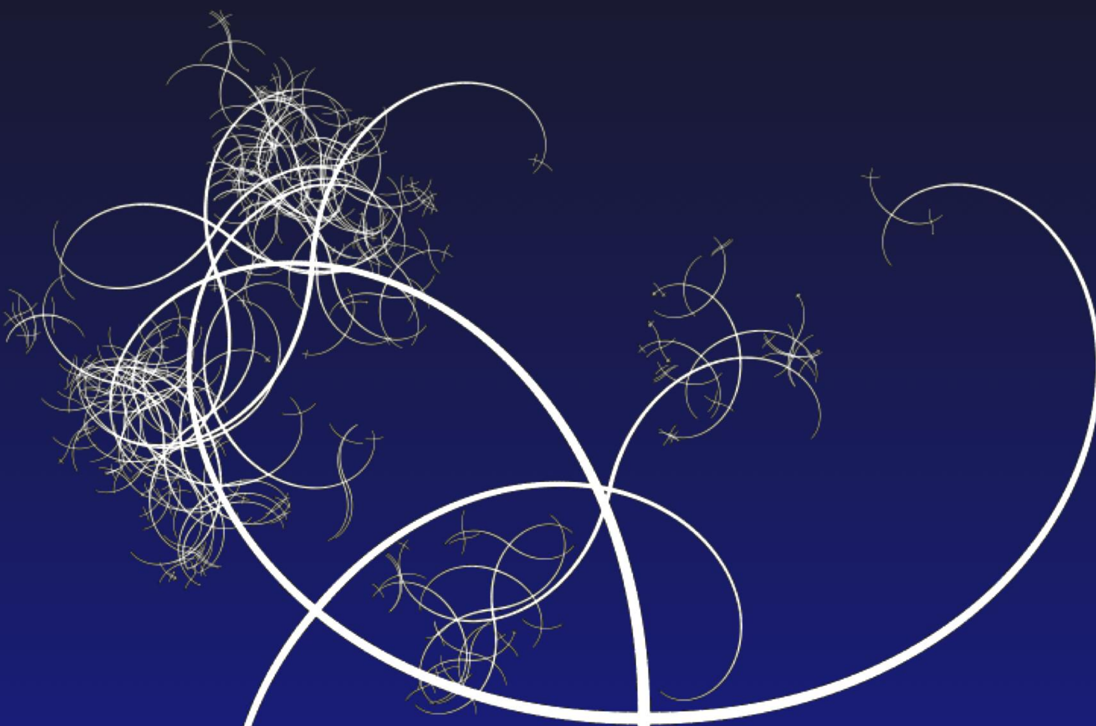
NISSUNA UMANA INVESTIGAZIONE SI PUO DIMANDARE VERA SCIENZIA
S'ESSA NON PASSA PER LE MATEMATICHE DIMOSTRAZIONI
LEONARDO DA VINCI



vol. 7 no. 1 2019

Mathematics and Mechanics of Complex Systems

Farid Messelmi

ON THE BLOCKING LIMIT OF STEADY-STATE FLOW OF HERSCHEL–BULKLEY FLUID

msp

MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS
Vol. 7, No. 1, 2019
dx.doi.org/10.2140/memocs.2019.7.63

ON THE BLOCKING LIMIT OF STEADY-STATE FLOW OF HERSCHEL–BULKLEY FLUID

FARID MESSELMI

This paper is devoted to the study of the blocking limit of Herschel–Bulkley fluid in the case of steady-state flow. To this aim, we consider a mathematical model which describes the steady-state flow of a Herschel–Bulkley fluid in a bounded domain. We give the mathematical formulation of the blockage phenomenon, and we establish the existence of the blocking limit. We also focus on behaviour of the flow with respect to the blocking limit.

1. Introduction

The rigid viscoplastic and incompressible fluid of Herschel and Bulkley has been investigated by mathematicians, physicists, and engineers as intensively as the Navier–Stokes equations though this model adequately describes a large class of flows. It has been used to model the flow of metals, plastic solids, and a variety of polymers. Physical experiments and numerical studies of the flow of Herschel–Bulkley fluids prove that when the yield stress increases, the rigid zones become larger and may completely block the flow. This property is called the blocking phenomenon. Due to existence of the yield limit, the model can capture phenomena connected with the development of discontinuous stresses. The literature concerning this topic is extensive; see, e.g., [Málek 2008; Málek et al. 2006; 2005; Messelmi 2017; Messelmi and Merouani 2013; 2010; Messelmi et al. 2010].

Our paper deals with the steady-state flow of Herschel and Bulkley. The main objective is the study of the behaviour of the flow. We provide a generalisation of a result obtained by Hild et al. [2002] for Bingham fluid to the steady-state flow of the Herschel–Bulkley model, ensuring the existence of the blocking limit. Moreover, we establish a result concerning the behaviour of the flow when the yield limit is near a minimal blocking limit.

The paper is organised as follows. In Section 2 we present the mechanical problem of the steady-state flow of Herschel–Bulkley fluid in a bounded domain $\Omega \subset \mathbb{R}^n$. We introduce some notation and preliminaries. In addition, we derive

Communicated by Carlo Marchioro.
MSC2010: 35J85, 76A05, 76E30.
Keywords: blocking limit, Herschel–Bulkley fluid, variational inequality.

the variational formulation of the problem. In Section 3, we show the mathematical formulation of the blockage phenomenon and we prove the existence of the blocking limit. Section 4 is devoted to the study of the behaviour of the flow with respect to the blocking limit.

2. Problem statement

We consider a mathematical problem modelling the steady-state flow of the rigid viscoplastic and incompressible Herschel–Bulkley fluid in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$, with the boundary Γ of class C^1. The fluid is acted upon by given volume forces of density $\boldsymbol{f}$. On Γ we suppose that the velocity is equal to zero.

We denote by S_n the space of symmetric tensors on $\mathbb{R}^n$. We define the inner product and the Euclidean norm on $\mathbb{R}^n$ and S_n, respectively, by

$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{v} &= u_i v_i \quad \text{for all } \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^n, & \boldsymbol{\sigma}\cdot\boldsymbol{\tau} &= \sigma_{ij}\tau_{ij} \quad \text{for all } \boldsymbol{\sigma},\boldsymbol{\tau}\in S_n,\\ |\boldsymbol{u}| &= (\boldsymbol{u}\cdot\boldsymbol{u})^{1/2} \quad \text{for all } \boldsymbol{u}\in\mathbb{R}^n, & |\boldsymbol{\sigma}| &= (\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})^{1/2} \quad \text{for all } \boldsymbol{\sigma}\in\mathbb{S}_n. \end{aligned}$$

Here and below, the indices i and j run from 1 to n and the summation convention over repeated indices is used. We denote by $\boldsymbol{\sigma}^D$ the deviator of $\boldsymbol{\sigma} = (\sigma_{ij})$ given by

$$\boldsymbol{\sigma}^D = (\sigma_{ij}^D), \quad \sigma_{ij}^D = \sigma_{ij} - \frac{\sigma_{kk}}{n}\delta_{ij},$$

where $\boldsymbol{\delta} = (\delta_{ij})$ denotes the identity tensor.

Let $1 < p \le 2$. We consider the rate-of-deformation operator defined for every $\boldsymbol{u} \in W^{1,p}(\Omega)^n$ by

$$D(\boldsymbol{u}) = (D_{ij}(\boldsymbol{u})), \quad D_{ij}(\boldsymbol{u}) = \tfrac{1}{2}(u_{i,j} + u_{j,i}).$$

The steady-state flow of Herschel–Bulkley fluid can be described by the following mechanical problem.

Problem $\mathbf{P}_1$. Find the velocity field $\boldsymbol{u} = (u_i) : \Omega \to \mathbb{R}^n$ and the stress field $\boldsymbol{\sigma} = (\sigma_{ij}) : \Omega \to S_n$ such that

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f} \quad \text{in } \Omega. \tag{2-1}$$

$$\left.\begin{aligned} \boldsymbol{\sigma}^D &= \mu|D(\boldsymbol{u})|^{p-2}D(\boldsymbol{u}) + g(D(\boldsymbol{u})/|D(\boldsymbol{u})|) && \text{if } |D(\boldsymbol{u})| \neq 0,\\ |\boldsymbol{\sigma}^D| &\le g && \text{if } |D(\boldsymbol{u})| = 0 \end{aligned}\right\} \quad \text{in } \Omega, \tag{2-2}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2-3}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma. \tag{2-4}$$

Here $\operatorname{div}\boldsymbol{\sigma} = (\sigma_{ij,j})$ and $\operatorname{div}\boldsymbol{u} = u_{i,i}$. The flow is given by (2-1) where the density is assumed equal to one. Equation (2-2) represents the constitutive law of Herschel–Bulkley fluid where $\mu > 0$ and $g \ge 0$ represent the consistency and yield limit of the fluid, respectively, and $1 < p \leq 2$ is the power law index. Equation (2-3) represents the incompressibility condition. Equation (2-4) gives the adherence condition on the boundary Γ.

Existence of weak solutions for this problem was proved in 1969 for $p \geq 3n/(n+2)$, for which the energy equality holds and higher differentiability techniques can be applied, in 1997 for $p \geq 2n/(n+1)$, and recently for $p > 2n/(n+2)$ using the Lipschitz truncation method. Moreover, in 2010 some existence results regarding the thermal flow were established for the case $p \geq 3n/(n+2)$ [Frehse et al. 2003; Lions 1969; Málek 2008; Málek et al. 2006; Messelmi et al. 2010]. Up to now, there are only a few results concerning the regularity of weak solutions, especially in three-dimensional domains. Further, the asymptotic behaviour of the unsteady flow was the subject of [Messelmi 2017].

Remark. (1) The Bingham fluid represents a particular case of Herschel–Bulkley fluid corresponding to $p = 2$.

(2) In the constitutive law of Herschel–Bulkley fluid (2-2), the viscosity and hydrostatic pressure are given, respectively, by

$$\eta = \mu |D(\boldsymbol{u})|^{p-2}, \qquad \pi = -\frac{1}{n}\sigma_{kk}. \tag{2-5}$$

Let us introduce the function spaces

$$W_{p,\text{div}} = \{\boldsymbol{v} \in W_0^{1,p}(\Omega)^n : \text{div}(\boldsymbol{v}) = 0 \text{ in } \Omega\}, \tag{2-6}$$

$$LD(\Omega) = \{\boldsymbol{v} \in L^1(\Omega)^n : D(\boldsymbol{v}) \in L^1(\Omega)^{n\times n}\}, \tag{2-7}$$

$$VD(\Omega) = \{\boldsymbol{v} \in LD(\Omega) : v = 0 \text{ on } \Gamma\}, \tag{2-8}$$

$$W = \{\boldsymbol{v} \in VD(\Omega) : \text{div}\, \boldsymbol{v} = 0 \text{ in } \Omega\}. \tag{2-9}$$

$W_{p,\text{div}}$ is a Banach space equipped with the norm

$$\|\boldsymbol{v}\|_{W_{p,\text{div}}} = \|\boldsymbol{v}\|_{W^{1,p}(\Omega)^n}. \tag{2-10}$$

Moreover, Korn's inequality holds in the space $W_{p,\text{div}}$ [Messelmi et al. 2010], which means that there exists a positive constant C_0 depending only on Ω and Γ such that

$$C_0 \|D(\boldsymbol{v})\|_{L^p(\Omega)^{n\times n}} \geq \|\boldsymbol{v}\|_{W_{p,\text{div}}} \quad \text{for all } \boldsymbol{v} \in W_{p,\text{div}}. \tag{2-11}$$

The space $LD(\Omega)$ was introduced by Temam [1985]. It is a Banach space equipped with the norm

$$\|\boldsymbol{v}\|_{LD(\Omega)} = \|\boldsymbol{v}\|_{L^1(\Omega)^n} + \|D(\boldsymbol{v})\|_{L^1(\Omega)^{n\times n}}, \tag{2-12}$$

which is not reflexive, and $W^{1,1}(\Omega)^n \subset LD(\Omega)$. Since Korn's inequality does not hold on $LD(\Omega)$ (see the remarks in [Temam 1985]), the space $W^{1,1}(\Omega)^n$ is a proper subspace of $LD(\Omega)$. $VD(\Omega)$ is a closed subspace of $LD(\Omega)$. W is also a Banach space equipped with the norm given by (2-12). Furthermore, Korn's inequality holds in W [Temam 1985], and thus, there exists a positive constant C_W depending only on Ω and Γ such that

$$C_W \|D(\boldsymbol{v})\|_{L^1(\Omega)^{n\times n}} \geq \|\boldsymbol{v}\|_{LD(\Omega)} \quad \text{for all } \boldsymbol{v} \in W. \tag{2-13}$$

Denoting by p' the conjugate of p, we introduce the convective operator

$$B : W_{p,\mathrm{div}} \times W_{p,\mathrm{div}} \times W_{p,\mathrm{div}} \to \mathbb{R}, \quad B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{w}\, dx. \tag{2-14}$$

We begin by recalling the following lemma [Messelmi et al. 2010], which gives some properties of the convective operator B.

Lemma. *Suppose that*

$$\frac{3n}{n+2} \leq p \leq 2. \tag{2-15}$$

Then B is trilinear and continuous on $W_{p,\mathrm{div}} \times W_{p,\mathrm{div}} \times W_{p,\mathrm{div}}$. Moreover, for all $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in W_{p,\mathrm{div}} \times W_{p,\mathrm{div}} \times W_{p,\mathrm{div}}$ we have $B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -B(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})$.

For the rest of this paper, we choose $3n/(n+2) \leq p \leq 2$. The use of Green's formula permits us to derive the following variational formulation of the mechanical problem P_1 [Messelmi et al. 2010].

Problem $\mathbf{P}_g$. For prescribed data $\boldsymbol{f} \in W'_{p,\mathrm{div}}$, find $\boldsymbol{u} \in W_{p,\mathrm{div}}$ satisfying the variational inequality

$$B(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + \mu \int_\Omega |D(\boldsymbol{u})|^{p-2} D(\boldsymbol{u}) \cdot D(\boldsymbol{v}-\boldsymbol{u})\, dx \\ + g \int_\Omega |D(\boldsymbol{v})|\, dx - g \int_\Omega |D(\boldsymbol{u})|\, dx \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v}-\boldsymbol{u})\, dx \quad \text{for all } \boldsymbol{v} \in W_{p,\mathrm{div}}. \tag{2-16}$$

By taking $\boldsymbol{v} = 0$ and $\boldsymbol{v} = 2\boldsymbol{u}$ in (2-16), respectively,

$$\mu \int_\Omega |D(\boldsymbol{u})|^p\, dx + g \int_\Omega |D(\boldsymbol{u})|\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}\, dx. \tag{2-17}$$

This implies using again (2-16)

$$B(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \mu \int_\Omega |D(\boldsymbol{u})|^{p-2} D(\boldsymbol{u}) \cdot D(\boldsymbol{v})\, dx + g \int_\Omega |D(\boldsymbol{v})|\, dx \\ \geq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \text{for all } \boldsymbol{v} \in W_{p,\mathrm{div}}. \tag{2-18}$$

Consequently, the steady-state flow of Herschel–Bulkley fluid can be also described by the system (2-17)–(2-18).

3. Blockage property

This section is dedicated to the study of the blockage property of Herschel–Bulkley fluid. To do this, let us recall the following standard definition [Hild et al. 2002].

Definition. We will say that the fluid is blocked in the domain Ω if $\boldsymbol{u} = 0$ a.e. in Ω is a solution to the variational problem P_g.

We prove the following proposition, which gives the variational interpretation of the blockage property.

Proposition. *The fluid is blocked in the domain* Ω *if and only if*

$$g \int_\Omega |D(\boldsymbol{v})|\, dx \geq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \textit{for all } \boldsymbol{v} \in W_{p,\mathrm{div}}. \tag{3-1}$$

Proof. The first implication is an immediate consequence of the definition of blockage property. For the second one, we proceed as follows. Suppose that (3-1) holds. In particular, we have

$$g \int_\Omega |D(\boldsymbol{u})|\, dx \geq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}\, dx. \tag{3-2}$$

Subtracting the inequalities (2-17) and (3-1), we find

$$\mu \int_\Omega |D(\boldsymbol{u})|^p\, dx \leq B(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \mu \int_\Omega |D(\boldsymbol{u})|^{p-2} D(\boldsymbol{u}) \cdot D(\boldsymbol{v})\, dx \\ + g \int_\Omega |D(\boldsymbol{v})|\, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \text{for all } \boldsymbol{v} \in W_{p,\mathrm{div}}. \tag{3-3}$$

Thus, the result can be obtained by setting $\boldsymbol{v} = 0$ as a test function in (3-3) and using Korn's inequality. □

Hence, the mathematical study of the blockage property consists of finding the relationship between the yield limit g and the density of volume forces $\boldsymbol{f}$ such that the inequality (3-1) holds.

We say that g is a blocking limit if the inequality (3-1) is satisfied.

We suppose from now on that

$$\boldsymbol{f} \in L^\infty(\Omega)^n. \tag{3-4}$$

The statement below ensures the existence of a blocking phase for large-enough yield limit.

Proposition. *If* (3-4) *holds*, *then*

$$g^* = \sup_{\boldsymbol{v} \in W_{p,\mathrm{div}} - \{0\}} \frac{\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx}{\int_\Omega |D(\boldsymbol{v})|\, dx} < +\infty. \tag{3-5}$$

In addition, *if* $g \geq g^*$, *then the blocking occurs*; *it means that* (3-1) *holds.*

Proof. Let us define the form $l \in W'_{p,\mathrm{div}}$ by

$$\langle l, \boldsymbol{v}\rangle_{W'_{p,\mathrm{div}} \times W_{p,\mathrm{div}}} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \text{for all } \boldsymbol{v} \in W_{p,\mathrm{div}}. \tag{3-6}$$

The fact that $\boldsymbol{f} \in L^\infty(\Omega)^n$ implies that $l \in W'$. Then, there exists $C_1 > 0$ such that

$$|\langle l, \boldsymbol{v}\rangle_{W'_{p,\mathrm{div}} \times W_{p,\mathrm{div}}}| \le C_1 \|\boldsymbol{v}\|_{LD(\Omega)} \quad \text{for all } \boldsymbol{v} \in LD(\Omega). \tag{3-7}$$

This yields, thanks to the Korn inequality (2-13)

$$|\langle l, \boldsymbol{v}\rangle_{W'_{p,\mathrm{div}} \times W_{p,\mathrm{div}}}| \le C_1 C_W \|D(\boldsymbol{v})\|_{L^1(\Omega)^{n\times n}} \quad \text{for all } \boldsymbol{v} \in LD(\Omega). \tag{3-8}$$

Consequently, via (3-7) and (3-8) we obtain $g^* \le C_1 C_W$.

Now, if $g \ge g^*$, then (3-5) gives

$$g \int_\Omega |D(\boldsymbol{v})|\, dx \ge g^* \int_\Omega |D(\boldsymbol{v})|\, dx \ge \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \text{for all } \boldsymbol{v} \in W_{p,\mathrm{div}} - \{0\},$$

which completes the proof, by observing that if $\boldsymbol{v} = 0$, the inequality above also remains satisfied. □

Here g^* is said to be the minimal blocking limit.

Let g be a blocking limit. We denote by C the set

$$C = \left\{ \boldsymbol{v} \in W_{p,\mathrm{div}} \;\middle|\; g \int_\Omega |D(\boldsymbol{v})|\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \right\}. \tag{3-9}$$

It is straightforward to verify that the set C is a convex cone in $W_{p,\mathrm{div}}$.

4. Behaviour of the flow

Let us introduce for $\varepsilon > 0$ the perturbed yield limit

$$g_\varepsilon = (1 - \varepsilon^{p-1}) g, \tag{4-1}$$

and denote by $\boldsymbol{u}_\varepsilon$ the solution of the corresponding problem, i.e.,

$$B(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon) + \mu \int_\Omega |D(\boldsymbol{u}_\varepsilon)|^{p-2} D(\boldsymbol{u}_\varepsilon) \cdot D(\boldsymbol{v} - \boldsymbol{u}_\varepsilon)\, dx + g_\varepsilon \int_\Omega |D(\boldsymbol{v})|\, dx \\ - g_\varepsilon \int_\Omega |D(\boldsymbol{u}_\varepsilon)|\, dx \ge \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}_\varepsilon)\, dx \quad \text{for all } \boldsymbol{v} \in W_{p,\mathrm{div}}. \tag{4-2}$$

The above inequality can be written in equivalent form

$$\mu \int_\Omega |D(\boldsymbol{u}_\varepsilon)|^p \, dx + g_\varepsilon \int_\Omega |D(\boldsymbol{u}_\varepsilon)| \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}_\varepsilon \, dx, \tag{4-3}$$

$$B(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{v}) + \mu \int_\Omega |D(\boldsymbol{u}_\varepsilon)|^{p-2} D(\boldsymbol{u}_\varepsilon) \cdot D(\boldsymbol{v}) \, dx + g_\varepsilon \int_\Omega |D(\boldsymbol{v})| \, dx \geq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx \quad \text{for all } \boldsymbol{v} \in W_{p,\mathrm{div}}. \tag{4-4}$$

Setting now

$$\boldsymbol{w}_\varepsilon = \frac{\boldsymbol{u}_\varepsilon}{\varepsilon} \quad \text{for all } \varepsilon > 0, \tag{4-5}$$

in the following we establish a convergence result for $(\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ when ε tends to 0.

Theorem. *Suppose that g is a blocking limit. Then $(\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ convergences strongly, when ε tends to* 0 *in $W_{p,\mathrm{div}}$, to the solution $\boldsymbol{w}$ of the variational inequality*

$$w \in C : \mu \int_\Omega |D(\boldsymbol{w})|^{p-2} D(\boldsymbol{w}) \cdot D(\boldsymbol{v} - \boldsymbol{w}) \, dx \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{w}) \, dx \ \textit{for all } \boldsymbol{v} \in C. \tag{4-6}$$

Proof. The system becomes, taking into account (4-5),

$$\mu \varepsilon^{p-1} \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^p \, dx + (1 - \varepsilon^{p-1}) g \int_\Omega |D(\boldsymbol{w}_\varepsilon)| \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}_\varepsilon \, dx, \tag{4-7}$$

$$\varepsilon^2 B(\boldsymbol{w}_\varepsilon, \boldsymbol{w}_\varepsilon, \boldsymbol{v}) + \mu \varepsilon^{p-1} \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^{p-2} D(\boldsymbol{w}_\varepsilon) \cdot D(\boldsymbol{v}) \, dx + (1 - \varepsilon^{p-1}) g \int_\Omega |D(\boldsymbol{v})| \, dx \geq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx \quad \text{for all } \boldsymbol{v} \in W_{p,\mathrm{div}}. \tag{4-8}$$

Equation (4-7) gives

$$\mu \varepsilon^{p-1} \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^p \, dx + (1 - \varepsilon^{p-1}) \left(g \int_\Omega |D(\boldsymbol{w}_\varepsilon)| \, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}_\varepsilon \, dx \right) = \varepsilon^{p-1} g \int_\Omega |D(\boldsymbol{w}_\varepsilon)| \, dx. \tag{4-9}$$

Suppose that g is a blocking limit; then (4-9) gives

$$\mu \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^p \, dx \leq g \int_\Omega |D(\boldsymbol{w}_\varepsilon)| \, dx. \tag{4-10}$$

We deduce making use of Korn's inequality and some algebraic manipulations that

$$\|\boldsymbol{w}_\varepsilon\|_{W_{p,\mathrm{div}}} \leq c. \tag{4-11}$$

Hence, we can extract a subsequence still denoted by $(\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ such that

$$\boldsymbol{w}_\varepsilon \to \boldsymbol{w} \quad \text{in } W_{p,\mathrm{div}} \text{ weakly}. \tag{4-12}$$

The Rellich–Kondrachov compactness theorem allows us to get after a new extraction

$$\boldsymbol{w}_\varepsilon \to \boldsymbol{w} \quad \text{in } L^p(\Omega)^n \text{ strongly and a.e. in } \Omega. \tag{4-13}$$

Therefore, (4-9) gives again

$$(1-\varepsilon^{p-1})g \int_\Omega |D(\boldsymbol{w}_\varepsilon)|\,dx \le \int_\Omega \boldsymbol{f}\cdot \boldsymbol{w}_\varepsilon\,dx,$$

thereby allowing us to find

$$g \liminf \int_\Omega |D(\boldsymbol{w}_\varepsilon)|\,dx \le \lim \int_\Omega \boldsymbol{f}\cdot \boldsymbol{w}_\varepsilon\,dx. \tag{4-14}$$

This yields

$$g \int_\Omega |D(\boldsymbol{w})|\,dx \le \int_\Omega \boldsymbol{f}\cdot \boldsymbol{w}\,dx. \tag{4-15}$$

Consequently, since g is a blocking limit,

$$g \int_\Omega |D(\boldsymbol{w})|\,dx = \int_\Omega \boldsymbol{f}\cdot \boldsymbol{w}\,dx. \tag{4-16}$$

Taking $\boldsymbol{w}$ as test function in inequality (4-8), it implies that

$$\varepsilon^2 B(\boldsymbol{w}_\varepsilon, \boldsymbol{w}_\varepsilon, \boldsymbol{w}) + \mu\varepsilon^{p-1} \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^{p-2} D(\boldsymbol{w}_\varepsilon)\cdot D(\boldsymbol{w})\,dx \\ + (1-\varepsilon^{p-1})g \int_\Omega |D(\boldsymbol{w})|\,dx \ge \int_\Omega \boldsymbol{f}\cdot \boldsymbol{w}\,dx.$$

This gives, making use of (4-16),

$$\varepsilon^{3-p} B(\boldsymbol{w}_\varepsilon, \boldsymbol{w}_\varepsilon, \boldsymbol{w}) + \mu \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^{p-2} D(\boldsymbol{w}_\varepsilon)\cdot D(\boldsymbol{w})\,dx \ge g \int_\Omega |D(\boldsymbol{w})|\,dx. \tag{4-17}$$

Moreover, the lemma on page 66 permits us to obtain the estimate

$$|B(\boldsymbol{w}_\varepsilon, \boldsymbol{w}_\varepsilon, \boldsymbol{w})| \le \|\boldsymbol{w}_\varepsilon\|^2_{W_{p,\mathrm{div}}} \|\boldsymbol{w}\|_{W_{p,\mathrm{div}}}. \tag{4-18}$$

On the other hand, it is well known that the nonlinear term $\mu \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^{p-2} D(\boldsymbol{w}_\varepsilon)\cdot D(\boldsymbol{w})\,dx$ converges to $\mu \int_\Omega |D(\boldsymbol{w})|^p\,dx$ [Lions 1969]. Consequently, by passing to the limit, one can find, keeping in mind (4-18),

$$\mu \int_\Omega |D(\boldsymbol{w})|^p\,dx \ge g \int_\Omega |D(\boldsymbol{w})|\,dx. \tag{4-19}$$

We get thanks to (4-10)

$$\liminf \mu \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^p\,dx \le g \liminf \int_\Omega |D(\boldsymbol{w}_\varepsilon)|\,dx.$$

So, using (4-14) we can infer that

$$\liminf \mu \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^p \, dx \le \lim \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}_\varepsilon \, dx,$$

which implies that

$$\mu \int_\Omega |D(\boldsymbol{w})|^p \, dx \le \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} \, dx. \tag{4-20}$$

Putting together (4-16), (4-19), and (4-20) we obtain

$$\mu \int_\Omega |D(\boldsymbol{w})|^p \, dx = g \int_\Omega |D(\boldsymbol{w})| \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} \, dx, \tag{4-21}$$

which implies in particular that $\boldsymbol{w} \in C$. Furthermore, by (4-8) we get

$$\varepsilon^{3-p} B(\boldsymbol{w}_\varepsilon, \boldsymbol{w}_\varepsilon, \boldsymbol{v}) + \mu \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^{p-2} D(\boldsymbol{w}_\varepsilon) \cdot D(\boldsymbol{v}) \, dx$$
$$+ \frac{1}{\varepsilon^{p-1}} \left(g \int_\Omega |D(\boldsymbol{v})| \, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx \right) \ge g \int_\Omega |D(\boldsymbol{v})| \, dx \quad \text{for all } \boldsymbol{v} \in W_{p,\mathrm{div}}.$$

By choosing $\boldsymbol{v} \in C$ in the above inequality, the passage to the limit leads to

$$\mu \int_\Omega |D(\boldsymbol{w})|^{p-2} D(\boldsymbol{w}) \cdot D(\boldsymbol{v}) \, dx \ge g \int_\Omega |D(\boldsymbol{v})| \, dx \quad \text{for all } \boldsymbol{v} \in C.$$

This yields

$$\mu \int_\Omega |D(\boldsymbol{w})|^{p-2} D(\boldsymbol{w}) \cdot D(\boldsymbol{v}) \, dx \ge \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx \quad \text{for all } \boldsymbol{v} \in C. \tag{4-22}$$

Combining (4-21) and (4-22) yields the inequality (4-6).

Our objective now is to prove the strong convergence. With this aim, we proceed as follows. The use of (4-7) and (4-8) permits us to affirm that for every $\boldsymbol{v} \in W_{p,\mathrm{div}}$

$$\varepsilon^2 B(\boldsymbol{w}_\varepsilon, \boldsymbol{w}_\varepsilon, \boldsymbol{v}) + \mu \varepsilon^{p-1} \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^{p-2} D(\boldsymbol{w}_\varepsilon) \cdot D(\boldsymbol{v} - \boldsymbol{w}_\varepsilon) \, dx$$
$$\ge \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{w}_\varepsilon) \, dx - (1 - \varepsilon^{p-1}) g \left(\int_\Omega |D(\boldsymbol{v})| \, dx - \int_\Omega |D(\boldsymbol{w}_\varepsilon)| \, dx \right)$$

It follows, by setting $\boldsymbol{v} = \boldsymbol{w}$, that

$$-\varepsilon^2 B(\boldsymbol{w}_\varepsilon, \boldsymbol{w}_\varepsilon, \boldsymbol{w}) + \mu \varepsilon^{p-1} \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^{p-2} D(\boldsymbol{w}_\varepsilon) \cdot D(\boldsymbol{w}_\varepsilon - \boldsymbol{w}) \, dx$$
$$\le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w}_\varepsilon - \boldsymbol{w}) \, dx + (1 - \varepsilon^{p-1}) g \int_\Omega (|D(\boldsymbol{w})| - |D(\boldsymbol{w}_\varepsilon)|) \, dx. \tag{4-23}$$

Further, since g is the blocking limit and $\boldsymbol{w} \in W_{p,\mathrm{div}}$, one can verify that

$$g \int_\Omega (|D(\boldsymbol{w})| - |D(\boldsymbol{w}_\varepsilon)|) \, dx \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{w}_\varepsilon) \, dx.$$

Consequently, inequality (4-23) becomes

$$\mu \int_\Omega |D(\boldsymbol{w}_\varepsilon)|^{p-2} D(\boldsymbol{w}_\varepsilon) \cdot D(\boldsymbol{w}_\varepsilon - \boldsymbol{w})\, dx \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w}_\varepsilon - \boldsymbol{w})\, dx + \varepsilon^{3-p} B(\boldsymbol{w}_\varepsilon, \boldsymbol{w}_\varepsilon, \boldsymbol{w}). \tag{4-24}$$

This becomes

$$\begin{aligned} \mu \int_\Omega (|D(\boldsymbol{w}_\varepsilon)|^{p-2} D(\boldsymbol{w}_\varepsilon) - |D(\boldsymbol{w})|^{p-2} D(\boldsymbol{w})) \cdot D(\boldsymbol{w}_\varepsilon - \boldsymbol{w})\, dx \\ \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w}_\varepsilon - \boldsymbol{w})\, dx - \mu \int_\Omega |D(\boldsymbol{w})|^{p-2} D(\boldsymbol{w}) D(\boldsymbol{w}_\varepsilon - \boldsymbol{w})\, dx \\ + c\varepsilon^{3-p} \|\boldsymbol{w}_\varepsilon\|^2_{W_{p,\mathrm{div}}} \|\boldsymbol{w}\|_{W_{p,\mathrm{div}}}. \end{aligned} \tag{4-25}$$

Let us observe now that for every $x, y \in \mathbb{R}^n$,

$$(|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \ge c \frac{|x-y|^2}{(|x| + |y|)^{2-p}}, \quad 1 < p \le 2.$$

So applying the above inequality, (4-25) can be rewritten as

$$\begin{aligned} \mu \int_\Omega \frac{|D(\boldsymbol{w}_\varepsilon - \boldsymbol{w})|^2}{(|D(\boldsymbol{w}_\varepsilon)| + |D(\boldsymbol{w})|)^{2-p}}\, dx \le c \left| \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w}_\varepsilon - \boldsymbol{w})\, dx \right| \\ + c\varepsilon^{3-p} \|\boldsymbol{w}_\varepsilon\|^2_{W_{p,\mathrm{div}}} \|\boldsymbol{w}\|_{W_{p,\mathrm{div}}} + c\mu \left| \int_\Omega |D(\boldsymbol{w})|^{p-2} D(\boldsymbol{w}) D(\boldsymbol{w}_\varepsilon - \boldsymbol{w})\, dx \right|, \end{aligned}$$

which gives, exploiting Korn's and Hölder's inequalities,

$$\begin{aligned} \|\boldsymbol{w}_\varepsilon - \boldsymbol{w}\|^p_{W_p} \le c \left(\int_\Omega (|D(\boldsymbol{w}_m)| + |D(\boldsymbol{w})|)^p\, dx \right)^{(2-p)/2} \left(\left| \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w}_\varepsilon - \boldsymbol{w})\, dx \right| \right. \\ \left. + \varepsilon^{3-p} \|\boldsymbol{w}_\varepsilon\|^2_{W_{p,\mathrm{div}}} \|\boldsymbol{w}\|_{W_{p,\mathrm{div}}} + \mu \int_\Omega |D(\boldsymbol{w})|^{p-2} D(\boldsymbol{w}) D(\boldsymbol{w}_\varepsilon - \boldsymbol{w})\, dx \right)^{p/2}. \end{aligned}$$

Passing to the limit, we conclude, using (4-12) and taking into account the fact that $|D(\boldsymbol{w})|^{p-2} D(\boldsymbol{w})$ is bounded in $L^{p'}(\Omega)^n$, that

$$\boldsymbol{w}_\varepsilon \to \boldsymbol{w} \quad \text{in } W_{p,\mathrm{div}} \text{ strongly},$$

which permits us to complete the proof. □

Corollary. *Denoting by $\boldsymbol{w}_0$ the unique solution of the variational equation given by*

$$\mu \int_\Omega |D(\boldsymbol{w}_0)|^{p-2} D(\boldsymbol{w}_0) \cdot D(\boldsymbol{v})\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \text{for all } \boldsymbol{v} \in W_{p,\mathrm{div}}, \tag{4-26}$$

then the following estimates hold:

$$\|D(\boldsymbol{w})\|_{L^p(\Omega)^{n\times n}} \le \|D(\boldsymbol{w}_0)\|_{L^p(\Omega)^{n\times n}}, \qquad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}\, dx \le \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}_0\, dx. \tag{4-27}$$

Proof. We can infer by setting $\boldsymbol{w}$ as a test function in (4-26) that

$$\mu \int_\Omega |D(\boldsymbol{w}_0)|^{p-2} D(\boldsymbol{w}_0) \cdot D(\boldsymbol{w})\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}\, dx.$$

This yields, using Hölder's inequality

$$\mu \| D(\boldsymbol{w}_0)\|^{p-1}_{L^p(\Omega)^{n\times n}} \| D(\boldsymbol{w})\|_{L^p(\Omega)^{n\times n}} \geq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}\, dx = \mu \| D(\boldsymbol{w})\|^{p}_{L^p(\Omega)^{n\times n}},$$

which allows us to get the first estimate. The second estimate becomes an immediate consequence of the first one. □

References

[Frehse et al. 2003] J. Frehse, J. Málek, and M. Steinhauer, "On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method", *SIAM J. Math. Anal.* **34**:5 (2003), 1064–1083.

[Hild et al. 2002] P. Hild, I. R. Ionescu, T. Lachand-Robert, and I. Roşca, "The blocking of an inhomogeneous Bingham fluid: applications to landslides", *M2AN Math. Model. Numer. Anal.* **36**:6 (2002), 1013–1026.

[Lions 1969] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.

[Málek 2008] J. Málek, "Mathematical properties of flows of incompressible power-law-like fluids that are described by implicit constitutive relations", *Electron. Trans. Numer. Anal.* **31** (2008), 110–125.

[Málek et al. 2005] J. Málek, M. Růžička, and V. V. Shelukhin, "Herschel–Bulkley fluids: existence and regularity of steady flows", *Math. Models Methods Appl. Sci.* **15**:12 (2005), 1845–1861.

[Málek et al. 2006] J. Málek, D. Pražák, and M. Steinhauer, "On the existence and regularity of solutions for degenerate power-law fluids", *Differ. Integral Equ.* **19**:4 (2006), 449–462.

[Messelmi 2017] F. Messelmi, "Asymptotic behavior of unsteady Herschel–Bulkley flow", *An. Univ. Oradea Fasc. Mat.* **24**:2 (2017), 181–194.

[Messelmi and Merouani 2010] F. Messelmi and B. Merouani, "Stationary thermal flow of a Bingham fluid whose viscosity, yield limit and friction depend on the temperature", *An. Univ. Oradea Fasc. Mat.* **17**:2 (2010), 59–74.

[Messelmi and Merouani 2013] F. Messelmi and A. Merouani, "Properties of the laminar flow of Herschel–Bulkley fluid", *An. Univ. Oradea Fasc. Mat.* **20**:1 (2013), 47–60.

[Messelmi et al. 2010] F. Messelmi, B. Merouani, and F. Bouzeghaya, "Steady-state thermal Herschel–Bulkley flow with Tresca's friction law", *Electron. J. Differential Equations* **2010**:46 (2010).

[Temam 1985] R. Temam, *Mathematical problems in plasticity*, Mathematical Methods of Information Science **12**, Gauthier-Villars, Paris, 1985.

Received 2 Sep 2018. Revised 10 Nov 2018. Accepted 11 Dec 2018.

FARID MESSELMI: foudimath@yahoo.fr
Department of Mathematics and LDMM Laboratory, University Ziane Achour of Djelfa, Djelfa, Algeria

MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS

msp.org/memocs

MEMOCS (ISSN 2325-3444 electronic, 2326-7186 printed) is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

Cover image: "Tangle" by © John Horigan; produced using the *Context Free* program (contextfreeart.org).

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS
vol. 7 no. 1 2019

MEMOCS is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

2326-7186(2019)7:1;1-6