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INVOLVING THE LAPLACE-BELTRAMI OPERATOR**



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In this paper we prove an error estimate for a model of heat conduction in composite materials having a microscopic structure arranged in a periodic array and thermally active membranes separating the heat-conductive phases.

1. Introduction

Heat and electrical conduction in composite materials has been widely investigated in recent years in the context of homogenization theory (see among others, e.g., [Amar et al. 2017a; 2017b; 2003; 2004; 2006; 2010; Auriault and Ene 1994; Buniou and Timofte 2016; Donato and Monsurrò 2001; Hummel 2000; Jose 2009; Timofte 2013]). In this paper we will focus on the study of models of heat conduction in composite materials used for encapsulation of electronic devices. This topic is attracting increasing interest among researchers, both from the point of view of applications and also in a more mathematical setting. In our previous paper [Amar and Gianni 2018b] (to which we refer for a more detailed physical description of the problem) a composite medium was taken into account, which was made of a hosting material with inclusions separated from their surroundings by a thermally active membrane.

Such a situation is consistent with many physical applications in which a material must be modified in a way such that its thermal conductivity is enhanced while preserving other material properties, e.g., ductility. This is, as stated above, the case of polymer encapsulation of electronic devices as well as, just to give an example, engine coolants. Specifically, in the first case, ductility of the material is required to fill the voids and the interstices among the electrical components by applying a moderate pressure. Polymers and rubbers have this property, but they do not display a satisfactory heat dissipation which, on the other hand, can be attained

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by adding highly conductive nanoparticles. In some situations, these nanoparticles are enclosed in a membrane separating them from the surrounding medium. It is therefore only natural to investigate the influence of these membranes on the overall conductivity of the composite medium under different assumptions on the thermal behavior of these interfaces. The case of perfect or imperfect thermal contact, though interesting from the point of view of applications, is mathematically well known; for this reason we focused on the case in which the membrane is thermally active, e.g., a tangential heat diffusion takes place. In [Amar and Gianni 2018b] a macroscopic model was deduced, via the unfolding homogenization technique, assuming the periodicity of the microscopic structure, whose characteristic length is described by a small parameter ε . We make use of a sensible mathematical description of the behavior of the interfaces which are modeled by means of the Laplace–Beltrami operator (see, e.g., [Allaire et al. 1996; Andreucci et al. 2003]).

In this paper we complete the research started in [Amar and Gianni 2018b] providing an “error estimate” which enables us to evaluate the rate of convergence, with respect to $\varepsilon \rightarrow 0$, of the solution u_ε of the microscopic (physical) problem to the solution u_0 of the macroscopic one. More precisely, we prove

$$\begin{aligned} \|u_\varepsilon - (u_0 + \varepsilon u_1)\|_{L^2(0,T;H^1(\Omega))} &\leq \gamma \sqrt{\varepsilon}, \\ \|u_\varepsilon - u_0\|_{L^2(\Omega_T)} &\leq \gamma \sqrt{\varepsilon}, \end{aligned}$$

for a proper constant $\gamma > 0$ independent of ε , where u_1 is the so-called first corrector and is defined in (3-13).

To obtain this estimate we follow the classical approach given by the asymptotic expansions due to Bensoussan, Lions, and Papanicolaou [Bensoussan et al. 1978] which, under extra-regularity assumptions, gives an H^1 -estimate for this error. The knowledge of the rate of convergence is a crucial tool for numerical applications. Moreover, we prove the symmetry and the strict positivity of the matrix describing the diffusivity of the macroscopic (homogenized) material. This last result is crucial to guarantee the well-posedness of the parabolic limit equation.

Though the results proved in this paper are along the same lines as other ones obtained in the framework of the homogenization theory, they are nevertheless of some mathematical interest due to the presence of the Laplace–Beltrami operator, which makes the computations a bit tricky.

The paper is organized as follows. In Section 2 we recall the definitions and some properties of the tangential operators (gradient, divergence, and Laplace–Beltrami operator), state our geometrical setting, and present our model. In Section 3, after having proved some energy inequalities, we follow the formal approach by Bensoussan, Lions, and Papanicolaou in order to introduce the cell functions and to guess the limit equation, proving the ellipticity of its principal part (see Theorem 3.1).

Finally, in [Section 4](#) taking advantage of the asymptotic expansions obtained in [Section 3](#), we provide the error estimate (see [Theorem 4.1](#)).

2. Preliminaries

2.1. Tangential derivatives. Let ϕ be a C^2 -function, Φ a C^2 -vector function, and S a smooth surface with normal unit vector n . We recall that the tangential gradient of ϕ is given by

$$\nabla^B \phi = \nabla \phi - (n \cdot \nabla \phi)n \quad (2-1)$$

and the tangential divergence of Φ is given by

$$\begin{aligned} \operatorname{div}^B \Phi &= \operatorname{div} \Phi - (n \cdot \nabla \Phi_i)n_i - (\operatorname{div} n)(n \cdot \Phi) \\ &= \operatorname{div}^B(\Phi - (n \cdot \Phi)n) = \operatorname{div}(\Phi - (n \cdot \Phi)n), \end{aligned} \quad (2-2)$$

where, taking into account the smoothness of S , the normal vector n can be naturally defined in a small neighborhood of S as $\nabla d/|\nabla d|$, where d is the signed distance from S . Moreover, we define the Laplace–Beltrami operator as

$$\Delta^B \phi = \operatorname{div}^B(\nabla^B \phi), \quad (2-3)$$

so that by (2-1) and (2-2) we get that the Laplace–Beltrami operator can be written as

$$\begin{aligned} \Delta^B \phi &= \Delta \phi - n^t \nabla^2 \phi n - (n \cdot \nabla \phi) \operatorname{div} n \\ &= (\delta_{ij} - n_i n_j) \partial_{ij}^2 \phi - n_j \partial_j \phi \partial_i n_i = (I - n \otimes n)_{ij} \partial_{ij}^2 \phi - (n \cdot \nabla \phi) \operatorname{div} n, \end{aligned} \quad (2-4)$$

where $\nabla^2 \phi$ stands for the Hessian matrix of ϕ . Finally, we recall that on a regular surface S with no boundary (i.e., when $\partial S = \emptyset$) we have

$$\int_S \operatorname{div}^B \Phi \, d\sigma = 0. \quad (2-5)$$

2.2. Geometrical setting. The typical periodic geometrical setting is displayed in [Figure 1](#). Here we give, for the sake of clarity, its detailed formal definition.

Let us introduce a periodic open subset E of \mathbb{R}^N , so that $E + z = E$ for all $z \in \mathbb{Z}^N$. We employ the notation $Y = (0, 1)^N$, and $E_{\text{int}} = E \cap Y$, $E_{\text{out}} = Y \setminus \bar{E}$, and $\Gamma = \partial E \cap \bar{Y}$. We assume that E_{out} is connected and $\Gamma \cap \partial Y = \emptyset$.

Let Ω be an open connected bounded subset of \mathbb{R}^N ; for all $\varepsilon > 0$ define $\Omega_{\text{int}}^\varepsilon = \Omega \cap \varepsilon E$ and $\Omega_{\text{out}}^\varepsilon = \Omega \setminus \varepsilon \bar{E}$, so that $\Omega = \Omega_{\text{int}}^\varepsilon \cup \Omega_{\text{out}}^\varepsilon \cup \Gamma^\varepsilon$, where $\Omega_{\text{int}}^\varepsilon$ and $\Omega_{\text{out}}^\varepsilon$ are two disjoint open subsets of Ω , and $\Gamma^\varepsilon = \partial \Omega_{\text{int}}^\varepsilon \cap \Omega = \partial \Omega_{\text{out}}^\varepsilon \cap \Omega$. The regions $\Omega_{\text{out}}^\varepsilon$ and $\Omega_{\text{int}}^\varepsilon$ correspond to the outer phase and the inclusions, respectively, while Γ^ε is the interface. We also assume that Ω and E have regular boundary, and we stipulate that $\operatorname{dist}(\Gamma^\varepsilon, \partial \Omega) \geq \gamma_0 \varepsilon$, for a suitable $\gamma_0 > 0$. To this purpose, for each ε , we are ready to remove the inclusions in all the cells which are not completely contained in Ω (see [Figure 1](#)). This assumption is in accordance with our previous

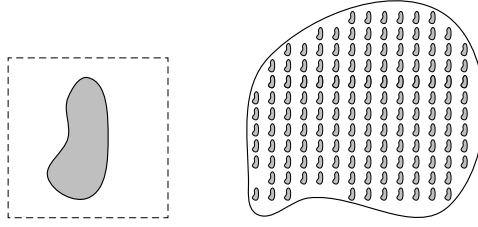


Figure 1. Left: the periodic cell Y . E_{int} is the shaded region, and E_{out} is the white region. Right: the region Ω .

papers [Amar et al. 2004; 2006; 2009a; 2009b; 2010; 2013], and maybe it can be dropped as in [Allaire and Murat 1993; Cioranescu et al. 2012]; nevertheless, we will not pursue this line of investigation in this paper.

Moreover, let ν denote the normal unit vector to Γ pointing into E_{out} , extended by periodicity to the whole of \mathbb{R}^N , so that $\nu_\varepsilon(x) = \nu(x/\varepsilon)$ denotes the normal unit vector to Γ^ε pointing into $\Omega_{\text{out}}^\varepsilon$.

Finally, given $T > 0$, we denote $\Omega_T = \Omega \times (0, T)$. More generally, for any spatial domain G , we denote $G_T = G \times (0, T)$.

2.3. Position of the problem. Let $\mu^\varepsilon, \lambda^\varepsilon : \Omega \rightarrow \mathbb{R}$ be defined as

$$\begin{aligned} \lambda^\varepsilon &= \lambda_{\text{int}} & \text{in } \Omega_{\text{int}}^\varepsilon, & & \lambda^\varepsilon &= \lambda_{\text{out}} & \text{in } \Omega_{\text{out}}^\varepsilon, \\ \mu^\varepsilon &= \mu_{\text{int}} & \text{in } \Omega_{\text{int}}^\varepsilon, & & \mu^\varepsilon &= \mu_{\text{out}} & \text{in } \Omega_{\text{out}}^\varepsilon. \end{aligned}$$

For every $\varepsilon > 0$, we consider the problem for $u_\varepsilon(x, t)$ given by

$$\mu^\varepsilon \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}(\lambda^\varepsilon \nabla u_\varepsilon) = 0 \quad \text{in } (\Omega_{\text{int}}^\varepsilon \cup \Omega_{\text{out}}^\varepsilon) \times (0, T), \quad (2-6)$$

$$[u_\varepsilon] = 0 \quad \text{on } \Gamma_T^\varepsilon, \quad (2-7)$$

$$\varepsilon \alpha \frac{\partial u_\varepsilon}{\partial t} - \varepsilon \beta \Delta^B u_\varepsilon = [\lambda^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] \quad \text{on } \Gamma_T^\varepsilon, \quad (2-8)$$

$$u_\varepsilon(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (2-9)$$

$$u_\varepsilon(x, 0) = \bar{u}_0(x) \quad \text{in } \Omega, \quad (2-10)$$

where we denote

$$[u_\varepsilon] = u_\varepsilon^{\text{out}} - u_\varepsilon^{\text{int}}, \quad (2-11)$$

and the same notation is employed also for other quantities. We assume that all the constants $\mu_{\text{int}}, \mu_{\text{out}}, \lambda_{\text{int}}, \lambda_{\text{out}}, \alpha, \beta$ involved in (2-6) and (2-8) are strictly positive.

Since problem (2-6)–(2-10) is not standard, in order to define a proper notion of weak solution, we will need to introduce some suitable function spaces. To this purpose and for later use, we will denote by $H_B^1(\Gamma^\varepsilon)$ the space of Lebesgue-measurable functions $u : \Gamma^\varepsilon \rightarrow \mathbb{R}$ such that $u \in L^2(\Gamma^\varepsilon)$ and $\nabla^B u \in L^2(\Gamma^\varepsilon)$. Let

us also set

$$\mathcal{X}_0^\varepsilon(\Omega) := H_0^1(\Omega) \cap H_B^1(\Gamma^\varepsilon). \quad (2-12)$$

Definition 2.1. We say that $u_\varepsilon \in L^2(0, T; \mathcal{X}_0^\varepsilon(\Omega))$ is a weak solution of problem (2-6)–(2-10) if

$$\begin{aligned} & - \int_0^T \int_\Omega \mu^\varepsilon u_\varepsilon \frac{\partial \phi}{\partial \tau} dx d\tau + \int_0^T \int_\Omega \lambda^\varepsilon \nabla u_\varepsilon \cdot \nabla \phi dx d\tau \\ & - \varepsilon \alpha \int_0^T \int_{\Gamma^\varepsilon} u_\varepsilon \frac{\partial \phi}{\partial \tau} d\sigma d\tau + \varepsilon \beta \int_0^T \int_{\Gamma^\varepsilon} \nabla^B u_\varepsilon \cdot \nabla^B \phi d\sigma d\tau \\ & = \int_\Omega \mu^\varepsilon \bar{u}_0 \phi(x, 0) dx + \varepsilon \alpha \int_{\Gamma^\varepsilon} \bar{u}_0 \phi(x, 0) d\sigma, \end{aligned} \quad (2-13)$$

for every test function $\phi \in C^\infty(\Omega_T)$ such that ϕ has compact support in Ω for every $t \in (0, T)$ and $\phi(\cdot, T) = 0$ in Ω .

If u_ε is smooth, by (2-4) it follows that (2-8) can be written in the form

$$\varepsilon \alpha \frac{\partial u_\varepsilon}{\partial t} - \varepsilon \beta (\Delta u_\varepsilon - v_\varepsilon^t \nabla^2 u_\varepsilon v_\varepsilon - (v_\varepsilon \cdot \nabla u_\varepsilon) \operatorname{div} v_\varepsilon) = [\lambda \nabla u_\varepsilon \cdot v_\varepsilon] \quad \text{on } \Gamma^\varepsilon, \quad (2-14)$$

where, as in (2-4), $\nabla^2 u_\varepsilon$ stands for the Hessian matrix of u_ε . By [Amar and Gianni 2018a, Theorem 4.2], for every $\varepsilon > 0$, problem (2-6)–(2-10) admits a unique solution $u_\varepsilon \in L^2(0, T; \mathcal{X}_0^\varepsilon(\Omega)) \cap C^0([0, T]; L^2(\Omega) \cap L^2(\Gamma^\varepsilon))$, if $\bar{u}_0 \in H_0^1(\Omega)$.

Finally, it will be useful in the sequel to also define $\mu, \lambda : Y \rightarrow \mathbb{R}$ as

$$\begin{aligned} \lambda &= \lambda_{\text{int}} \quad \text{in } E_{\text{int}}, & \lambda &= \lambda_{\text{out}} \quad \text{in } E_{\text{out}}, \\ \mu &= \mu_{\text{int}} \quad \text{in } E_{\text{int}}, & \mu &= \mu_{\text{out}} \quad \text{in } E_{\text{out}}. \end{aligned}$$

3. Homogenization of the microscopic problem

In the following, we will assume that the initial data satisfies

$$\bar{u}_0 \in H_0^1(\Omega) \cap H^2(\Omega). \quad (3-1)$$

By the trace inequality [Amar and Gianni 2018b, Proposition 1; Amar et al. 2004, proof of Lemma 7.1] we get that \bar{u}_0 satisfies

$$\varepsilon \int_{\Gamma^\varepsilon} |\bar{u}_0|^2 d\sigma \leq \gamma, \quad \varepsilon \int_{\Gamma^\varepsilon} |\nabla^B \bar{u}_0|^2 d\sigma \leq \gamma, \quad (3-2)$$

where $\gamma > 0$ is independent of ε . Notice that, for our purposes, it should be enough to assume that $\bar{u}_0 \in H_0^1(\Omega)$ and satisfies (3-2), but we prefer to assume (3-1) since it is reasonable to choose \bar{u}_0 not depending on ε .

We are interested in understanding the limiting behavior of the heat potential u_ε when $\varepsilon \rightarrow 0$; this leads us to look at the homogenization limit of problem (2-6)–(2-10).

To this purpose, we first obtain some energy estimates for the heat potential u_ε . Multiplying (2-6) by u_ε and integrating, formally, by parts, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^t \int_\Omega \mu^\varepsilon \frac{\partial u_\varepsilon^2}{\partial \tau} dx d\tau + \int_0^t \int_\Omega \lambda^\varepsilon |\nabla u_\varepsilon|^2 dx d\tau \\ & + \frac{\varepsilon\alpha}{2} \int_0^t \int_{\Gamma^\varepsilon} \frac{\partial u_\varepsilon^2}{\partial \tau} d\sigma d\tau + \varepsilon\beta \int_0^t \int_{\Gamma^\varepsilon} |\nabla^B u_\varepsilon|^2(x) d\sigma d\tau = 0. \end{aligned} \quad (3-3)$$

Then, evaluating the time integral and taking into account the initial condition (2-10), we obtain, for all $0 < t < T$,

$$\begin{aligned} & \frac{1}{2} \int_\Omega \mu^\varepsilon u_\varepsilon^2(t) dx + \int_0^t \int_\Omega \lambda^\varepsilon |\nabla u_\varepsilon|^2 dx d\tau + \frac{\varepsilon\alpha}{2} \int_{\Gamma^\varepsilon} u_\varepsilon^2(t) d\sigma + \varepsilon\beta \int_0^t \int_{\Gamma^\varepsilon} |\nabla^B u_\varepsilon|^2 d\sigma d\tau \\ & = \frac{1}{2} \int_\Omega \mu^\varepsilon \bar{u}_0^2 dx + \frac{\varepsilon\alpha}{2} \int_{\Gamma^\varepsilon} \bar{u}_0^2 d\sigma. \end{aligned} \quad (3-4)$$

By (3-2) the right-hand side of (3-4) is stable as $\varepsilon \rightarrow 0$; hence,

$$\begin{aligned} & \sup_{t \in (0, T)} \int_\Omega u_\varepsilon^2(t) dx + \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 dx d\tau \\ & + \sup_{t \in (0, T)} \varepsilon \int_{\Gamma^\varepsilon} u_\varepsilon^2(t) d\sigma + \varepsilon \int_0^T \int_{\Gamma^\varepsilon} |\nabla^B u_\varepsilon|^2 d\sigma d\tau \leq \gamma, \end{aligned} \quad (3-5)$$

where γ is a constant independent of ε .

Notice that inequality (3-5) implies that there exists a function u belonging to $L^2(0, T; H_0^1(\Omega))$ such that, up to a subsequence, $u_\varepsilon \rightharpoonup u$ weakly in $L^2(0, T; H_0^1(\Omega))$. It will be our purpose to characterize the limit function u .

3.1. The two-scale expansion. We summarize here, to establish the notation, some well known asymptotic expansions needed in the two-scale method (see, e.g., [Bensoussan et al. 1978; Sánchez-Palencia 1980]), when applied to stationary or evolutive problems involving second-order partial differential equations. Introduce the microscopic variables $y \in Y$ and $y = x/\varepsilon$, and assume

$$u_\varepsilon = u_\varepsilon(x, y, t) = u_0(x, y, t) + \varepsilon u_1(x, y, t) + \varepsilon^2 u_2(x, y, t) + \dots \quad (3-6)$$

Note that u_0, u_1, u_2 are periodic in y , and u_1, u_2 are assumed to have zero integral average over Y . Recalling that

$$\operatorname{div} = \frac{1}{\varepsilon} \operatorname{div}_y + \operatorname{div}_x, \quad \nabla = \frac{1}{\varepsilon} \nabla_y + \nabla_x, \quad (3-7)$$

we compute

$$\nabla u_\varepsilon = \frac{1}{\varepsilon} \nabla_y u_0 + (\nabla_x u_0 + \nabla_y u_1) + \varepsilon (\nabla_y u_2 + \nabla_x u_1) + \dots, \quad (3-8)$$

and

$$\Delta u_\varepsilon = \frac{1}{\varepsilon^2} A_0 u_0 + \frac{1}{\varepsilon} (A_0 u_1 + A_1 u_0) + (A_0 u_2 + A_1 u_1 + A_2 u_0) + \dots, \quad (3-9)$$

where

$$A_0 = \Delta_y, \quad A_1 = \operatorname{div}_y \nabla_x + \operatorname{div}_x \nabla_y, \quad A_2 = \Delta_x. \quad (3-10)$$

Moreover, recalling (2-3) and taking into account that the normal vector ν_ε depends only on the microscopic variable, we also obtain

$$\Delta^B u_\varepsilon = \frac{1}{\varepsilon^2} A_0^B u_0 + \frac{1}{\varepsilon} (A_0^B u_1 + A_1^B u_0) + (A_0^B u_2 + A_1^B u_1 + A_2^B u_0) + \dots, \quad (3-11)$$

where

$$\begin{aligned} A_0^B &= \Delta_y^B, & A_2^B &= \Delta_x^B \\ A_1^B &= \operatorname{div}_x^B \nabla_y^B + \operatorname{div}_y^B \nabla_x^B = 2(I - \nu \otimes \nu)_{ij} \partial_{x_i}^2 \partial_{y_j}^2 - (\operatorname{div}_y \nu) \nu \cdot \nabla_x. \end{aligned} \quad (3-12)$$

Substituting in (2-6)–(2-10) the expansion (3-6), and using (3-7)–(3-12), one readily obtains, by matching corresponding powers of ε , that u_0 solves $[u_0] = 0$ on Γ , and

$$\mathcal{P}_0[u_0] : \begin{cases} -\lambda \Delta_y u_0 = 0 & \text{in } E_{\text{int}}, E_{\text{out}}, \\ \beta \Delta_y^B u_0 + [\lambda \nabla_y u_0 \cdot \nu] = 0 & \text{on } \Gamma. \end{cases}$$

By the equality

$$\begin{aligned} 0 &= \int_Y \lambda |\nabla_y u_0|^2 dy + \int_\Gamma [\lambda \nabla_y u_0 \cdot \nu] u_0 d\sigma \\ &= \int_Y \lambda |\nabla_y u_0|^2 dy - \int_\Gamma \beta \Delta_y^B u_0 u_0 d\sigma \\ &= \int_Y \lambda |\nabla_y u_0|^2 dy + \int_\Gamma \beta |\nabla_y^B u_0|^2 d\sigma, \end{aligned}$$

we obtain that u_0 is independent of y , i.e., $u_0 = u_0(x, t)$.

Moreover, u_1 satisfies $[u_1] = 0$ on Γ , and

$$\mathcal{P}_1[u_1] : \begin{cases} -\lambda \Delta_y u_1 = 0 & \text{in } E_{\text{int}}, E_{\text{out}}, \\ \beta \Delta_y^B u_1 + [\lambda \nabla_y u_1 \cdot \nu] = -\beta (\operatorname{div}_y^B \nabla_x^B u_0) - [\lambda \nabla_x u_0 \cdot \nu] & \text{on } \Gamma. \end{cases}$$

Following a classical approach, we introduce the factorization

$$u_1(x, y, t) = -\chi(y) \cdot \nabla_x u_0(x, t) = -\chi_h(y) \frac{\partial u_0}{\partial x_h}(x, t), \quad h = 1, \dots, N, \quad (3-13)$$

for a vector function $\chi : Y \rightarrow \mathbb{R}^N$, whose components χ_h satisfy

$$-\lambda \operatorname{div}_y (\nabla_y \chi_h - \mathbf{e}_h) = 0 \quad \text{in } E_{\text{int}}, E_{\text{out}}, \quad (3-14)$$

$$\beta \Delta_y^B (\chi_h - y_h) = -[\lambda (\nabla_y \chi_h - \mathbf{e}_h) \cdot \nu] \quad \text{on } \Gamma, \quad (3-15)$$

$$[\chi_h] = 0 \quad \text{on } \Gamma. \quad (3-16)$$

The functions χ_h are also required to be periodic in Y , with zero integral average on Y (here, \mathbf{e}_h denotes the h vector of the canonical basis of \mathbb{R}^N). We note that [Amar and Gianni 2018a, Theorem 5.1 and Remark 5.3] assures existence and uniqueness of the cell functions $\chi_h \in C_{\#}^{\infty}(Y)$, for $h = 1, \dots, N$ (here and in the following, the subscript # denotes the Y -periodicity).

Finally, u_2 solves $[u_2] = 0$ on Γ , and

$$\mathcal{P}_2[u_2] : \begin{cases} -\lambda \Delta_y u_2 = -\mu u_{0t} + \lambda \Delta_x u_0 + 2\lambda \frac{\partial^2 u_1}{\partial x_j \partial y_j} & \text{in } E_{\text{int}}, E_{\text{out}}, \\ \beta \Delta_y^B u_2 + [\lambda \nabla_y u_2 \cdot \nu] = \alpha u_{0t} - \beta \Delta_x^B u_0 - \beta \operatorname{div}_x^B \nabla_y^B u_1 \\ \quad - \beta \operatorname{div}_y^B \nabla_x^B u_1 - [\lambda \nabla_x u_1 \cdot \nu] & \text{on } \Gamma. \end{cases}$$

The limiting equation for u_0 is finally obtained as a compatibility condition for $\mathcal{P}_2[u_2]$, and amounts to

$$\begin{aligned} & \int_Y \left(-\mu u_{0t} + \lambda \Delta_x u_0 + 2\lambda \frac{\partial^2 u_1}{\partial x_j \partial y_j} \right) dy = \int_{\Gamma} [\lambda \nabla_y u_2 \cdot \nu] d\sigma \\ & = \int_{\Gamma} (\alpha u_{0t} - [\lambda \nabla_x u_1 \cdot \nu] - \beta \Delta_y^B u_2 - \beta \Delta_x^B u_0 - \beta \operatorname{div}_x^B \nabla_y^B u_1 - \beta \operatorname{div}_y^B \nabla_x^B u_1) d\sigma. \end{aligned} \quad (3-17)$$

We now replace the factorization (3-13) in the previous equality, and we take into account that

$$2 \int_Y \lambda \frac{\partial^2 u_1}{\partial x_j \partial y_j} dy = -2 \int_{\Gamma} [\lambda \nabla_x u_1 \cdot \nu] d\sigma, \quad (3-18)$$

$$- \int_{\Gamma} [\lambda \nabla_x u_1 \cdot \nu] d\sigma = \operatorname{div} \left(\left(\int_{\Gamma} [\lambda] (\nu \otimes \chi) d\sigma \right) \nabla u_0 \right), \quad (3-19)$$

$$- \int_{\Gamma} \beta \Delta_y^B u_2 d\sigma = 0, \quad (3-20)$$

$$- \int_{\Gamma} \beta \Delta_x^B u_0 d\sigma = -\beta |\Gamma| \Delta u_0 + \operatorname{div} \left(\left(\int_{\Gamma} \beta (\nu \otimes \nu) d\sigma \right) \nabla u_0 \right), \quad (3-21)$$

$$- \int_{\Gamma} \beta \operatorname{div}_x^B \nabla_y^B u_1 d\sigma = \operatorname{div} \left(\left(\int_{\Gamma} \beta (I - \nu \otimes \nu) \nabla_y \chi d\sigma \right) \nabla u_0 \right), \quad (3-22)$$

$$- \int_{\Gamma} \beta \operatorname{div}_y^B \nabla_x^B u_1 d\sigma = 0, \quad (3-23)$$

where (3-23) follows from (2-5), since Γ has no boundary. Hence, we obtain for the homogenized solution u_0 the parabolic equation

$$\tilde{\mu}u_{0t} - \operatorname{div}((\lambda_0 I + A^{\text{hom}})\nabla u_0) = 0 \quad \text{in } \Omega_T, \quad (3-24)$$

where

$$\begin{aligned} \tilde{\mu} &= \mu_{\text{int}}|E_{\text{int}}| + \mu_{\text{out}}|E_{\text{out}}| + \alpha|\Gamma|, \\ \lambda_0 &= \lambda_{\text{int}}|E_{\text{int}}| + \lambda_{\text{out}}|E_{\text{out}}|, \\ A^{\text{hom}} &= \int_{\Gamma} [\lambda](\nu \otimes \chi) \, d\sigma + \beta \int_{\Gamma} ((I - \nu \otimes \nu) + (\nu \otimes \nu)\nabla_y \chi - \nabla_y \chi) \, d\sigma \\ &= \int_{\Gamma} [\lambda](\nu \otimes \chi) \, d\sigma - \beta \int_{\Gamma} \nabla_y^B(\chi - y) \, d\sigma. \end{aligned} \quad (3-25)$$

Clearly, (3-24) must be complemented with a boundary and an initial condition which are $u_0 = 0$ on $\partial\Omega \times (0, T)$ and $u_0(x, 0) = \bar{u}_0(x)$ in Ω , respectively, as follows from the microscopic problem (2-6)–(2-10). Indeed, by (3-5) we obtain that $\{u_\varepsilon\}$ converges weakly in $L^2(0, T; H_0^1(\Omega))$, which implies the weak convergence of the trace on $\partial\Omega$, while the initial data is already included in the weak formulation of the problem.

Theorem 3.1. *The matrix $\lambda_0 I + A^{\text{hom}}$ is symmetric and positive-definite.*

Proof. We first prove the symmetry. By (2-1), we have

$$- \int_{\Gamma} \nabla_y^B y_h \cdot \nabla_y^B \chi_j \, d\sigma = - \int_{\Gamma} (\mathbf{e}_h - \nu_h \nu) \cdot \nabla_y^B \chi_j \, d\sigma = - \int_{\Gamma} (\nabla_y^B \chi_j)_h \, d\sigma; \quad (3-26)$$

then, taking into account (3-14)–(3-16), we obtain

$$\begin{aligned} 0 &= - \int_Y \lambda \Delta_y(\chi_h - y_h)\chi_j \, dy \\ &= \int_Y \lambda \nabla_y(\chi_h - y_h) \cdot \nabla_y \chi_j \, dy - \beta \int_{\Gamma} \Delta_y^B(\chi_h - y_h)\chi_j \, d\sigma \\ &= \int_Y \lambda \nabla_y \chi_h \cdot \nabla_y \chi_j \, dy - \int_Y \lambda \mathbf{e}_h \cdot \nabla_y \chi_j \, dy + \beta \int_{\Gamma} \nabla_y^B(\chi_h - y_h) \cdot \nabla_y^B \chi_j \, d\sigma \\ &= \int_Y \lambda \nabla_y \chi_h \cdot \nabla_y \chi_j \, dy + \int_{\Gamma} [\lambda] \nu_h \chi_j \, d\sigma \\ &\quad + \beta \int_{\Gamma} \nabla_y^B \chi_h \nabla_y^B \chi_j \, d\sigma - \beta \int_{\Gamma} \nabla_y^B y_h \nabla_y^B \chi_j \, d\sigma \\ &= \int_Y \lambda \nabla_y \chi_h \cdot \nabla_y \chi_j \, dy + \int_{\Gamma} [\lambda] \nu_h \chi_j \, d\sigma \\ &\quad + \beta \int_{\Gamma} \nabla_y^B \chi_h \nabla_y^B \chi_j \, d\sigma - \beta \int_{\Gamma} (\nabla_y^B \chi_j)_h \, d\sigma. \end{aligned} \quad (3-27)$$

From (3-25) and (3-27), we can rewrite

$$A^{\text{hom}} = \int_{\Gamma} \beta(I - \nu \otimes \nu) \, d\sigma - \int_Y \lambda(\nabla_y \chi \otimes \nabla_y \chi) \, dy - \int_{\Gamma} \beta(\nabla_y^B \chi \otimes \nabla_y^B \chi) \, d\sigma,$$

which gives the symmetry of the matrix A^{hom} and hence the symmetry of the whole matrix $\lambda_0 I + A^{\text{hom}}$.

Let us now prove that it is also positive-definite. Firstly, we observe that, using (3-26) and (3-27), we obtain

$$\begin{aligned} & \int_Y \lambda \nabla(\chi_h - y_h) \cdot \nabla(\chi_j - y_j) \, dy + \beta \int_{\Gamma} \nabla_y^B(\chi_h - y_h) \nabla_y^B(\chi_j - y_j) \, d\sigma \\ &= \int_Y \lambda \nabla \chi_h \cdot \nabla \chi_j \, dy + \int_Y \lambda \mathbf{e}_h \cdot \mathbf{e}_j \, dy - \int_Y \lambda \nabla \chi_h \cdot \mathbf{e}_j \, dy - \int_Y \lambda \nabla \chi_j \cdot \mathbf{e}_h \, dy \\ & \quad + \beta \int_{\Gamma} \nabla^B \chi_h \cdot \nabla^B \chi_j \, d\sigma + \beta \int_{\Gamma} \nabla^B y_h \cdot \nabla^B y_j \, d\sigma \\ & \quad - \beta \int_{\Gamma} \nabla^B \chi_h \cdot \nabla^B y_j \, d\sigma - \beta \int_{\Gamma} \nabla^B \chi_j \cdot \nabla^B y_h \, d\sigma \\ &= \int_Y \lambda \nabla \chi_h \cdot \nabla \chi_j \, dy + \int_Y \lambda \delta_{hj} \, dy + \int_{\Gamma} [\lambda] \chi_h \nu_j \, d\sigma + \int_{\Gamma} [\lambda] \chi_j \nu_h \, d\sigma \\ & \quad + \beta \int_{\Gamma} \nabla^B \chi_h \cdot \nabla^B \chi_j \, d\sigma + \beta \int_{\Gamma} \nabla^B y_h \cdot \nabla^B y_j \, d\sigma \\ & \quad - \beta \int_{\Gamma} (\nabla_y^B \chi_h)_j \, d\sigma - \beta \int_{\Gamma} (\nabla_y^B \chi_j)_h \, d\sigma \\ &= \int_Y \lambda \nabla \chi_h \cdot \nabla \chi_j \, dy + \int_Y \lambda \delta_{hj} \, dy + \beta \int_{\Gamma} \nabla^B \chi_h \cdot \nabla^B \chi_j \, d\sigma + \beta \int_{\Gamma} \nabla^B y_h \cdot \nabla^B y_j \, d\sigma \\ & \quad - 2 \int_Y \lambda \nabla \chi_h \cdot \nabla \chi_j \, dy - 2\beta \int_{\Gamma} \nabla^B \chi_h \nabla^B \chi_j \, d\sigma \\ &= \int_Y \lambda \delta_{hj} \, dy - \int_Y \lambda \nabla \chi_h \cdot \nabla \chi_j \, dy + \beta \int_{\Gamma} (\delta_{hj} - \nu_h \nu_j) \, d\sigma - \beta \int_{\Gamma} \nabla^B \chi_h \nabla^B \chi_j \, d\sigma. \end{aligned}$$

Then, we can rewrite

$$\begin{aligned} (\lambda_0 I + A^{\text{hom}})_{hj} &= \int_Y \lambda \delta_{hj} \, dy + \int_{\Gamma} \beta \delta_{hj} \, d\sigma - \int_{\Gamma} \beta \nu_h \nu_j \, d\sigma \\ & \quad - \int_Y \lambda \nabla \chi_h \cdot \nabla \chi_j \, dy - \int_{\Gamma} \beta \nabla^B \chi_h \cdot \nabla^B \chi_j \, d\sigma \\ &= \int_Y \lambda \nabla(\chi_h - y_h) \cdot \nabla(\chi_j - y_j) \, dy + \int_{\Gamma} \beta \nabla^B(\chi_h - y_h) \cdot \nabla^B(\chi_j - y_j) \, d\sigma. \end{aligned}$$

Finally, setting $\lambda_{\min} = \min(\lambda_{\text{int}}, \lambda_{\text{out}})$ and using Jensen's inequality, we obtain

$$\begin{aligned}
 \sum_{h,j=1}^N (\lambda_0 I + A^{\text{hom}})_{hj} \xi_h \xi_j &= \int_Y \sum_{h,j=1}^N \lambda (\nabla \chi_h \xi_h - \mathbf{e}_h \xi_h) \cdot (\nabla \chi_j \xi_j - \mathbf{e}_j \xi_j) \, dy \\
 &\quad + \int_{\Gamma} \sum_{h,j=1}^N \beta \nabla^B (\chi_h \xi_h - y_h \xi_h) \cdot \nabla^B (\chi_j \xi_j - y_j \xi_j) \, d\sigma \\
 &\geq \lambda_{\min} \int_Y \left| \sum_{h=1}^N (\nabla \chi_h \xi_h - \mathbf{e}_h \xi_h) \right|^2 \, dy + \beta \int_{\Gamma} \left| \sum_{h=1}^N \nabla^B (\chi_h \xi_h - y_h \xi_h) \right|^2 \, d\sigma \\
 &\geq \lambda_{\min} \left| \int_Y \sum_{h=1}^N (\nabla \chi_h \xi_h - \mathbf{e}_h \xi_h) \, dy \right|^2 + \beta |\Gamma| \left| \frac{1}{|\Gamma|} \int_{\Gamma} \sum_{h=1}^N \nabla^B (\chi_h \xi_h - y_h \xi_h) \, d\sigma \right|^2 \\
 &\geq \lambda_{\min} \sum_{j=1}^N \left(\sum_{h=1}^N \left(\xi_h \int_Y \frac{\partial \chi_h}{\partial y_j} \, dy - \delta_{hj} \xi_h \right) \right)^2 + \frac{\beta}{|\Gamma|} \left| \sum_{h=1}^N \int_{\Gamma} \nabla^B (\chi_h \xi_h - y_h \xi_h) \, d\sigma \right|^2 \\
 &\geq \lambda_{\min} \sum_{j=1}^N \left(\sum_{h=1}^N \xi_h \int_{\partial Y} \chi_h n_j \, d\sigma - \xi_j \right)^2 = \lambda_{\min} |\xi|^2
 \end{aligned}$$

where we have denoted by $n = (n_1, \dots, n_N)$ the outward unit normal to ∂Y . Moreover, we remark that the last integral vanishes because of the periodicity of the cell function χ_h .

This proves that the homogenized matrix is positive-definite. \square

Remark 3.2. We note that the homogenized matrix is positive-definite independently of the strict positivity of β .

Once [Theorem 3.1](#) has been proved, the existence of a unique solution for (3-24) complemented with suitable initial and boundary conditions is standard. The next proposition states the regularity of this solution, which is a property needed in order to obtain the error estimate.

Proposition 3.3. *Assume $\bar{u}_0 \in C_c^\infty(\Omega)$ (i.e., \bar{u}_0 has compact support in Ω). Then, the solution u_0 to (3-24) satisfying the homogeneous boundary condition on $\partial\Omega \times [0, T]$ and the initial condition $u(x, 0) = \bar{u}_0(x)$ in Ω belongs to $C^\infty(\bar{\Omega} \times [0, T])$.*

Proof. The result can be obtained applying [[Friedman 1964](#), Theorem 12 in §5]. \square

Remark 3.4. Actually, the asserted C^∞ -regularity of the homogenized solution u_0 is far from being optimal in order to obtain the error estimate proved in [Section 4](#). Indeed, to this purpose, it is enough to have that $u_0 \in C^0([0, T]; C^3(\bar{\Omega}))$ and this is

guaranteed if, for instance, $\bar{u}_0 \in C^4(\bar{\Omega})$ and satisfies the compatibility conditions

$$L_{\text{hom}}\bar{u}_0(x) = 0, \quad L_{\text{hom}}^2\bar{u}_0(x) := L_{\text{hom}}(L_{\text{hom}}\bar{u}_0(x)) = 0 \quad \text{on } \partial\Omega, \quad (3-28)$$

where $L_{\text{hom}} = -\text{div}((\lambda_0 I + A^{\text{hom}})\nabla)$, with λ_0 and A^{hom} defined in (3-25). However, we prefer the simpler assumptions of Proposition 3.3, since we are not interested in stating which are the minimal conditions to be satisfied by the initial data in order to obtain the optimal regularity of the homogenized solution.

For further use (taking into account the system satisfied by u_2 and (3-24)), we introduce the factorization of the function u_2 in terms of the homogenized solution u_0 ; i.e.,

$$u_2(x, y, t) = \tilde{\chi}_{ij}(y) \frac{\partial^2 u_0}{\partial x_i \partial x_j}(x, t), \quad i, j = 1, \dots, N, \quad (3-29)$$

where the functions $\tilde{\chi}_{ij} : Y \rightarrow \mathbb{R}$ satisfy

$$-\lambda \Delta_y \tilde{\chi}_{ij} = -\frac{\mu}{\tilde{\mu}}(\lambda_0 \delta_{ij} + a_{ij}^{\text{hom}}) + \lambda \delta_{ij} - 2\lambda \frac{\partial \chi_i}{\partial y_j} =: F \quad \text{in } E_{\text{int}}, E_{\text{out}}, \quad (3-30)$$

$$\begin{aligned} \beta \Delta_y^B \tilde{\chi}_{ij} + [\lambda \nabla_y \tilde{\chi}_{ij} \cdot \nu] &= \frac{\alpha}{\tilde{\mu}}(\lambda_0 \delta_{ij} + a_{ij}^{\text{hom}}) - \beta(\delta_{ij} - (\nu \otimes \nu)_{ij}) \\ + 2\beta(I - (\nu \otimes \nu))_i \cdot \nabla \chi_j - \beta \nu_j \chi_i \text{div } \nu + [\lambda \nu_i] \chi_j &=: G \quad \text{on } \Gamma, \end{aligned} \quad (3-31)$$

$$[\tilde{\chi}_{ij}] = 0 \quad \text{on } \Gamma. \quad (3-32)$$

The functions $\tilde{\chi}_{ij}$ are also required to be periodic in Y , with zero integral average on Y . In order to obtain (3-30)–(3-32) we have taken into account (3-12), which gives

$$\text{div}_x^B(\nabla_y^B \phi) + \text{div}_y^B(\nabla_x^B \phi) = 2(\delta_{ij} - \nu_i \nu_j) \frac{\partial^2 \phi}{\partial x_i \partial y_j} - \nu_j \frac{\partial \nu_i}{\partial y_i} \frac{\partial \phi}{\partial x_j},$$

with $\phi(x, y, t) = u_1(x, y, t) = -\chi(y) \cdot \nabla_x u_0(x, t)$ and the usual summation convention for repeated indexes. By [Amar and Gianni 2018a, Theorem 5.1 and Remark 5.3], problem (3-30)–(3-32) admits a unique solution $\tilde{\chi}_{ij} \in C_{\#}^{\infty}(Y)$, for $i, j = 1, \dots, N$, since it is easy to check that

$$\int_Y F \, dy = \int_{\Gamma} G \, d\sigma.$$

4. Error estimate

In this section we prove that the limit u of the sequence $\{u_{\varepsilon}\}$ of the solutions of problem (2-6)–(2-10) coincides with the solution u_0 of (3-24). In order to achieve this result, we will state an error estimate for the sequence $\{u_{\varepsilon}\}$, which gives the rate of convergence of such a sequence to the homogenized function u_0 , in a suitable norm, thus obtaining a stronger convergence result with respect to the one

obtained in our previous paper [Amar and Gianni 2018b]. However, this result needs extra-regularity assumptions on the initial data $\bar{u}_0(x)$ (see Proposition 3.3 and Remark 3.4), which assure more regularity of the homogenized solution u_0 .

Theorem 4.1. *Assume that $\bar{u}_0 \in C_c^\infty(\Omega)$. Let u_0 be the smooth solution of (3-24), satisfying the initial condition $u_0(x, 0) = \bar{u}_0(x)$ in Ω and the boundary condition $u_0(x, t) = 0$ on $\partial\Omega \times (0, T)$; moreover, let u_1 be the function defined in (3-13). Then*

$$\|u_\varepsilon - (u_0 + \varepsilon u_1)\|_{L^2(0, T; H^1(\Omega))} \leq \gamma \sqrt{\varepsilon}, \quad (4-1)$$

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \leq \gamma \sqrt{\varepsilon}, \quad (4-2)$$

for a proper constant $\gamma > 0$, independent of ε .

Proof. Let us define the rest function

$$r_\varepsilon(x, t) = (u_\varepsilon(x, t) - u_0(x, t) - \varepsilon u_1(x, x/\varepsilon, t))\varepsilon^{-1}, \quad x \in \Omega, \quad t > 0.$$

Separately in $\Omega_{\text{int}}^\varepsilon$ and in $\Omega_{\text{out}}^\varepsilon$, we get

$$\begin{aligned} \mu^\varepsilon \frac{\partial r_\varepsilon}{\partial t} - \operatorname{div}(\lambda^\varepsilon \nabla r_\varepsilon) &= \frac{1}{\varepsilon} \left\{ -\mu^\varepsilon \frac{\partial u_0}{\partial t} + \operatorname{div}(\lambda^\varepsilon \nabla u_0) - \mu^\varepsilon \varepsilon \frac{\partial u_1}{\partial t} + \varepsilon \operatorname{div}(\lambda^\varepsilon \nabla u_1) \right\} \\ &= \frac{1}{\varepsilon} \left\{ -\mu^\varepsilon \frac{\partial u_0}{\partial t} + \lambda^\varepsilon \Delta_x u_0 + 2\lambda^\varepsilon u_{1x_h y_h} \right\} - \mu^\varepsilon \frac{\partial u_1}{\partial t} + \lambda^\varepsilon \Delta_x u_1 + \frac{1}{\varepsilon^2} \lambda^\varepsilon \Delta_y u_1 \\ &= -\frac{1}{\varepsilon} \lambda^\varepsilon \Delta_y u_2 - \mu^\varepsilon \frac{\partial u_1}{\partial t} + \lambda^\varepsilon \Delta_x u_1 =: E_\varepsilon - \mu^\varepsilon \frac{\partial u_1}{\partial t}. \end{aligned}$$

Moreover,

$$[r_\varepsilon] = 0, \quad r_\varepsilon(x, 0) = -u_1(x, x/\varepsilon, 0) = \chi(x/\varepsilon) \cdot \nabla_x u_0(x, 0) = \chi(x/\varepsilon) \cdot \nabla_x \bar{u}_0(x, 0),$$

and

$$\begin{aligned} \varepsilon \alpha \frac{\partial r_\varepsilon}{\partial t} - \varepsilon \beta \Delta^B r_\varepsilon &= \frac{1}{\varepsilon} \left\{ \varepsilon \alpha \frac{\partial u_\varepsilon}{\partial t} - \varepsilon \beta \Delta^B u_\varepsilon - \varepsilon \alpha \frac{\partial u_0}{\partial t} + \varepsilon \beta \Delta^B u_0 \right\} - \left\{ \varepsilon \alpha \frac{\partial u_1}{\partial t} - \varepsilon \beta \Delta^B u_1 \right\} \\ &= \frac{1}{\varepsilon} [\lambda^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] - \alpha \frac{\partial u_0}{\partial t} + \beta \Delta_x^B u_0 + \beta \operatorname{div}_x^B \nabla_y^B u_1 + \beta \operatorname{div}_y^B \nabla_x^B u_1 \\ &\quad - \varepsilon \alpha \frac{\partial u_1}{\partial t} + \varepsilon \beta \Delta_x^B u_1 + \frac{1}{\varepsilon} (\beta \Delta_y^B u_1 + \beta \operatorname{div}_y^B \nabla_x^B u_0 + \beta \operatorname{div}_x^B \nabla_y^B u_0) + \frac{1}{\varepsilon^2} \beta \Delta_y^B u_0 \\ &= \frac{1}{\varepsilon} [\lambda^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] - [\lambda^\varepsilon (\nabla_x u_1 + \nabla_y u_2) \cdot \nu_\varepsilon] - \beta \Delta_y^B u_2 \\ &\quad - \varepsilon \left(\alpha \frac{\partial u_1}{\partial t} - \beta \Delta_x^B u_1 \right) - \frac{1}{\varepsilon} [\lambda^\varepsilon (\nabla_x u_0 + \nabla_y u_1) \cdot \nu_\varepsilon] \\ &= [\lambda^\varepsilon \nabla r_\varepsilon \cdot \nu_\varepsilon] - \varepsilon \left(\alpha \frac{\partial u_1}{\partial t} - \beta \Delta_x^B u_1 \right) - [\lambda^\varepsilon \nabla_y u_2 \cdot \nu_\varepsilon] - \beta \Delta_y^B u_2, \end{aligned}$$

where we have taken into account the problems satisfied by u_1 and u_2 (u_1 and u_2 are defined in [Section 3.1](#)) and the fact that $\operatorname{div}_x^B \nabla_y^B u_0 = 0$ and $\Delta_y^B u_0 = 0$.

Let us now introduce the corrected rest function

$$\tilde{r}_\varepsilon = r_\varepsilon + u_1 \phi_\varepsilon,$$

where ϕ_ε is a cut-off function equal to 1 in a neighborhood of $\partial\Omega$, and such that

$$\phi_\varepsilon(x) = 0 \quad \text{if } \operatorname{dist}(x, \partial\Omega) \geq \gamma_0 \varepsilon.$$

Clearly, $\phi_\varepsilon \equiv 0$ on Γ^ε (since $\operatorname{dist}(\Gamma^\varepsilon, \partial\Omega) \geq \gamma_0 \varepsilon$, by the assumptions made in [Section 2.2](#)), so that $r_\varepsilon = \tilde{r}_\varepsilon$ on Γ^ε . We may assume $0 \leq \phi_\varepsilon \leq 1$, $|\nabla \phi_\varepsilon| \leq \gamma/\varepsilon$. The function \tilde{r}_ε satisfies $[\tilde{r}_\varepsilon] = 0$ on Γ^ε and

$$\mu^\varepsilon \frac{\partial \tilde{r}_\varepsilon}{\partial t} - \lambda^\varepsilon \Delta \tilde{r}_\varepsilon = E_\varepsilon - \mu^\varepsilon \frac{\partial u_1}{\partial t} + \mu^\varepsilon \phi_\varepsilon \frac{\partial u_1}{\partial t} - \lambda^\varepsilon \Delta(u_1 \phi_\varepsilon) \quad \text{in } \Omega_{\text{int}}^\varepsilon, \Omega_{\text{out}}^\varepsilon, \quad (4-3)$$

$$\tilde{r}_\varepsilon(x, 0) = \chi(x/\varepsilon) \cdot \nabla_x \bar{u}_0(x, 0)(1 - \phi_\varepsilon) \quad \text{on } \Omega, \quad (4-4)$$

$$\tilde{r}_\varepsilon = 0 \quad \text{on } \partial\Omega, \quad (4-5)$$

and on Γ^ε

$$\begin{aligned} \varepsilon \alpha \frac{\partial \tilde{r}_\varepsilon}{\partial t} - \varepsilon \beta \Delta^B \tilde{r}_\varepsilon &= [\lambda^\varepsilon \nabla r_\varepsilon \cdot \nu_\varepsilon] - \varepsilon \left(\alpha \frac{\partial u_1}{\partial t} - \beta \Delta_x^B u_1 \right) - [\lambda^\varepsilon \nabla_y u_2 \cdot \nu_\varepsilon] - \beta \Delta_y^B u_2 \\ &= [\lambda^\varepsilon \nabla \tilde{r}_\varepsilon \cdot \nu_\varepsilon] - \varepsilon \left(\alpha \frac{\partial u_1}{\partial t} - \beta \Delta_x^B u_1 \right) - [\lambda^\varepsilon \nabla_y u_2 \cdot \nu_\varepsilon] - \beta \Delta_y^B u_2. \end{aligned} \quad (4-6)$$

Note that the correction $u_1 \phi_\varepsilon$ has been introduced precisely in order to guarantee (4-5). Multiply (4-3) by \tilde{r}_ε and integrate by parts; by virtue of (4-5), we get

$$\begin{aligned} &\int_0^t \int_\Omega \{E_\varepsilon - \lambda^\varepsilon \Delta(u_1 \phi_\varepsilon)\} \tilde{r}_\varepsilon \, dx \, d\tau - \int_0^t \int_\Omega \left\{ \mu^\varepsilon \frac{\partial u_1}{\partial \tau} (1 - \phi_\varepsilon) \right\} \tilde{r}_\varepsilon \, dx \, d\tau \\ &= \frac{1}{2} \int_0^t \int_\Omega \mu^\varepsilon \frac{\partial \tilde{r}_\varepsilon^2}{\partial \tau} \, dx \, d\tau + \int_0^t \int_\Omega \lambda^\varepsilon |\nabla \tilde{r}_\varepsilon|^2 \, dx \, d\tau + \int_0^t \int_{\Gamma^\varepsilon} [\lambda^\varepsilon \nabla \tilde{r}_\varepsilon \cdot \nu_\varepsilon] \tilde{r}_\varepsilon \, d\sigma \, d\tau \\ &= \frac{1}{2} \int_\Omega \mu^\varepsilon \tilde{r}_\varepsilon^2(x, t) \, dx - \frac{1}{2} \int_\Omega \mu^\varepsilon \tilde{r}_\varepsilon^2(x, 0) \, dx + \int_0^t \int_\Omega \lambda^\varepsilon |\nabla \tilde{r}_\varepsilon|^2 \, dx \, d\tau \\ &+ \frac{\varepsilon}{2} \int_{\Gamma^\varepsilon} \alpha \tilde{r}_\varepsilon^2(x, t) \, d\sigma - \frac{\varepsilon}{2} \int_{\Gamma^\varepsilon} \alpha \tilde{r}_\varepsilon^2(x, 0) \, d\sigma + \varepsilon \beta \int_0^t \int_{\Gamma^\varepsilon} |\nabla^B \tilde{r}_\varepsilon|^2 \, d\sigma \, d\tau \\ &+ \varepsilon \int_0^t \int_{\Gamma^\varepsilon} \left(\alpha \frac{\partial u_1}{\partial t} - \beta \Delta_x^B u_1 \right) \tilde{r}_\varepsilon \, d\sigma \, d\tau + \int_0^t \int_{\Gamma^\varepsilon} (\beta \Delta_y^B u_2 + [\lambda \nabla_y u_2 \cdot \nu_\varepsilon]) \tilde{r}_\varepsilon \, d\sigma \, d\tau. \end{aligned} \quad (4-7)$$

This implies

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega} \mu^\varepsilon \tilde{r}_\varepsilon^2(x, t) \, dx + \frac{\varepsilon}{2} \int_{\Gamma^\varepsilon} \alpha \tilde{r}_\varepsilon^2(x, t) \, d\sigma + \int_0^t \int_{\Omega} \lambda^\varepsilon |\nabla \tilde{r}_\varepsilon|^2 \, dx \, d\tau \\
 & \quad + \varepsilon \beta \int_0^t \int_{\Gamma^\varepsilon} |\nabla^B \tilde{r}_\varepsilon|^2 \, d\sigma \, d\tau \\
 & = \frac{1}{2} \int_{\Omega} \mu^\varepsilon \tilde{r}_\varepsilon^2(x, 0) \, dx + \frac{\varepsilon}{2} \int_{\Gamma^\varepsilon} \alpha \tilde{r}_\varepsilon^2(x, 0) \, d\sigma \\
 & \quad - \varepsilon \int_0^t \int_{\Gamma^\varepsilon} \left(\alpha \frac{\partial u_1}{\partial \tau} - \beta \Delta_x^B u_1 \right) \tilde{r}_\varepsilon \, d\sigma \, d\tau \\
 & \quad - \int_0^t \int_{\Gamma^\varepsilon} (\beta \Delta_y^B u_2 + [\lambda^\varepsilon \nabla_y u_2 \cdot \nu_\varepsilon]) \tilde{r}_\varepsilon \, d\sigma \, d\tau \\
 & \quad + \int_0^t \int_{\Omega} \{E_\varepsilon - \lambda^\varepsilon \Delta(u_1 \phi_\varepsilon)\} \tilde{r}_\varepsilon \, dx \, d\tau - \int_0^t \int_{\Omega} \left\{ \mu^\varepsilon \frac{\partial u_1}{\partial \tau} (1 - \phi_\varepsilon) \right\} \tilde{r}_\varepsilon \, dx \, d\tau.
 \end{aligned}$$

Next, compute

$$\begin{aligned}
 \int_0^t \int_{\Omega} E_\varepsilon \tilde{r}_\varepsilon \, dx \, d\tau & = \int_0^t \int_{\Omega} \lambda^\varepsilon \left\{ -\frac{1}{\varepsilon} \Delta_y u_2 + \Delta_x u_1 \right\} \tilde{r}_\varepsilon \, dx \, d\tau \\
 & = \int_0^t \int_{\Omega} \lambda^\varepsilon \left\{ -\frac{1}{\varepsilon} \Delta_y u_2 - \operatorname{div}_x(\nabla_y u_2) \right\} \tilde{r}_\varepsilon \, dx \, d\tau \\
 & \quad + \int_0^t \int_{\Omega} \lambda^\varepsilon \{ \operatorname{div}_x(\nabla_y u_2) + \Delta_x u_1 \} \tilde{r}_\varepsilon \, dx \, d\tau \\
 & = - \int_0^t \int_{\Omega} \operatorname{div}(\lambda^\varepsilon \nabla_y u_2) \tilde{r}_\varepsilon \, dx \, d\tau \\
 & \quad + \int_0^t \int_{\Omega} \{ \lambda^\varepsilon \operatorname{div}_x(\nabla_y u_2) + \lambda^\varepsilon \Delta_x u_1 \} \tilde{r}_\varepsilon \, dx \, d\tau \\
 & = \int_0^t \int_{\Gamma^\varepsilon} [\lambda^\varepsilon \nabla_y u_2 \cdot \nu_\varepsilon] \tilde{r}_\varepsilon \, d\sigma \, d\tau + \int_0^t \int_{\Omega} \lambda^\varepsilon \nabla_y u_2 \cdot \nabla \tilde{r}_\varepsilon \, dx \, d\tau \\
 & \quad + \int_0^t \int_{\Omega} \{ \lambda^\varepsilon \operatorname{div}_x(\nabla_y u_2) + \lambda^\varepsilon \Delta_x u_1 \} \tilde{r}_\varepsilon \, dx \, d\tau. \quad (4-8)
 \end{aligned}$$

Note that the last integral in (4-8) can be bounded in the following way:

$$\int_0^t \int_{\Omega} \{ \lambda^\varepsilon \operatorname{div}_x(\nabla_y u_2) + \lambda^\varepsilon \Delta_x u_1 \} \tilde{r}_\varepsilon \, dx \, d\tau \leq \gamma(\delta) + \delta \int_0^t \int_{\Omega} \tilde{r}_\varepsilon^2 \, dx \, d\tau,$$

where $\delta > 0$ will be chosen in the following. We exploit here the estimate

$$\int_0^t \int_{\Omega} (u_{2x_i y_i}^2 + u_{1x_i x_i}^2) dx d\tau \leq \gamma, \quad (4-9)$$

which is a consequence of the regularity of the cell functions χ and $\tilde{\chi}$ (recall (3-13)–(3-16) and (3-29)–(3-32)) and of the homogenized function u_0 . Similarly, for $\delta' = \min(\lambda_{\text{int}}, \lambda_{\text{out}})/2$,

$$\begin{aligned} - \int_0^t \int_{\Omega} \lambda^\varepsilon \Delta(u_1 \phi_\varepsilon) \tilde{r}_\varepsilon dx d\tau &= \int_0^t \int_{\Omega} \lambda^\varepsilon \nabla(u_1 \phi_\varepsilon) \cdot \nabla \tilde{r}_\varepsilon dx d\tau \\ &\leq \delta' \int_0^t \int_{\Omega} |\nabla \tilde{r}_\varepsilon|^2 dx d\tau + \frac{\gamma(\delta')}{\varepsilon^2} |\{x \in \Omega \mid \text{dist}(x, \partial\Omega) \leq \gamma_0 \varepsilon\}| \\ &\leq \delta' \int_0^t \int_{\Omega} |\nabla \tilde{r}_\varepsilon|^2 dx d\tau + \frac{\gamma(\delta')}{\varepsilon}, \end{aligned} \quad (4-10)$$

where, again due to the stated regularity of χ and u_0 , we used

$$\sup_{x \in \Omega, y \in Y, 0 < t < T} \{|u_1| + |\nabla_x u_1| + |\nabla_y u_1|\}(x, y, t) < +\infty. \quad (4-11)$$

Moreover, for δ'' which will be chosen later, we obtain

$$\begin{aligned} \int_0^t \int_{\Gamma^\varepsilon} (\beta \Delta_y^B u_2) \tilde{r}_\varepsilon d\sigma d\tau &= \varepsilon \beta \int_0^t \int_{\Gamma^\varepsilon} \left(\frac{1}{\varepsilon} \text{div}_y^B \nabla_y^B u_2 + \text{div}_x^B \nabla_y^B u_2 \right) \tilde{r}_\varepsilon d\sigma d\tau \\ &\quad - \varepsilon \beta \int_0^t \int_{\Gamma^\varepsilon} (\text{div}_x^B \nabla_y^B u_2) \tilde{r}_\varepsilon d\sigma d\tau \\ &= -\varepsilon \beta \int_0^t \int_{\Gamma^\varepsilon} \nabla_y^B u_2 \nabla^B \tilde{r}_\varepsilon d\sigma d\tau - \varepsilon \beta \int_0^t \int_{\Gamma^\varepsilon} (\text{div}_x^B \nabla_y^B u_2) \tilde{r}_\varepsilon d\sigma d\tau \\ &= \gamma(\delta'') + \delta'' \varepsilon \int_0^t \int_{\Gamma^\varepsilon} |\nabla^B \tilde{r}_\varepsilon|^2 d\sigma d\tau + \gamma(\delta'') + \delta'' \varepsilon \int_0^t \int_{\Gamma^\varepsilon} \tilde{r}_\varepsilon^2 d\sigma d\tau. \end{aligned}$$

Here, we use

$$\varepsilon \int_0^t \int_{\Gamma^\varepsilon} (|\nabla_y^B u_2|^2 + |\text{div}_x^B \nabla_y^B u_2|^2) d\sigma d\tau \leq \gamma,$$

which is again a consequence of the regularity of $\tilde{\chi}$ and u_0 .

Combining the previous estimates, we have

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \mu^\varepsilon \tilde{r}_\varepsilon^2(x, t) \, dx + \frac{\varepsilon}{2} \int_{\Gamma^\varepsilon} \alpha \tilde{r}_\varepsilon^2(x, t) \, d\sigma + \int_0^t \int_{\Omega} \lambda^\varepsilon |\nabla \tilde{r}_\varepsilon|^2 \, dx \, d\tau \\
& \quad + \varepsilon \beta \int_0^t \int_{\Gamma^\varepsilon} |\nabla^B \tilde{r}_\varepsilon|^2 \, d\sigma \, d\tau \\
& \leq \frac{1}{2} \int_{\Omega} \mu^\varepsilon \tilde{r}_\varepsilon^2(x, 0) \, dx + \frac{\varepsilon}{2} \int_{\Gamma^\varepsilon} \alpha \tilde{r}_\varepsilon^2(x, 0) \, d\sigma - \varepsilon \int_0^t \int_{\Gamma^\varepsilon} \left(\alpha \frac{\partial u_1}{\partial \tau} - \beta \Delta_x^B u_1 \right) \tilde{r}_\varepsilon \, d\sigma \, d\tau \\
& \quad + \gamma(\delta'') + \delta'' \varepsilon \int_0^t \int_{\Gamma^\varepsilon} |\nabla^B \tilde{r}_\varepsilon|^2 \, d\sigma \, d\tau + \delta'' \varepsilon \int_0^t \int_{\Gamma^\varepsilon} \tilde{r}_\varepsilon^2 \, d\sigma \, d\tau \\
& \quad - \int_0^t \int_{\Gamma^\varepsilon} [\lambda^\varepsilon \nabla_y u_2 \cdot \nu_\varepsilon] \tilde{r}_\varepsilon \, d\sigma \, d\tau + \int_0^t \int_{\Gamma^\varepsilon} [\lambda^\varepsilon \nabla_y u_2 \cdot \nu_\varepsilon] \tilde{r}_\varepsilon \, d\sigma \, d\tau \\
& \quad + \int_0^t \int_{\Omega} \lambda^\varepsilon \nabla_y u_2 \cdot \nabla \tilde{r}_\varepsilon \, dx \, d\tau + \gamma(\delta) + \delta \int_0^t \int_{\Omega} \tilde{r}_\varepsilon^2 \, dx \, d\tau \\
& \quad + \delta' \int_0^t \int_{\Omega} |\nabla \tilde{r}_\varepsilon|^2 \, dx \, d\tau + \frac{\gamma(\delta')}{\varepsilon} - \int_0^t \int_{\Omega} \left\{ \mu^\varepsilon \frac{\partial u_1}{\partial \tau} (1 - \phi_\varepsilon) \right\} \tilde{r}_\varepsilon \, dx \, d\tau \\
& \leq \gamma + \gamma(\delta''') + \varepsilon \delta''' \int_0^t \int_{\Gamma^\varepsilon} \tilde{r}_\varepsilon^2 \, d\sigma \, d\tau \\
& \quad + \gamma(\delta'') + \delta'' \varepsilon \int_0^t \int_{\Gamma^\varepsilon} |\nabla^B \tilde{r}_\varepsilon|^2 \, d\sigma \, d\tau + \delta'' \varepsilon \int_0^t \int_{\Gamma^\varepsilon} \tilde{r}_\varepsilon^2 \, d\sigma \, d\tau \\
& \quad + \gamma(\delta''') + \delta''' \int_0^t \int_{\Omega} |\nabla \tilde{r}_\varepsilon|^2 \, dx \, d\tau + \gamma(\delta) + \delta \int_0^t \int_{\Omega} \tilde{r}_\varepsilon^2 \, dx \, d\tau \\
& \quad + \delta' \int_0^t \int_{\Omega} |\nabla \tilde{r}_\varepsilon|^2 \, dx \, d\tau + \frac{\gamma(\delta')}{\varepsilon} + \gamma(\delta''') + \delta''' \int_0^t \int_{\Omega} \tilde{r}_\varepsilon^2 \, dx \, d\tau, \quad (4-12)
\end{aligned}$$

where δ''' will be chosen later. Finally, using Poincaré's inequality, Gronwall's lemma, and absorbing the gradient term in (4-12) into the left-hand side (which is possible choosing $\delta, \delta', \delta'', \delta'''$ sufficiently small), we get

$$\int_0^t \int_{\Omega} |\nabla \tilde{r}_\varepsilon|^2 \, dx \, d\tau \leq \frac{\gamma}{\varepsilon}. \quad (4-13)$$

On recalling the definition of \tilde{r}_ε , and invoking Poincaré's inequality again, we obtain

$$\int_0^t \int_{\Omega} (u_\varepsilon - u_0 - \varepsilon u_1 (1 - \phi_\varepsilon))^2 \, dx \, d\tau \leq \gamma \varepsilon. \quad (4-14)$$

Moreover, taking into account that $r_\varepsilon = \tilde{r}_\varepsilon - u_1 \phi_\varepsilon$ and using (4-13), it follows that

$$\int_0^t \int_{\Omega} |\nabla r_\varepsilon|^2 \, dx \, d\tau \leq \gamma \left[\int_0^t \int_{\Omega} |\nabla \tilde{r}_\varepsilon|^2 \, dx \, d\tau + \int_0^t \int_{\Omega} |\nabla (u_1 \phi_\varepsilon)|^2 \, dx \, d\tau \right] \leq \frac{\gamma}{\varepsilon}, \quad (4-15)$$

where we recall the estimate for $\nabla(u_1\phi_\varepsilon)$ done in (4-10). Hence, by (4-14) and (4-15), we obtain (4-1). Finally, (4-2) can be obtained making use of (4-14) and taking into account that

$$\int_0^t \int_{\Omega} (\varepsilon u_1(1 - \phi_\varepsilon))^2 dx d\tau \leq \gamma \varepsilon^2. \quad \square$$

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