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CORRELATION INEQUALITIES FOR THE POTTS MODEL

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Dedicated in friendship to Lucio Russo

Correlation inequalities are presented for ferromagnetic Potts models with external field, using the random-cluster representation of Fortuin and Kasteleyn, together with the FKG inequality. These results extend and simplify earlier inequalities of Ganikhodjaev and Razak, and also of Schonmann, and include GKS-type inequalities when the spin space is taken as the set of q-th roots of unity.

1. Introduction

Correlation inequalities are key to the classical theory of interacting systems in statistical mechanics. The Ising model, especially, has a plethora of associated inequalities that have played significant roles in the development of a coherent theory of phase transition (see, for example, the books [7; 22]). These inequalities are frequently named after their discoverers, and include inequalities of Griffiths [14; 15; 16], Griffiths, Kelly, and Sherman (GKS) [20], Griffiths, Hurst, and Sherman (GHS) [17], Ginibre [13], Simon and Lieb [21; 24], and so on.

A more probabilistic theory of Ising/Potts models has emerged since around 1970, initiated partly by the work of Fortuin and Kasteleyn [8; 9; 10] on the *random-cluster* representation of the Potts model and the *random-current method* championed by Aizenman [1] and co-authors. Probably the principle inequality in the probabilistic formulation is that of Fortuin, Kasteleyn, and Ginibre (FKG) [11].

Inequalities are rarer for the Potts model, and our purpose in this note is to derive certain correlation inequalities for a ferromagnetic Potts model with external field, akin to the GKS inequalities for the Ising model. The main technique used here is the random-cluster representation of this model and particularly the FKG inequality.

Our results generalize and simplify the work of Ganikhodjaev and Razak [12], who have shown how to formulate and prove GKS-type inequalities for the Potts

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model with a general number q of local states. Furthermore, our Theorems 3.5 and 3.7 extend the two correlation inequalities of Schonmann [23], which in turn extended inequalities of [6]. Some of the arguments given here may be known to others.

The structure of this paper is as follows. The Potts and random-cluster models are introduced in Section 2, and the results of the paper (Theorems 3.5–3.7) follow in Section 3. The proofs are given in Sections 4, 5, and 6.

2. The Potts model with external field

Let $G = (V, E)$ be a finite graph, and let $J = (J_e : e \in E)$ and $h = (h_v : v \in V)$ be vectors of nonnegative reals and $q \in \{2, 3, \dots\}$. An edge $e \in E$ joins two distinct vertices x and y, and we write $e = \langle x, y\rangle$.

We take the "local state space" for the q-state Potts model to be the set $\mathcal{Q} := \{0, 1, \dots, q-1\}$ of "spins". The configuration space of the model is the product space $\Sigma := \mathcal{Q}^V$, and a typical configuration is written $\sigma = (\sigma_v : v \in V) \in \Sigma$. The Potts measure on G with parameters J and h has sample space Σ and probability measure given by

$$\pi(\sigma) = \frac{1}{Z} \exp\left\{ \sum_{e=\langle x,y\rangle \in E} J_e \delta_e(\sigma) + \sum_{v \in V} h_v \delta_v(\sigma) \right\}, \quad \sigma \in \Sigma,$$

where $\delta_e(\sigma) = \delta_{\sigma_x,\sigma_y}$ and $\delta_v(\sigma) = \delta_{\sigma_v,0}$ are Kronecker delta functions and Z is the appropriate normalizing constant. Thus, the J_e are *edge-coupling constants*, and the h_v are *external fields* relative to the local state 0. The Potts measure is said to be *ferromagnetic* since $J_e \ge 0$ for $e \in E$.

We shall make use of the random-cluster representation, for a recent account and bibliography of which we refer the reader to [18]. The graph G is augmented by adding a "ghost" vertex g, which is joined by edges $\langle g, v\rangle$ to each vertex $v \in V$; the ensuing graph is denoted $G^+ = (V^+, E^+)$. The relevant sample space is the product space $\Omega := \{0, 1\}^{E^+}$. For $\omega = (\omega_e : e \in E^+) \in \Omega$, an edge e is called *open* if $\omega_e = 1$ and *closed* otherwise.

An edge $e \in E$ is assigned parameter $p_e = 1 - e^{-J_e}$, and an edge of the form $\langle g, v\rangle$ is assigned parameter $p_v = 1 - e^{-h_v}$. The random-cluster probability measure ϕ on G has sample space Ω and is given by

$$\phi(\omega) = \frac{1}{Z_{\mathrm{RC}}} \left\{ \prod_{e=\langle x,y\rangle \in E^+} p_e^{\omega_e} (1 - p_e)^{1-\omega_e} \right\} q^{k(\omega)}, \quad \omega \in \Omega,$$

where $k(\omega)$ is the number of connected components of the graph with vertex set V^+ and edge set $\eta(\omega) := \{e \in E^+ : \omega_e = 1\}$.

The relationship between the Potts model and the random-cluster model is explained in [18, §1.4], where it is shown in particular that $Z_{\mathrm{RC}} = e^{-|E|} Z$.

The measures π and ϕ may be coupled as follows. Suppose ω is sampled from Ω according to ϕ, and let C_v be the connected component of $(V, \eta(\omega))$ containing $v \in V^+$; the C_v are called *open clusters*. Every vertex in C_g is allocated spin 0. To an open cluster of ω other than C_g, we allocate a uniformly chosen spin from $\mathcal{Q}$ such that every vertex in the cluster receives this spin and the spins of different clusters are independent. The ensuing spin vector $\sigma = \sigma(\omega)$ has law π. See [18, Theorem 1.3] for a proof of this standard fact and for references to the original work of Fortuin and Kasteleyn.

This paper will make use of the FKG inequality and the comparison inequalities for the random-cluster model. These are presented in a number of places already and are not repeated here. The reader is referred instead to [18, Theorem 3.8] for the FKG inequality and to [18, Theorem 3.21] for the comparison inequalities.

3. The correlation inequalities

We begin with a space of functions. Let $\mathcal{F}_q$ be the set of functions $f : \mathcal{Q} \to \mathbb{C}$ such that, for all integers $m, n \geq 0$,

$$\mathbb{E}(f(X)^m) \text{ is real and nonnegative,} \tag{3.1}$$

$$\mathbb{E}(f(X)^{m+n}) \geq \mathbb{E}(f(X)^m)\mathbb{E}(f(X)^n), \tag{3.2}$$

where X is a uniformly distributed random variable on $\mathcal{Q}$. The above conditions may be written out as follows. We have that $f \in \mathcal{F}_q$ if, for $m, n \geq 0$,

$$S_m := \sum_{x \in \mathcal{Q}} f(x)^m \text{ is real and nonnegative,}$$

$$q S_{m+n} \geq S_m S_n.$$

For $I \in \mathcal{Q}$, let $\mathcal{F}_q^I$ be the subset of $\mathcal{F}_q$ containing all f such that

$$f(I) = \max\{|f(x)| : x \in \mathcal{Q}\}. \tag{3.3}$$

This condition entails that $f(I)$ is real and nonnegative.

Let $f : \mathcal{Q} \to \mathbb{C}$. For $\sigma \in \Sigma$, let

$$f(\sigma)^R := \prod_{v \in R} f(\sigma_v), \qquad R \subseteq V. \tag{3.4}$$

Thinking of σ as a random vector with law π, we write $\langle f(\sigma)^R \rangle$ for the mean value of $f(\sigma)^R$.

Theorem 3.5. *Let $f \in \mathcal{F}_q^0$. For $R \subseteq V$, the mean $\langle f(\sigma)^R \rangle$ is real-valued and nondecreasing in the vectors J and h and satisfies $\langle f(\sigma)^R \rangle \geq 0$. For $R, S \subseteq V$, we have*

$$\langle f(\sigma)^R f(\sigma)^S \rangle \geq \langle f(\sigma)^R \rangle \langle f(\sigma)^S \rangle.$$

If there is no external field, in that $h \equiv 0$, it suffices for the above that $f \in \mathcal{F}_q$ in place of $f \in \mathcal{F}_q^0$.

Here are three classes of functions belonging to $\mathcal{F}_q^0$.

Theorem 3.6. *Let $q \geq 2$. The following functions $f : \mathcal{Q} \to \mathbb{C}$ belong to $\mathcal{F}_q^0$:*

(a) $f(x) = \frac{1}{2}(q-1) - x$,

(b) $f(x) = e^{2\pi i x/q}$, *a q-th root of unity, and*

(c) $f : \mathcal{Q} \to [0, \infty)$, *with $f(x) \leq f(0)$ for $x \in \mathcal{Q}$.*

When combined with Theorem 3.5, case (a) yields the inequalities of Ganikhodjaev and Razak [12], but with simpler proofs. When $q = 2$, the latter reduce to the GKS inequalities for the Ising model; see [14; 15; 16; 20]. We do not know if the implications of Theorem 3.5 with case (b) are either known or useful. Perhaps they are examples of the results of Ginibre [13]. In case (c) with $f(x) = \delta_{x,0}$, Theorem 3.5 yields the first correlation inequality of Schonmann [23].

Our second main result follows next.

Theorem 3.7. *Let $q \geq 2$ and $f_0 \in \mathcal{F}_q^0$, and let $f_1 : \mathcal{Q} \to \mathbb{C}$ satisfy (3.1). If f_0 and f_1 have disjoint support in that $f_0 f_1 \equiv 0$, then for $R, S \subseteq V$,*

$$\langle f_0(\sigma)^R f_1(\sigma)^S \rangle \leq \langle f_0(\sigma)^R \rangle \langle f_1(\sigma)^S \rangle.$$

If $h \equiv 0$, it is enough to assume $f_0 \in \mathcal{F}_q$ in place of $f_0 \in \mathcal{F}_q^0$.

Two correlation inequalities were proved in [23]: a "positive" inequality that is implied by Theorems 3.5 and 3.6(c) and a "negative" inequality that is obtained as a special case of Theorem 3.7 on setting $f_0(x) = \delta_{x,0}$ and $f_1(x) = \delta_{x,1}$. Recall that Schonmann's inequalities were themselves (partial) generalizations of correlation inequalities of [6].

Amongst the feasible extensions of the above theorems that come to mind, we mention the classical space-time models used to study the quantum Ising/Potts models [2; 3; 4; 5; 19].

4. Proof of Theorem 3.5

We use the coupling of the random-cluster and Potts model described in Section 2. Let $\omega \in \Omega$, and let $A_g, A_1, A_2, \ldots, A_k$ be the vertex sets of the open clusters of ω, where A_g is that of the open cluster C_g containing g.

Let $R \subseteq V$, and let $f \in \mathcal{F}_q^0$. By (3.4),

$$f(\sigma)^R = f(0)^{|R \cap A_g|} \prod_{r=1}^{k} f(X_r)^{|R \cap A_r|},$$

where X_r is the random spin assigned to A_r. This has conditional expectation $g_R : \Omega \to \mathbb{C}$ given by

$$\begin{aligned} g_R(\omega) &:= \mathbb{E}(f(\sigma)^R \mid \omega) \\ &= f(0)^{|R \cap A_g|} \prod_{r=1}^{k} \mathbb{E}(f(X)^{|R \cap A_r|} \mid \omega). \end{aligned}$$

By (3.1) and (3.3), $g_R(\omega)$ is real and nonnegative, whence so is its mean $\phi(g_R) = \langle f(\sigma)^R \rangle$. (It will be convenient to use $\phi(Y)$ to denote the expectation of a random variable $Y : \Omega \to \mathbb{R}$.)

We show next that g_R is a nondecreasing function on the partially ordered set Ω. It suffices to consider the case when the configuration ω' is obtained from ω by adding an edge between two clusters of ω. In this case, by (3.2) and (3.3), $g_R(\omega') \geq g_R(\omega)$. That $\langle f(\sigma)^R \rangle = \phi(g_R)$ is nondecreasing in J and h follows by the appropriate comparison inequality for the random-cluster measure ϕ [18, Theorem 3.21].

Now,

$$\mathbb{E}(f(\sigma)^R f(\sigma)^S \mid \omega) = f(0)^{|R \cap A_g| + |S \cap A_g|} \prod_{r=1}^{k} \mathbb{E}(f(X)^{|R \cap A_r| + |S \cap A_r|} \mid \omega).$$

By (3.2),

$$\mathbb{E}(f(\sigma)^R f(\sigma)^S \mid \omega) \geq g_R(\omega) g_S(\omega).$$

By the FKG property of ϕ [18, Theorem 3.8],

$$\begin{aligned} \langle f(\sigma)^R f(\sigma)^S \rangle &= \phi(\mathbb{E}(f(\sigma)^R f(\sigma)^S \mid \omega)) \\ &\geq \langle f(\sigma)^R \rangle \langle f(\sigma)^S \rangle, \end{aligned}$$

as required.

When $h \equiv 0$, the terms in $f(0)$ do not appear in the above, and it therefore suffices that $f \in \mathcal{F}_q$.

5. Proof of Theorem 3.6

We shall use the elementary fact that, if T is a nonnegative random variable,

$$\mathbb{E}(T^{m+n}) \geq \mathbb{E}(T^m)\mathbb{E}(T^n), \quad m, n \geq 0. \tag{5.1}$$

This trivial inequality may be proved in several ways, one of which is the following. Let T_1 and T_2 be independent copies of T. Clearly,

$$(T_1^m - T_2^m)(T_1^n - T_2^n) \geq 0 \tag{5.2}$$

since either $0 \leq T_1 \leq T_2$ or $0 \leq T_2 \leq T_1$. Inequality (5.1) follows by multiplying out (5.2) and averaging.

(a) Inequality (3.3) with $I = 0$ is a triviality. Since $f(X)$ is real-valued, with the same distribution as $-f(X)$, $\mathbb{E}(f(X)^m) = 0$ when m is odd and is positive when m is even. When $m+n$ is even, (3.2) follows from (5.1) with $T = f(X)^2$, and both sides of (3.2) are 0 otherwise.

(b) An easy calculation shows that

$$\mathbb{E}(f(X)^m) = \begin{cases} 1 & \text{if } q \text{ divides } m, \\ 0 & \text{otherwise,} \end{cases}$$

and (3.1) and (3.2) follow.

(c) Inequality (3.2) follows by (5.1) with $T = f(X)$.

6. Proof of Theorem 3.7

We may as well assume that $f_0 \not\equiv 0$ so that $f_0(0) > 0$ and $f_1(0) = 0$. We use the notation of Section 4, and let $F_i : \Omega \to \mathbb{C}$ be given by

$$F_0(\omega) = f_0(0)^{|R\cap A_g|} \prod_{r=1}^{k} \mathbb{E}(f_0(X)^{|R\cap A_r|} \mid \omega), \tag{6.1}$$

$$F_1(\omega) = \prod_{r=1}^{k} \mathbb{E}(f_1(X)^{|S\cap A_r|} \mid \omega). \tag{6.2}$$

By (3.1), F_0 and F_1 are real-valued and nonnegative. Since $f_0 \in \mathcal{F}_q^0$, F_0 is nondecreasing (as in Section 4).

Since $f_0 f_1 \equiv 0$,

$$\mathbb{E}(f_0(\sigma)^R f_1(\sigma)^S \mid \omega) = 1_Z(\omega) F_0(\omega) F_1(\omega),$$

where 1_Z is the indicator function of the event $Z = \{S \nleftrightarrow R \cup \{g\}\}$. Here, as usual, we write $A \leftrightarrow B$ if there exists an open path in ω from some vertex of A to some vertex of B. Let T be the subset of V^+ containing all vertices joined to S by open paths, and write ω_T for the configuration ω restricted to T. Using conditional expectation,

$$\begin{aligned} \langle f_0(\sigma)^R f_1(\sigma)^S \rangle &= \phi(1_Z F_0 F_1) \\ &= \phi(1_Z F_1 \phi(F_0 \mid T, \omega_T)), \end{aligned} \tag{6.3}$$

where we have used the fact that 1_Z and F_1 are functions of the pair T, ω_T only. On the event Z, F_0 is a nondecreasing function of the configuration restricted to $V^+ \setminus T$. Furthermore, given T, the conditional measure on $V^+ \setminus T$ is the corresponding random-cluster measure. It follows that

$$\phi(F_0 \mid T, \omega_T) \le \phi(F_0) \quad \text{on the event } Z$$

by [18, Theorem 3.21]. By (6.3),

$$\begin{aligned} \langle f_0(\sigma)^R f_1(\sigma)^S \rangle &\le \phi(1_Z F_1 \phi(F_0)) \\ &\le \phi(F_0)\phi(F_1) \\ &= \langle f_0(\sigma)^R \rangle \langle f_1(\sigma)^S \rangle, \end{aligned}$$

and the theorem is proved.

When $h \equiv 0$, $A_g = \{g\}$ in (6.1), and it suffices that $f_0 \in \mathcal{F}_q$.

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