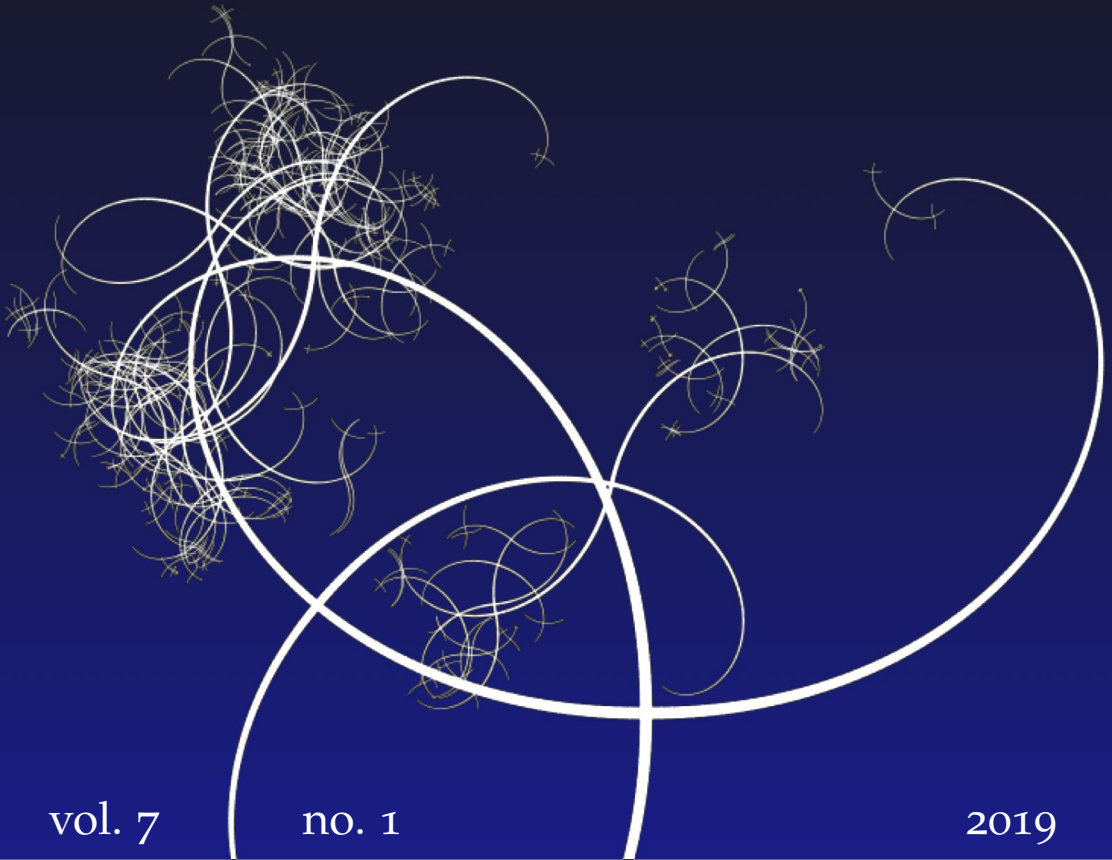


NISSUNA UMANA INVESTIGAZIONE SI PUO DIMANDARE VERA SCIENZA  
S'ESSA NON PASSA PER LE MATEMATICHE DIMOSTRAZIONI  
LEONARDO DA VINCI



vol. 7

no. 1

2019

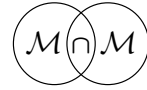
MATHEMATICS AND MECHANICS  
*of*  
**Complex Systems**

FARID MESSELM

ON THE BLOCKING LIMIT OF STEADY-STATE FLOW  
OF HERSCHEL–BULKLEY FLUID







## ON THE BLOCKING LIMIT OF STEADY-STATE FLOW OF HERSCHEL–BULKLEY FLUID

FARID MESSELMİ

This paper is devoted to the study of the blocking limit of Herschel–Bulkley fluid in the case of steady-state flow. To this aim, we consider a mathematical model which describes the steady-state flow of a Herschel–Bulkley fluid in a bounded domain. We give the mathematical formulation of the blockage phenomenon, and we establish the existence of the blocking limit. We also focus on behaviour of the flow with respect to the blocking limit.

### 1. Introduction

The rigid viscoplastic and incompressible fluid of Herschel and Bulkley has been investigated by mathematicians, physicists, and engineers as intensively as the Navier–Stokes equations though this model adequately describes a large class of flows. It has been used to model the flow of metals, plastic solids, and a variety of polymers. Physical experiments and numerical studies of the flow of Herschel–Bulkley fluids prove that when the yield stress increases, the rigid zones become larger and may completely block the flow. This property is called the blocking phenomenon. Due to existence of the yield limit, the model can capture phenomena connected with the development of discontinuous stresses. The literature concerning this topic is extensive; see, e.g., [Málek 2008; Málek et al. 2006; 2005; Messelmi 2017; Messelmi and Merouani 2013; 2010; Messelmi et al. 2010].

Our paper deals with the steady-state flow of Herschel and Bulkley. The main objective is the study of the behaviour of the flow. We provide a generalisation of a result obtained by Hild et al. [2002] for Bingham fluid to the steady-state flow of the Herschel–Bulkley model, ensuring the existence of the blocking limit. Moreover, we establish a result concerning the behaviour of the flow when the yield limit is near a minimal blocking limit.

The paper is organised as follows. In Section 2 we present the mechanical problem of the steady-state flow of Herschel–Bulkley fluid in a bounded domain  $\Omega \subset \mathbb{R}^n$ . We introduce some notation and preliminaries. In addition, we derive

---

**Communicated by Carlo Marchioro.**

*MSC2010:* 35J85, 76A05, 76E30.

*Keywords:* blocking limit, Herschel–Bulkley fluid, variational inequality.

the variational formulation of the problem. In Section 3, we show the mathematical formulation of the blockage phenomenon and we prove the existence of the blocking limit. Section 4 is devoted to the study of the behaviour of the flow with respect to the blocking limit.

## 2. Problem statement

We consider a mathematical problem modelling the steady-state flow of the rigid viscoplastic and incompressible Herschel–Bulkley fluid in a bounded domain  $\Omega \subset \mathbb{R}^n$  ( $n = 2, 3$ ), with the boundary  $\Gamma$  of class  $C^1$ . The fluid is acted upon by given volume forces of density  $\mathbf{f}$ . On  $\Gamma$  we suppose that the velocity is equal to zero.

We denote by  $S_n$  the space of symmetric tensors on  $\mathbb{R}^n$ . We define the inner product and the Euclidean norm on  $\mathbb{R}^n$  and  $S_n$ , respectively, by

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_i v_i & \text{for all } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n, & & \boldsymbol{\sigma} \cdot \boldsymbol{\tau} &= \sigma_{ij} \tau_{ij} & \text{for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in S_n, \\ |\mathbf{u}| &= (\mathbf{u} \cdot \mathbf{u})^{1/2} & \text{for all } \mathbf{u} \in \mathbb{R}^n, & & |\boldsymbol{\sigma}| &= (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{1/2} & \text{for all } \boldsymbol{\sigma} \in S_n. \end{aligned}$$

Here and below, the indices  $i$  and  $j$  run from 1 to  $n$  and the summation convention over repeated indices is used. We denote by  $\boldsymbol{\sigma}^D$  the deviator of  $\boldsymbol{\sigma} = (\sigma_{ij})$  given by

$$\boldsymbol{\sigma}^D = (\sigma_{ij}^D), \quad \sigma_{ij}^D = \sigma_{ij} - \frac{\sigma_{kk}}{n} \delta_{ij},$$

where  $\boldsymbol{\delta} = (\delta_{ij})$  denotes the identity tensor.

Let  $1 < p \leq 2$ . We consider the rate-of-deformation operator defined for every  $\mathbf{u} \in W^{1,p}(\Omega)^n$  by

$$D(\mathbf{u}) = (D_{ij}(\mathbf{u})), \quad D_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

The steady-state flow of Herschel–Bulkley fluid can be described by the following mechanical problem.

**Problem P<sub>1</sub>.** Find the velocity field  $\mathbf{u} = (u_i) : \Omega \rightarrow \mathbb{R}^n$  and the stress field  $\boldsymbol{\sigma} = (\sigma_{ij}) : \Omega \rightarrow S_n$  such that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} \quad \text{in } \Omega. \quad (2-1)$$

$$\left. \begin{aligned} \boldsymbol{\sigma}^D &= \mu |D(\mathbf{u})|^{p-2} D(\mathbf{u}) + g(D(\mathbf{u})/|D(\mathbf{u})|) & \text{if } |D(\mathbf{u})| \neq 0, \\ |\boldsymbol{\sigma}^D| &\leq g & \text{if } |D(\mathbf{u})| = 0 \end{aligned} \right\} \quad \text{in } \Omega, \quad (2-2)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (2-3)$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma. \quad (2-4)$$

Here  $\operatorname{div} \boldsymbol{\sigma} = (\sigma_{ij,j})$  and  $\operatorname{div} \mathbf{u} = u_{i,i}$ . The flow is given by (2-1) where the density is assumed equal to one. Equation (2-2) represents the constitutive law of Herschel–Bulkley fluid where  $\mu > 0$  and  $g \geq 0$  represent the consistency and yield limit of the

fluid, respectively, and  $1 < p \leq 2$  is the power law index. Equation (2-3) represents the incompressibility condition. Equation (2-4) gives the adherence condition on the boundary  $\Gamma$ .

Existence of weak solutions for this problem was proved in 1969 for  $p \geq 3n/(n+2)$ , for which the energy equality holds and higher differentiability techniques can be applied, in 1997 for  $p \geq 2n/(n+1)$ , and recently for  $p > 2n/(n+2)$  using the Lipschitz truncation method. Moreover, in 2010 some existence results regarding the thermal flow were established for the case  $p \geq 3n/(n+2)$  [Frehse et al. 2003; Lions 1969; Málek 2008; Málek et al. 2006; Messelmi et al. 2010]. Up to now, there are only a few results concerning the regularity of weak solutions, especially in three-dimensional domains. Further, the asymptotic behaviour of the unsteady flow was the subject of [Messelmi 2017].

- Remark.** (1) The Bingham fluid represents a particular case of Herschel–Bulkley fluid corresponding to  $p = 2$ .
- (2) In the constitutive law of Herschel–Bulkley fluid (2-2), the viscosity and hydrostatic pressure are given, respectively, by

$$\eta = \mu |D(\mathbf{u})|^{p-2}, \quad \pi = -\frac{1}{n} \sigma_{kk}. \quad (2-5)$$

Let us introduce the function spaces

$$W_{p,\text{div}} = \{\mathbf{v} \in W_0^{1,p}(\Omega)^n : \text{div}(\mathbf{v}) = 0 \text{ in } \Omega\}, \quad (2-6)$$

$$LD(\Omega) = \{\mathbf{v} \in L^1(\Omega)^n : D(\mathbf{v}) \in L^1(\Omega)^{n \times n}\}, \quad (2-7)$$

$$VD(\Omega) = \{\mathbf{v} \in LD(\Omega) : \mathbf{v} = 0 \text{ on } \Gamma\}, \quad (2-8)$$

$$W = \{\mathbf{v} \in VD(\Omega) : \text{div } \mathbf{v} = 0 \text{ in } \Omega\}. \quad (2-9)$$

$W_{p,\text{div}}$  is a Banach space equipped with the norm

$$\|\mathbf{v}\|_{W_{p,\text{div}}} = \|\mathbf{v}\|_{W^{1,p}(\Omega)^n}. \quad (2-10)$$

Moreover, Korn's inequality holds in the space  $W_{p,\text{div}}$  [Messelmi et al. 2010], which means that there exists a positive constant  $C_0$  depending only on  $\Omega$  and  $\Gamma$  such that

$$C_0 \|D(\mathbf{v})\|_{L^p(\Omega)^{n \times n}} \geq \|\mathbf{v}\|_{W_{p,\text{div}}} \quad \text{for all } \mathbf{v} \in W_{p,\text{div}}. \quad (2-11)$$

The space  $LD(\Omega)$  was introduced by Temam [1985]. It is a Banach space equipped with the norm

$$\|\mathbf{v}\|_{LD(\Omega)} = \|\mathbf{v}\|_{L^1(\Omega)^n} + \|D(\mathbf{v})\|_{L^1(\Omega)^{n \times n}}, \quad (2-12)$$

which is not reflexive, and  $W^{1,1}(\Omega)^n \subset LD(\Omega)$ . Since Korn's inequality does not hold on  $LD(\Omega)$  (see the remarks in [Temam 1985]), the space  $W^{1,1}(\Omega)^n$  is a proper

subspace of  $LD(\Omega)$ .  $VD(\Omega)$  is a closed subspace of  $LD(\Omega)$ .  $W$  is also a Banach space equipped with the norm given by (2-12). Furthermore, Korn's inequality holds in  $W$  [Temam 1985], and thus, there exists a positive constant  $C_W$  depending only on  $\Omega$  and  $\Gamma$  such that

$$C_W \|D(\mathbf{v})\|_{L^1(\Omega)^{n \times n}} \geq \|\mathbf{v}\|_{LD(\Omega)} \quad \text{for all } \mathbf{v} \in W. \quad (2-13)$$

Denoting by  $p'$  the conjugate of  $p$ , we introduce the convective operator

$$B : W_{p,\text{div}} \times W_{p,\text{div}} \times W_{p,\text{div}} \rightarrow \mathbb{R}, \quad B(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} \, dx. \quad (2-14)$$

We begin by recalling the following lemma [Messelmi et al. 2010], which gives some properties of the convective operator  $B$ .

**Lemma.** *Suppose that*

$$\frac{3n}{n+2} \leq p \leq 2. \quad (2-15)$$

*Then  $B$  is trilinear and continuous on  $W_{p,\text{div}} \times W_{p,\text{div}} \times W_{p,\text{div}}$ . Moreover, for all  $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in W_{p,\text{div}} \times W_{p,\text{div}} \times W_{p,\text{div}}$  we have  $B(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -B(\mathbf{u}, \mathbf{w}, \mathbf{v})$ .*

For the rest of this paper, we choose  $3n/(n+2) \leq p \leq 2$ . The use of Green's formula permits us to derive the following variational formulation of the mechanical problem  $P_1$  [Messelmi et al. 2010].

**Problem  $P_g$ .** For prescribed data  $\mathbf{f} \in W'_{p,\text{div}}$ , find  $\mathbf{u} \in W_{p,\text{div}}$  satisfying the variational inequality

$$\begin{aligned} & B(\mathbf{u}, \mathbf{u}, \mathbf{v} - \mathbf{u}) + \mu \int_{\Omega} |D(\mathbf{u})|^{p-2} D(\mathbf{u}) \cdot D(\mathbf{v} - \mathbf{u}) \, dx \\ & + g \int_{\Omega} |D(\mathbf{v})| \, dx - g \int_{\Omega} |D(\mathbf{u})| \, dx \geq \int_{\Omega} \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) \, dx \quad \text{for all } \mathbf{v} \in W_{p,\text{div}}. \end{aligned} \quad (2-16)$$

By taking  $\mathbf{v} = 0$  and  $\mathbf{v} = 2\mathbf{u}$  in (2-16), respectively,

$$\mu \int_{\Omega} |D(\mathbf{u})|^p \, dx + g \int_{\Omega} |D(\mathbf{u})| \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, dx. \quad (2-17)$$

This implies using again (2-16)

$$\begin{aligned} & B(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \mu \int_{\Omega} |D(\mathbf{u})|^{p-2} D(\mathbf{u}) \cdot D(\mathbf{v}) \, dx + g \int_{\Omega} |D(\mathbf{v})| \, dx \\ & \geq \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx \quad \text{for all } \mathbf{v} \in W_{p,\text{div}}. \end{aligned} \quad (2-18)$$

Consequently, the steady-state flow of Herschel–Bulkley fluid can be also described by the system (2-17)–(2-18).

### 3. Blockage property

This section is dedicated to the study of the blockage property of Herschel–Bulkley fluid. To do this, let us recall the following standard definition [Hild et al. 2002].

**Definition.** We will say that the fluid is blocked in the domain  $\Omega$  if  $\mathbf{u} = 0$  a.e. in  $\Omega$  is a solution to the variational problem  $P_g$ .

We prove the following proposition, which gives the variational interpretation of the blockage property.

**Proposition.** *The fluid is blocked in the domain  $\Omega$  if and only if*

$$g \int_{\Omega} |D(\mathbf{v})| dx \geq \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \quad \text{for all } \mathbf{v} \in W_{p,\text{div}}. \quad (3-1)$$

*Proof.* The first implication is an immediate consequence of the definition of blockage property. For the second one, we proceed as follows. Suppose that (3-1) holds. In particular, we have

$$g \int_{\Omega} |D(\mathbf{u})| dx \geq \int_{\Omega} \mathbf{f} \cdot \mathbf{u} dx. \quad (3-2)$$

Subtracting the inequalities (2-17) and (3-1), we find

$$\begin{aligned} \mu \int_{\Omega} |D(\mathbf{u})|^p dx \leq B(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \mu \int_{\Omega} |D(\mathbf{u})|^{p-2} D(\mathbf{u}) \cdot D(\mathbf{v}) dx \\ + g \int_{\Omega} |D(\mathbf{v})| dx - \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \quad \text{for all } \mathbf{v} \in W_{p,\text{div}}. \end{aligned} \quad (3-3)$$

Thus, the result can be obtained by setting  $\mathbf{v} = 0$  as a test function in (3-3) and using Korn's inequality.  $\square$

Hence, the mathematical study of the blockage property consists of finding the relationship between the yield limit  $g$  and the density of volume forces  $\mathbf{f}$  such that the inequality (3-1) holds.

We say that  $g$  is a blocking limit if the inequality (3-1) is satisfied.

We suppose from now on that

$$\mathbf{f} \in L^{\infty}(\Omega)^n. \quad (3-4)$$

The statement below ensures the existence of a blocking phase for large-enough yield limit.

**Proposition.** *If (3-4) holds, then*

$$g^* = \sup_{\mathbf{v} \in W_{p,\text{div}} - \{0\}} \frac{\int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx}{\int_{\Omega} |D(\mathbf{v})| dx} < +\infty. \quad (3-5)$$

*In addition, if  $g \geq g^*$ , then the blocking occurs; it means that (3-1) holds.*

*Proof.* Let us define the form  $l \in W'_{p,\text{div}}$  by

$$\langle l, \mathbf{v} \rangle_{W'_{p,\text{div}} \times W_{p,\text{div}}} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx \quad \text{for all } \mathbf{v} \in W_{p,\text{div}}. \quad (3-6)$$

The fact that  $\mathbf{f} \in L^{\infty}(\Omega)^n$  implies that  $l \in W'$ . Then, there exists  $C_1 > 0$  such that

$$|\langle l, \mathbf{v} \rangle_{W'_{p,\text{div}} \times W_{p,\text{div}}}| \leq C_1 \|\mathbf{v}\|_{LD(\Omega)} \quad \text{for all } \mathbf{v} \in LD(\Omega). \quad (3-7)$$

This yields, thanks to the Korn inequality (2-13)

$$|\langle l, \mathbf{v} \rangle_{W'_{p,\text{div}} \times W_{p,\text{div}}}| \leq C_1 C_W \|D(\mathbf{v})\|_{L^1(\Omega)^{n \times n}} \quad \text{for all } \mathbf{v} \in LD(\Omega). \quad (3-8)$$

Consequently, via (3-7) and (3-8) we obtain  $g^* \leq C_1 C_W$ .

Now, if  $g \geq g^*$ , then (3-5) gives

$$g \int_{\Omega} |D(\mathbf{v})| \, dx \geq g^* \int_{\Omega} |D(\mathbf{v})| \, dx \geq \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx \quad \text{for all } \mathbf{v} \in W_{p,\text{div}} - \{0\},$$

which completes the proof, by observing that if  $\mathbf{v} = 0$ , the inequality above also remains satisfied.  $\square$

Here  $g^*$  is said to be the minimal blocking limit.

Let  $g$  be a blocking limit. We denote by  $C$  the set

$$C = \left\{ \mathbf{v} \in W_{p,\text{div}} \mid g \int_{\Omega} |D(\mathbf{v})| \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx \right\}. \quad (3-9)$$

It is straightforward to verify that the set  $C$  is a convex cone in  $W_{p,\text{div}}$ .

#### 4. Behaviour of the flow

Let us introduce for  $\varepsilon > 0$  the perturbed yield limit

$$g_{\varepsilon} = (1 - \varepsilon^{p-1})g, \quad (4-1)$$

and denote by  $\mathbf{u}_{\varepsilon}$  the solution of the corresponding problem, i.e.,

$$\begin{aligned} B(\mathbf{u}_{\varepsilon}, \mathbf{u}_{\varepsilon}, \mathbf{v} - \mathbf{u}_{\varepsilon}) + \mu \int_{\Omega} |D(\mathbf{u}_{\varepsilon})|^{p-2} D(\mathbf{u}_{\varepsilon}) \cdot D(\mathbf{v} - \mathbf{u}_{\varepsilon}) \, dx + g_{\varepsilon} \int_{\Omega} |D(\mathbf{v})| \, dx \\ - g_{\varepsilon} \int_{\Omega} |D(\mathbf{u}_{\varepsilon})| \, dx \geq \int_{\Omega} \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}_{\varepsilon}) \, dx \quad \text{for all } \mathbf{v} \in W_{p,\text{div}}. \end{aligned} \quad (4-2)$$



The above inequality can be written in equivalent form

$$\mu \int_{\Omega} |D(\mathbf{u}_{\varepsilon})|^p dx + g_{\varepsilon} \int_{\Omega} |D(\mathbf{u}_{\varepsilon})| dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_{\varepsilon} dx, \quad (4-3)$$

$$\begin{aligned} B(\mathbf{u}_{\varepsilon}, \mathbf{u}_{\varepsilon}, \mathbf{v}) + \mu \int_{\Omega} |D(\mathbf{u}_{\varepsilon})|^{p-2} D(\mathbf{u}_{\varepsilon}) \cdot D(\mathbf{v}) dx + g_{\varepsilon} \int_{\Omega} |D(\mathbf{v})| dx \\ \geq \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \quad \text{for all } \mathbf{v} \in W_{p,\text{div}}. \end{aligned} \quad (4-4)$$

Setting now

$$\mathbf{w}_{\varepsilon} = \frac{\mathbf{u}_{\varepsilon}}{\varepsilon} \quad \text{for all } \varepsilon > 0, \quad (4-5)$$

in the following we establish a convergence result for  $(\mathbf{w}_{\varepsilon})_{\varepsilon>0}$  when  $\varepsilon$  tends to 0.

**Theorem.** *Suppose that  $g$  is a blocking limit. Then  $(\mathbf{w}_{\varepsilon})_{\varepsilon>0}$  converges strongly, when  $\varepsilon$  tends to 0 in  $W_{p,\text{div}}$ , to the solution  $\mathbf{w}$  of the variational inequality*

$$\mathbf{w} \in C : \mu \int_{\Omega} |D(\mathbf{w})|^{p-2} D(\mathbf{w}) \cdot D(\mathbf{v} - \mathbf{w}) dx \geq \int_{\Omega} \mathbf{f} \cdot (\mathbf{v} - \mathbf{w}) dx \quad \text{for all } \mathbf{v} \in C. \quad (4-6)$$

*Proof.* The system becomes, taking into account (4-5),

$$\mu \varepsilon^{p-1} \int_{\Omega} |D(\mathbf{w}_{\varepsilon})|^p dx + (1 - \varepsilon^{p-1}) g \int_{\Omega} |D(\mathbf{w}_{\varepsilon})| dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_{\varepsilon} dx, \quad (4-7)$$

$$\begin{aligned} \varepsilon^2 B(\mathbf{w}_{\varepsilon}, \mathbf{w}_{\varepsilon}, \mathbf{v}) + \mu \varepsilon^{p-1} \int_{\Omega} |D(\mathbf{w}_{\varepsilon})|^{p-2} D(\mathbf{w}_{\varepsilon}) \cdot D(\mathbf{v}) dx \\ + (1 - \varepsilon^{p-1}) g \int_{\Omega} |D(\mathbf{v})| dx \geq \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \quad \text{for all } \mathbf{v} \in W_{p,\text{div}}. \end{aligned} \quad (4-8)$$

Equation (4-7) gives

$$\begin{aligned} \mu \varepsilon^{p-1} \int_{\Omega} |D(\mathbf{w}_{\varepsilon})|^p dx + (1 - \varepsilon^{p-1}) \left( g \int_{\Omega} |D(\mathbf{w}_{\varepsilon})| dx - \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_{\varepsilon} dx \right) \\ = \varepsilon^{p-1} g \int_{\Omega} |D(\mathbf{w}_{\varepsilon})| dx. \end{aligned} \quad (4-9)$$

Suppose that  $g$  is a blocking limit; then (4-9) gives

$$\mu \int_{\Omega} |D(\mathbf{w}_{\varepsilon})|^p dx \leq g \int_{\Omega} |D(\mathbf{w}_{\varepsilon})| dx. \quad (4-10)$$

We deduce making use of Korn's inequality and some algebraic manipulations that

$$\|\mathbf{w}_{\varepsilon}\|_{W_{p,\text{div}}} \leq c. \quad (4-11)$$

Hence, we can extract a subsequence still denoted by  $(\mathbf{w}_{\varepsilon})_{\varepsilon>0}$  such that

$$\mathbf{w}_{\varepsilon} \rightarrow \mathbf{w} \quad \text{in } W_{p,\text{div}} \text{ weakly.} \quad (4-12)$$

The Rellich–Kondrachov compactness theorem allows us to get after a new extraction

$$\mathbf{w}_\varepsilon \rightarrow \mathbf{w} \quad \text{in } L^p(\Omega)^n \text{ strongly and a.e. in } \Omega. \quad (4-13)$$

Therefore, (4-9) gives again

$$(1 - \varepsilon^{p-1})g \int_{\Omega} |D(\mathbf{w}_\varepsilon)| dx \leq \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_\varepsilon dx,$$

thereby allowing us to find

$$g \liminf \int_{\Omega} |D(\mathbf{w}_\varepsilon)| dx \leq \lim \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_\varepsilon dx. \quad (4-14)$$

This yields

$$g \int_{\Omega} |D(\mathbf{w})| dx \leq \int_{\Omega} \mathbf{f} \cdot \mathbf{w} dx. \quad (4-15)$$

Consequently, since  $g$  is a blocking limit,

$$g \int_{\Omega} |D(\mathbf{w})| dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} dx. \quad (4-16)$$

Taking  $\mathbf{w}$  as test function in inequality (4-8), it implies that

$$\begin{aligned} \varepsilon^2 B(\mathbf{w}_\varepsilon, \mathbf{w}_\varepsilon, \mathbf{w}) + \mu \varepsilon^{p-1} \int_{\Omega} |D(\mathbf{w}_\varepsilon)|^{p-2} D(\mathbf{w}_\varepsilon) \cdot D(\mathbf{w}) dx \\ + (1 - \varepsilon^{p-1})g \int_{\Omega} |D(\mathbf{w})| dx \geq \int_{\Omega} \mathbf{f} \cdot \mathbf{w} dx. \end{aligned}$$

This gives, making use of (4-16),

$$\varepsilon^{3-p} B(\mathbf{w}_\varepsilon, \mathbf{w}_\varepsilon, \mathbf{w}) + \mu \int_{\Omega} |D(\mathbf{w}_\varepsilon)|^{p-2} D(\mathbf{w}_\varepsilon) \cdot D(\mathbf{w}) dx \geq g \int_{\Omega} |D(\mathbf{w})| dx. \quad (4-17)$$

Moreover, the lemma on page 66 permits us to obtain the estimate

$$|B(\mathbf{w}_\varepsilon, \mathbf{w}_\varepsilon, \mathbf{w})| \leq \|\mathbf{w}_\varepsilon\|_{W_{p,\text{div}}}^2 \|\mathbf{w}\|_{W_{p,\text{div}}}. \quad (4-18)$$

On the other hand, it is well known that the nonlinear term  $\mu \int_{\Omega} |D(\mathbf{w}_\varepsilon)|^{p-2} D(\mathbf{w}_\varepsilon) \cdot D(\mathbf{w}) dx$  converges to  $\mu \int_{\Omega} |D(\mathbf{w})|^p dx$  [Lions 1969]. Consequently, by passing to the limit, one can find, keeping in mind (4-18),

$$\mu \int_{\Omega} |D(\mathbf{w})|^p dx \geq g \int_{\Omega} |D(\mathbf{w})| dx. \quad (4-19)$$

We get thanks to (4-10)

$$\liminf \mu \int_{\Omega} |D(\mathbf{w}_\varepsilon)|^p dx \leq g \liminf \int_{\Omega} |D(\mathbf{w}_\varepsilon)| dx.$$

So, using (4-14) we can infer that

$$\liminf \mu \int_{\Omega} |D(\mathbf{w}_{\varepsilon})|^p dx \leq \lim \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_{\varepsilon} dx,$$

which implies that

$$\mu \int_{\Omega} |D(\mathbf{w})|^p dx \leq \int_{\Omega} \mathbf{f} \cdot \mathbf{w} dx. \quad (4-20)$$

Putting together (4-16), (4-19), and (4-20) we obtain

$$\mu \int_{\Omega} |D(\mathbf{w})|^p dx = g \int_{\Omega} |D(\mathbf{w})| dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} dx, \quad (4-21)$$

which implies in particular that  $\mathbf{w} \in C$ . Furthermore, by (4-8) we get

$$\begin{aligned} & \varepsilon^{3-p} B(\mathbf{w}_{\varepsilon}, \mathbf{w}_{\varepsilon}, \mathbf{v}) + \mu \int_{\Omega} |D(\mathbf{w}_{\varepsilon})|^{p-2} D(\mathbf{w}_{\varepsilon}) \cdot D(\mathbf{v}) dx \\ & + \frac{1}{\varepsilon^{p-1}} \left( g \int_{\Omega} |D(\mathbf{v})| dx - \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \right) \geq g \int_{\Omega} |D(\mathbf{v})| dx \quad \text{for all } \mathbf{v} \in W_{p,\text{div}}. \end{aligned}$$

By choosing  $\mathbf{v} \in C$  in the above inequality, the passage to the limit leads to

$$\mu \int_{\Omega} |D(\mathbf{w})|^{p-2} D(\mathbf{w}) \cdot D(\mathbf{v}) dx \geq g \int_{\Omega} |D(\mathbf{v})| dx \quad \text{for all } \mathbf{v} \in C.$$

This yields

$$\mu \int_{\Omega} |D(\mathbf{w})|^{p-2} D(\mathbf{w}) \cdot D(\mathbf{v}) dx \geq \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \quad \text{for all } \mathbf{v} \in C. \quad (4-22)$$

Combining (4-21) and (4-22) yields the inequality (4-6).

Our objective now is to prove the strong convergence. With this aim, we proceed as follows. The use of (4-7) and (4-8) permits us to affirm that for every  $\mathbf{v} \in W_{p,\text{div}}$

$$\begin{aligned} & \varepsilon^2 B(\mathbf{w}_{\varepsilon}, \mathbf{w}_{\varepsilon}, \mathbf{v}) + \mu \varepsilon^{p-1} \int_{\Omega} |D(\mathbf{w}_{\varepsilon})|^{p-2} D(\mathbf{w}_{\varepsilon}) \cdot D(\mathbf{v} - \mathbf{w}_{\varepsilon}) dx \\ & \geq \int_{\Omega} \mathbf{f} \cdot (\mathbf{v} - \mathbf{w}_{\varepsilon}) dx - (1 - \varepsilon^{p-1}) g \left( \int_{\Omega} |D(\mathbf{v})| dx - \int_{\Omega} |D(\mathbf{w}_{\varepsilon})| dx \right) \end{aligned}$$

It follows, by setting  $\mathbf{v} = \mathbf{w}$ , that

$$\begin{aligned} & -\varepsilon^2 B(\mathbf{w}_{\varepsilon}, \mathbf{w}_{\varepsilon}, \mathbf{w}) + \mu \varepsilon^{p-1} \int_{\Omega} |D(\mathbf{w}_{\varepsilon})|^{p-2} D(\mathbf{w}_{\varepsilon}) \cdot D(\mathbf{w}_{\varepsilon} - \mathbf{w}) dx \\ & \leq \int_{\Omega} \mathbf{f} \cdot (\mathbf{w}_{\varepsilon} - \mathbf{w}) dx + (1 - \varepsilon^{p-1}) g \int_{\Omega} (|D(\mathbf{w})| - |D(\mathbf{w}_{\varepsilon})|) dx. \quad (4-23) \end{aligned}$$

Further, since  $g$  is the blocking limit and  $\mathbf{w} \in W_{p,\text{div}}$ , one can verify that

$$g \int_{\Omega} (|D(\mathbf{w})| - |D(\mathbf{w}_{\varepsilon})|) dx \leq \int_{\Omega} \mathbf{f} \cdot (\mathbf{w} - \mathbf{w}_{\varepsilon}) dx.$$

Consequently, inequality (4-23) becomes

$$\mu \int_{\Omega} |D(\mathbf{w}_{\varepsilon})|^{p-2} D(\mathbf{w}_{\varepsilon}) \cdot D(\mathbf{w}_{\varepsilon} - \mathbf{w}) \, dx \leq \int_{\Omega} \mathbf{f} \cdot (\mathbf{w}_{\varepsilon} - \mathbf{w}) \, dx + \varepsilon^{3-p} B(\mathbf{w}_{\varepsilon}, \mathbf{w}_{\varepsilon}, \mathbf{w}). \quad (4-24)$$

This becomes

$$\begin{aligned} \mu \int_{\Omega} (|D(\mathbf{w}_{\varepsilon})|^{p-2} D(\mathbf{w}_{\varepsilon}) - |D(\mathbf{w})|^{p-2} D(\mathbf{w})) \cdot D(\mathbf{w}_{\varepsilon} - \mathbf{w}) \, dx \\ \leq \int_{\Omega} \mathbf{f} \cdot (\mathbf{w}_{\varepsilon} - \mathbf{w}) \, dx - \mu \int_{\Omega} |D(\mathbf{w})|^{p-2} D(\mathbf{w}) D(\mathbf{w}_{\varepsilon} - \mathbf{w}) \, dx \\ + c\varepsilon^{3-p} \|\mathbf{w}_{\varepsilon}\|_{W_{p,\text{div}}}^2 \|\mathbf{w}\|_{W_{p,\text{div}}}. \end{aligned} \quad (4-25)$$

Let us observe now that for every  $x, y \in \mathbb{R}^n$ ,

$$(|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \geq c \frac{|x - y|^2}{(|x| + |y|)^{2-p}}, \quad 1 < p \leq 2.$$

So applying the above inequality, (4-25) can be rewritten as

$$\begin{aligned} \mu \int_{\Omega} \frac{|D(\mathbf{w}_{\varepsilon} - \mathbf{w})|^2}{(|D(\mathbf{w}_{\varepsilon})| + |D(\mathbf{w})|)^{2-p}} \, dx \leq c \left| \int_{\Omega} \mathbf{f} \cdot (\mathbf{w}_{\varepsilon} - \mathbf{w}) \, dx \right| \\ + c\varepsilon^{3-p} \|\mathbf{w}_{\varepsilon}\|_{W_{p,\text{div}}}^2 \|\mathbf{w}\|_{W_{p,\text{div}}} + c\mu \left| \int_{\Omega} |D(\mathbf{w})|^{p-2} D(\mathbf{w}) D(\mathbf{w}_{\varepsilon} - \mathbf{w}) \, dx \right|, \end{aligned}$$

which gives, exploiting Korn's and Hölder's inequalities,

$$\begin{aligned} \|\mathbf{w}_{\varepsilon} - \mathbf{w}\|_{W_p}^p \leq c \left( \int_{\Omega} (|D(\mathbf{w}_{\varepsilon})| + |D(\mathbf{w})|)^p \, dx \right)^{(2-p)/2} \left( \left| \int_{\Omega} \mathbf{f} \cdot (\mathbf{w}_{\varepsilon} - \mathbf{w}) \, dx \right| \right. \\ \left. + \varepsilon^{3-p} \|\mathbf{w}_{\varepsilon}\|_{W_{p,\text{div}}}^2 \|\mathbf{w}\|_{W_{p,\text{div}}} + \mu \int_{\Omega} |D(\mathbf{w})|^{p-2} D(\mathbf{w}) D(\mathbf{w}_{\varepsilon} - \mathbf{w}) \, dx \right)^{p/2}. \end{aligned}$$

Passing to the limit, we conclude, using (4-12) and taking into account the fact that  $|D(\mathbf{w})|^{p-2} D(\mathbf{w})$  is bounded in  $L^{p'}(\Omega)^n$ , that

$$\mathbf{w}_{\varepsilon} \rightarrow \mathbf{w} \quad \text{in } W_{p,\text{div}} \text{ strongly,}$$

which permits us to complete the proof.  $\square$

**Corollary.** Denoting by  $\mathbf{w}_0$  the unique solution of the variational equation given by

$$\mu \int_{\Omega} |D(\mathbf{w}_0)|^{p-2} D(\mathbf{w}_0) \cdot D(\mathbf{v}) \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx \quad \text{for all } \mathbf{v} \in W_{p,\text{div}}, \quad (4-26)$$

then the following estimates hold:

$$\|D(\mathbf{w})\|_{L^p(\Omega)^{n \times n}} \leq \|D(\mathbf{w}_0)\|_{L^p(\Omega)^{n \times n}}, \quad \int_{\Omega} \mathbf{f} \cdot \mathbf{w} \, dx \leq \int_{\Omega} \mathbf{f} \cdot \mathbf{w}_0 \, dx. \quad (4-27)$$

*Proof.* We can infer by setting  $\mathbf{w}$  as a test function in (4-26) that

$$\mu \int_{\Omega} |D(\mathbf{w}_0)|^{p-2} D(\mathbf{w}_0) \cdot D(\mathbf{w}) \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} \, dx.$$

This yields, using Hölder's inequality

$$\mu \|D(\mathbf{w}_0)\|_{L^p(\Omega)^{n \times n}}^{p-1} \|D(\mathbf{w})\|_{L^p(\Omega)^{n \times n}} \geq \int_{\Omega} \mathbf{f} \cdot \mathbf{w} \, dx = \mu \|D(\mathbf{w})\|_{L^p(\Omega)^{n \times n}}^p,$$

which allows us to get the first estimate. The second estimate becomes an immediate consequence of the first one.  $\square$

## References

- [Frehse et al. 2003] J. Frehse, J. Málek, and M. Steinhauer, “On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method”, *SIAM J. Math. Anal.* **34**:5 (2003), 1064–1083.
- [Hild et al. 2002] P. Hild, I. R. Ionescu, T. Lachand-Robert, and I. Roşca, “The blocking of an inhomogeneous Bingham fluid: applications to landslides”, *M2AN Math. Model. Numer. Anal.* **36**:6 (2002), 1013–1026.
- [Lions 1969] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
- [Málek 2008] J. Málek, “Mathematical properties of flows of incompressible power-law-like fluids that are described by implicit constitutive relations”, *Electron. Trans. Numer. Anal.* **31** (2008), 110–125.
- [Málek et al. 2005] J. Málek, M. Růžička, and V. V. Shelukhin, “Herschel–Bulkley fluids: existence and regularity of steady flows”, *Math. Models Methods Appl. Sci.* **15**:12 (2005), 1845–1861.
- [Málek et al. 2006] J. Málek, D. Pražák, and M. Steinhauer, “On the existence and regularity of solutions for degenerate power-law fluids”, *Differ. Integral Equ.* **19**:4 (2006), 449–462.
- [Messelmi 2017] F. Messelmi, “Asymptotic behavior of unsteady Herschel–Bulkley flow”, *An. Univ. Oradea Fasc. Mat.* **24**:2 (2017), 181–194.
- [Messelmi and Merouani 2010] F. Messelmi and B. Merouani, “Stationary thermal flow of a Bingham fluid whose viscosity, yield limit and friction depend on the temperature”, *An. Univ. Oradea Fasc. Mat.* **17**:2 (2010), 59–74.
- [Messelmi and Merouani 2013] F. Messelmi and A. Merouani, “Properties of the laminar flow of Herschel–Bulkley fluid”, *An. Univ. Oradea Fasc. Mat.* **20**:1 (2013), 47–60.
- [Messelmi et al. 2010] F. Messelmi, B. Merouani, and F. Bouzgehaya, “Steady-state thermal Herschel–Bulkley flow with Tresca's friction law”, *Electron. J. Differential Equations* **2010**:46 (2010).
- [Temam 1985] R. Temam, *Mathematical problems in plasticity*, Mathematical Methods of Information Science **12**, Gauthier-Villars, Paris, 1985.

Received 2 Sep 2018. Revised 10 Nov 2018. Accepted 11 Dec 2018.

FARID MESSELMİ: foudimath@yahoo.fr

Department of Mathematics and LDMM Laboratory, University Ziane Achour of Djelfa, Djelfa, Algeria





# MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS

[msp.org/memocs](http://msp.org/memocs)

## EDITORIAL BOARD

ANTONIO CARCATERRA	Università di Roma "La Sapienza", Italia
ERIC A. CARLEN	Rutgers University, USA
FRANCESCO DELL'ISOLA	(CO-CHAIR) Università di Roma "La Sapienza", Italia
RAFFAELE ESPOSITO	(TREASURER) Università dell'Aquila, Italia
ALBERT FANNJIANG	University of California at Davis, USA
GILLES A. FRANCFORT	(CO-CHAIR) Université Paris-Nord, France
PIERANGELO MARCATI	Università dell'Aquila, Italy
JEAN-JACQUES MARIGO	École Polytechnique, France
PETER A. MARKOWICH	DAMTP Cambridge, UK, and University of Vienna, Austria
MARTIN OSTOJA-STARZEWSKI	(CHAIR MANAGING EDITOR) Univ. of Illinois at Urbana-Champaign, USA
PIERRE SEPPECHER	Université du Sud Toulon-Var, France
DAVID J. STEIGMANN	University of California at Berkeley, USA
PAUL STEINMANN	Universität Erlangen-Nürnberg, Germany
PIERRE M. SUQUET	LMA CNRS Marseille, France

## MANAGING EDITORS

MICOL AMAR	Università di Roma "La Sapienza", Italia
ANGELA MADEO	Université de Lyon-INSA (Institut National des Sciences Appliquées), France
MARTIN OSTOJA-STARZEWSKI	(CHAIR MANAGING EDITOR) Univ. of Illinois at Urbana-Champaign, USA

## ADVISORY BOARD

ADNAN AKAY	Carnegie Mellon University, USA, and Bilkent University, Turkey
HOLM ALTENBACH	Otto-von-Guericke-Universität Magdeburg, Germany
MICOL AMAR	Università di Roma "La Sapienza", Italia
HARM ASKES	University of Sheffield, UK
TEODOR ATANACKOVIĆ	University of Novi Sad, Serbia
VICTOR BERDICHEVSKY	Wayne State University, USA
GUY BOUCHITTÉ	Université du Sud Toulon-Var, France
ANDREA BRAIDES	Università di Roma Tor Vergata, Italia
ROBERTO CAMASSA	University of North Carolina at Chapel Hill, USA
MAURO CARFORE	Università di Pavia, Italia
ERIC DARVE	Stanford University, USA
FELIX DARVE	Institut Polytechnique de Grenoble, France
ANNA DE MASI	Università dell'Aquila, Italia
GIANPIETRO DEL PIERO	Università di Ferrara and International Research Center MEMOCS, Italia
EMMANUELE DI BENEDETTO	Vanderbilt University, USA
VICTOR A. EREMEYEV	Gdansk University of Technology, Poland
BERNOLD FIEDLER	Freie Universität Berlin, Germany
IRENE M. GAMBA	University of Texas at Austin, USA
DAVID Y. GAO	Federation University and Australian National University, Australia
SERGEY GAVRILYUK	Université Aix-Marseille, France
TIMOTHY J. HEALEY	Cornell University, USA
DOMINIQUE JEULIN	École des Mines, France
ROGER E. KHAYAT	University of Western Ontario, Canada
CORRADO LATTANZIO	Università dell'Aquila, Italy
ROBERT P. LIPTON	Louisiana State University, USA
ANGELO LUONGO	Università dell'Aquila, Italia
ANGELA MADEO	Université de Lyon-INSA (Institut National des Sciences Appliquées), France
JUAN J. MANFREDI	University of Pittsburgh, USA
CARLO MARCHIORO	Università di Roma "La Sapienza", Italia
ANIL MISRA	University of Kansas, USA
ROBERTO NATALINI	Istituto per le Applicazioni del Calcolo "M. Picone", Italy
PATRIZIO NEFF	Universität Duisburg-Essen, Germany
THOMAS J. PENCE	Michigan State University, USA
ANDREY PIATNITSKI	Narvik University College, Norway, Russia
ERRICO PRESUTTI	Università di Roma Tor Vergata, Italy
MARIO PULVIRENTI	Università di Roma "La Sapienza", Italia
LUCIO RUSSO	Università di Roma "Tor Vergata", Italia
MIGUEL A. F. SANJUAN	Universidad Rey Juan Carlos, Madrid, Spain
PATRICK SELVADURAI	McGill University, Canada
MIROSLAV ŠILHAVÝ	Academy of Sciences of the Czech Republic
GUIDO SWEERS	Universität zu Köln, Germany
ANTOINETTE TORDSILLAS	University of Melbourne, Australia
LEV TRUSKINOVSKY	École Polytechnique, France
JUAN J. L. VELÁZQUEZ	Bonn University, Germany
VINCENZO VESPRI	Università di Firenze, Italia
ANGELO VULPIANI	Università di Roma La Sapienza, Italia

MEMOCS (ISSN 2325-3444 electronic, 2326-7186 printed) is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

Cover image: "Tangle" by © John Horigan; produced using the *Context Free* program ([contextfreeart.org](http://contextfreeart.org)).

PUBLISHED BY



**mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

- Homogenization of nonlinear inextensible pantographic structures      1  
by  $\Gamma$ -convergence  
    Jean-Jacques Alibert and Alessandro Della Corte
- A note on Couette flow of micropolar fluids according to Eringen's      25  
theory  
    Wilhelm Rickert, Elena N. Vilchevskaya and Wolfgang H.  
    Müller
- Analytical solutions for the natural frequencies of rectangular      51  
symmetric angle-ply laminated plates  
    Florence Browning and Harm Askes
- On the blocking limit of steady-state flow of Herschel–Bulkley      63  
fluid  
    Farid Messelmi
- Continuum theory for mechanical metamaterials with a cubic      75  
lattice substructure  
    Simon R. Eugster, Francesco dell'Isola and David J.  
    Steigmann

*MEMOCS* is a journal of the International Research Center for  
the Mathematics and Mechanics of Complex Systems  
at the Università dell'Aquila, Italy.

