Sets of inhomogeneous linear forms can be not isotropically winning

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We give an example of irrational vector $\theta \in \mathbb{R}^2$ such that the set
\[
\text{Bad}_\theta := \{(\eta_1, \eta_2) : \inf_{x \in \mathbb{N}} x^{1/2} \max_{i=1,2} \|x\theta_i - \eta_i\| > 0\}
\]
is not absolutely winning with respect to McMullen’s game.

1. Introduction

We consider a problem related to inhomogeneous Diophantine approximation. Given $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ we study the set of pairs $(\eta_1, \eta_2) \in \mathbb{R}^2$ such that the system of two linear forms
\[
\|x\theta_1 - \eta_1\|, \quad \|x\theta_2 - \eta_2\|
\]
where $\|\cdot\|$ stands for the distance to the nearest integer, is badly approximable. We prove a statement complementary to our recent result from [Bengoechea et al. 2017]. We construct $\theta$ such that the set
\[
\text{Bad}_\theta := \{(\eta_1, \eta_2) : \inf_{x \in \mathbb{N}} x^{1/2} \max_{i=1,2} \|x\theta_i - \eta_i\| > 0\}
\]
is not isotropically winning.

Our paper is organized as follows. In Section 2 we discuss different games appearing in Diophantine problems. In Section 3 we give a brief survey on inhomogeneous badly approximable systems of linear forms and formulate our main result, Theorem 3.1. Sections 4 and 5 are devoted to some auxiliary observations. In Sections 6, 7, and 8 we give a proof for Theorem 3.1.

2. Schmidt’s game and its generalizations

The following game was introduced by Schmidt [1966; 1969; 1980]. Let $0 < \alpha, \beta < 1$. Suppose that two players A and B choose in turn a nested sequence of closed balls:
\[
B_1 \supset A_1 \supset B_2 \supset A_2 \supset \cdots
\]
with the property that the diameters $|A_i|, |B_i|$ of the balls $A_i, B_i$ satisfy
\[
|A_i| = \alpha |B_i|, \quad |B_{i+1}| = \beta |A_i| \quad \text{for all } i = 1, 2, 3, \ldots
\]
for fixed $0 < \alpha, \beta < 1$. A set $E \subset \mathbb{R}^n$ is called $(\alpha, \beta)$-winning if player A has a strategy which guarantees that intersection $\bigcap A_i$ meets $E$ regardless of the way B chooses to play. A set $E \supset \mathbb{R}^n$ is called an $\alpha$-winning set if it is $(\alpha, \beta)$-winning for all $0 < \beta < 1$.

There are different modifications of Schmidt’s game: the strong game and absolute game introduced in [McMullen 2010], the hyperplane absolute game introduced in [Kleinbock and Weiss 2010], the potential game considered in [Fishman et al. 2013], and some others. In [Bengoechea et al. 2017], we introduced isotropically winning sets. Let us describe here some of these generalizations in more detail.

The definition of an absolutely winning set was given in [McMullen 2010]. Consider the following game. Suppose $A$ and $B$ choose in turn a sequence of balls $A_i$ and $B_i$ such that the sets $B_1 \supset (B_1 \setminus A_1) \supset B_2 \supset (B_2 \setminus A_2) \supset B_3 \supset \cdots$ are nested. For fixed $0 < \beta < \frac{1}{3}$ we suppose $|B_{i+1}| \geq \beta |B_i|$, $|A_i| \leq \beta |B_i|$. We say $E$ is an absolute winning set if for all $\beta \in (0, \frac{1}{3})$, player A has a strategy which guarantees that $\bigcap B_i$ meets $E$ regardless of how B chooses to play. McMullen proved that an absolute winning set is $\alpha$-winning for all $\alpha < \frac{1}{2}$. Several examples of absolute winning sets were exhibited by McMullen [2010]. In particular, a set of badly approximable numbers in $\mathbb{R}$ is absolutely winning. However the set of simultaneously badly approximable vectors in $\mathbb{R}^n$ for $n > 1$ is not absolutely winning.

In [Bengoechea et al. 2017] another strong variant of the winning property was given. We say that a set $E \subset \mathbb{R}^n$ is isotropically winning if for each $d \leq n$ and for each $d$-dimensional affine subspace $A \subset \mathbb{R}^n$ the intersection $E \cap A$ is $\frac{1}{2}$-winning for Schmidt’s game considered as a game in $A$. It is clear that an absolute winning set is isotropically winning for each $\alpha \leq \frac{1}{2}$.

### 3. Inhomogeneous approximations

The first important result on inhomogeneous approximations in the one-dimensional case is due to Khin-chine [1926]. He proved that there exists an absolute constant $\gamma$ such that for every $\theta \in \mathbb{R}$ there exists $\eta \in \mathbb{R}$ such that

$$\inf_{q \in \mathbb{Z}} q \|q\theta - \eta\| > \gamma.$$  

Later (see [Khinchin 1937; 1948]) he proved that for given positive numbers $n, m \in \mathbb{Z}$ there exists a positive constant $\gamma_{nm}$ such that for any $m \times n$ real matrix $\theta$ there exists a vector $\eta \in \mathbb{R}^n$ such that

$$\inf_{x \in \mathbb{Z}^m \setminus \{0\}} (\|\theta x - \eta\|_{\mathbb{Z}^n})^n \|x\|^m > \gamma_{nm}$$

(here $\|\cdot\|_{\mathbb{Z}^n}$ stands for the distance to the nearest integral point in sup-norm). These results are presented in a wonderful book by Cassels [1957].

Jarník [1941], proved a generalization of this statement. Suppose $\psi(t)$ is a function decreasing to zero as $t \to +\infty$. Let $\rho(t)$ be the function inverse to the function $t \mapsto 1/\psi(t)$. Suppose that for all $t > 1$ one has $\psi_\theta(t) \leq \psi(t)$. Then there exists a vector $\eta \in \mathbb{R}^n$ such that

$$\inf_{x \in \mathbb{Z}^m \setminus \{0\}} (\|\theta x - \eta\|_{\mathbb{Z}^n}) \cdot \rho(8m \cdot \|x\|) > \gamma$$
with appropriate $\gamma = \gamma(n, m)$.

Denote by
\[ \text{Bad}_\theta = \{ \alpha \in [0, 1] : \inf_{q \in \mathbb{N}} q \cdot \|q \theta - \alpha\| > 0 \}. \]

It happens that the winning property of this inhomogeneous Diophantine set was considered quite recently. Tseng [2009] showed that $\text{Bad}_\theta$ is winning for all real numbers $\theta$ in classical Schmidt’s sense. For the corresponding multidimensional sets
\[ \text{Bad}(n, m) = \left\{ \theta \in \text{Mat}_{n \times m}(\mathbb{R}) : \inf_{q \in \mathbb{Z}_{n \times m}^{\neq 0}} \max_{1 \leq i \leq n} (|q|^{m/n} \|\theta_i(q)\|) > 0 \right\}. \]

the winning property is shown, for example, in [Einsiedler and Tseng 2011; Moshchevitin 2011]. In [Broderick et al. 2013] it was shown that the set $\text{Bad}(n, m)$ is hyperplane absolutely winning. The methods used in [Broderick et al. 2013] come from [Broderick et al. 2011].

Further generalizations deal with the twisted sets
\[ \text{Bad}(i, j) = \left\{ (\theta_1, \theta_2) \in \mathbb{R}^2 : \inf_{q \in \mathbb{N}} \max_{1 \leq i \leq n} (|q|^i \|\theta_1(q)\|, q^j \|\theta_2(q)\|) > 0 \right\}, \]

where $i, j$ are real positive numbers satisfying $i + j = 1$, introduced by Schmidt. In [An 2016] it was proved that $\text{Bad}(i, j)$ is winning for the standard Schmidt game. In higher dimension, we fix an $n$-tuple $k = (k_1, \ldots, k_n)$ of real numbers satisfying
\[ k_1, \ldots, k_n > 0 \quad \text{and} \quad \sum_{i=1}^{n} k_i = 1, \quad (1) \]

and define
\[ \text{Bad}(k, n, m) = \left\{ \theta \in \text{Mat}_{n \times m}(\mathbb{R}) : \inf_{q \in \mathbb{Z}_{n \times m}^{\neq 0}} \max_{1 \leq i \leq n} (|q|^{m k_i} \|\theta_i(q)\|) > 0 \right\}. \]

Here, $|\cdot|$ denotes the supremum norm, $\theta = (\theta_{ij})$, and $\theta_i(q)$ is the product of the $i$-th line of $\theta$ with the vector $q$, i.e.,
\[ \theta_i(q) = \sum_{j=1}^{m} q_j \theta_{ij}. \]

In the twisted setting, much less is known. In particular up to now the winning property of the set $\text{Bad}(k, n, m)$ in dimension greater that two is not proved.

Given $\theta \in \text{Mat}_{n \times m}(\mathbb{R})$, we define
\[ \text{Bad}_\theta(k, n, m) = \left\{ x \in \mathbb{R}^n : \inf_{q \in \mathbb{Z}_{n \times m}^{\neq 0}} \max_{1 \leq i \leq n} (|q|^{m k_i} \|\theta_i(q) - x_i\|) > 0 \right\}. \]

Harrap and Moshchevitin [2017] showed that this set is winning provided that $\theta \in \text{Bad}(k, n, m)$.

In [Bengoechea et al. 2017] it was proved that if we suppose that $\theta \in \text{Bad}(k, n, m)$, the set $\text{Bad}_\theta(k, n, m)$ is isotropically winning.\footnote{In fact, the approach from [Bengoechea et al. 2017] gives a little bit more. Instead of property that for any subspace $\mathcal{A}$ the intersection $E \cap \mathcal{A}$ is $\frac{1}{2}$-winning in $\mathcal{A}$, one can see that it is $\alpha$-winning for all $\alpha \in (0, \frac{1}{2})$. It is not completely clear for the author if these two properties are equivalent. (For a closely related problem, see [Dremov 2002].)}

We should note that even in the case $n = 2, m = 1$ it is not known if the set $\text{Bad}_\theta(k, 2, 1)$ is $\alpha$-winning for some positive $\alpha$ without the condition $\theta \in \text{Bad}(k, 2, 1)$.
In this article we show that the condition $\theta$ be from Bad($k, n, m$) is essential for the isotropically winning property, and prove the following theorem.

**Theorem 3.1.** There exists a vector $\theta = (\theta_1, \theta_2)$ such that:

1. $1, \theta_1, \theta_2$ are linearly independent over $\mathbb{Z}$.
2. Bad$_\theta := \{ (\eta_1, \eta_2) : \inf_{x \in \mathbb{N}} x^{1/2} \max_{i=1,2} \| x\theta_i - \eta_i \| > 0 \}$ is not isotropically winning.

4. Some more remarks

In the sequel, $x = (x_0, x_1, x_2)$ is a vector in $\mathbb{R}^3$, $| \cdot |$ stands for the Euclidean norm of the vector, and by $(\mathbf{w}, \mathbf{t})$ we denote the inner product of vectors $\mathbf{w}$ and $\mathbf{t}$.

The proof of Theorem 3.1 we will give in Section 6. There we will construct a special $\theta$ and a one-dimensional affine subspace $P$ such that $\theta \in P$ and for the segment $D = P \cap \{ |z - \theta| \leq 1 \}$ one has $D \cap \text{Bad}_\theta = \emptyset$. Moreover, given an arbitrary positive function $\omega(t)$ monotonically (slowly) increasing to infinity we can ensure that for all $\eta = (\eta_1, \eta_2) \in D$ there exist infinitely many $x \in \mathbb{Z}$ such that

$$\max_{i=1,2} \| x\theta_i - \eta_i \| < \frac{\omega(x)}{x}.$$ 

To explain the construction of the proof it is useful to consider the case when $\theta_1, \theta_2, 1$ are linearly dependent. This case we will discuss in Section 5.

**Remark 4.1.** From the result of the paper [Bengoechea et al. 2017] it follows that the vector $\theta$ constructed in Theorem 3.1 does not belong to the set

$$\text{Bad} = \{ (\theta_1, \theta_2) \mid \inf_{x \in \mathbb{N}} x^{1/2} \max(\| \theta_1 x \|, \| \theta_2 x \|) > 0 \}.$$ 

**Remark 4.2.** Let $\theta = (a_1/q, a_2/q)$ be rational. Let $\eta = (\eta_1, \eta_2) \notin \frac{1}{q} \cdot \mathbb{Z}^2$; then for any $x \in \mathbb{Z}$,

$$\max_{i=1,2} \left\| x \frac{a_i}{q} - \eta_i \right\| \geq \text{dist}(\eta, \frac{1}{q} \cdot \mathbb{Z}^2) > 0.$$ 

So the set

$$B = \left\{ \eta : \inf_{x \in \mathbb{Z}} \max_{i=1,2} \left\| x \frac{a_i}{q} - \eta_i \right\| > 0 \right\}$$

contains $\mathbb{R}^2 \setminus \frac{1}{q} \cdot \mathbb{Z}^2$ and is trivially winning. It is clear that for any one-dimensional affine subspace $\ell$ we have $B \cap \ell \supset \left( \mathbb{R}^2 \setminus \frac{1}{q} \cdot \mathbb{Z}^2 \right) \cap \ell$. So obviously $B \cap \ell$ is also winning in $\ell$.

5. Linearly dependent case

Let $1, \theta_1, \theta_2$ be linearly dependent and at least one of $\theta_j$ is irrational. This means that there exists $z = (z_0, z_1, z_2) \in \mathbb{Z}^3$ such that $(z, \theta) = 0$. Let us consider the two-dimensional rational subspace

$$\pi = \{ x \in \mathbb{R}^3 : (x, z) = 0 \},$$

so $\theta \in \pi$.

Let us define the one-dimensional subspace $P = \{ (x_1, x_2) : (1, x_1, x_2) \in \pi \} \subset \mathbb{R}^2$. 
We will prove that there exists a constant $\gamma$ such that for any $\eta = (\eta_1, \eta_2) \in \mathcal{P}$ the inequality

$$\max_{i=1,2} \| \theta_i x - \eta_i \| < \frac{\gamma}{x}$$

has infinitely many solutions in $x \in \mathbb{N}$. (This statement is similar to Chebyshev's theorem [Khinchin 1964, Theorem 24, Chapter 2].)

Denote by $\Lambda = \pi \cap \mathbb{Z}^3$ the integer lattice with the determinant $d := \det \Lambda = |z|$. Denote by $\{g_v = (q_v, a_{1v}, a_{2v})\}_{v=1,2,3,...} \subset \Lambda$ the sequence of the best approximations of $\theta$ by the lattice $\Lambda$ and the corresponding parallelograms

$$\Pi_v = \{ x = (x_0, x_1, x_2) \in \pi : 0 \leq x_0 \leq q_v, \text{ dist}(x, l(\theta)) \leq \text{dist}(g_{v-1}, l(\theta)) \},$$

which contains a fundamental domain of the two-dimensional $\Lambda$. Obviously, $\text{vol} \Pi_v \leq 4d$. So,

$$\text{dist}(g_{v-1}, l(\theta)) \ll \frac{d}{q_v}, \quad (2)$$

with an absolute constant in the sign $\ll$. It is clear that for any point $\eta \in \pi$, the shift $\eta + \Pi_v$ contains a point of $\Lambda$.

For any $\eta = (\eta_1, \eta_2) \in \mathcal{P}$ and for any positive integer $v$ the planar domain $\tilde{\eta} + \Pi_v$, $\tilde{\eta} = (1, -\eta_1, -\eta_2)$ contains an integer point $y = (x, y_1, y_2) \in \Lambda$.

It is clear that

$$1 \leq x \leq 1 + q_v \quad (3)$$

and

$$\max_{i=1,2} \| \theta_i x - \eta_i \| \ll \text{dist}(y, l(\theta) + \tilde{\eta}) \ll \text{dist}(l(\theta), g_{v-1}),$$

and by (2),

$$\max_{i=1,2} \| \theta_i x - \eta_i \| \ll \frac{d}{q_v}, \quad (4)$$

From (3), (4) it follows that the inequality

$$\max_{i=1,2} \| \theta_i x - \eta_i \| \ll \frac{d}{x}$$

has infinitely many solutions and everything is proved.

6. Inductive construction of integer points

Let $\omega(t)$ be arbitrary positive function monotonically (slowly) increasing to infinity. Here we describe the inductive construction of integer points $z_v = (q_v, z_{1v}, z_{2v})$. The base of the induction process is trivial. One can take an arbitrary primitive pair of integer vectors that can be completed to a basis of $\mathbb{Z}^3$.

Suppose that we have two primitive integer vectors

$$z_{v-1} = (q_{v-1}, z_{1v-1}, z_{2v-1}) \in \mathbb{Z}^3, \quad z_v = (q_v, z_{1v}, z_{2v}) \in \mathbb{Z}^3.$$

Now we explain how to construct the next integer vector $z_{v+1}$.

We consider the two-dimensional subspace

$$\pi_v = \langle z_{v-1}, z_v \rangle_{\mathbb{R}}.$$
The pair of vectors $z_{v-1}$ and $z_v$ is primitive, so the lattice spanned by them is
\[ \Lambda_v := (z_{v-1}, z_v) \mathbb{Z} = \pi_v \cap \mathbb{Z}^3. \]
By $d_v = \det \Lambda_v$ we denote the two-dimensional fundamental volume of the lattice $\Lambda_v$. Now we define the vector $n_v = (n_{0v}, n_{1v}, n_{2v}) \in \mathbb{R}^3$ from the conditions
\[ \pi_v = \{ x \in \mathbb{R}^3 : (x, n_v) = 0 \}, \quad |n_v| = 1. \]
Put
\[ \sigma_v = \text{dist}(z_{v-1}, l(z_v)). \] (5)
Obviously, $|z_v| \asymp q_v$ and
\[ \sigma_v \asymp \frac{d_v}{q_v}. \] (6)
We define a vector $e_v$ from the conditions
\[ e_v \in \pi_v, \quad |e_v| = 1, \quad (e_v, z_v) = 0, \] (7)
so $e_v$ is parallel to $\pi_v$ and orthogonal to $z_v$.

Define the rectangle
\[ \Pi_v = \{ x = (x_0, x_1, x_2) : x = tz_v + re_v, \ 0 \leq t \leq |z_v|, \ |r| \leq \sigma_v \}. \]
It is clear that rectangle $\Pi_v \subset \pi_v$ contains a fundamental domain of the lattice $\Lambda_v$. We need two axillary vectors $z^a_v$ and $z^b_v$ defined as
\[ z^a_v = z_v + a_v e_v, \quad z^b_v = z_v + b_v n_v, \]
where positive $a_v$ is chosen in such a way that
\[ a_v d_v^2 \leq v^{-1} \omega \left( \frac{q_v^2}{d_v^2}, \frac{1}{a_v} \right) \] (8)
and
\[ b_v = a_v \min \left( 1, \frac{d_v}{q_v} \right). \] (9)
From the construction, it follows that
\[ |z^a_v| \asymp |z^b_v| \asymp |z_v| \asymp q_v. \] (10)
The integer lattice $\mathbb{Z}^3$ splits into levels with respect to the two-dimensional sublattice $\Lambda_v$ in such a way that
\[ \mathbb{Z}^3 = \bigsqcup_{i \in \mathbb{Z}} \Lambda_{v,i}, \]
where $\Lambda_{v,j} = \Lambda_v + jz', \ j \in \mathbb{Z}$ and integer vector $z'$ completes the couple $z_{v-1}, z_v$ to the basis in $\mathbb{Z}^3$. We consider the affine subspace $\pi^1_v = \pi_v + z' \supset \Lambda_{v,1}$, which is parallel to $\pi_v$. It is clear that dist($\pi_v, \pi^1_v$) = 1/d_v.

We need to determine the next integer point $z_{v+1}$. Denote by $\mathfrak{P}$ the central projection with center 0 onto the affine subspace $\pi^1_v$. We consider the triangle $\Delta$ with vertices $z_v, z^a_v, z^b_v$ and its image $\mathfrak{P}\Delta$ under
The projection $\mathcal{P}$ (Figure 1).

Define

$$Z = \mathcal{P}z^b_v.$$  \hfill (11)

One can see that

$$|Z| \asymp \frac{q_v}{d_v}.$$  \hfill (12)

Define rays

$$R_1 = \{z = Z + tz_v : t \geq 0\} \quad \text{and} \quad R_2 = \{z = Z + tz_v^a : t \geq 0\}.$$  

It is clear that $R_1 \cap R_2 = \{Z\}$ and $R_1, R_2 \subset \pi^1_v$. Moreover, the whole convex angle bounded by rays $R_1, R_2$ form the image of the triangle $\Delta$ under the projection $\mathcal{P}$:

$$\mathcal{P}\Delta = \text{conv}(R_1 \cup R_2).$$

The affine subspace $\pi^1_v$ contains the affine lattice $\Lambda^1_v = \Lambda_v + z'$ which is congruent to the lattice $\Lambda_v$. Thus, for any $\xi \in \pi^1_v$, the shift $\Pi_v + \xi$ contains an integer point from $\Lambda^1_v$.

Put

$$\tau_v = \frac{2\sigma_v |z_v|}{a_v}.$$  \hfill (13)

Consider the point

$$\xi_v = Z + \tau_v z_v + \sigma_v e_v \in \pi^1_v,$$

and the rectangle

$$\Pi^1_v = \Pi_v + \xi_v \subset \pi^1_v.$$  

It is clear that

$$\Pi^1_v \subset \mathcal{P}\Delta$$

(here $Z$ was defined in (11), $e_v$ was defined in (7), and the parameters $\sigma_v, \tau_v$ come from (5) and (13)).

Now we take the integer point

$$z_{v+1} = (q_{v+1}, z_1 v+1, z_2 v+1) \in \Lambda^1_v \cap \Pi^1_v.$$
From the construction it follows that

$$q_{v+1} \asymp |z_{v+1}| \asymp |z| + \tau_v|z_v| + |z_v| \asymp q_v \left(1 + \frac{1}{d_v b_v} + \frac{\sigma_v}{a_v}\right) \asymp q_v \left(1 + \frac{1}{d_v b_v}\right) + \frac{d_v}{a_v} \asymp \frac{q_v}{d_v b_v}. $$

(Here we use (6), (9), (10), (12), and (13).) From (9) we see that

$$q_{v+1} \gg \left(\frac{q_v}{d_v}\right)^2 \frac{1}{a_v}. \quad (14)$$

Now we are able to define the next two-dimensional lattice

$$\Lambda_{v+1} = \langle z_v, z_{v+1} \rangle \mathbb{Z}. $$

Let $d_{v+1}$ be its fundamental volume. We will estimate the value of $d_{v+1}$ taking into account (9) as

$$d_{v+1} \ll q_v \cdot \text{dist}(z_{v+1}, l(z_v)) \ll \frac{q_v}{d_v} \cdot \frac{a_v}{b_v} \ll \left(\frac{q_v}{d_v}\right)^2 \ll q_v^2. \quad (15)$$

From (14) and (15), we deduce that

$$d_{v+1} \ll a_v d_v^2 q_{v+1}. $$

By the choice of $a_v$ (by formula (8)) we have

$$d_{v+1} \leq \frac{\omega(q_{v+1})}{v}. \quad (16)$$

7. The vector $\theta$

Now we define

$$\theta_v = (\theta_1 v, \theta_2 v), \quad \theta_j v = \frac{q_{j v}}{q_v}. $$

We consider the angles between the successive vectors $n_v$ and $n_{v+1}$:

$$\alpha_v = \angle(n_v, n_{v+1}) \asymp \tan \angle(n_v, n_{v+1}). $$

Since $z_{v+1} \in \mathcal{F} \Delta$ (see Figure 2), we have

$$\tan \angle(n_v, n_{v+1}) \leq \frac{b_v}{a_v}. $$

![Figure 2](attachment:image.png)  

**Figure 2.** The vector $z_{v+1}$ intersects the interior of the triangle $\Delta = z_v z^a_v z^b_v$. 
and so
\[ \alpha_v \ll \frac{b_v}{a_v}. \tag{17} \]

As \( z_{v+1} \in \mathcal{P} \Delta \), we have
\[ |\theta_v - \theta_{v+1}| \ll \sqrt{\frac{a_v^2 + b_v^2}{q_v}} \ll \frac{a_v}{q_v} \tag{18} \]
by the same argument. There exist limits
\[ \lim_{v \to \infty} \theta_v = \theta = (\theta_1, \theta_2) \quad \text{and} \quad \lim_{v \to \infty} n_v = n, \]
and from (17) and (18) we deduce that
\[ 0 < |\theta - \theta_v| \ll \frac{a_v}{q_v} \tag{19} \]
and
\[ \text{angle}(n, n_v) \ll \frac{b_v}{a_v} \tag{20} \]
It is clear that \( \theta \notin \mathbb{Q}^2 \). A slight modification\(^2\) of the procedure of choosing vectors \( z_v \) ensures the condition that \( 1, \theta_1, \theta_2 \) are linearly independent over \( \mathbb{Z} \). Define \( \pi = \{ x \in \mathbb{R}^3 : (x, n) = 0 \} \). Then \( \theta \in \pi \) by continuity and we can assume that \( n \notin \mathbb{Q}^3 \).

8. Winning property
Consider the one-dimensional affine subspaces
\[ \mathcal{P}_v = \{(x_1, x_2) \in \mathbb{R}^2 : (1, x_1, x_2) \in \pi_v \} \subset \mathbb{R}^2 \]
and
\[ \mathcal{P} = \{(x_1, x_2) \in \mathbb{R}^2 : (1, x_1, x_2) \in \pi \} \subset \mathbb{R}^2, \]
where \( \pi \) was defined at the end of the previous section. Let
\[ B_1(\theta) = \{ \xi \in \mathbb{R}^2 : \text{dist}(\xi, \theta) < 1 \}. \]
We will show that for any \( \eta = (\eta_1, \eta_2) \in \mathcal{P} \cap B_1(\theta) \) there exists infinitely many solutions of the inequality
\[ \max_{i=1,2} \| \theta_i x - \eta_i \| < \frac{\omega(x)}{x} \]
in integers \( x \). Denote by \( \eta_v = (\eta_{1v}, \eta_{2v}) \) the orthogonal projection of \( \eta \) onto \( \mathcal{P}_v \). From (20) we see that
\[ |\eta - \eta_v| \ll \frac{b_v}{a_v}. \tag{21} \]
\(^2\)A similar procedure was explained in [Moshchevitin 2012]. There, the author provides the linear independence of coordinates of the limit vector by “going away from all rational subspaces” (the beginning of the proof of Theorem 1 in the case \( k = 1 \), p. 132 and the beginning of §5, p. 146).
For any \( \eta_v = (\eta_{1v}, \eta_{2v}) \in \mathcal{P}_v \) the planar domain \( \bar{\eta}_v + \Pi_v, \bar{\eta}_v = (1, -\eta_{1v}, -\eta_{2v}) \) contains an integer point \( y_v = (x_v, y_{1v}, y_{2v}) \in \Lambda_v \). It is clear that

\[
|x_v| \ll q_v \tag{22}
\]

and

\[
\max_{i=1,2} |\theta_{iv}x_v - \eta_{iv} - y_{iv}| \ll \frac{d_v}{q_v} \tag{23}
\]

By (19), (21), (22), and (23) we have

\[
\max_{i=1,2} \|\theta_{ix_v} - \eta_i\| \leq |x_v| \max_{i=1,2} |\theta_i - \theta_{iv}| + \max_{i=1,2} \|\theta_{iv}x_v - \eta_{iv}\| + \max_{i=1,2} |\eta_i - \eta_{iv}| \ll a_v + \frac{d_v}{q_v} + \frac{b_v}{a_v} \ll \frac{d_v}{q_v}.
\]

In the last inequality we use (9). By (16) we have

\[
\max_{i=1,2} \|\theta_{ix_v} - \eta_i\| \leq \frac{\omega(q_v)}{q_v}
\]

for large \( v \). As \( \bar{\eta} \in \pi \) and \( y_v \in \pi_v \), \( \max_{i=1,2} \|\theta_{ix_v} - \eta_i\| \neq 0 \) infinitely often (in fact for all large \( v \)).

References


SETS OF INHOMOGENEOUS LINEAR FORMS CAN BE NOT ISOTROPICALLY WINNING


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