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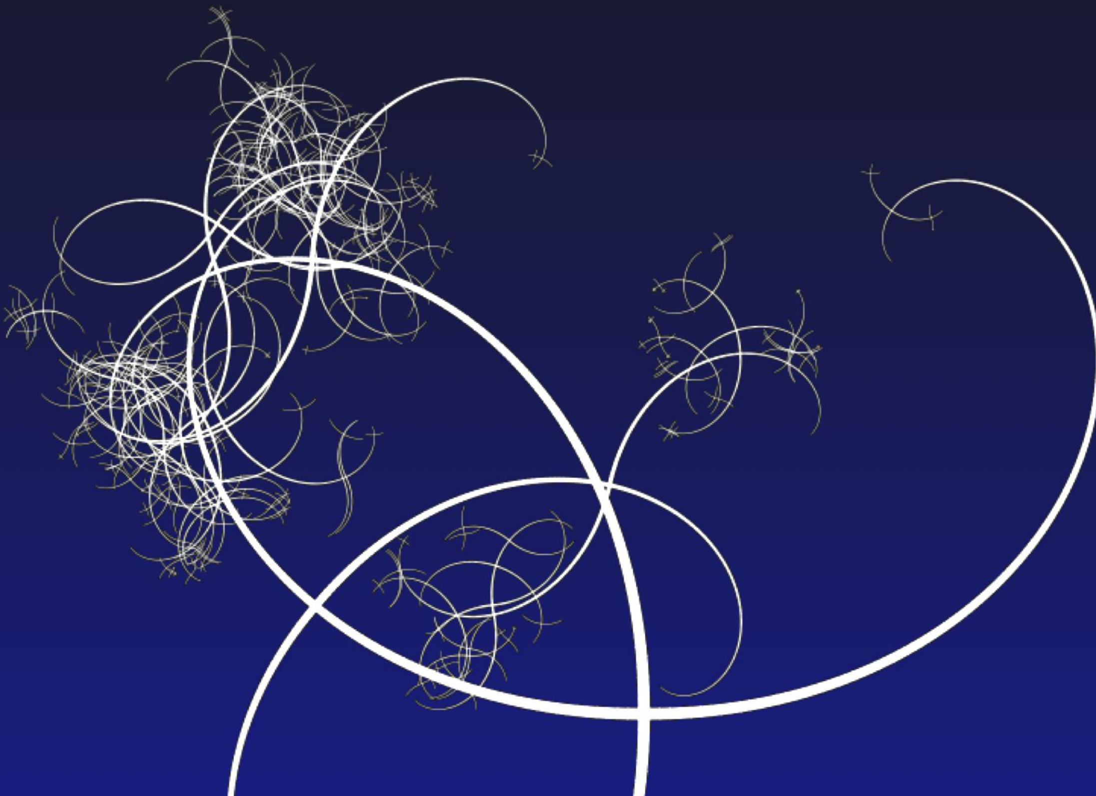

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CONTINUUM THEORY FOR MECHANICAL METAMATERIALS WITH A CUBIC LATTICE SUBSTRUCTURE

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A three-dimensional continuum theory for fibrous mechanical metamaterials is proposed, in which the fibers are assumed to be spatial Kirchhoff rods whose mechanical response is controlled by a deformation field and a rotation field, the former accounting for strain of the rod and the latter for flexure and twist of the rod as it deforms. This leads naturally to a model based on Cosserat elasticity. Rigidity constraints are introduced that effectively reduce the model to a variant of second-gradient elasticity theory.

1. Introduction

The advent of 3D printing and associated microfabrication technologies has facilitated the design and realization of a range of mechanical metamaterials. These lightweight artificial materials exhibit stiffness and energy-absorbing properties far exceeding those of conventional bulk materials [Barchiesi et al. 2019; Mieszala et al. 2017; Vangelatos et al. 2019]. A unit cell of such a material — typically of microscopic dimensions — consists of a lattice of beam-like or rod-like fibers interacting at internal connections. The intrinsic extensional, flexural, and torsional stiffnesses of the fibers combine with the architecture of the lattice to confer high stiffness on the material at the macroscale together with enhanced energy absorption via microscale buckling.

These technologies provide impetus for the development of a continuum theory for the analysis of the macroscale response of metamaterials with lattice-like substructures. Toward this end, we outline a Cosserat model in which the deformation and rotation fields account respectively for the strains and orientations of the fibers, regarded as spatial Kirchhoff rods. In this preliminary work we confine attention to the simplest case of a cubic lattice architecture in which the constituent fibers are

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initially orthogonal. Further, in view of the solid nature of the internal connections in typical metamaterials [Vangelatos et al. 2019], we assume the fibers to be rigid in the sense that they remain mutually orthogonal in the course of deformation. They are free to extend or contract and to bend and twist, however, and these modes of deformation are modeled explicitly.

In Section 2 we review Kirchhoff's one-dimensional theory of rods as a prelude to the development, in Section 3, of an analogous three-dimensional Cosserat continuum model. We show that the rigidity constraint imposed in the continuum theory effectively determines the Cosserat rotation in terms of the gradient of the deformation field. Because the gradient of the rotation field is involved in the constitutive functions, this has the consequence that the Cosserat model reduces to a particular second-gradient theory of elasticity [Spencer and Soldatos 2007]. Details of this reduction are given in Section 4. Finally, in Section 5 we apply the theory to predict the response of a block to finite flexure. This solution serves to illustrate certain unusual features of the proposed model.

2. Kirchoff rods

In Kirchhoff's theory the rod is regarded as a spatial curve endowed with an elastic strain-energy function that depends on curvature and twist [Landau and Lifshitz 1986; Dill 1992; Antman 2005]. In Dill's derivation from conventional three-dimensional nonlinear elasticity, this theory also accommodates small axial strain along the rod, whereas this effect is suppressed in derivations based on asymptotic analysis or the method of gamma convergence. We include it here. In the present section we outline the basic elements of Kirchhoff's theory, including the kinematics, the constitutive theory, and the variational derivation of the equilibrium equations. Although this theory is well known, we review it here to facilitate the interpretation of the ensuing continuum theory of metamaterials.

2.1. ***Kinematics.*** The basic kinematic variables in the theory are a deformation field $\boldsymbol{r}(s)$, where $s \in [0, l]$ and l is the length of the rod in a reference configuration, and a right-handed, orthonormal triad $\{\boldsymbol{d}_i(s)\}$ in which $\boldsymbol{d}_3 = \boldsymbol{d}$, where $\boldsymbol{d}$ is the unit vector defined by [Dill 1992; Steigmann 1996; Antman 2005]

$$\begin{aligned} \boldsymbol{r}'(s) &= \lambda \boldsymbol{d}, \\ \lambda &= |\boldsymbol{r}'(s)|. \end{aligned} \tag{1}$$

Here λ is the *stretch* of the rod. Thus, $\boldsymbol{d}$ is the unit tangent to the rod in a deformed configuration and $\boldsymbol{d}_\alpha$ ($\alpha = 1, 2$) span its cross-sectional plane at arclength station s.

A central aspect of Kirchhoff's theory is that each cross section deforms as a rigid disc. Accordingly, there is a rotation field $\boldsymbol{R}(s)$ such that $\boldsymbol{d}_i = \boldsymbol{R}\boldsymbol{D}_i$, where

$\boldsymbol{D}_i(s)$ are the values of $\boldsymbol{d}_i(s)$ in a reference configuration; thus,

$$\boldsymbol{R} = \boldsymbol{d}_i \otimes \boldsymbol{D}_i. \tag{2}$$

The curvature and twist of the rod are computed from the derivatives $\boldsymbol{d}_i'(s)$, where

$$\boldsymbol{d}_i' = \boldsymbol{R}'\boldsymbol{D}_i + \boldsymbol{R}\boldsymbol{D}_i'. \tag{3}$$

Let $\{\boldsymbol{E}_i\}$ be a fixed right-handed background frame. Then $\boldsymbol{D}_i(s) = \boldsymbol{A}(s)\boldsymbol{E}_i$ for some rotation field $\boldsymbol{A}$, yielding

$$\boldsymbol{d}_i' = \boldsymbol{W}\boldsymbol{d}_i = \boldsymbol{w} \times \boldsymbol{d}_i, \tag{4}$$

where

$$\boldsymbol{W} = \boldsymbol{R}'\boldsymbol{R}^t + \boldsymbol{R}\boldsymbol{A}'\boldsymbol{A}^t\boldsymbol{R}^t \tag{5}$$

is a skew tensor and $\boldsymbol{w}$ is its axial vector. If the rod is straight and untwisted in the reference configuration, i.e., if $\boldsymbol{D}_i' = \boldsymbol{0}$, then $\boldsymbol{W} = \boldsymbol{R}'\boldsymbol{R}^t$.

2.2. *Strain-energy function.* We assume the strain energy E stored in a segment $[l_1, l_2] \subset [0, l]$ of a rod of length l to be expressible as

$$E = \int_{l_1}^{l_2} U\,ds, \tag{6}$$

where U, the strain energy per unit length, is a function of the list $\{\boldsymbol{R}, \boldsymbol{R}', \boldsymbol{r}'\}$, possibly depending explicitly on s. Explicit s-dependence may arise from the initial curvature or twist of the rod, or from nonuniform material properties.

We require U to be Galilean invariant and hence that its values be unaffected by the substitution $\{\boldsymbol{R}, \boldsymbol{R}', \boldsymbol{r}'\} \to \{\boldsymbol{Q}\boldsymbol{R}, \boldsymbol{Q}\boldsymbol{R}', \boldsymbol{Q}\boldsymbol{r}'\}$, where $\boldsymbol{Q}$ is an arbitrary *uniform* rotation. Because U is defined pointwise, to derive a necessary condition we select the rotation $\boldsymbol{Q} = \boldsymbol{R}^t_{|s}$ and conclude that U is determined by the list $\{\boldsymbol{R}^t\boldsymbol{R}', \boldsymbol{R}^t\boldsymbol{r}'\}$. This list is trivially Galilean invariant. It is equivalent to $\{\boldsymbol{R}^t\boldsymbol{W}\boldsymbol{R} - \boldsymbol{A}'\boldsymbol{A}^t, \lambda\boldsymbol{D}\}$, where $\boldsymbol{D} = \boldsymbol{D}_3$ and $\boldsymbol{R}^t\boldsymbol{W}\boldsymbol{R} - \boldsymbol{A}'\boldsymbol{A}^t$ is a Galilean-invariant measure of the *relative* flexure and twist of the rod due to deformation. Here $\boldsymbol{D}$ and $\boldsymbol{A}'\boldsymbol{A}^t$ are independent of the deformation and serve to confer an explicit s-dependence on the strain-energy function; accordingly, we write $U = U(\boldsymbol{R}^t\boldsymbol{W}\boldsymbol{R}, \lambda; s)$. If the rod is initially straight and untwisted, as we assume hereafter, then $\boldsymbol{D}_i' = \boldsymbol{0}$ and any explicit s-dependence of the energy is due solely to nonuniformity of the material properties. Henceforth, we assume material properties to be uniform.

In the present circumstances we have

$$\boldsymbol{R}^t\boldsymbol{W}\boldsymbol{R} = W_{ij}\boldsymbol{D}_i \otimes \boldsymbol{D}_j, \quad \text{with } W_{ij} = \boldsymbol{d}_i \cdot \boldsymbol{W}\boldsymbol{d}_j = \boldsymbol{d}_i \cdot \boldsymbol{d}_j'. \tag{7}$$

Thus,

$$U = W(\lambda, \boldsymbol{\kappa}), \tag{8}$$

where $\boldsymbol{\kappa}(=\kappa_i \boldsymbol{D}_i)$ is the axial vector of $\boldsymbol{R}^t \boldsymbol{W} \boldsymbol{R}$; i.e.,

$$\kappa_i = \tfrac{1}{2} e_{ijk} \boldsymbol{d}_k \cdot \boldsymbol{d}'_j. \tag{9}$$

Here e_{ijk} is the Levi-Civita permutation symbol ($e_{123} = +1$, etc.), κ_3 is the twist of the rod, and κ_α are the curvatures. Moreover, it follows easily from (4) and (9) that

$$\boldsymbol{w} = \boldsymbol{R}\boldsymbol{\kappa} = \kappa_i \boldsymbol{d}_i \tag{10}$$

if the rod is initially straight and untwisted.

For example, in the classical theory [Dill 1992; Steigmann 1996; Antman 2005], the strain-energy function is

$$W(\lambda, \boldsymbol{\kappa}) = \tfrac{1}{2} E(\lambda - 1)^2 + \tfrac{1}{2} F \kappa_\alpha \kappa_\alpha + \tfrac{1}{2} T \tau^2, \tag{11}$$

where $\tau(=\kappa_3)$ is the twist, E is the extensional stiffness (Young's modulus times the cross-sectional area), F is the flexural stiffness (Young's modulus times the second moment of area of the cross section), and T is the torsional stiffness (the shear modulus times the polar moment of the cross section).

The terms involving curvature and twist in this expression are appropriate for rods of circular cross section composed of isotropic materials [Landau and Lifshitz 1986]. The homogeneous quadratic dependence of the energy on these terms may be understood in terms of a local length scale such as the diameter of a fiber cross section. The curvature-twist vector, when nondimensionalized by this local scale, is typically small in applications. For example, the minimum radius of curvature of a bent fiber is typically much larger than the fiber diameter. If the bending and twisting moments vanish when the rod is straight and untwisted, then the leading-order contribution of the curvature-twist vector to the strain energy is quadratic; this is reflected in (11). In general the flexural and torsional stiffnesses in this expression may depend on fiber stretch, but in the small-extensional-strain regime contemplated here, they are approximated at leading order by constants in the case of a uniform rod.

2.3. ***Variational theory.*** The equilibrium equations of the Kirchhoff theory are well known and easily derived from elementary considerations, but it is instructive to review their variational derivation here as a prelude to the considerations that follow.

We assume that equilibria of the rod are such as to satisfy the virtual-power statement

$$\dot{E} = P, \tag{12}$$

where P is the virtual power of the loads — the explicit form of which is deduced below — and the superposed dot is used to identify a variational derivative. These are derivatives, with respect to ϵ, of the one-parameter deformation and rotation fields $\boldsymbol{r}(s;\epsilon)$ and $\boldsymbol{R}(s;\epsilon)$, respectively, where $\boldsymbol{r}(s)=\boldsymbol{r}(s;0)$ and $\boldsymbol{R}(s)=\boldsymbol{R}(s;0)$ are equilibrium fields, and (see (11))

$$\dot{U}=\dot{W}=W_\lambda\dot{\lambda}+\mu_i\dot{\kappa}_i, \tag{13}$$

where

$$W_\lambda=\frac{\partial W}{\partial\lambda}\quad\text{and}\quad\mu_i=\frac{\partial W}{\partial\kappa_i} \tag{14}$$

are evaluated at equilibrium, corresponding to $\epsilon=0$.

From (1) we have

$$\dot{\lambda}\boldsymbol{d}+\boldsymbol{\omega}\times\boldsymbol{r}'=\boldsymbol{u}', \tag{15}$$

where $\boldsymbol{u}(s)=\dot{\boldsymbol{r}}$ is the virtual translational velocity and $\boldsymbol{\omega}(s)$ is the axial vector of the skew tensor $\dot{\boldsymbol{R}}\boldsymbol{R}^t$; i.e.,

$$\dot{\boldsymbol{d}}_i=\boldsymbol{\omega}\times\boldsymbol{d}_i. \tag{16}$$

It follows from (9) and (16) that

$$\begin{aligned}\dot{\kappa}_i&=\tfrac{1}{2}e_{ijk}(\dot{\boldsymbol{d}}_k\cdot\boldsymbol{d}_j'+\boldsymbol{d}_k\cdot\dot{\boldsymbol{d}}_j')\\&=\tfrac{1}{2}e_{ijk}[\boldsymbol{\omega}\times\boldsymbol{d}_k\cdot\boldsymbol{d}_j'+\boldsymbol{d}_k\cdot(\boldsymbol{\omega}'\times\boldsymbol{d}_j+\boldsymbol{\omega}\times\boldsymbol{d}_j')],\end{aligned} \tag{17}$$

in which the terms involving $\boldsymbol{\omega}$ cancel; the $e-\delta$ identity $\frac{1}{2}e_{ijk}e_{mjk}=\delta_{im}$ (the Kronecker delta), combined with $\boldsymbol{d}_j\times\boldsymbol{d}_k=e_{mjk}\boldsymbol{d}_m$, then yields

$$\dot{\kappa}_i=\boldsymbol{d}_i\cdot\boldsymbol{\omega}'. \tag{18}$$

Thus,

$$\dot{E}=I[\boldsymbol{u},\boldsymbol{\omega}], \tag{19}$$

where

$$I[\boldsymbol{u},\boldsymbol{\omega}]=\int_{l_1}^{l_2}(W_\lambda\boldsymbol{d}\cdot\boldsymbol{u}'+\boldsymbol{\mu}\cdot\boldsymbol{\omega}')\,ds, \tag{20}$$

with

$$\boldsymbol{\mu}=\mu_i\boldsymbol{d}_i. \tag{21}$$

Further, from (1) we have the orthogonality constraints

$$\boldsymbol{r}'\cdot\boldsymbol{d}_\alpha=0 \tag{22}$$

for $\alpha=1,2$. To accommodate these in the variational formulation, we introduce the energy

$$E^*=E+\int_{l_1}^{l_2}f_\alpha\boldsymbol{r}'\cdot\boldsymbol{d}_\alpha\,ds, \tag{23}$$

where $f_\alpha(s)$ are Lagrange multipliers. This is an extension to arbitrary deformations of the actual energy, the latter being defined only for the class of deformations defined by the constraints. Moreover, (12) is replaced by

$$\dot{E}^* = P, \tag{24}$$

where

$$\dot{E}^* = \int_{l_1}^{l_2} [(W_\lambda \boldsymbol{d} + f_\alpha \boldsymbol{d}_\alpha) \cdot \boldsymbol{u}' + \boldsymbol{\mu} \cdot \boldsymbol{\omega}' + f_\alpha \boldsymbol{d}_\alpha \times \boldsymbol{r}' \cdot \boldsymbol{\omega}]\, ds. \tag{25}$$

We do not make variations with respect to the multipliers f_α explicit as these merely return the constraints (22).

We conclude that (24) reduces to

$$(\boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{\mu} \cdot \boldsymbol{\omega})|_{l_1}^{l_2} - \int_{l_1}^{l_2} [\boldsymbol{u} \cdot \boldsymbol{f}' + \boldsymbol{\omega} \cdot (\boldsymbol{\mu}' - \boldsymbol{f} \times \boldsymbol{r}')]\, ds = P, \tag{26}$$

where

$$\boldsymbol{f} = W_\lambda \boldsymbol{d} + f_\alpha \boldsymbol{d}_\alpha. \tag{27}$$

This implies that the virtual power is expressible in the form

$$P = (\boldsymbol{t} \cdot \boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega})|_{l_1}^{l_2} + \int_{l_1}^{l_2} (\boldsymbol{u} \cdot \boldsymbol{g} + \boldsymbol{\omega} \cdot \boldsymbol{\pi})\, ds, \tag{28}$$

in which $\boldsymbol{t}$ and $\boldsymbol{c}$ represent forces and couples acting at the ends of the segment and $\boldsymbol{g}$ and $\boldsymbol{\pi}$ are force and couple distributions acting in the interior.

By the fundamental lemma, the Euler equations holding at points in the interior of the rod are

$$\boldsymbol{f}' + \boldsymbol{g} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{\mu}' + \boldsymbol{\pi} = \boldsymbol{f} \times \boldsymbol{r}', \tag{29}$$

and the endpoint conditions are

$$\boldsymbol{f} = \boldsymbol{t} \quad \text{and} \quad \boldsymbol{\mu} = \boldsymbol{c}, \tag{30}$$

provided that neither position nor section orientation is assigned at the endpoints. These are the equilibrium conditions of classical rod theory in which $\boldsymbol{f}$ and $\boldsymbol{\mu}$ respectively are the cross-sectional force and moment transmitted by the segment $(s, l]$ on the part $[0, s]$. Equations (27) and $(30)_1$ justify the interpretation of the Lagrange multipliers f_α as transverse shear forces acting on a fiber cross section.

Other boundary conditions are, of course, feasible. For example, if the tangent direction $\boldsymbol{d}$ is assigned at a boundary point, then its variation $\boldsymbol{\omega} \times \boldsymbol{d}$ vanishes there, leaving $\boldsymbol{\omega} = \omega \boldsymbol{d}$ in which ω is arbitrary. In this case (26) and (28) furnish the boundary condition

$$\boldsymbol{\mu} \cdot \boldsymbol{d} = c, \tag{31}$$

in which $c = \boldsymbol{c} \cdot \boldsymbol{d}$ is the twisting moment applied at the boundary.

For the strain-energy function (11), we have

$$W_\lambda = E(\lambda - 1), \tag{32}$$

together with $\mu_3 \boldsymbol{d}_3 = T\tau\boldsymbol{d}$ and $\mu_\alpha \boldsymbol{d}_\alpha = F\kappa_\alpha \boldsymbol{d}_\alpha$. To reduce the second expression we use (9), together with $\boldsymbol{d}\cdot\boldsymbol{d}'_\mu = -\boldsymbol{d}_\mu\cdot\boldsymbol{d}'$, to derive $\kappa_\alpha = e_{\alpha 3\mu}\boldsymbol{d}_\mu\cdot\boldsymbol{d}'$. From $\boldsymbol{d}\cdot\boldsymbol{d}' = 0$ we have $\boldsymbol{d}' = (\boldsymbol{d}_\alpha\cdot\boldsymbol{d}')\boldsymbol{d}_\alpha$ and $\boldsymbol{d}\times\boldsymbol{d}' = (\boldsymbol{d}_\alpha\cdot\boldsymbol{d}')\boldsymbol{d}\times\boldsymbol{d}_\alpha = (e_{\beta 3\alpha}\boldsymbol{d}_\alpha\cdot\boldsymbol{d}')\boldsymbol{d}_\beta$; thus, $\kappa_\beta\boldsymbol{d}_\beta = \boldsymbol{d}\times\boldsymbol{d}'$ and (21) becomes

$$\boldsymbol{\mu} = T\tau\boldsymbol{d} + F\boldsymbol{d}\times\boldsymbol{d}'. \tag{33}$$

From (1) we have that $(29)_2$ is equivalent to the system

$$\boldsymbol{d}\cdot(\boldsymbol{\mu}' + \boldsymbol{\pi}) = 0 \quad \text{and} \quad \boldsymbol{d}\times(\boldsymbol{\mu}' + \boldsymbol{\pi}) = \lambda\boldsymbol{d}\times(\boldsymbol{f}\times\boldsymbol{d}). \tag{34}$$

In the second of these we combine the identity $\boldsymbol{d}\times(\boldsymbol{f}\times\boldsymbol{d}) = \boldsymbol{f} - (\boldsymbol{d}\cdot\boldsymbol{f})\boldsymbol{d}$ with (27) to obtain

$$\boldsymbol{f} = W_\lambda\boldsymbol{d} + \lambda^{-1}\boldsymbol{d}\times(\boldsymbol{\mu}' + \boldsymbol{\pi}), \tag{35}$$

which may be combined with $(29)_1$ and $(34)_1$ to provide an alternative set of equilibrium equations. The latter form of Kirchhoff's theory furnishes a more natural analog to the system derived for a lattice of rods in Sections 3 and 4.

2.4. *Three-dimensional lattice.* We suppose the three-dimensional continuum to be composed of a continuous distribution of orthogonal rods of the kind discussed in the foregoing. Every point of the continuum is regarded as a point of intersection of three fibers. These are assumed to be aligned, prior to deformation, with the *uniform*, right-handed orthonormal triad $\{\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}\}$. The lattice of fibers is assumed to be rigid in the sense that it remains orthogonal, and similarly oriented, in the course of the deformation. That is, the set $\{\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}\}$ of material vectors is stretched and rotated to the (generally nonuniform) set $\{\lambda_l\boldsymbol{l}, \lambda_m\boldsymbol{m}, \lambda_n\boldsymbol{n}\}$, where $\{\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}\}$ is a right-handed orthonormal triad and $\{\lambda_l, \lambda_m, \lambda_n\}$ are the fiber stretches.

Accordingly, the orientation of the deformed lattice is specified by the rotation field

$$\boldsymbol{R} = \boldsymbol{l}\otimes\boldsymbol{L} + \boldsymbol{m}\otimes\boldsymbol{M} + \boldsymbol{n}\otimes\boldsymbol{N}. \tag{36}$$

This furnishes the curvature-twist vectors $\boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m, \boldsymbol{\kappa}_n$ of the constituent fibers in accordance with (9); thus, for example, the curvature-twist vector of a fiber initially aligned with $\boldsymbol{L}$ is $\boldsymbol{\kappa}_l = \kappa_{(l)i}\boldsymbol{L}_i$, where $\{\boldsymbol{L}_i\} = \{\boldsymbol{L}_\alpha, \boldsymbol{L}\}$ with $\{\boldsymbol{L}_\alpha\} = \{\boldsymbol{M}, \boldsymbol{N}\}$, and

$$\kappa_{(l)i} = \tfrac{1}{2}e_{ijk}\boldsymbol{l}_k\cdot\boldsymbol{l}'_j, \tag{37}$$

where $\{\boldsymbol{l}_i\} = \{\boldsymbol{l}_\alpha, \boldsymbol{l}\}$ with $\{\boldsymbol{l}_\alpha\} = \{\boldsymbol{m}, \boldsymbol{n}\}$. Here, $\boldsymbol{l}'_j = (\nabla\boldsymbol{l}_j)\boldsymbol{L}$ is the directional derivative along the $\boldsymbol{L}$-fiber. Because $\boldsymbol{l}_i = \boldsymbol{R}\boldsymbol{L}_i$ it is evident that $\boldsymbol{\kappa}_l$ is determined by the rotation field $\boldsymbol{R}$ and its gradient.

In the same way, $\boldsymbol{\kappa}_m = \kappa_{(m)i}\boldsymbol{M}_i$, where $\{\boldsymbol{M}_i\} = \{\boldsymbol{M}_\alpha, \boldsymbol{M}\}$ with $\{\boldsymbol{M}_\alpha\} = \{\boldsymbol{N}, \boldsymbol{L}\}$, and

$$\kappa_{(m)i} = \tfrac{1}{2}e_{ijk}\boldsymbol{m}_k \cdot \boldsymbol{m}'_j, \tag{38}$$

where $\{\boldsymbol{m}_i\} = \{\boldsymbol{R}\boldsymbol{M}_i\} = \{\boldsymbol{m}_\alpha, \boldsymbol{m}\}$, with $\{\boldsymbol{m}_\alpha\} = \{\boldsymbol{n}, \boldsymbol{l}\}$, and where $\boldsymbol{m}'_j = (\nabla \boldsymbol{m}_j)\boldsymbol{M}$ is now the directional derivative along the $\boldsymbol{M}$-fiber. Finally, $\boldsymbol{\kappa}_n = \kappa_{(n)i}\boldsymbol{N}_i$, where $\{\boldsymbol{N}_i\} = \{\boldsymbol{N}_\alpha, \boldsymbol{N}\}$ with $\{\boldsymbol{N}_\alpha\} = \{\boldsymbol{L}, \boldsymbol{M}\}$, and

$$\kappa_{(n)i} = \tfrac{1}{2}e_{ijk}\boldsymbol{n}_k \cdot \boldsymbol{n}'_j, \tag{39}$$

where $\{\boldsymbol{n}_i\} = \{\boldsymbol{R}\boldsymbol{N}_i\} = \{\boldsymbol{n}_\alpha, \boldsymbol{n}\}$, with $\{\boldsymbol{n}_\alpha\} = \{\boldsymbol{l}, \boldsymbol{m}\}$ and $\boldsymbol{n}'_j = (\nabla \boldsymbol{n}_j)\boldsymbol{N}$. Thus, all three curvature-twist vectors are determined by the single rotation field $\boldsymbol{R}$ and its gradient.

Because the fibers are convected as material curves, we have $\lambda_l \boldsymbol{l} = \boldsymbol{F}\boldsymbol{L}$, etc., where $\boldsymbol{F} = \nabla\boldsymbol{\chi}$ is the gradient of the deformation $\boldsymbol{\chi}(\boldsymbol{X})$ of the continuum. Here $\boldsymbol{X}$ is the position of a material point in a reference configuration, κ say. The orthonormality of $\{\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}\}$ then furnishes

$$\boldsymbol{F} = \lambda_l \boldsymbol{l} \otimes \boldsymbol{L} + \lambda_m \boldsymbol{m} \otimes \boldsymbol{M} + \lambda_n \boldsymbol{n} \otimes \boldsymbol{N}, \tag{40}$$

where $\lambda_l = |\boldsymbol{F}\boldsymbol{L}|$, etc. Evidently,

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}, \tag{41}$$

where

$$\boldsymbol{U} = \lambda_l \boldsymbol{L} \otimes \boldsymbol{L} + \lambda_m \boldsymbol{M} \otimes \boldsymbol{M} + \lambda_n \boldsymbol{N} \otimes \boldsymbol{N} \tag{42}$$

is positive definite and symmetric. The Cosserat rotation (36) thus coincides with the rotation in the polar factorization of the deformation gradient in which $\boldsymbol{U}$ is the associated right-stretch tensor. Because $\boldsymbol{R}$ is uniquely determined by $\boldsymbol{F}$ in this case, the curvature-twist vectors of the fibers are ultimately determined by the first and second deformation gradients $\nabla\boldsymbol{\chi}$ and $\nabla\nabla\boldsymbol{\chi}$. It is this fact which yields the reduction, detailed in Section 4, of the Cosserat continuum model outlined in Section 3 to a special second-gradient model of elasticity. Moreover, the present model furnishes a rare example of a material for which the principal axes of strain are fixed in the body.

It may be observed that the kinematical structure of the present three-dimensional framework is not entirely analogous to that of rod theory. This is due to the *partial* coupling between deformation and rotation implied by (1), whereas in the present three-dimensional theory the relevant rotation field is controlled *entirely* by the continuum deformation.

We note that if the fibers are inextensible, i.e., if $\lambda_l = \lambda_m = \lambda_n = 1$, then $\boldsymbol{U} = \boldsymbol{I}$, $\nabla\boldsymbol{\chi} = \boldsymbol{R}$, and the deformation is necessarily rigid [Gurtin 1981]. Here $\boldsymbol{I}$ is the three-dimensional identity. Thus, nontrivial deformations necessarily entail fiber extension or contraction. Accordingly, we expect a lattice material of the kind envisaged to be extremely stiff — the raison d'être of mechanical metamaterials — if the extensional stiffnesses of the fibers are large. In contrast, Kirchhoff's theory accommodates nonrigid inextensional deformations.

3. Cosserat elasticity

3.1. ***Kinematics.*** In the foregoing we have argued that the lattice material may be regarded as a Cosserat continuum endowed with a rotation field $\boldsymbol{R}(\boldsymbol{X})$. This rotation is determined by the deformation $\boldsymbol{\chi}(\boldsymbol{X})$. However, in the present section we regard these fields as being independent in the spirit of the conventional Cosserat theory. The rotation and deformation are ultimately reconnected in Section 4. Existence theory for Cosserat elasticity is discussed in [Neff 2006].

Thus, we introduce a referential energy density $U(\boldsymbol{F}, \boldsymbol{R}, \nabla\boldsymbol{R}; \boldsymbol{X})$, where $\boldsymbol{F}$ is the usual deformation gradient and $\nabla\boldsymbol{R}$ is the rotation gradient; i.e.,

$$\boldsymbol{F} = F_{iA}\boldsymbol{e}_i \otimes \boldsymbol{E}_A, \qquad \boldsymbol{R} = R_{iA}\boldsymbol{e}_i \otimes \boldsymbol{E}_A, \quad \text{and} \quad \nabla\boldsymbol{R} = R_{iA,B}\boldsymbol{e}_i \otimes \boldsymbol{E}_A \otimes \boldsymbol{E}_B \tag{43}$$

with

$$F_{iA} = \chi_{i,A}, \tag{44}$$

where $(\cdot)_{,A} = \partial(\cdot)/\partial X_A$ and we use a Cartesian index notation that emphasizes the two-point character of the deformation gradient and rotation fields. Here $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{E}_A\}$ are fixed orthonormal bases associated with the Cartesian coordinates x_i and X_A, where $x_i = \chi_i(X_A)$.

3.2. ***Strain-energy function.*** We suppose the strain energy to be Galilean invariant and thus require that

$$U(\boldsymbol{F}, \boldsymbol{R}, \nabla\boldsymbol{R}; \boldsymbol{X}) = U(\boldsymbol{QF}, \boldsymbol{QR}, \boldsymbol{Q}\nabla\boldsymbol{R}; \boldsymbol{X}), \tag{45}$$

where $\boldsymbol{Q}$ is an arbitrary spatially uniform rotation and $(\boldsymbol{Q}\nabla\boldsymbol{R})_{iAB} = (Q_{ij}R_{jA})_{,B} = Q_{ij}R_{jA,B}$. The restriction

$$U(\boldsymbol{F}, \boldsymbol{R}, \nabla\boldsymbol{R}; \boldsymbol{X}) = W(\boldsymbol{E}, \boldsymbol{\Gamma}; \boldsymbol{X}), \tag{46}$$

where [Pietraszkiewicz and Eremeyev 2009; Eremeyev and Pietraszkiewicz 2012; Steigmann 2012; 2015]

$$\boldsymbol{E} = \boldsymbol{R}^t\boldsymbol{F} = E_{AB}\boldsymbol{E}_A \otimes \boldsymbol{E}_B, \quad E_{AB} = R_{iA}F_{iB}, \tag{47}$$

$$\boldsymbol{\Gamma} = \Gamma_{DC}\boldsymbol{E}_D \otimes \boldsymbol{E}_C, \qquad \Gamma_{DC} = \tfrac{1}{2}e_{BAD}R_{iA}R_{iB,C}, \tag{48}$$

with W a suitable function and e_{ABC} the permutation symbol, furnishes the necessary and sufficient condition for Galilean invariance. Sufficiency is obvious; necessity follows by choosing $\boldsymbol{Q} = \boldsymbol{R}^t_{|X}$, where X is the material point in question, and making use of the fact that, for each fixed $C \in \{1, 2, 3\}$, the matrix $R_{iA}R_{iB,C}$ is skew. This follows by differentiating $R_{iA}R_{iB} = \delta_{AB}$ (the Kronecker delta). The associated axial vectors $\boldsymbol{\gamma}_C$ have components

$$\gamma_{D(C)} = \tfrac{1}{2}e_{BAD}R_{iA}R_{iB,C}, \tag{49}$$

yielding [Pietraszkiewicz and Eremeyev 2009]

$$\boldsymbol{\Gamma} = \boldsymbol{\gamma}_C \otimes \boldsymbol{E}_C, \tag{50}$$

and so $\boldsymbol{\Gamma}$ — the *wryness tensor* — stands in one-to-one relation to the Cosserat strain measure $\boldsymbol{R}^t\nabla\boldsymbol{R}$.

3.3. ***Virtual power and equilibrium.*** We define equilibria to be states that satisfy the virtual-power statement

$$\dot{E}_\pi = P_\pi, \tag{51}$$

where π is an arbitrary subvolume of κ, P_π is the virtual power of the loads acting thereon,

$$E_\pi = \int_\pi U\,dv \tag{52}$$

is the strain energy contained in π, and superposed dots identify variational derivatives, as in Section 2. Thus,

$$\dot{U} = \dot{W} = \boldsymbol{\sigma}\cdot\dot{\boldsymbol{E}} + \boldsymbol{\mu}\cdot\dot{\boldsymbol{\Gamma}}, \tag{53}$$

where

$$\boldsymbol{\sigma} = W_{\boldsymbol{E}} \quad \text{and} \quad \boldsymbol{\mu} = W_{\boldsymbol{\Gamma}} \tag{54}$$

are evaluated at equilibrium and the variational derivatives are evaluated at an equilibrium state. The dots interposed between the terms in (53) represent the standard Euclidean inner product on the linear space of tensors. The explicit form of P_π is deduced below.

From (47) we have

$$\dot{\boldsymbol{E}} = \boldsymbol{R}^t(\nabla\boldsymbol{u} - \boldsymbol{\Omega}\boldsymbol{F}), \quad \text{where } \boldsymbol{u} = \dot{\boldsymbol{\chi}} \text{ and } \boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^t. \tag{55}$$

Then,

$$\boldsymbol{\sigma}\cdot\dot{\boldsymbol{E}} = \boldsymbol{R}\boldsymbol{\sigma}\cdot\nabla\boldsymbol{u} - \boldsymbol{\Omega}\cdot\mathrm{Skw}(\boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{F}^t). \tag{56}$$

Here, of course, $\boldsymbol{\Omega}$ is skew. Let $\boldsymbol{\omega} = ax\boldsymbol{\Omega}$ be its axial vector, defined, for arbitrary $\boldsymbol{v}$, by $\boldsymbol{\omega}\times\boldsymbol{v} = \boldsymbol{\Omega}\boldsymbol{v}$. If $\boldsymbol{\alpha}$ is a skew tensor and $\boldsymbol{a} = ax\boldsymbol{\alpha}$, then, as is well known, $\boldsymbol{\Omega}\cdot\boldsymbol{\alpha} = 2\boldsymbol{\omega}\cdot\boldsymbol{a}$. Further, $\boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{F}^t = \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{E}^t\boldsymbol{R}^t$ and $\mathrm{Skw}(\boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{E}^t\boldsymbol{R}^t) = \boldsymbol{R}(\mathrm{Skw}\,\boldsymbol{\sigma}\boldsymbol{E}^t)\boldsymbol{R}^t$ so that, finally,

$$\boldsymbol{\sigma}\cdot\dot{\boldsymbol{E}} = \boldsymbol{R}\boldsymbol{\sigma}\cdot\nabla\boldsymbol{u} - 2ax[\boldsymbol{R}(\mathrm{Skw}\,\boldsymbol{\sigma}\boldsymbol{E}^t)\boldsymbol{R}^t]\cdot\boldsymbol{\omega}. \tag{57}$$

To reduce $\boldsymbol{\mu}\cdot\dot{\boldsymbol{\Gamma}}$ we make use of the formula [Steigmann 2012]

$$\boldsymbol{\mu}\cdot\dot{\boldsymbol{\Gamma}} = \bar{\omega}_E(\mu_{EC,C} + e_{EDB}\mu_{BC}\Gamma_{DC}) - (\bar{\omega}_E\mu_{EC})_{,C}, \tag{58}$$

where $\bar{\boldsymbol{\omega}} = -\boldsymbol{R}^t\boldsymbol{\omega}$ is the axial vector of $\dot{\boldsymbol{R}}^t\boldsymbol{R}$. This may be recast in the form

$$\boldsymbol{\mu}\cdot\dot{\boldsymbol{\Gamma}} = \mathrm{Div}(\boldsymbol{\mu}^t\boldsymbol{R}^t\boldsymbol{\omega}) - \boldsymbol{\omega}\cdot\mathrm{Div}(\boldsymbol{R\mu}) + \omega_i\mu_{EC}(R_{iE,C} - e_{BDE}R_{iB}\Gamma_{DC}). \tag{59}$$

The inverse of $(48)_2$ is

$$e_{BDE}\Gamma_{DC} = R_{jB}R_{jE,C}; \quad \text{thus,} \quad e_{BDE}R_{iB}\Gamma_{DC} = \delta_{ij}R_{jE,C}, \tag{60}$$

implying that the last term of (59) vanishes and hence that

$$\boldsymbol{\mu}\cdot\dot{\boldsymbol{\Gamma}} = \boldsymbol{R\mu}\cdot\nabla\boldsymbol{\omega}. \tag{61}$$

Substitution of (57) and (61) into (51) furnishes

$$P_\pi = \int_{\partial\pi}\boldsymbol{R\mu\nu}\cdot\boldsymbol{\omega}\,da - \int_\pi\{\boldsymbol{R\sigma}\cdot\nabla\boldsymbol{u} + \boldsymbol{\omega}\cdot[\mathrm{Div}(\boldsymbol{R\mu}) + 2ax(\boldsymbol{R}(\mathrm{Skw}\,\boldsymbol{\sigma}\boldsymbol{E}^t)\boldsymbol{R}^t)]\}\,dv, \tag{62}$$

where $\boldsymbol{\nu}$ is the exterior unit normal to the piecewise smooth surface $\partial\pi$. The virtual power is thus of the form

$$P_\pi = \int_{\partial\pi}(\boldsymbol{t}\cdot\boldsymbol{u} + \boldsymbol{c}\cdot\boldsymbol{\omega})\,da + \int_\pi(\boldsymbol{g}\cdot\boldsymbol{u} + \boldsymbol{\pi}\cdot\boldsymbol{\omega})\,dv, \tag{63}$$

where $\boldsymbol{t}$ and $\boldsymbol{c}$ are densities of force and couple acting on $\partial\pi$, and $\boldsymbol{g}$ and $\boldsymbol{\pi}$ are densities of force and couple acting in π.

If $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are independent and if there are no kinematical constraints, then by the fundamental lemma,

$$\boldsymbol{t} = \boldsymbol{R\sigma\nu} \quad \text{and} \quad \boldsymbol{c} = \boldsymbol{R\mu\nu} \quad \text{on } \partial\pi, \tag{64}$$

and

$$\boldsymbol{g} = -\,\mathrm{Div}(\boldsymbol{R\sigma}) \quad \text{and} \quad \boldsymbol{\pi} = -\,\mathrm{Div}(\boldsymbol{R\mu}) - 2ax[\boldsymbol{R}(\mathrm{Skw}\,\boldsymbol{\sigma}\boldsymbol{E}^t)\boldsymbol{R}^t] \quad \text{in } \pi. \tag{65}$$

These are the equilibrium conditions for a standard Cosserat continuum in which the deformation $\boldsymbol{\chi}$ and rotation $\boldsymbol{R}$ are independent kinematical fields. The use of the axial vector $\boldsymbol{\omega}$ in their derivation yields a simpler set of equations than that derived in [Reissner 1975; 1987; Steigmann 2012; 2015] on the basis of the axial vector $\bar{\boldsymbol{\omega}}$.

3.4. ***Specialization to an orthogonal lattice.*** The curvature-twist vector $\boldsymbol{\kappa}_l$ of a fiber initially aligned with $\boldsymbol{L}$ may be described in the present framework by using (36) and (37) to write

$$\kappa_{(l)i} = \tfrac{1}{2}e_{ijk}\boldsymbol{L}_k\cdot\boldsymbol{R}^t\boldsymbol{R}'\boldsymbol{L}_j, \tag{66}$$

where $(\cdot)'$ is the directional derivative along $\boldsymbol{L}$ and we have assumed that the fibers are initially straight and untwisted, i.e., that $\boldsymbol{L}_j' = \boldsymbol{0}$. Here we use $R_{iA}' = R_{iA,B}L_B$ to derive (see $(60)_1$)

$$\boldsymbol{R}^t\boldsymbol{R}' = R_{iC}R_{iA,B}L_B\boldsymbol{E}_C \otimes \boldsymbol{E}_A = e_{ACD}\Gamma_{DB}L_B\boldsymbol{E}_C \otimes \boldsymbol{E}_A, \tag{67}$$

which implies that $\boldsymbol{\kappa}_l = \kappa_{(l)i}\boldsymbol{L}_i$ is determined by $\boldsymbol{\Gamma}$ via $\boldsymbol{\Gamma}\boldsymbol{L}$. In the same way, $\boldsymbol{\kappa}_m$ and $\boldsymbol{\kappa}_n$ are determined by $\boldsymbol{\Gamma}$ via $\boldsymbol{\Gamma}\boldsymbol{M}$ and $\boldsymbol{\Gamma}\boldsymbol{N}$, respectively.

Moreover, from (41) and (47) we find that $\boldsymbol{E} = \boldsymbol{U}$ in the case of an orthogonal lattice. The fiber stretches are thus given by

$$\lambda_l = \boldsymbol{L} \otimes \boldsymbol{L} \cdot \boldsymbol{E}, \qquad \lambda_m = \boldsymbol{M} \otimes \boldsymbol{M} \cdot \boldsymbol{E}, \qquad \lambda_m = \boldsymbol{N} \otimes \boldsymbol{N} \cdot \boldsymbol{E}. \tag{68}$$

In a discrete lattice consisting of spatial rods interacting at interior nodes as in [Steigmann 1996], the strain energy is the sum of the strain energies of the individual rods. This motivates the assumption of an additive decomposition of the strain energy in the continuum lattice model; i.e.,

$$W(\boldsymbol{E}, \boldsymbol{\Gamma}) = W_l(\lambda_l, \boldsymbol{\kappa}_l) + W_m(\lambda_m, \boldsymbol{\kappa}_m) + W_n(\lambda_n, \boldsymbol{\kappa}_n), \tag{69}$$

in which $W_{l,m,n}$ are the strain energies, per unit initial volume, of the three fiber families.

Using $(54)_1$ and (68) we then derive

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_l \otimes \boldsymbol{L} + \boldsymbol{\sigma}_m \otimes \boldsymbol{M} + \boldsymbol{\sigma}_n \otimes \boldsymbol{N}, \tag{70}$$

with

$$\boldsymbol{\sigma}_l = \frac{\partial W_l}{\partial \lambda_l}\boldsymbol{L}, \qquad \boldsymbol{\sigma}_m = \frac{\partial W_m}{\partial \lambda_m}\boldsymbol{M}, \qquad \boldsymbol{\sigma}_n = \frac{\partial W_n}{\partial \lambda_n}\boldsymbol{N}, \tag{71}$$

yielding

$$\boldsymbol{\sigma}\boldsymbol{E}^t = \lambda_l\frac{\partial W_l}{\partial \lambda_l}\boldsymbol{L} \otimes \boldsymbol{L} + \lambda_m\frac{\partial W_m}{\partial \lambda_m}\boldsymbol{M} \otimes \boldsymbol{M} + \lambda_n\frac{\partial W_n}{\partial \lambda_n}\boldsymbol{N} \otimes \boldsymbol{N}. \tag{72}$$

We thus have $\mathrm{Skw}(\boldsymbol{\sigma}\boldsymbol{E}^t) = \boldsymbol{0}$ and conclude that the associated interaction term vanishes in $(65)_2$. In general the latter may be interpreted as a distributed moment transmitted to the fibers by a matrix material in which the fibers are embedded [Steigmann 2012]. However, the relatively simple model discussed here does not take account of an underlying matrix.

To derive the relevant expression for the couple stress $\boldsymbol{\mu}$ we use (18), for the fiber family initially aligned with $\boldsymbol{L}$, in the form

$$\dot{\kappa}_{(l)i} = \boldsymbol{l}_i \cdot (\nabla\boldsymbol{\omega})\boldsymbol{L} = \boldsymbol{R}\boldsymbol{L}_i \otimes \boldsymbol{L} \cdot \nabla\boldsymbol{\omega}. \tag{73}$$

Thus,

$$\frac{\partial W_l}{\partial \kappa_{(l)i}}\dot{\kappa}_{(l)i} = \boldsymbol{R}\boldsymbol{\mu}_l \otimes \boldsymbol{L} \cdot \nabla\boldsymbol{\omega}, \quad \text{with } \boldsymbol{\mu}_l = \frac{\partial W_l}{\partial \kappa_{(l)i}}\boldsymbol{L}_i. \tag{74}$$

Proceeding from $(54)_2$ and (61) in the same way, we find that

$$W_{\boldsymbol{\Gamma}} \cdot \dot{\boldsymbol{\Gamma}} = \boldsymbol{R}(\boldsymbol{\mu}_l \otimes \boldsymbol{L} + \boldsymbol{\mu}_m \otimes \boldsymbol{M} + \boldsymbol{\mu}_n \otimes \boldsymbol{N}) \cdot \nabla \boldsymbol{\omega}, \tag{75}$$

with

$$\boldsymbol{\mu}_m = \frac{\partial W_m}{\partial \kappa_{(m)i}} \boldsymbol{M}_i \quad \text{and} \quad \boldsymbol{\mu}_n = \frac{\partial W_n}{\partial \kappa_{(n)i}} \boldsymbol{N}_i, \tag{76}$$

and comparison with (61) furnishes

$$\boldsymbol{\mu} = \boldsymbol{\mu}_l \otimes \boldsymbol{L} + \boldsymbol{\mu}_m \otimes \boldsymbol{M} + \boldsymbol{\mu}_n \otimes \boldsymbol{N}. \tag{77}$$

In the virtual-power statement (51) we then have

$$\dot{E}_\pi = \int_{\partial \pi} \boldsymbol{R}\boldsymbol{\mu}\boldsymbol{\nu} \cdot \boldsymbol{\omega}\, da + \int_\pi [\boldsymbol{R}\boldsymbol{\sigma} \cdot \nabla \boldsymbol{u} - \boldsymbol{\omega} \cdot \mathrm{Div}(\boldsymbol{R}\boldsymbol{\mu})]\, dv, \tag{78}$$

where

$$\mathrm{Div}(\boldsymbol{R}\boldsymbol{\sigma}) = [\nabla(\boldsymbol{R}\boldsymbol{\sigma}_l)]\boldsymbol{L} + [\nabla(\boldsymbol{R}\boldsymbol{\sigma}_m)]\boldsymbol{M} + [\nabla(\boldsymbol{R}\boldsymbol{\sigma}_n)]\boldsymbol{N} \tag{79}$$

and

$$\mathrm{Div}(\boldsymbol{R}\boldsymbol{\mu}) = [\nabla(\boldsymbol{R}\boldsymbol{\mu}_l)]\boldsymbol{L} + [\nabla(\boldsymbol{R}\boldsymbol{\mu}_m)]\boldsymbol{M} + [\nabla(\boldsymbol{R}\boldsymbol{\mu}_n)]\boldsymbol{N}. \tag{80}$$

The equilibrium equations are *not* obtained by substituting into (64) and (65), however, because the virtual translational velocity $\boldsymbol{u}$ and rotational velocity $\boldsymbol{\omega}$ are not independent.

Suppose, for example, that all three fiber families have identical uniform properties, each with a strain-energy function of the form (11). Then (see $(71)_1$ and $(74)_2$),

$$\boldsymbol{R}\boldsymbol{\sigma}_l = E(\lambda_l - 1)\boldsymbol{l} \quad \text{and} \quad \boldsymbol{R}\boldsymbol{\mu}_l = T\tau_l \boldsymbol{l} + F\boldsymbol{l} \times (\nabla \boldsymbol{l})\boldsymbol{L}, \quad \text{etc.}, \tag{81}$$

where $\tau_l = \kappa_{(l)3}$, etc., in which E, T, and F respectively are the constant extensional, torsional, and flexural stiffnesses of the fibers. Again we note that the torsional and flexural stiffnesses could conceivably depend on fiber stretch. However, for small extensional strains they are approximated at leading order by constants.

4. Reduction to second-gradient elasticity

4.1. ***Reducing a linear form in the rotational virtual velocity to a linear form in the gradient of the translational virtual velocity.*** To effect the reduction of the Cosserat model to a second-gradient elasticity model [Toupin 1964; Mindlin and Tiersten 1962; Koiter 1964], we proceed from (36) to write the virtual spin tensor $\boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^t$ in the form

$$\boldsymbol{\Omega} = \dot{\boldsymbol{l}} \otimes \boldsymbol{l} + \dot{\boldsymbol{m}} \otimes \boldsymbol{m} + \dot{\boldsymbol{n}} \otimes \boldsymbol{n}, \tag{82}$$

where, from (40),

$$\lambda_l \dot{\boldsymbol{l}} = (\nabla \boldsymbol{u})\boldsymbol{L} - \dot{\lambda}_l \boldsymbol{l}, \quad \text{etc.} \tag{83}$$

Its axial vector $\boldsymbol{\omega}$ is such that

$$\boldsymbol{\omega} \times \boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{l})\dot{\boldsymbol{l}} + (\boldsymbol{v} \cdot \boldsymbol{m})\dot{\boldsymbol{m}} + (\boldsymbol{v} \cdot \boldsymbol{n})\dot{\boldsymbol{n}} \tag{84}$$

for every vector $\boldsymbol{v}$. Using the well known identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}, \tag{85}$$

we derive

$$(\boldsymbol{v} \cdot \boldsymbol{l})\dot{\boldsymbol{l}} = (\boldsymbol{l} \times \dot{\boldsymbol{l}}) \times \boldsymbol{v} + (\boldsymbol{v} \cdot \dot{\boldsymbol{l}})\boldsymbol{l}, \quad \text{etc.}, \tag{86}$$

yielding

$$\boldsymbol{\omega} \times \boldsymbol{v} = (\boldsymbol{l} \times \dot{\boldsymbol{l}} + \boldsymbol{m} \times \dot{\boldsymbol{m}} + \boldsymbol{n} \times \dot{\boldsymbol{n}}) \times \boldsymbol{v} + \boldsymbol{\Omega}^t \boldsymbol{v}, \tag{87}$$

and thus conclude that

$$\boldsymbol{\omega} = \tfrac{1}{2}(\boldsymbol{l} \times \dot{\boldsymbol{l}} + \boldsymbol{m} \times \dot{\boldsymbol{m}} + \boldsymbol{n} \times \dot{\boldsymbol{n}}). \tag{88}$$

In the virtual-power statement (78) we have two linear expressions of the form $\boldsymbol{a} \cdot \boldsymbol{\omega}$. We write these as

$$\boldsymbol{a} \cdot \boldsymbol{\omega} = \tfrac{1}{2}(\boldsymbol{a} \times \boldsymbol{l} \cdot \dot{\boldsymbol{l}} + \boldsymbol{a} \times \boldsymbol{m} \cdot \dot{\boldsymbol{m}} + \boldsymbol{a} \times \boldsymbol{n} \cdot \dot{\boldsymbol{n}}), \tag{89}$$

where

$$\boldsymbol{a} \times \boldsymbol{l} \cdot \dot{\boldsymbol{l}} = \lambda_l^{-1} \boldsymbol{a} \times \boldsymbol{l} \cdot (\nabla \boldsymbol{u})\boldsymbol{L} = \lambda_l^{-1} (\boldsymbol{a} \times \boldsymbol{l}) \otimes \boldsymbol{L} \cdot \nabla \boldsymbol{u}, \quad \text{etc.} \tag{90}$$

Altogether,

$$\boldsymbol{a} \cdot \boldsymbol{\omega} = \tfrac{1}{2}[\lambda_l^{-1}(\boldsymbol{a} \times \boldsymbol{l}) \otimes \boldsymbol{L} + \lambda_m^{-1}(\boldsymbol{a} \times \boldsymbol{m}) \otimes \boldsymbol{M} + \lambda_n^{-1}(\boldsymbol{a} \times \boldsymbol{n}) \otimes \boldsymbol{N}] \cdot \nabla \boldsymbol{u}. \tag{91}$$

Every linear scalar-valued function of $\boldsymbol{\omega}$ may thus be expressed as a linear function of $\nabla \boldsymbol{u}$. We thereby reduce (78) to

$$\dot{E}_\pi = \int_\pi \boldsymbol{P} \cdot \nabla \boldsymbol{u} \, dv + \int_{\partial \pi} \boldsymbol{Q} \cdot \nabla \boldsymbol{u} \, da, \tag{92}$$

where

$$\boldsymbol{P} = \boldsymbol{R}\boldsymbol{\sigma} - \tfrac{1}{2}[\lambda_l^{-1}(\boldsymbol{a} \times \boldsymbol{l}) \otimes \boldsymbol{L} + \lambda_m^{-1}(\boldsymbol{a} \times \boldsymbol{m}) \otimes \boldsymbol{M} + \lambda_n^{-1}(\boldsymbol{a} \times \boldsymbol{n}) \otimes \boldsymbol{N}], \quad \text{with } \boldsymbol{a} = \mathrm{Div}(\boldsymbol{R}\boldsymbol{\mu}), \tag{93}$$

and

$$\boldsymbol{Q} = \tfrac{1}{2}[\lambda_l^{-1}(\boldsymbol{b} \times \boldsymbol{l}) \otimes \boldsymbol{L} + \lambda_m^{-1}(\boldsymbol{b} \times \boldsymbol{m}) \otimes \boldsymbol{M} + \lambda_n^{-1}(\boldsymbol{b} \times \boldsymbol{n}) \otimes \boldsymbol{N}], \quad \text{with } \boldsymbol{b} = \boldsymbol{R}\boldsymbol{\mu}\boldsymbol{\nu}. \tag{94}$$

4.2. ***Constraints on the virtual velocity gradient and the extended variational problem.*** The translational virtual velocity gradient is subject to further constraints associated with the rigid connectivity of the lattice. This implies that (see (36) and (40))

$$\boldsymbol{FL}\cdot\boldsymbol{FM}=0,\qquad \boldsymbol{FL}\cdot\boldsymbol{FN}=0,\quad\text{and}\quad \boldsymbol{FM}\cdot\boldsymbol{FN}=0. \tag{95}$$

To accommodate these we replace the virtual-power statement (51) by

$$\dot{E}^*_\pi = P_\pi, \tag{96}$$

where

$$E^*_\pi = E_\pi + \int_\pi \tfrac{1}{2}\boldsymbol{\Lambda}\cdot\boldsymbol{C}\,dv + \int_{\partial\pi}\tfrac{1}{2}\boldsymbol{\Xi}\cdot\boldsymbol{C}\,da, \tag{97}$$

is the extended energy, $\boldsymbol{C}=\boldsymbol{F}^t\boldsymbol{F}$ is the right Cauchy–Green deformation tensor, and

$$\begin{aligned}\boldsymbol{\Lambda}&=\Lambda_{LM}(\boldsymbol{L}\otimes\boldsymbol{M}+\boldsymbol{M}\otimes\boldsymbol{L})+\Lambda_{LN}(\boldsymbol{L}\otimes\boldsymbol{N}+\boldsymbol{N}\otimes\boldsymbol{L})+\Lambda_{MN}(\boldsymbol{M}\otimes\boldsymbol{N}+\boldsymbol{N}\otimes\boldsymbol{M}),\\ \boldsymbol{\Xi}&=\Xi_{LM}(\boldsymbol{L}\otimes\boldsymbol{M}+\boldsymbol{M}\otimes\boldsymbol{L})+\Xi_{LN}(\boldsymbol{L}\otimes\boldsymbol{N}+\boldsymbol{N}\otimes\boldsymbol{L})+\Xi_{MN}(\boldsymbol{M}\otimes\boldsymbol{N}+\boldsymbol{N}\otimes\boldsymbol{M}),\end{aligned} \tag{98}$$

in which Λ_{LM}, Ξ_{LM}, etc., are Lagrange multipliers. We require multipliers on $\partial\pi$ because the gradient of $\boldsymbol{u}$ thereon, which figures in the virtual-work statement (see (92)), is restricted by (95). Their role in the theory is illustrated in Section 5.

Equation (96) is treated as an unconstrained variational problem in which the additional terms have the variational derivatives

$$\tfrac{1}{2}\boldsymbol{\Lambda}\cdot\dot{\boldsymbol{C}}=\boldsymbol{F}\boldsymbol{\Lambda}\cdot\nabla\boldsymbol{u}\quad\text{and}\quad \tfrac{1}{2}\boldsymbol{\Xi}\cdot\dot{\boldsymbol{C}}=\boldsymbol{F}\boldsymbol{\Xi}\cdot\nabla\boldsymbol{u} \tag{99}$$

at fixed values of the multipliers, whereas variations with respect to the latter merely return the constraints and, as before, are not made explicit. Finally, in (96) we have

$$\dot{E}_\pi=\int_\pi \boldsymbol{T}\cdot\nabla\boldsymbol{u}\,dv+\int_{\partial\pi}\boldsymbol{S}\cdot\nabla\boldsymbol{u}\,da, \tag{100}$$

where

$$\boldsymbol{T}=\boldsymbol{P}+\boldsymbol{F}\boldsymbol{\Lambda}\quad\text{and}\quad \boldsymbol{S}=\boldsymbol{Q}+\boldsymbol{F}\boldsymbol{\Xi} \tag{101}$$

and it is understood that these are evaluated with the constraints (95) in force.

It is useful to observe, from (101), (93), and (70), that

$$\boldsymbol{T}=\boldsymbol{T}_l\otimes\boldsymbol{L}+\boldsymbol{T}_m\otimes\boldsymbol{M}+\boldsymbol{T}_n\otimes\boldsymbol{N}, \tag{102}$$

where

$$\begin{aligned}\boldsymbol{T}_l&=\boldsymbol{R}\boldsymbol{\sigma}_l+\lambda_m\Lambda_{LM}\boldsymbol{m}+\lambda_n\Lambda_{LN}\boldsymbol{n}+\tfrac{1}{2}\lambda_l^{-1}\boldsymbol{l}\times\boldsymbol{a},\\ \boldsymbol{T}_m&=\boldsymbol{R}\boldsymbol{\sigma}_m+\lambda_l\Lambda_{LM}\boldsymbol{l}+\lambda_n\Lambda_{MN}\boldsymbol{n}+\tfrac{1}{2}\lambda_m^{-1}\boldsymbol{m}\times\boldsymbol{a},\\ \boldsymbol{T}_n&=\boldsymbol{R}\boldsymbol{\sigma}_n+\lambda_l\Lambda_{LN}\boldsymbol{l}+\lambda_m\Lambda_{MN}\boldsymbol{m}+\tfrac{1}{2}\lambda_n^{-1}\boldsymbol{n}\times\boldsymbol{a},\end{aligned} \tag{103}$$

and $\boldsymbol{a}$ is given by $(93)_2$. Similarly,

$$\boldsymbol{S} = \boldsymbol{S}_l \otimes \boldsymbol{L} + \boldsymbol{S}_m \otimes \boldsymbol{M} + \boldsymbol{S}_n \otimes \boldsymbol{N}, \tag{104}$$

where

$$\begin{aligned}
\boldsymbol{S}_l &= \lambda_m \Xi_{LM} \boldsymbol{m} + \lambda_n \Xi_{LN} \boldsymbol{n} + \tfrac{1}{2}\lambda_l^{-1} \boldsymbol{b} \times \boldsymbol{l}, \\
\boldsymbol{S}_m &= \lambda_l \Xi_{LM} \boldsymbol{l} + \lambda_n \Xi_{MN} \boldsymbol{n} + \tfrac{1}{2}\lambda_m^{-1} \boldsymbol{b} \times \boldsymbol{m}, \\
\boldsymbol{S}_n &= \lambda_l \Xi_{LN} \boldsymbol{l} + \lambda_m \Xi_{MN} \boldsymbol{m} + \tfrac{1}{2}\lambda_n^{-1} \boldsymbol{b} \times \boldsymbol{n},
\end{aligned} \tag{105}$$

and $\boldsymbol{b}$ is given by $(94)_2$.

4.3. *Equilibrium conditions.* Equation (100) involves the restriction to the boundary of the gradient of the translational virtual velocity. To treat this we introduce the surface parametrization $\boldsymbol{X}(\theta^\alpha)$ of $\partial\pi$, where θ^α $(\alpha = 1, 2)$ is a system of convected curvilinear coordinates. This induces the tangent basis $\boldsymbol{A}_\alpha = \boldsymbol{X}_{,\alpha}$ and dual tangent basis $\boldsymbol{A}^\alpha$, which we use to decompose $\nabla \boldsymbol{u}_{|\partial\pi}$ as

$$\nabla \boldsymbol{u} = \boldsymbol{u}_{,\alpha} \otimes \boldsymbol{A}^\alpha + \boldsymbol{u}_\nu \otimes \boldsymbol{\nu}, \tag{106}$$

where $\boldsymbol{u}_{,\alpha} = \partial \boldsymbol{u}(\boldsymbol{X}(\theta^\beta))/\partial\theta^\alpha = (\nabla \boldsymbol{u})\boldsymbol{A}_\alpha$ are the tangential derivatives of $\boldsymbol{u}$ and $\boldsymbol{u}_\nu = (\nabla \boldsymbol{u})\boldsymbol{\nu}$ is the normal derivative. Thus,

$$\boldsymbol{S} \cdot \nabla \boldsymbol{u} = \boldsymbol{S}\boldsymbol{\nu} \cdot \boldsymbol{u}_\nu + \boldsymbol{S}^\alpha \cdot \boldsymbol{u}_{,\alpha}, \tag{107}$$

where $\boldsymbol{S}^\alpha = \boldsymbol{S}\boldsymbol{A}^\alpha$.

Because $\partial\pi$ is piecewise smooth it is the union of a finite number of smooth subsurfaces ω_i that intersect at edges e_i. Applying Stokes' theorem to each of these subsurfaces, we find that

$$\int_{\partial\pi} \boldsymbol{S}^\alpha \cdot \boldsymbol{u}_{,\alpha}\, da = \sum \int_{\partial\omega_i} \boldsymbol{S}^\alpha \xi_{(i)\alpha} \cdot \boldsymbol{u}\, ds - \int_{\partial\pi} \boldsymbol{S}^\alpha_{|\alpha} \cdot \boldsymbol{u}\, da, \tag{108}$$

where $\boldsymbol{\xi}_i = \xi_{(i)\alpha}\boldsymbol{A}^\alpha$ is the unit normal to the curve $\partial\omega_i$ such that $\{\boldsymbol{\nu}_i \boldsymbol{\xi}_i \boldsymbol{\tau}_i\}$ forms a right-handed orthonormal triad, where $\boldsymbol{\tau}_i$ is the unit tangent to $\partial\omega_i$ and s is arc-length measured in the direction of $\boldsymbol{\tau}_i$, and where $\boldsymbol{S}^\alpha_{|\alpha}$ is the covariant divergence on $\partial\pi$, defined by

$$\boldsymbol{S}^\alpha_{|\alpha} = A^{-1/2}(A^{1/2}\boldsymbol{S}^\alpha)_{,\alpha}, \tag{109}$$

with $A = \det(\boldsymbol{A}_\alpha \cdot \boldsymbol{A}_\beta)$. It is understood that each curve $\partial\omega_i$ in (108) is traversed counterclockwise as the smooth subsurface ω_i is viewed from the side of ω_i into which its surface normal $\boldsymbol{\nu}_i$ is directed. We elaborate further below.

Accordingly (100) is reduced to

$$\dot{E}^* = \int_{\partial\pi} [(\boldsymbol{T}\boldsymbol{\nu} - \boldsymbol{S}^\alpha_{|\alpha}) \cdot \boldsymbol{u} + \boldsymbol{S}\boldsymbol{\nu} \cdot \boldsymbol{u}_\nu]\, da + \sum \int_{\partial\omega_i} \boldsymbol{S}\boldsymbol{\xi}_i \cdot \boldsymbol{u}\, ds - \int_\pi \boldsymbol{u} \cdot \operatorname{Div} \boldsymbol{T}\, dv, \tag{110}$$

and (96) implies that the virtual power has the form

$$P_\pi = \int_{\partial\pi} (\boldsymbol{t}\cdot\boldsymbol{u} + \boldsymbol{s}\cdot\boldsymbol{u}_\nu)\,da + \sum \int_{e_i} \boldsymbol{f}_i\cdot\boldsymbol{u}\,ds + \int_\pi \boldsymbol{g}\cdot\boldsymbol{u}\,dv, \tag{111}$$

where e_i is the i-th edge of $\partial\pi$.

If no further kinematic constraints are operative, then because $\boldsymbol{u}$ and $\boldsymbol{u}_\nu$ may be specified independently on $\partial\pi$, the fundamental lemma yields the traction and double force

$$\boldsymbol{t} = \boldsymbol{T}\boldsymbol{\nu} - \boldsymbol{S}^{\alpha}_{|\alpha} \quad \text{and} \quad \boldsymbol{s} = \boldsymbol{S}\boldsymbol{\nu}, \tag{112}$$

respectively, on $\partial\pi$, and the body force

$$\boldsymbol{g} = -\operatorname{Div}\boldsymbol{T} \tag{113}$$

in the interior of π. By choosing $\boldsymbol{\nu} = \boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}$ in succession, we may conclude, from $(112)_1$, (102), and (103), that the Lagrange multipliers Λ_{LM}, etc., are proportional to transverse shear stresses acting on the fiber "cross sections" (see (27)). Similarly, Ξ_{LM}, etc., are proportional to transverse double forces.

Concerning the edge forces $\boldsymbol{f}_i$, we observe that an edge e is the intersection of two subsurfaces ω_+ and ω_-, say. Accordingly, in (111) e is traversed twice: once in the sense of $\boldsymbol{\tau}_+$ and once in the sense of $\boldsymbol{\tau}_- = -\boldsymbol{\tau}_+$. With (112) and (113) in force the fundamental lemma then furnishes the edge force density

$$\boldsymbol{f} = [\boldsymbol{S}\boldsymbol{\xi}] \quad \text{on } e, \tag{114}$$

where $[\,\cdot\,]$ is the difference of the limits of the enclosed quantity on e when approached from ω_+ and ω_-, i.e, $[\,\cdot\,] = (\,\cdot\,)_+ - (\,\cdot\,)_-$.

Fuller discussions of edge forces, and of the wedge forces operating at vertices in continua of grade higher than two, may be found in [Mindlin 1965; dell'Isola and Seppecher 1995; 1997; dell'Isola et al. 2012; Fosdick 2016].

4.4. ***Rigid-body variations.*** In classical rigid-body mechanics the relevant actions are the net force and couple acting on the body. To deduce their forms in the present model, we specialize the virtual-power statement to rigid-body virtual translations and rotations. In view of the invariance of the strain energy under such variations (see (45)), this statement reduces to $P_\pi = 0$ for all deformations of the form

$$\boldsymbol{\chi}(\boldsymbol{X};\epsilon) = \boldsymbol{Q}(\epsilon)\boldsymbol{x} + \boldsymbol{d}(\boldsymbol{\epsilon}), \tag{115}$$

where $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X})$ is an equilibrium deformation field, $\boldsymbol{Q}(\epsilon)$ is a one-parameter family of rotations with $\boldsymbol{Q}(0) = \boldsymbol{I}$, and $\boldsymbol{d}(\epsilon)$ is a family of vectors with $\boldsymbol{d}(0) = \boldsymbol{0}$. Again using superposed dots to denote derivatives with respect to ϵ, evaluated at $\epsilon = 0$, we compute the virtual translational velocity $\boldsymbol{u}(\boldsymbol{X}) = \boldsymbol{\omega}\times\boldsymbol{x} + \dot{\boldsymbol{d}}$, where $\boldsymbol{\omega}$ is the axial vector of $\dot{\boldsymbol{Q}}$.

Because $\boldsymbol{\omega}$ and $\dot{\boldsymbol{d}}$ are arbitrary, the virtual-work statement (see (111)) is satisfied if and only if

$$\int_{\partial\pi} \boldsymbol{t}\, da + \int_{\pi} \boldsymbol{g}\, dv + \sum \int_{e_i} \boldsymbol{f}_i\, ds = \boldsymbol{0} \tag{116}$$

and

$$\int_{\partial\pi} (\boldsymbol{x} \times \boldsymbol{t} + \boldsymbol{c})\, da + \int_{\pi} \boldsymbol{x} \times \boldsymbol{g}\, dv + \sum \int_{e_i} \boldsymbol{x} \times \boldsymbol{f}_i\, ds = \boldsymbol{0}, \tag{117}$$

where

$$\boldsymbol{c} = \boldsymbol{x}_\nu \times \boldsymbol{s} \tag{118}$$

and

$$\boldsymbol{x}_\nu = \boldsymbol{F}\boldsymbol{\nu} \tag{119}$$

is the normal derivative of the equilibrium deformation on $\partial\pi$.

These are respectively the force and moment balances for the arbitrary part $\pi \subset \kappa$ of the body, the latter implying that $\boldsymbol{c}$ is a distribution of couples acting on $\partial\pi$. Because these conditions were derived using a special form of $\boldsymbol{u}$, they are necessary for equilibrium. Indeed, it may be shown that they follow from (112) and (113). However, they are not sufficient — the arbitrariness of π notwithstanding — because $(112)_2$ involves the entire double force on $\partial\pi$, whereas (118) is insensitive to that part of the double force which is parallel to $\boldsymbol{x}_\nu$. This situation stands in contrast to first-gradient elasticity, in which (116) and (117) (with $\boldsymbol{c}$ and $\boldsymbol{f}_i$ equal to zero) are both necessary *and* sufficient for equilibrium. The utility of the variational approach to our subject can thus hardly be overestimated [Germain 1973]. This perspective is amplified and extended in a recent revival [Eugster and dell'Isola 2017; 2018a; 2018b; Eugster and Glocker 2017] of Hellinger's approach to continuum mechanics.

5. Example: bending a block to a cylinder

To illustrate the model we use it to solve the classical problem of bending a block to a cylindrical annulus [Ogden 1984]. The conventional treatment of this problem relies on the use of first-gradient elasticity. Here we highlight the additional flexibility in its solution afforded by the present model.

We choose the fibers to be aligned initially with a Cartesian coordinate system (X, Y, Z), so that

$$\boldsymbol{X} = X\boldsymbol{L} + Y\boldsymbol{M} + Z\boldsymbol{N}. \tag{120}$$

The block occupies the volume defined by $A < X < A + W$, $-H/2 < Y < H/2$, and $-D/2 < Z < D/2$, where A is a positive constant, W is the width of the block, H is the height, and D is the depth. The deformed position is

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}) = r(X)\boldsymbol{e}_r(\theta(Y)) + Z\boldsymbol{N}, \tag{121}$$

where

$$\boldsymbol{e}_r(\theta) = \cos\theta \boldsymbol{L} + \sin\theta \boldsymbol{M}. \tag{122}$$

Thus, the deformation maps vertical planes $X = \text{const.}$ to cylinders $r = \text{const.}$, and horizontal planes $Y = \text{const.}$ to radial planes $\theta = \text{const.}$. There is no displacement along the Z-axis.

The deformation gradient is

$$\boldsymbol{F} = r'\boldsymbol{e}_r \otimes \boldsymbol{L} + r\theta'\boldsymbol{e}_\theta \otimes \boldsymbol{M} + \boldsymbol{N} \otimes \boldsymbol{N}, \tag{123}$$

where $\boldsymbol{e}_\theta = \boldsymbol{N} \times \boldsymbol{e}_r$, and we assume, as in the classical treatment, that $\theta(Y) = \alpha Y$, with α a positive constant. Accordingly, $\det \boldsymbol{F} = \alpha r r'$, and the usual restriction $\det \boldsymbol{F} > 0$ implies that $r(X)$ is an increasing function, i.e., $r' > 0$. It follows immediately that

$$\boldsymbol{R} = \boldsymbol{e}_r \otimes \boldsymbol{L} + \boldsymbol{e}_\theta \otimes \boldsymbol{M} + \boldsymbol{N} \otimes \boldsymbol{N} \quad \text{and} \quad \boldsymbol{E} = \boldsymbol{U} = r'\boldsymbol{L} \otimes \boldsymbol{L} + \alpha r \boldsymbol{M} \otimes \boldsymbol{M} + \boldsymbol{N} \otimes \boldsymbol{N}, \tag{124}$$

and hence, from (36) and (42), that

$$\begin{aligned} &\boldsymbol{l} = \boldsymbol{e}_r(\theta), \qquad &&\boldsymbol{m} = \boldsymbol{e}_\theta(\theta), \qquad &&\boldsymbol{n} = \boldsymbol{N}, \\ &\lambda_l = r'(X), \qquad &&\lambda_m = \alpha r(X), \qquad &&\lambda_n = 1. \end{aligned} \tag{125}$$

Clearly the rigidity constraints (95) are satisfied.

Using (37)–(39) we find that all fiber twists $\tau_{l,m,n} (= \kappa_{(l,m,n)3})$ vanish. Assuming the fiber constitutive relations $(81)_2$, we deduce that

$$\begin{aligned} \boldsymbol{R}\boldsymbol{\mu}_l &= \boldsymbol{R}\boldsymbol{\mu}_n = \boldsymbol{0}, \\ \boldsymbol{R}\boldsymbol{\mu}_m &= \alpha F \boldsymbol{N}, \end{aligned} \tag{126}$$

so that $\boldsymbol{a} = \boldsymbol{0}$ in (93) and (103), whereas

$$\boldsymbol{b} = \alpha F(\boldsymbol{M} \cdot \boldsymbol{\nu})\boldsymbol{N} \tag{127}$$

in (94). Guided by the structure of solutions to rod theory for uniformly curved rods, we seek a solution in which the various Λ — the transverse shear stresses acting on the fiber cross sections — vanish. In this case (81) and (103) imply that

$$\begin{aligned} \boldsymbol{T}_l &= f(\lambda_l)\boldsymbol{e}_r, \\ \boldsymbol{T}_m &= f(\lambda_m)\boldsymbol{e}_\theta, \\ \boldsymbol{T}_n &= \boldsymbol{0}, \end{aligned} \tag{128}$$

where (see (81)) $f(\lambda) = E(\lambda - 1)$. We use (79)–(81) to compute

$$\operatorname{Div} \boldsymbol{T} = E[r'' - \alpha(\alpha r - 1)]\boldsymbol{e}_r, \tag{129}$$

and conclude, from (113) (with vanishing body force), that $r(X)$ satisfies the simple linear differential equation

$$r'' - \alpha^2 r = -\alpha, \tag{130}$$

the general solution to which is

$$r(X) = \alpha^{-1} + C_1 \exp(\alpha X) + C_2 \exp(-\alpha X), \tag{131}$$

where $C_{1,2}$ are constants.

To complete the solution we proceed as in the classical case and impose zero traction (and double-force) conditions at $X = A$ and $X = A + W$. With $\boldsymbol{\nu} = \pm \boldsymbol{L}$ we find that $\boldsymbol{b}$ vanishes. We also assume that the various Ξ vanish on these surfaces, and hence that $\boldsymbol{S}$ also vanishes. The double force then vanishes, as required, and the tractions are $\pm \boldsymbol{T}_l$. The condition of zero traction thus requires that $f(\lambda_l)$ vanish, and hence that $r' = 1$ at $X = A$ and at $X = A + W$. We thus obtain

$$\begin{aligned} C_1 &= \alpha^{-1} \exp(-\alpha B)[1 + \exp(-\alpha W)]^{-1}, \\ C_2 &= -\alpha^{-1} \exp(\alpha B)[1 + \exp(\alpha W)]^{-1}, \end{aligned} \tag{132}$$

where $B = A + W$, and verify that the admissibility condition $r'(X) > 0$ is satisfied.

On the planes $Z = \pm D/2$ we again assume that the various Ξ vanish, finding that the tractions and double forces also vanish on these surfaces. From (114) we also find that the edge forces vanish on the edges defined by $(X, Z) = (A, \pm D/2)$ and $(X, Z) = (A + W, \pm D/2)$.

The situation is different on the planes $Y = \pm H/2$. For example, at $Y = H/2$ we have $\boldsymbol{\nu} = \boldsymbol{M}$, yielding

$$\begin{aligned} \boldsymbol{T}\boldsymbol{\nu} &= E(\alpha r - 1)\boldsymbol{e}_\theta \\ \boldsymbol{s} = \boldsymbol{S}_m &= (r' \Xi^+_{LM} - F/2r)\boldsymbol{e}_r + \Xi^+_{MN}\boldsymbol{N}, \end{aligned} \tag{133}$$

where the Ξ^+ are the values of the Ξ at $Y = +H/2$. Further,

$$\boldsymbol{S}^\alpha_{|\alpha} = \boldsymbol{S}_{l,X} + \boldsymbol{S}_{n,Z}, \tag{134}$$

where

$$\begin{aligned} \boldsymbol{S}_l &= \alpha(r\Xi^+_{LM} + F/2r')\boldsymbol{e}_\theta + \Xi^+_{LN}\boldsymbol{N}, \\ \boldsymbol{S}_n &= r'\Xi^+_{LN}\boldsymbol{e}_r + \alpha r \Xi^+_{MN}\boldsymbol{e}_\theta. \end{aligned} \tag{135}$$

Then (112) and (128) deliver the traction

$$\begin{aligned} \boldsymbol{t} = [E(\alpha r - 1) - \alpha(r\Xi^+_{LM} + F/2r')_{,X}]\boldsymbol{e}_\theta \\ - \Xi^+_{LN,X}\boldsymbol{N} - r'\Xi^+_{LN,Z}\boldsymbol{e}_r - \alpha r \Xi^+_{MN,Z}\boldsymbol{e}_\theta. \end{aligned} \tag{136}$$

Accordingly, and in contrast to the classical treatment [Ogden 1984], we may impose a zero-traction condition on this surface, provided that

$$E(r-\alpha^{-1})-(r\,\Xi^{+}_{LM}+F/2r')_{,X}-r\,\Xi^{+}_{MN,Z}=0, \tag{137}$$

together with

$$\Xi^{+}_{LN,X}=0 \quad \text{and} \quad \Xi^{+}_{LN,Z}=0. \tag{138}$$

We assume that $\Xi^{+}_{MN,Z}=0$ and use (130) to reduce (137) to

$$E\alpha^{-2}r''(X)-(r\,\Xi^{+}_{LM}+F/2r')_{,X}=0, \tag{139}$$

concluding that

$$E\alpha^{-2}r'-(r\,\Xi^{+}_{LM}+F/2r')=G^{+}(Z), \tag{140}$$

for some function G^{+}.

The couple distribution $\boldsymbol{c}^{+}$ on $Y=H/2$ is given by (118), with $\boldsymbol{x}_{\nu}=\boldsymbol{F}\boldsymbol{M}=\alpha r\boldsymbol{e}_{\theta}$. Thus,

$$\boldsymbol{c}^{+}=\alpha r\boldsymbol{e}_{\theta}\times\boldsymbol{S}_{m}=\alpha r'(F/2r'-r\,\Xi^{+}_{LM})\boldsymbol{N}+\alpha r\,\Xi^{+}_{MN}\boldsymbol{e}_{r}. \tag{141}$$

A solution with $\boldsymbol{c}^{+}$ parallel to the cylinder axis $\boldsymbol{N}$ and independent of Z is obtained by taking $\Xi^{+}_{MN}=0$ and G^{+} to be constant. From $(133)_2$ this is seen to be tantamount to the assignment of the double-force distribution on $Y=H/2$. The edge forces operating at the edges defined by $(X,Y)=(A,H/2)$ and $(X,Y)=(A+W,H/2)$ are found, using (114), to be

$$\boldsymbol{f}=\mp[\alpha(r\,\Xi^{+}_{LM}+F/2)\boldsymbol{e}_{\theta}+\Xi^{+}_{LN}\boldsymbol{N}], \tag{142}$$

respectively, where Ξ^{+}_{LM} and Ξ^{+}_{LN} are evaluated at $X=A,\,A+W$, respectively, and we have used the condition $r'=1$ at both edges. These edge forces vanish if and only if Ξ^{+}_{LN} and $r\,\Xi^{+}_{LM}+F/2$ vanish at $X=A,\,A+W$. Combing the latter with (140), we find that

$$G^{+}=E\alpha^{-2}, \tag{143}$$

and (140) then delivers

$$\Xi^{+}_{LM}(X)=r^{-1}[E\alpha^{-2}(r'-1)-F/2r'] \tag{144}$$

with $r(X)$ given by (131) and (132). The couple distribution on $Y=H/2$ is

$$\boldsymbol{c}^{+}=[\alpha F+E\alpha^{-1}r'(r'-1)]\boldsymbol{N}. \tag{145}$$

Moreover, because Ξ^{+}_{LN} is independent of X (see $(138)_1$), it vanishes everywhere on this surface.

Finally, the edge forces acting on the edges defined by $(Y,Z)=(H/2,\pm D/2)$ are found to be

$$\boldsymbol{f}=\pm\alpha r\,\Xi^{+}_{MN}\boldsymbol{e}_{\theta}, \tag{146}$$

respectively. These vanish provided that Ξ^{+}_{MN} vanishes at $Z = \pm D/2$. Because this function was assumed to be independent of Z in the course of constructing the solution, it then vanishes everywhere on the plane $Y = H/2$. We conclude that $\boldsymbol{S}_n$ vanishes there, and that $\boldsymbol{S}_l$ and the double force are given respectively by

$$\boldsymbol{S}_l = E\alpha^{-1}(r'-1)\boldsymbol{e}_\theta \quad \text{and} \quad \boldsymbol{S}_m = r^{-1}[E\alpha^{-2}r'(r'-1) - F]\boldsymbol{e}_r. \tag{147}$$

The situation on the surface $Y = -H/2$ is similar and so we leave the remaining details to the interested reader. A novel feature of the present model is the prediction that the tractions transmitted by the initially vertical fibers to the surfaces $Y = \pm H/2$ can be nullified by the rigid joints of the intersecting fibers via the associated Lagrange multipliers.

We have said nothing about the stability of this solution. In particular, the vertical fibers near $X = A$ may become susceptible to buckling as the flexure angle α increases. However, we defer the analysis of buckling — of considerable importance in the mechanics of metamaterials — to a future investigation.

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