}$$

The expansion (1-7) has positive radius of convergence $r_{\underline{n}}$, $\underline{n} \in \mathbb{Z}^s$, by the boundedness of V, which implies

$$r_{\underline{n}} \geq \frac{d(\lambda_{\underline{n}})\|V\|}{2}, \quad d(\lambda_{\underline{n}}) = \operatorname{dist}(\operatorname{Spec}[(T) \setminus \{\lambda_{\underline{n}}\}], \lambda_{\underline{n}}). \tag{1-8}$$

The vanishing of the convergence radius $r(\underline{n}) \downarrow 0$ as $d(\lambda_{\underline{n}}) \downarrow 0$ not only may cause the divergence of the perturbation expansion but may prevent its very existence also for a bounded perturbation V; see Remarks 1.3 and 2.1. Then the purpose of this paper is to explicit determine, under the above assumptions, the dependence of the perturbation series on the vanishing rate of the isolation distance $d(\lambda_n)$ by proving the following quantitative estimate.

Theorem 1.1. *Let T and V fulfill assumptions (A1)–(A3). Set*

$$R(\Lambda, \alpha, \gamma) := \frac{\Lambda}{\alpha^\gamma}. \tag{1-9}$$

Then the following $\underline{n}$-independent estimate holds:

$$|B_\ell(\underline{n})| < R(\Lambda, \alpha, \gamma)^\ell (4\ell)^{\ell+1}[\gamma(\ell-1)]! \quad \textit{for all } \underline{n} \in \mathbb{Z}^s. \tag{1-10}$$

The uniform estimate (1-10) makes it possible to establish the uniform asymptotic nature to all orders of the perturbation expansion.

Corollary 1.2. *The perturbation expansion $\lambda_{\underline{n}} + \sum_{\ell=1}^{\infty} B_\ell(\underline{n}) g^\ell$ represents an asymptotic expansion to all orders of the eigenvalue $\lambda_{\underline{n}}(g)$ uniformly with respect to $\underline{n} \in \mathbb{N}^s$; i.e, for any fixed $N \in \mathbb{N}$,*

$$\lim_{|g| \to 0} \frac{\left|\lambda_{\underline{n}}(g) - \sum_{\ell=1}^{N} B_\ell(\underline{n}) g^\ell\right|}{|g|^N} = 0 \tag{1-11}$$

uniformly with respect to $\underline{n} \in \mathbb{N}^s$.

Remark 1.3. The very existence of perturbation theory at the vanishing of the isolation distance, i.e., at the limit $\underline{n} \to \infty$ in the present case, requires the validity of estimates independent of $\underline{n}$ on the coefficients $B_{\ell,\underline{n}}$. The conditions (1-5) and (1-6) imply the existence of $\bar{g}(\underline{n}) > 0$ such that

$$|g| \sum_{\substack{\underline{m} \in \mathbb{Z}^s \\ \underline{m} \neq \underline{n}}} \frac{|\langle \psi_{\underline{m}}, V\psi_{\underline{n}} \rangle|}{|\lambda_{\underline{m}} - \lambda_{\underline{n}}|} < 1, \quad |g| < \bar{g}(\underline{n}). \tag{1-12}$$

Now

$$\sup_{\underline{m}\neq\underline{n}} \frac{1}{|\lambda_{\underline{m}} - \lambda_{\underline{n}}|} \leq \frac{2}{d_{\underline{n}}}.$$

Moreover, since V is bounded and symmetric,

$$\|V\| = \max\biggl(\sup_{\underline{m}\in\mathbb{Z}^s} \sum_{n\in\mathbb{Z}^s} |\langle\psi_{\underline{m}}, V\psi_{\underline{n}}\rangle|, \sup_{\underline{n}\in\mathbb{Z}^s} \sum_{\underline{m}\in\mathbb{Z}^s} |\langle\psi_{\underline{m}}, V\psi_{\underline{n}}\rangle|\biggr),$$

whence

$$\sum_{\substack{\underline{m}\in\mathbb{Z}^s\\ \underline{m}\neq\underline{n}}} \frac{|\langle\psi_{\underline{m}}, V\psi_{\underline{n}}\rangle|}{|\lambda_{\underline{m}} - \lambda_{\underline{n}}|} \leq \frac{2}{d_{\underline{n}}}\|V\|.$$

Thus, if $\underline{n}$ is fixed, i.e., if $\lambda_{\underline{n}}$ is *isolated* with isolation distance $d_{\underline{n}} > 0$, the standard convergence criterion valid for the perturbation series of the isolated eigenvalue $\lambda_{\underline{n}}$ under the bounded perturbation V (see, e.g., [Kato 1976, §VII.2])

$$\frac{2|g|}{d_{\underline{n}}}\|V\| < 1 \tag{1-13}$$

implies a fortiori the inequality (1-12). On the other hand, as $\underline{n} \to \infty$, i.e., $d_{\underline{n}} \to 0$, in general the inequality (1-13) has a meaning only for $g = 0$, while (1-12) can be rewritten (again by (1-5) and (1-6), which are an adaptation of the small-denominator conditions of classical perturbation theory) in the form of an inequality independent of $\underline{n}$:

$$|g|A\Lambda\biggl[\frac{d^\gamma}{d\alpha^\gamma}(2e^{\alpha\gamma})\biggr]^s < 1. \tag{1-14}$$

This inequality is the starting point for the $\underline{n}$-independent estimates of Theorem 1.1.

Remark 1.4 (notation). The underlining operation always transforms into vector indices with s components the corresponding scalar ones. Namely,

$$\underline{m} := (m_1, \dots, m_s) \in \mathbb{Z}^s, \quad m_i \in \mathbb{Z},\ i = 1, \dots, s. \tag{1-15}$$

The star operation transforms a positive integer index into a nonnegative one; i.e., $q \in \mathbb{N}_*$ means $q = 0, 1, \dots$. Furthermore, $|\underline{m}|$ denotes the length of the multi-index $\underline{m}$:

$$|\underline{m}| := |m_1| + \cdots + |m_s|. \tag{1-16}$$

Products and powers of multi-indices abbreviate products and powers of the composing indices:

$$\begin{aligned} \underline{m}! &:= m_1! \cdots m_s!, \\ \underline{z}^{\underline{m}} &:= z_1^{m_1} \cdots z_s^{m_s}, \\ \Gamma(\alpha\underline{z}) &:= \Gamma(\alpha z_1) \cdots \Gamma(\alpha z_s), \quad \alpha \in \mathbb{R}. \end{aligned} \tag{1-17}$$

Example 1.5. Let T be the Schrödinger operator in $L^2(\mathbb{R}^s)$ with domain and action defined as

$$\begin{aligned} D(T) &= H^2(\mathbb{R}^s) \cap L^2_2(\mathbb{R}^s), \\ Tu &= -\tfrac{1}{2}\Delta u + \frac{1}{2}\sum_{k=1}^{s}[\omega_k^2 x_k^2 - s/2]u, \quad u \in D(T). \end{aligned} \tag{1-18}$$

T is the self-adjoint, compact-resolvent Schrödinger operator generated by the p-dimensional quantum harmonic oscillator, with frequencies $1 \geq \omega_k > 0$, $k = 1, \ldots, s$. Thus, condition (A1) is fulfilled.

The rescaling map $(U_{\underline{\omega}} f)(\underline{x}) = (\omega_1 \cdots \omega_s)^{1/2} f(\underline{\omega x})$, $\underline{\omega x} := (\omega_1 x_1, \ldots, \omega_s x_s)$, is unitary in $L^2(\mathbb{R}^s)$, and by an abuse of notation, we still denote by T the unitary image $U_{\underline{\omega}} T U_{\underline{\omega}}^{-1}$. Hence, the action Tu becomes

$$Tu = \frac{1}{2}\sum_{k=1}^{s}\omega_k\left[-\frac{d^2u}{dx_k^2} + x_k^2 u - Iu\right].$$

The corresponding eigenvalues are

$$\lambda_{\underline{n}}(\omega) = \sum_{k=1}^{s}\omega_k n_k := \langle \underline{\omega}, \underline{n}\rangle, \quad \underline{n} \in (\mathbb{N}_*)^s. \tag{1-19}$$

Since $\omega_i > 0$ and $n_i > 0$, $i = 1, \ldots, s$, the difference $|\lambda_{\underline{r}} - \lambda_{\underline{n}}| = \langle \underline{\omega}, (\underline{r} - \underline{n})\rangle$ can vanish only if $|\underline{r} - \underline{n}| \to \infty$, with at least two of the components $r_j - n_j$, $j = 1, \ldots, s$, having different sign. Hence, condition (A2) is fulfilled. Assume now irrational independence of the frequencies, i.e.,

$$\omega_1\nu_1 + \cdots + \omega_s\nu_s = 0, \ \nu_k \in \mathbb{Z}, \quad \text{if and only if} \quad \nu_k = 0, \ k = 1, \ldots, s. \tag{1-20}$$

Then all eigenvalues $\lambda_{\underline{n}}(\underline{\omega})$ are simple, and condition (A3) is equivalent to requiring the *diophantine condition*

$$|\langle \underline{\omega}, \underline{\nu}\rangle| > \Lambda^{-1}|\underline{\nu}|^{-\gamma}, \quad \gamma > s - 1, \ \underline{m} - \underline{n} := \underline{\nu} \neq 0 \tag{1-21}$$

on the frequencies $\underline{\omega}$. The set of the diophantine values of $\underline{\omega}$ is dense in $[0, 1]^s$.

By condition (1-20), the eigenvalues $\lambda_{\underline{n}}(\underline{\omega})$ are simple. The corresponding normalized eigenvectors are

$$\psi_{\underline{n}}(\underline{x}) = \frac{1}{\sqrt{2^{\underline{n}}\underline{n}!}} e^{-\underline{x}^2/2} H_{\underline{n}}(\underline{x}) := |\underline{n}\rangle, \tag{1-22}$$

where $x \mapsto H_n(x)$, $n = 0, 1, \ldots,$ is the n-th Hermite polynomial in $\mathbb{R}$ and

$$\begin{aligned} \underline{x} &= (x_1, \ldots, x_s) \in \mathbb{R}^s, & 2^{\underline{n}} &= 2^{n_1 + \cdots + n_s}, \\ e^{-\underline{x}^2} &= e^{-x_1^2 - \cdots - x_s^2}, & H_{\underline{n}}(\underline{x}) &= H_{n_1}(x_1) \cdots H_{n_s}(x_s). \end{aligned}$$

Now let $2 < q < 6$. Consider the function $\mathcal{V}(\underline{x}) \in C^\infty(\mathbb{R}^s;\mathbb{R})$ such that

$$\mathcal{V}(\underline{x}) = e^{-|\underline{x}|^q}\Phi(\underline{x}), \qquad \sup_{\underline{x}\in\mathbb{R}^s}|\Phi(\underline{x})| \le 1,\ \ |\underline{x}|^q := |x_1|^q + \cdots + |x_s|^q. \tag{1-23}$$

Denote by $V(\underline{x})$ the maximal multiplication operator by $\mathcal{V}(\underline{x})$ in $L^2(\mathbb{R}^s)$. Then $\|V\|_{L^2\to L^2} \le 1$. Thus, the operator $H = T + V$ defined on $D(T)$ is self-adjoint in $L^2(\mathbb{R}^s)$ with compact resolvent. In Proposition 2.6 below we will prove the estimate

$$|\langle\psi_{\underline{m}}, V\psi_{\underline{n}}\rangle_{L^2(\mathbb{R}^s)}| \le e^{-\Sigma(|\underline{m}|+|\underline{n}|)},$$

with $\Sigma > 0$ independent of $(\underline{m},\underline{n})$. Hence, (A4) is fulfilled in this example.

2. Proof of the result

We recall the basic definitions of (Rayleigh–Schrödinger) eigenvalue perturbation theory [Kato 1976, §II.1.5 and §§VII.1–3] in this context. Consider an eigenvalue $\lambda_{\underline{n}}$, $\underline{n} \in \mathbb{Z}^s$, of T corresponding to the (normalized) eigenvector $\psi_{\underline{n}}$, henceforth abbreviated $|\underline{n}\rangle$.

- $P_{\underline{n}}$ denotes the (one-dimensional) orthogonal projection operator from X to the one-dimensional subspace spanned by $|\underline{n}\rangle$.
- S denotes the reduced resolvent of T, i.e., $S(\lambda) := \displaystyle\sum_{\underline{k}\neq\underline{n}\in\mathbb{Z}^s}\frac{P_{\underline{k}}}{\lambda_{\underline{k}}-\lambda}$.
- $S_{\underline{n}}$ is the reduced resolvent evaluated at $\lambda = \lambda_{\underline{n}}$, i.e., $S_{\underline{n}} := \displaystyle\sum_{\underline{k}\neq\underline{n}\in\mathbb{Z}^s}\frac{P_{\underline{k}}}{\lambda_{\underline{k}}-\lambda_{\underline{n}}}$.
- $$[S_{\underline{n}}]^\ell := \sum_{\underline{k}\neq\underline{n}\in\mathbb{Z}^s}\frac{P_{\underline{k}}}{(\lambda_{\underline{k}}-\lambda_{\underline{n}})^\ell}, \qquad \ell \ge 1, \qquad [S_{\underline{n}}]^0 = P_{\underline{n}}. \tag{2-1}$$

In this situation we can use the explicit expressions (2.32)–(2.33) in [Kato 1976, Chapter II] for the coefficients $B_\ell(\underline{n})$ of the perturbation series (1-7):

$$B_\ell(\underline{n}) = \sum_{p=1}^{\ell}\frac{(-1)^p}{p}\sum_{\substack{k_1+\cdots+k_p=p-1\\ k_i=0,1,\dots\\ i=1,\dots,p-1}}\operatorname{Tr}[V[S_{\underline{n}}]^{k_1}V\cdots V[S_{\underline{n}}]^{k_p}]. \tag{2-2}$$

Since

$$[S_{\underline{n}}]^{k_p}|\underline{n}\rangle = 0, \quad k_p > 0, \qquad [S_{\underline{n}}]^0|\underline{n}\rangle = |\underline{n}\rangle,$$
$$\operatorname{Tr}[V[S_{\underline{n}}]^{k_1}V\cdots V[S_{\underline{n}}]^{k_p}] = \langle n, V[S_{\underline{n}}]^{k_1}V\cdots V^{k_p}n\rangle, \tag{2-3}$$

(2-2) becomes

$$B_\ell(\underline{n}) = \sum_{p=1}^{\ell} \frac{(-1)^p}{p} B_{\ell,p}(\underline{n}), \tag{2-4}$$

where

$$B_{\ell,p}(\underline{n}) = \sum_{k_1+\cdots+k_{p-1}=p-1} \langle \underline{n}, V[S_{\underline{n}}]^{k_1} V \cdots V[S_{\underline{n}}]^{k_{p-1}} V\underline{n}\rangle. \tag{2-5}$$

Remark 2.1. For $\ell = 2$, we have $p = 2$ and the above formulas yield the standard second-order term of the Rayleigh–Schrd̈inger expansion:

$$B_{2,2}(\underline{n}) = \sum_{\underline{k}\neq\underline{n}} \frac{|\langle \underline{n}, V\underline{k}\rangle|^2}{\lambda_{\underline{k}} - \lambda_{\underline{n}}}.$$

In the absence of a condition controlling the vanishing of $|\lambda_{\underline{k}} - \lambda_{\underline{n}}|$ as $|\underline{k} - \underline{n}| \to \infty$, the above series can of course diverge even if

$$\sup_{\underline{n}\in\mathbb{N}^s} \sum_{\underline{k}\in\mathbb{N}^s} \langle |\underline{n}, V\underline{k}|\rangle^2 < +\infty.$$

On the other hand, this last inequality is implied by the standard Schur condition ensuring the boundedness of V^2 and hence of V.

We can rewrite in more detail the factor $\langle \underline{n}, V S_{\underline{n}}^{k_1} \cdots V S_{\underline{n}}^{k_{p-1}} V\underline{n}\rangle$ making explicit all factors with at least one $k_i = 0$. Namely, consider the q-partition

$$p - 1 = j_1 + \cdots + j_q, \quad j_s \geq 1,\ 1 \leq q \leq p-1. \tag{2-6}$$

The number of the q-partitions is (see, e.g., [Andrews 1976]) $N(p,q) = \binom{p-1}{q-1}$. Furthermore, let $m(p,q) = p-1-q$ be the total number of zeros in the q-partition of $(k_1, \ldots, k_{p-1})$. We can thus rearrange the sequence $(k_1, \ldots, k_{p-1})$ as

$$k_1+\cdots+k_{p-1} = p-1 = m_1+j_1+m_2+j_2+m_q+j_q, \quad m = m_1+\cdots+m_q. \tag{2-7}$$

As a consequence,

$$\begin{aligned} V[S_{\underline{n}}]^{k_1} \cdots V[S_{\underline{n}}]^{k_{p-1}} V = \sum_{q=1}^{(p-1)} N(p,q) \prod_{h=1}^{m_1} [V P_{\underline{n}}]^h V[S_{\underline{n}}]^{j_1} \\ \times \prod_{h=1}^{m_2} [V P_{\underline{n}}]^h \cdot V[S_{\underline{n}}]^{j_2} \cdots \prod_{h=1}^{m_q} [V P_{\underline{n}}]^h \cdot V[S_{\underline{n}}]^{j_q} V, \end{aligned} \tag{2-8}$$

whence:

Lemma 2.2. *Let $k_1, \dots, k_p, m_1, \dots, m_q, j_1, \dots, j_q$ be related through (2-7). Then*

$$B_{\ell,p}(\underline{n}) = \sum_{k_1+\cdots+k_{p-1}=p-1} \langle \underline{n}, V[S_{\underline{n}}]^{k_1} \cdots V[S_{\underline{n}}]^{k_{p-1}} V\underline{n}\rangle$$

$$= \sum_{q=1}^{p-1} N(p,q)\langle \underline{n}, V\underline{n}\rangle^{m_1+\cdots+m_q} \sum_{\substack{\underline{r}_1\neq n,\dots,\underline{r}_q\neq \underline{n}\\ j_1+\cdots+j_q=p-1}} \frac{|\langle \underline{n}, V\underline{r}_1\rangle|^2}{(\lambda_{\underline{r}_1}-\lambda_{\underline{n}})^{j_1}} \cdot \frac{|\langle \underline{n}, V\underline{r}_2\rangle|^2}{(\lambda_{\underline{r}_2}-\lambda_{\underline{n}})^{j_2}} \cdots \frac{|\langle \underline{n}, V\underline{r}_q\rangle|^2}{(\lambda_{\underline{r}_q}-\lambda_{\underline{n}})^{j_q}}$$

$$= \sum_{q=1}^{p-1} N(p,q)\langle \underline{n}, V\underline{n}\rangle^{p-1-q} \sum_{\substack{\underline{r}_1\neq \underline{n},\dots,\underline{r}_q\neq \underline{n}\\ j_1+\cdots+j_q=p-1}} \frac{|\langle \underline{n}, V\underline{r}_1\rangle|^2}{(\lambda_{\underline{r}_1}-\lambda_{\underline{n}})^{j_1}} \cdot \frac{|\langle \underline{n}, V\underline{r}_2\rangle|^2}{(\lambda_{\underline{r}_2}-\lambda_{\underline{n}})^{j_2}} \cdots \frac{|\langle \underline{n}, V\underline{r}_q\rangle|^2}{(\lambda_{\underline{r}_q}-\lambda_{\underline{n}})^{j_q}}.$$

Proof. The product (2-8) is unchanged if $P_{\underline{n}}$ is replaced by $P_{\underline{n}}^2$. Recalling that

$$S^j = \sum_{\underline{r}\neq \underline{n}}^{\infty} \frac{P_{\underline{r}}}{(\lambda_{\underline{r}} - \lambda_{\underline{n}})^j}$$

and denoting

$$\Omega_1(m, j; n) := \prod_{h=1}^{m} [V P_{\underline{n}}^2]^h V S^j,$$

$$\Omega_2(m-1, j; n) := \prod_{h=1}^{m-1} [V P_{\underline{n}}^2]^h V P_{\underline{n}} P_{\underline{n}} V S^j,$$

$$\Omega_3(m-1, j; n, r) := \prod_{h=1}^{m-1} [V P_{\underline{n}}^2]^h V P_{\underline{n}} P_{\underline{n}} V P_{\underline{r}} P_{\underline{r}},$$

this yields

$$B_{\ell,p}(\underline{n}) = \sum_{q=1}^{p-1} N(p,q)\langle \underline{n}, \Omega_1(m_1, j_1; n) \cdots \Omega_1(m_q, j_q; n) V\underline{n}\rangle$$

$$= \sum_{q=1}^{p-1} N(p,q)\Big\langle \underline{n}, \prod_{h=1}^{m_1-1} \Omega_2(m_1-1, j_1; n) \cdots \Omega_2(m_q-1, j_q; n) V\underline{n}\Big\rangle$$

$$= \sum_{q=1}^{p-1} N(p,q)(\lambda_{\underline{r}_1} - \lambda_{\underline{n}})^{-j_1} \cdot (\lambda_{\underline{r}_2} - \lambda_{\underline{n}})^{-j_2} \cdots (\lambda_{\underline{r}_q} - \lambda_{\underline{n}})^{-} j_q$$

$$\times \sum_{\substack{\underline{r}_1\neq \underline{n},\dots,\underline{r}_q\neq \underline{n}\\ j_1+\cdots+j_q=p-1}} \langle \underline{n}, \Omega_3(m_1-1, j_1; n, \underline{r}_1) \cdots \Omega_3(m_q-1, j_q; n, \underline{r}_q) V\underline{n}\rangle.$$

Hence, by (2-8),

$$\sum_{k_1+\cdots+k_{p-1}=p-1} \langle n, V[S_{\underline{n}}]^{k_1} \cdots V[S_{\underline{n}}]^{k_{p-1}} V n\rangle$$

$$= \sum_{q=1}^{p-1} N(p,q) \Big\langle \underline{n}, \prod_{h=1}^{m_1-1} [V P_{\underline{n}}^2]^h V P_{\underline{n}} P_{\underline{n}} V P_{\underline{r}_1} P_{\underline{r}_1} \times \prod_{h=1}^{m_2-1} [V P_{\underline{n}}^2]^h V P_{\underline{n}} P_{\underline{n}} \cdot V P_{\underline{r}_2} P_{\underline{r}_2} \cdots \prod_{h=1}^{m_q} V P_{\underline{r}_q} P \underline{r}_q V \underline{n} \Big\rangle$$

$$= \sum_{q=1}^{p-1} N(p,q) \langle \underline{n}, V\underline{n}\rangle^{m_1} \cdots \langle \underline{n}, V\underline{n}\rangle^{m_q} \times \sum_{\underline{r}_1 \neq \underline{n}, \ldots, \underline{r}_q \neq \underline{n}} \frac{\langle \underline{n}, V\underline{r}_1\rangle\langle \underline{r}_1, V\underline{n}\rangle}{(\lambda_{\underline{r}_1} - \lambda_{\underline{n}})^{j_1}} \cdot \frac{\langle \underline{n}, V\underline{r}_2\rangle\langle \underline{r}_2, V\underline{n}\rangle}{(\lambda_{\underline{r}_2} - \lambda_{\underline{n}})^{j_2}} \cdots \frac{\langle \underline{n}, V\underline{r}_q\rangle\langle \underline{r}_q, V\underline{n}\rangle}{(\lambda_{\underline{r}_q} - \lambda_{\underline{n}})^{j_q}}$$

$$= \sum_{q=1}^{p-1} N(p,q) \langle \underline{n}, V\underline{n}\rangle^{p-1-q} \sum_{\substack{r_1 \neq \underline{n}, \ldots, r_q \neq \underline{n} \\ j_1+\cdots+j_q=p-1}} \frac{|\langle \underline{n}, V\underline{r}_1\rangle|^2}{(\lambda_{\underline{r}_1} - \lambda_{\underline{n}})^{j_1}} \cdot \frac{|\langle \underline{n}, V\underline{r}_2\rangle|^2}{(\lambda_{\underline{r}_2} - \lambda_{\underline{n}})^{j_2}} \cdots \frac{|\langle \underline{n}, V\underline{r}_q\rangle|^2}{(\lambda_{\underline{r}_q} - \lambda_{\underline{n}})^{j_q}}.$$

This concludes the proof. □

The first step in estimating the coefficients $B_\ell(n)$ is therefore estimating the fractions $|\langle \underline{n}, V\underline{r}\rangle|^2/(\lambda_{\underline{r}} - \lambda_{\underline{n}})^j$. In turn, this requires an analysis of the vanishing mechanism of the denominators $(\lambda_{\underline{r}} - \lambda_{\underline{n}})^j$. A preliminary remark is:

Lemma 2.3. *With the assumptions of Lemma 2.2*,

$$\sum_{\underline{r} \neq \underline{n} \in \mathbb{Z}^s} \frac{|\langle \underline{n}, V\underline{r}\rangle^2|}{|\lambda_{\underline{r}} - \lambda_{\underline{n}}|^j} \leq \frac{pA}{\alpha^p} \left(\frac{p\Lambda}{\alpha^\gamma}\right)^j (\gamma j)!. \tag{2-9}$$

Proof. Equation (2-9) is a direct consequence of assumptions (A3) and (A4) because

$$\sum_{\underline{r} \neq \underline{n} \in \mathbb{Z}^s} \frac{|\langle \underline{n}, V\underline{r}\rangle^2|}{|\lambda_{\underline{r}} - \lambda_{\underline{n}}|^j} \leq A\Lambda^j \sum_{\underline{r} \neq \underline{n} \in \mathbb{Z}^s} |\underline{r} - \underline{n}|^{\gamma j} e^{-\alpha|\underline{r}-\underline{n}|} = A\Lambda^j \sum_{\underline{x} \neq 0} |\underline{x}|^{\gamma j} e^{-\alpha|\underline{x}|},$$

where

$$\sum_{\underline{x} \neq 0} |\underline{x}|^{\gamma j} e^{-\alpha|\underline{x}|} := \sum_{\underline{x} \neq 0} [|x_1|^\gamma + \cdots + |x_p|^\gamma]^j e^{-\alpha|\underline{x}|} \leq p^j (|x_1|^{\gamma j} + \cdots + |x_p|^{\gamma j}) e^{-\alpha|\underline{x}|}.$$

Hence,

$$\sum_{\underline{x} \neq 0} |\underline{x}|^{\gamma j} e^{-\alpha|\underline{x}|} \leq p^{j+1} \sum_{x_1 \neq 0} |x_1|^{\gamma j} e^{-\alpha|\underline{x}|} \leq \frac{p^{j+1} (\gamma j)!}{\alpha^{\gamma j + p}}$$

and summing up we get

$$\sum_{\underline{r}\neq\underline{n}\in\mathbb{Z}^s}\frac{|\langle\underline{n},V\underline{r}\rangle^2|}{|\lambda_{\underline{r}}-\lambda_{\underline{n}}|^j}\leq\frac{pA}{\alpha^p}\left(\frac{p\Lambda}{\alpha^\gamma}\right)^j(\gamma j)!,$$

and this proves the lemma. □

Corollary 2.4. *Recalling that* $k_1+\cdots+k_{p-1}=p-1$, $p=2,\ldots,$ *in the assumptions of Lemma 2.3, the following bounds hold*:

$$\begin{aligned}|B_{\ell,p}(\underline{n})|&\leq\sum_{k_1+\cdots+k_{p-1}=p-1}|\langle n,V[S_{\underline{n}}]^{k_1}\cdots V[S_{\underline{n}}]^{k_{p-1}}Vn\rangle|\\&\leq p\left(\frac{2A}{\alpha}\right)^p\left(\frac{p\Lambda}{\alpha^\gamma}\right)^{p-1}\sum_{q=1}^{p-1}p^q\sum_{j_1+\cdots+j_q=p-1}(\gamma j_1)!\cdots(\gamma j_q)!.\end{aligned}\tag{2-10}$$

Proof. It is enough to insert (2-9) in the statement of Lemma 2.2 on account of the bounds $N(p,q)<2^p$, $|\langle\underline{n},V\underline{n}\rangle|\leq 1$ and the fact that $j_1+\cdots+j_q=p-1$. □

We can now state and prove the main estimate.

Proposition 2.5. *Under assumptions* (A1)–(A3),

$$|B_\ell(\underline{n})|\leq(4\ell^2)^{\ell+1}R(\Lambda,\alpha,\gamma)^\ell[\gamma(\ell-1)]!.\tag{2-11}$$

Proof. We have, by (2-4),

$$|B_\ell(\underline{n})|\leq\sum_{p=1}^{\ell}\frac{B_{\ell,p}(\underline{n})}{p}.$$

Clearly,

$$\sum_{j_1+\cdots+j_q=p-1}(\gamma j_1)!\cdots(\gamma j_q)!\leq(p-1)[\gamma(p-1)]!.$$

Moreover,

$$\sum_{q=1}^{p-1}p^q\leq p^p.$$

Therefore, by Corollary 2.4,

$$|B_{\ell,p}(\underline{n})|<(2p)^p(p-1)p\left(\frac{2A}{\alpha}\right)^p\left(\frac{p\Lambda}{\alpha^\gamma}\right)^{p-1}[\gamma(p-1)]!,$$

whence since $A<\alpha$

$$\begin{aligned}|B_\ell(\underline{n})|&\leq\sum_{p=1}^{\ell}B_{\ell,p}(\underline{n})\leq(2\ell)^\ell\ell^3 2^\ell\ell^{\ell-1}\left(\frac{\Lambda}{\alpha^\gamma}\right)^{\ell-1}[\gamma(\ell-1)]!\\&\leq(4\ell^2)^{\ell+1}R(\Lambda,\alpha,\gamma)^\ell[\gamma(\ell-1)]!,\end{aligned}$$

where

$$R(\Lambda, \alpha, \gamma) := \frac{\Lambda}{\alpha^{\gamma}}. \tag{2-12}$$

Thus, the proof of the proposition is complete. □

Proof of Theorem 1.1. The assertion is just (2-11). □

Proof of Corollary 1.2. The validity of (1-11) is a direct consequence of the existence of the perturbation expansion for all $\underline{n} \in \mathbb{N}^p$. The uniformity with respect to $\underline{n}$ follows from the $\underline{n}$-independent bound (1-10). □

Example 1.5 (continued). Consider again the normalized eigenvectors of T:

$$\psi_{\underline{n}}(\underline{x}) = \prod_{k=1}^{s} \psi_{n_k}(x_k), \qquad \psi_n(x) := \frac{1}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x),\ n = 0, 1, \dots, \tag{2-13}$$

where $H_n(x)$, $x \in \mathbb{R}$, is the n-th Hermite polynomial. The vectors $\{\psi_{\underline{n}}(\underline{x}) : \underline{n} \in \mathbb{N}^s\}$ form an orthonormal basis in $L^2(\mathbb{R}^s)$. Recall that

$$\mathcal{V}(\underline{x}) := e^{-|\underline{x}|^q} \Phi(\underline{x}).$$

Consequently, considering the potential $\mathcal{V}(\underline{x})$ and the corresponding maximal multiplication operator V in $L^2(\mathbb{R}^s)$, we have:

Proposition 2.6. *Condition (A4) is fulfilled in this example*; *i.e., there is* $\Sigma(q) > 0$ *such that*

$$\langle \psi_{\underline{m}}, V\psi_{\underline{n}} \rangle := \langle \underline{m}, \mathcal{V}(\underline{x})\underline{n} \rangle \le e^{-\Sigma[|\underline{m}|+|\underline{n}|]}. \tag{2-14}$$

Proof. Consider first the case $s = 1$. Recall the formula

$$H_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^{n-2k}}{k!\,(n-2k)!},$$

where as usual $[n/2]$ is the integer part of n. We can thus write

$$\psi_m(x)\psi_n(x) = \sqrt{2^{-(m+n)/2} m!\, n!} \sum_{h=0}^{[m/2]} \sum_{k=0}^{[n/2]} \frac{(-1)^{h+k} (2x)^{m-2h} (2x)^{n-2k}}{h!\,(m-2h)!\,k!\,(n-2k)!},$$

whence

$$\begin{aligned} |\langle m, \mathcal{V}n \rangle| &\le \sqrt{2^{(m+n)/2} m!\, n!} \sum_{h=0}^{[m/2]} \sum_{k=0}^{[n/2]} \frac{\int_R x^{m+n-2(h+k)} e^{-|x|^q}\, dx}{h!\,(m-2h)!\,k!\,(n-2k)!} \\ &< \sqrt{2^{(m+n)/2} m!\, n!} \sum_{h=0}^{[m/2]} \sum_{k=0}^{[n/2]} \frac{\Gamma([m+n-2(h+k)+1]/q)}{h!\,(m-2h)!\,k!\,(n-2k)!}. \end{aligned}$$

Now,

$$\min_{0\leq h\leq [m/2]}[h!\,(m-2h)!]=\Gamma(m/3+1)^2,$$

$$\min_{0\leq k\leq [n/2]}[h!\,(n-2k)!]=\Gamma(n/3+1)^2,$$

$$\max_{\substack{0\leq h\leq [m/2]\\ 0\leq k\leq [n/2]}}[\Gamma([m+n-2(h+k)+1]/q)]=\Gamma([m+n+1]/q),$$

and this implies

$$|\langle m,\mathcal{V}n\rangle|\leq\sqrt{2^{(m+n)/2}m!\,n!}\cdot\left[\frac{m}{2}\right]\cdot\left[\frac{n}{2}\right]\frac{\Gamma([m+n+1]/q)}{\Gamma(m/3+1)^2\cdot\Gamma(n/3+1)^2}. \tag{2-15}$$

Now apply the Stirling formula. Since

$$6\pi\,2^{(m+n)/4}e^{-(m+n)/2}e^{-[m+n+1]/q}e^{(m+n)/3}\cdot\left[\frac{m}{2}\right]\cdot\left[\frac{n}{2}\right]\leq 1$$

for $m+n$ large enough,

$$|\langle m,\mathcal{V}n\rangle|\leq\frac{((m+q)/q)^{(m+q)/q}}{m^{m/6}n^{n/6}}.$$

Without loss, we can take $m=n+k$, $k\geq 0$. Then

$$|\langle m,\mathcal{V}n\rangle|=|\langle n,\mathcal{V}(n+k)\rangle|\leq\frac{[(2n+k)/q]^{(2n+k)/q}}{(n+k)^{(n+k)/6}n^{n/6}}.$$

Now $(n+k)>(2n+k)/q>(n+k)/6$ if $2<q<6$ and hence there is $0<L<1$ such that

$$|\langle n,\mathcal{V}(n+k)\rangle|\leq\frac{[(2n+k)/q]^{(n+k)/6}}{(n+k)^{(n+k)/6}n^{n/6}}<\frac{L^{n+k}}{n^{n/6}}=\frac{L^m}{n^{n/6}},$$

whence, a fortiori, with $L=e^{-\Sigma}$

$$|\langle m,\mathcal{V}n\rangle|\leq e^{-\Sigma(m+n)}.$$

This concludes the proof for $s=1$. The general case follows through an immediate product argument. □

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