Sets of inhomogeneous linear forms can be not isotropically winning

Natalia Dyakova
Sets of inhomogeneous linear forms
can be not isotropically winning

Natalia Dyakova

We give an example of irrational vector $\theta \in \mathbb{R}^2$ such that the set
\[
\text{Bad}_\theta := \{(\eta_1, \eta_2) : \inf_{x \in \mathbb{N}} x^{1/2} \max_{i=1,2} \|x\theta_i - \eta_i\| > 0\}
\]
is not absolutely winning with respect to McMullen’s game.

1. Introduction

We consider a problem related to inhomogeneous Diophantine approximation. Given $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ we study the set of pairs $(\eta_1, \eta_2) \in \mathbb{R}^2$ such that the system of two linear forms
\[
\|x\theta_1 - \eta_1\|, \quad \|x\theta_2 - \eta_2\|
\]
where $\|\cdot\|$ stands for the distance to the nearest integer, is badly approximable. We prove a statement complementary to our recent result from [Bengoechea et al. 2017]. We construct $\theta$ such that the set
\[
\text{Bad}_\theta := \{(\eta_1, \eta_2) : \inf_{x \in \mathbb{N}} x^{1/2} \max_{i=1,2} \|x\theta_i - \eta_i\| > 0\}
\]
is not isotropically winning.

Our paper is organized as follows. In Section 2 we discuss different games appearing in Diophantine problems. In Section 3 we give a brief survey on inhomogeneous badly approximable systems of linear forms and formulate our main result, Theorem 3.1. Sections 4 and 5 are devoted to some auxiliary observations. In Sections 6, 7, and 8 we give a proof for Theorem 3.1.

2. Schmidt’s game and its generalizations

The following game was introduced by Schmidt [1966; 1969; 1980]. Let $0 < \alpha, \beta < 1$. Suppose that two players A and B choose in turn a nested sequence of closed balls:
\[
B_1 \supset A_1 \supset B_2 \supset A_2 \supset \cdots
\]
with the property that the diameters $|A_i|, |B_i|$ of the balls $A_i, B_i$ satisfy
\[
|A_i| = \alpha |B_i|, \quad |B_{i+1}| = \beta |A_i| \quad \text{for all } i = 1, 2, 3, \ldots,
\]

The author is supported by RFBR Grant No. 18-01-00886a.


Keywords: inhomogeneous diophantine approximation, winning sets.
for fixed $0 < \alpha, \beta < 1$. A set $E \subset \mathbb{R}^n$ is called $(\alpha, \beta)$-winning if player A has a strategy which guarantees that intersection $\bigcap A_i$ meets $E$ regardless of the way B chooses to play. A set $E \supset \mathbb{R}^n$ is called an $\alpha$-winning set if it is $(\alpha, \beta)$-winning for all $0 < \beta < 1$.

There are different modifications of Schmidt’s game: the strong game and absolute game introduced in [McMullen 2010], the hyperplane absolute game introduced in [Kleinbock and Weiss 2010], the potential game considered in [Fishman et al. 2013], and some others. In [Bengoechea et al. 2017], we introduced isotropically winning sets. Let us describe here some of these generalizations in more detail.

The definition of an absolutely winning set was given in [McMullen 2010]. Consider the following game. Suppose $A$ and $B$ choose in turn a sequence of balls $A_i$ and $B_i$ such that the sets $B_1 \supset (B_1 \setminus A_1) \supset B_2 \supset (B_2 \setminus A_2) \supset B_3 \supset \cdots$ are nested. For fixed $0 < \beta < \frac{1}{3}$ we suppose

$$|B_{i+1}| \geq \beta |B_i|, \quad |A_i| \leq \beta |B_i|.$$  

We say $E$ is an absolute winning set if for all $\beta \in (0, \frac{1}{3})$, player A has a strategy which guarantees that $\cap B_i$ meets $E$ regardless of how B chooses to play. Mcmullen proved that an absolute winning set is $\alpha$-winning for all $\alpha < \frac{1}{2}$. Several examples of absolute winning sets were exhibited by McMullen [2010]. In particular, a set of badly approximable numbers in $\mathbb{R}$ is absolutely winning. However the set of simultaneously badly approximable vectors in $\mathbb{R}^n$ for $n > 1$ is not absolutely winning.

In [Bengoechea et al. 2017] another strong variant of the winning property was given. We say that a set $E \subset \mathbb{R}^n$ is isotropically winning if for each $d \leq n$ and for each $d$-dimensional affine subspace $A \subset \mathbb{R}^n$ the intersection $E \cap A$ is $\frac{1}{2}$-winning for Schmidt’s game considered as a game in $A$. It is clear that an absolute winning set is isotropically winning for each $\alpha \leq \frac{1}{2}$.

3. Inhomogeneous approximations

The first important result on inhomogeneous approximations in the one-dimensional case is due to Khinchine [1926]. He proved that there exists an absolute constant $\gamma$ such that for every $\theta \in \mathbb{R}$ there exists $\eta \in \mathbb{R}$ such that

$$\inf_{q \in \mathbb{Z}} q \|q\theta - \eta\| > \gamma.$$  

Later (see [Khinchin 1937; 1948]) he proved that for given positive numbers $n, m \in \mathbb{Z}$ there exists a positive constant $\gamma_{nm}$ such that for any $m \times n$ real matrix $\theta$ there exists a vector $\eta \in \mathbb{R}^n$ such that

$$\inf_{x \in \mathbb{Z}^n \setminus \{0\}} (\|\theta x - \eta\|_{\mathbb{Z}^n})^n \|x\|^m > \gamma_{nm}$$

(here $\|\cdot\|_{\mathbb{Z}^n}$ stands for the distance to the nearest integral point in sup-norm). These results are presented in a wonderful book by Cassels [1957].

Jarník [1941], proved a generalization of this statement. Suppose $\psi(t)$ is a function decreasing to zero as $t \to +\infty$. Let $\rho(t)$ be the function inverse to the function $t \mapsto 1/\psi(t)$. Suppose that for all $t > 1$ one has $\psi_\theta(t) \leq \psi(t)$. Then there exists a vector $\eta \in \mathbb{R}^n$ such that

$$\inf_{x \in \mathbb{Z}^n \setminus \{0\}} (\|\theta x - \eta\|_{\mathbb{Z}^n}) \cdot \rho(8m \cdot \|x\|) > \gamma$$.
with appropriate \( \gamma = \gamma(n, m) \).

Denote by
\[
\text{Bad}_\theta = \{ \alpha \in [0, 1) : \inf_{q \in \mathbb{N}} q \cdot \|q\theta - \alpha\| > 0 \}.
\]

It happens that the winning property of this inhomogeneous Diophantine set was considered quite recently. Tseng [2009] showed that \( \text{Bad}_\theta \) is winning for all real numbers \( \theta \) in classical Schmidt’s sense. For the corresponding multidimensional sets
\[
\text{Bad}(n, m) = \{ \theta \in \text{Mat}_{n \times m}(\mathbb{R}) : \inf_{q \in \mathbb{Z}^m_{\neq 0}} \max_{1 \leq i \leq n} (|q|^{|m|/n} \|\theta_i(q)\|) > 0 \}.
\]

the winning property is shown, for example, in [Einsiedler and Tseng 2011; Moshchevitin 2011]. In [Broderick et al. 2013] it was shown that the set \( \text{Bad}(n, m) \) is hyperplane absolutely winning. The methods used in [Broderick et al. 2013] come from [Broderick et al. 2011].

Further generalizations deal with the twisted sets
\[
\text{Bad}(i, j) = \{ (\theta_1, \theta_2) \in \mathbb{R}^2 : \inf_{q \in \mathbb{N}} \max(q^i \|q\theta_1\|, q^j \|q\theta_2\|) > 0 \},
\]
where \( i, j \) are real positive numbers satisfying \( i + j = 1 \), introduced by Schmidt. In [An 2016] it was proved that \( \text{Bad}(i, j) \) is winning for the standard Schmidt game. In higher dimension, we fix an \( n \)-tuple \( k = (k_1, \ldots, k_n) \) of real numbers satisfying
\[
k_1, \ldots, k_n > 0 \quad \text{and} \quad \sum_{i=1}^n k_i = 1,
\]
and define
\[
\text{Bad}(k, n, m) = \{ \theta \in \text{Mat}_{n \times m}(\mathbb{R}) : \inf_{q \in \mathbb{Z}^m_{\neq 0}} \max_{1 \leq i \leq n} (|q|^{mk_i} \|\theta_i(q)\|) > 0 \}.
\]
Here, \( |\cdot| \) denotes the supremum norm, \( \theta = (\theta_{ij}) \), and \( \theta_i(q) \) is the product of the \( i \)-th line of \( \theta \) with the vector \( q \), i.e.,
\[
\theta_i(q) = \sum_{j=1}^m q_j \theta_{ij}.
\]

In the twisted setting, much less is known. In particular up to now the winning property of the set \( \text{Bad}(k, n, m) \) in dimension greater that two is not proved.

Given \( \theta \in \text{Mat}_{n \times m}(\mathbb{R}) \), we define
\[
\text{Bad}_\theta(k, n, m) = \{ x \in \mathbb{R}^n : \inf_{q \in \mathbb{Z}^m_{\neq 0}} \max_{1 \leq i \leq n} (|q|^{mk_i} \|\theta_i(q) - x_i\|) > 0 \}.
\]
Harrap and Moshchevitin [2017] showed that this set is winning provided that \( \theta \in \text{Bad}(k, n, m) \). In [Bengoechea et al. 2017] it was proved that if we suppose that \( \theta \in \text{Bad}(k, n, m) \), the set \( \text{Bad}_\theta(k, n, m) \) is isotropically winning.\(^1\)

We should note that even in the case \( n = 2, m = 1 \) it is not known if the set \( \text{Bad}_\theta(k, 2, 1) \) is \( \alpha \)-winning for some positive \( \alpha \) without the condition \( \theta \in \text{Bad}(k, 2, 1) \).

\(^1\)In fact, the approach from [Bengoechea et al. 2017] gives a little bit more. Instead of property that for any subspace \( \mathcal{A} \) the intersection \( E \cap \mathcal{A} \) is \( \frac{1}{2} \)-winning in \( \mathcal{A} \), one can see that it is \( \alpha \)-winning for all \( \alpha \in (0, \frac{1}{2}) \). It is not completely clear for the author if these two properties are equivalent. (For a closely related problem, see [Dremov 2002].)
In this article we show that the condition $\theta$ be from Bad($k, n, m$) is essential for the isotropically winning property, and prove the following theorem.

**Theorem 3.1.** There exists a vector $\theta = (\theta_1, \theta_2)$ such that:

1. $\theta_1, \theta_2$ are linearly independent over $\mathbb{Z}$.
2. $\text{Bad}_\theta := \left\{ (\eta_1, \eta_2) : \inf_{x \in \mathbb{N}} x^{1/2} \max_{i=1,2} \| x\theta_i - \eta_i \| > 0 \right\}$ is not isotropically winning.

### 4. Some more remarks

In the sequel, $x = (x_0, x_1, x_2)$ is a vector in $\mathbb{R}^3$, $|\cdot|$ stands for the Euclidean norm of the vector, and by $(w, t)$ we denote the inner product of vectors $w$ and $t$.

The proof of Theorem 3.1 we will give in Section 6. There we will construct a special $\theta$ and a one-dimensional affine subspace $P$ such that $\theta \in P$ and for the segment $D = P \cap \{|z - \theta| \leq 1\}$ one has $D \cap \text{Bad}_\theta = \emptyset$. Moreover, given an arbitrary positive function $\omega(t)$ monotonically (slowly) increasing to infinity we can ensure that for all $\eta = (\eta_1, \eta_2) \in D$ there exist infinitely many $x \in \mathbb{Z}$ such that

$$\max_{i=1,2} \| x\theta_i - \eta_i \| < \frac{\omega(x)}{x}.$$ 

To explain the construction of the proof it is useful to consider the case when $\theta_1, \theta_2, 1$ are linearly dependent. This case we will discuss in Section 5.

**Remark 4.1.** From the result of the paper [Bengoechea et al. 2017] it follows that the vector $\theta$ constructed in Theorem 3.1 does not belong to the set

$$\text{Bad} = \left\{ (\theta_1, \theta_2) \mid \inf_{x \in \mathbb{N}} x^{1/2} \max(\|\theta_1 x\|, \|\theta_2 x\|) > 0 \right\}.$$ 

**Remark 4.2.** Let $\theta = (a_1/q, a_2/q)$ be rational. Let $\eta = (\eta_1, \eta_2) \not\in \frac{1}{q} \cdot \mathbb{Z}^2$; then for any $x \in \mathbb{Z}$,

$$\max_{i=1,2} \left\| x \frac{a_i}{q} - \eta_i \right\| \geq \text{dist}(\eta, \frac{1}{q} \cdot \mathbb{Z}^2) > 0.$$ 

So the set

$$\mathcal{B} = \left\{ \eta : \inf_{x \in \mathbb{Z}} \max_{i=1,2} \left\| x \frac{a_i}{q} - \eta_i \right\| > 0 \right\}$$

contains $\mathbb{R}^2 \setminus \frac{1}{q} \cdot \mathbb{Z}^2$ and is trivially winning. It is clear that for any one-dimensional affine subspace $\ell$ we have $\mathcal{B} \cap \ell \supset \left((\mathbb{R}^2 \setminus \frac{1}{q} \cdot \mathbb{Z}^2) \right) \cap \ell$. So obviously $\mathcal{B} \cap \ell$ is also winning in $\ell$.

### 5. Linearly dependent case

Let $1, \theta_1, \theta_2$ be linearly dependent and at least one of $\theta_j$ is irrational. This means that there exists $z = (z_0, z_1, z_2) \in \mathbb{Z}^3$ such that $(z, \theta) = 0$. Let us consider the two-dimensional rational subspace

$$\pi = \{ x \in \mathbb{R}^3 : (x, z) = 0 \},$$

so $\theta \in \pi$.

Let us define the one-dimensional subspace $P = \{ (x_1, x_2) : (1, x_1, x_2) \in \pi \} \subset \mathbb{R}^2$. 
We will prove that there exists a constant $\gamma$ such that for any $\eta = (\eta_1, \eta_2) \in \mathcal{P}$ the inequality

$$\max_{i=1,2} \| \theta_i x - \eta_i \| < \frac{\gamma}{x}$$

has infinitely many solutions in $x \in \mathbb{N}$. (This statement is similar to Chebyshev’s theorem [Khinchin 1964, Theorem 24, Chapter 2].)

Denote by $\Lambda = \pi \cap \mathbb{Z}^3$ the integer lattice with the determinant $d := \det \Lambda = |\mathbb{z}|$. Denote by $\{g_v = (q_v, a_{1v}, a_{2v})\}_{v=1,2,3,\ldots} \subset \Lambda$ the sequence of the best approximations of $\theta$ by the lattice $\Lambda$ and the corresponding parallelograms

$$\Pi_v = \{ x = (x_0, x_1, x_2) \in \pi : 0 \leq x_0 \leq q_v, \ \text{dist}(x, l(\theta)) \leq \text{dist}(g_{v-1}, l(\theta)) \},$$

which contains a fundamental domain of the two-dimensional $\Lambda$. Obviously, $\text{vol} \Pi_v \leq 4d$. So,

$$\text{dist}(g_{v-1}, l(\theta)) \ll \frac{d}{q_v}, \quad (2)$$

with an absolute constant in the sign $\ll$. It is clear that for any point $\eta \in \pi$, the shift $\eta + \Pi_v$ contains a point of $\Lambda$.

For any $\eta = (\eta_1, \eta_2) \in \mathcal{P}$ and for any positive integer $v$ the planar domain $\eta + \Pi_v$, $\eta = (1, -\eta_1, -\eta_2)$ contains an integer point $y = (x, y_1, y_2) \in \Lambda$.

It is clear that

$$1 \leq x \leq 1 + q_v \quad (3)$$

and

$$\max_{i=1,2} \| \theta_i x - \eta_i \| \ll \text{dist}(y, l(\theta) + \bar{\eta}) \ll \text{dist}(l(\theta), g_{v-1}),$$

and by (2),

$$\max_{i=1,2} \| \theta_i x - \eta_i \| \ll \frac{d}{q_v}. \quad (4)$$

From (3), (4) it follows that the inequality

$$\max_{i=1,2} \| \theta_i x - \eta_i \| \ll \frac{d}{x}$$

has infinitely many solutions and everything is proved.

### 6. Inductive construction of integer points

Let $\omega(t)$ be arbitrary positive function monotonically (slowly) increasing to infinity. Here we describe the inductive construction of integer points $z_v = (q_v, z_{1v}, z_{2v})$. The base of the induction process is trivial. One can take an arbitrary primitive pair of integer vectors that can be completed to a basis of $\mathbb{Z}^3$.

Suppose that we have two primitive integer vectors

$$z_{v-1} = (q_{v-1}, z_{1v-1}, z_{2v-1}) \in \mathbb{Z}^3, \quad z_v = (q_v, z_{1v}, z_{2v}) \in \mathbb{Z}^3.$$

Now we explain how to construct the next integer vector $z_{v+1}$.

We consider the two-dimensional subspace

$$\pi_v = (z_{v-1}, z_v)_{\mathbb{R}}.$$
The pair of vectors \( z_{v-1} \) and \( z_v \) is primitive, so the lattice spanned by them is
\[
\Lambda_v := \langle z_{v-1}, z_v \rangle_{\mathbb{Z}} = \pi_v \cap \mathbb{Z}^3.
\]
By \( d_v = \det \Lambda_v \) we denote the two-dimensional fundamental volume of the lattice \( \Lambda_v \). Now we define the vector \( n_v = (n_{0v}, n_{1v}, n_{2v}) \in \mathbb{R}^3 \) from the conditions
\[
\pi_v = \{ x \in \mathbb{R}^3 : (x, n_v) = 0 \}, \quad |n_v| = 1.
\]
Put
\[
\sigma_v = \text{dist}(z_{v-1}, l(z_v)).
\]
Obviously, \(|z_v| \asymp q_v\) and
\[
\sigma_v \asymp \frac{d_v}{q_v}.
\]
We define a vector \( e_v \) from the conditions
\[
e_v \in \pi_v, \quad |e_v| = 1, \quad (e_v, z_v) = 0,
\]
so \( e_v \) is parallel to \( \pi_v \) and orthogonal to \( z_v \).

Define the rectangle
\[
\Pi_v = \{ x = (x_0, x_1, x_2) : x = tz_v + re_v, \ 0 \leq t \leq |z_v|, \ |r| \leq \sigma_v \}.
\]
It is clear that rectangle \( \Pi_v \subset \pi_v \) contains a fundamental domain of the lattice \( \Lambda_v \). We need two axillary vectors \( z^a_v \) and \( z^b_v \) defined as
\[
z^a_v = z_v + a_v e_v, \quad z^b_v = z^a_v + b_v n_v,
\]
where positive \( a_v \) is chosen in such a way that
\[
a_v d_v^2 \leq v^{-1} \omega \left( \frac{q_v^2}{d_v^2} \cdot \frac{1}{a_v} \right)
\]
and
\[
b_v = a_v \min \left( 1, \frac{d_v}{q_v} \right).
\]

From the construction, it follows that
\[
|z^a_v| \asymp |z^b_v| \asymp |z_v| \asymp q_v.
\]
The integer lattice \( \mathbb{Z}^3 \) splits into levels with respect to the two-dimensional sublattice \( \Lambda_v \) in such a way that
\[
\mathbb{Z}^3 = \bigcup_{i \in \mathbb{Z}} \Lambda_{v,i},
\]
where \( \Lambda_{v,j} = \Lambda_v + j z', \ j \in \mathbb{Z} \) and integer vector \( z' \) completes the couple \( z_{v-1}, z_v \) to the basis in \( \mathbb{Z}^3 \). We consider the affine subspace \( \pi^1_v = \pi_v + z' \supset \Lambda_{v,1} \), which is parallel to \( \pi_v \). It is clear that \( \text{dist}(\pi_v, \pi^1_v) = 1/d_v \).

We need to determine the next integer point \( z_{v+1} \). Denote by \( \mathcal{P} \) the central projection with center 0 onto the affine subspace \( \pi^1_v \). We consider the triangle \( \Delta \) with vertices \( z_v, z^a_v, z^b_v \) and its image \( \mathcal{P}\Delta \) under
SETS OF INHOMOGENEOUS LINEAR FORMS CAN BE NOT ISOTROPICALLY WINNING

Figure 1. The central projection $\mathcal{P}$.

the projection $\mathcal{P}$ (Figure 1).

Define

$$ Z = \mathcal{P}z_v^b. \quad (11) $$

One can see that

$$ |Z| \asymp \frac{q_v}{d_v b_v}. \quad (12) $$

Define rays

$$ \mathcal{R}_1 = \{z = Z + t z_v : t \geq 0\} \quad \text{and} \quad \mathcal{R}_2 = \{z = Z + t z_v^a : t \geq 0\}. $$

It is clear that $\mathcal{R}_1 \cap \mathcal{R}_2 = \{Z\}$ and $\mathcal{R}_1, \mathcal{R}_2 \subset \pi_1^v$. Moreover, the whole convex angle bounded by rays $\mathcal{R}_1, \mathcal{R}_2$ form the image of the triangle $\Delta$ under the projection $\mathcal{P}$:

$$ \mathcal{P}\Delta = \text{conv}(\mathcal{R}_1 \cup \mathcal{R}_2). $$

The affine subspace $\pi_1^v$ contains the affine lattice $\Lambda_1^v = \Lambda_v + z'$ which is congruent to the lattice $\Lambda_v$. Thus, for any $\xi \in \pi_1^v$, the shift $\Pi_v + \xi$ contains an integer point from $\Lambda_1^v$.

Put

$$ \tau_v = \frac{2\sigma_v |z_v|}{a_v}. \quad (13) $$

Consider the point

$$ \xi_v = Z + \tau_v z_v + \sigma_v e_v \in \pi_1^v, $$

and the rectangle

$$ \Pi_1^v = \Pi_v + \xi_v \subset \pi_1^v. $$

It is clear that

$$ \Pi_1^v \subset \mathcal{P}\Delta $$

(here $Z$ was defined in (11), $e_v$ was defined in (7), and the parameters $\sigma_v, \tau_v$ come from (5) and (13)).

Now we take the integer point

$$ z_{v+1} = (q_{v+1}, z_{1 v+1}, z_{2 v+1}) \in \Lambda_1^v \cap \Pi_1^v. $$
From the construction it follows that
\[
q_{\nu+1} \asymp |z_{\nu+1}| \asymp |z| + \tau_{\nu} |z_{\nu}| + |z_{\nu}| \asymp q_{\nu} \left( 1 + \frac{1}{d_{\nu} b_{\nu}} + \frac{\sigma_{\nu}}{a_{\nu}} \right) \asymp q_{\nu} \left( 1 + \frac{1}{d_{\nu} b_{\nu}} \right) + \frac{d_{\nu}}{a_{\nu}} \asymp \frac{q_{\nu}}{d_{\nu} b_{\nu}}.
\]
(Here we use (6), (9), (10), (12), and (13).) From (9) we see that
\[
q_{\nu+1} \gg \left( \frac{q_{\nu}}{d_{\nu}} \right)^2 \frac{1}{a_{\nu}}. \tag{14}
\]
Now we are able to define the next two-dimensional lattice
\[
\Lambda_{\nu+1} = \langle z_{\nu}, z_{\nu+1} \rangle \mathbb{Z}.
\]
Let \( d_{\nu+1} \) be its fundamental volume. We will estimate the value of \( d_{\nu+1} \) taking into account (9) as
\[
d_{\nu+1} \ll q_{\nu} \cdot \text{dist}(z_{\nu+1}, l(z_{\nu})) \ll \frac{q_{\nu}}{d_{\nu}} \cdot \frac{a_{\nu}}{b_{\nu}} \ll \left( \frac{q_{\nu}}{d_{\nu}} \right)^2 \ll q_{\nu}^2. \tag{15}
\]
From (14) and (15), we deduce that
\[
d_{\nu+1} \ll a_{\nu} d_{\nu}^2 q_{\nu+1}.
\]
By the choice of \( a_{\nu} \) (by formula (8)) we have
\[
d_{\nu+1} \leq \frac{\omega(q_{\nu+1})}{v}. \tag{16}
\]

7. The vector \( \theta \)

Now we define
\[
\theta_{\nu} = (\theta_{1\nu}, \theta_{2\nu}), \quad \theta_{j\nu} = \frac{q_{j\nu}}{q_{\nu}}.
\]
We consider the angles between the successive vectors \( n_{\nu} \) and \( n_{\nu+1} \):
\[
\alpha_{\nu} = \text{angle}(n_{\nu}, n_{\nu+1}) \asymp \text{tan angle}(n_{\nu}, n_{\nu+1}).
\]
Since \( z_{\nu+1} \in \mathcal{P} \Delta \) (see Figure 2), we have
\[
\text{tan angle}(n_{\nu}, n_{\nu+1}) \leq \frac{b_{\nu}}{a_{\nu}},
\]

**Figure 2.** The vector \( z_{\nu+1} \) intersects the interior of the triangle \( \Delta = z_{\nu} z_{\nu}^a z_{\nu}^b \).
and so
\[ \alpha_v \ll \frac{b_v}{a_v}. \]  \hspace{1cm} (17)

As \( z_{v+1} \in \mathcal{P} \Delta \), we have
\[ |\theta_v - \theta_{v+1}| \ll \frac{\sqrt{a_v^2 + b_v^2}}{q_v} \ll \frac{a_v}{q_v} \]  \hspace{1cm} (18)
by the same argument. There exist limits
\[ \lim_{v \to \infty} \theta_v = \theta = (\theta_1, \theta_2) \quad \text{and} \quad \lim_{v \to \infty} n_v = n, \]
and from (17) and (18) we deduce that
\[ 0 < |\theta - \theta_v| \ll \frac{a_v}{q_v} \]  \hspace{1cm} (19)
and
\[ \angle(n, n_v) \ll \frac{b_v}{a_v}. \]  \hspace{1cm} (20)

It is clear that \( \theta \not\in \mathbb{Q}^2 \). A slight modification\(^2\) of the procedure of choosing vectors \( z_v \) ensures the condition that \( 1, \theta_1, \theta_2 \) are linearly independent over \( \mathbb{Z} \). Define \( \pi = \{ x \in \mathbb{R}^3 : (x, n) = 0 \} \). Then \( \theta \in \pi \) by continuity and we can assume that \( n \not\in \mathbb{Q}^3 \).

8. Winning property

Consider the one-dimensional affine subspaces
\[ \mathcal{P}_v = \{(x_1, x_2) \in \mathbb{R}^2 : (1, x_1, x_2) \in \pi_v \} \subset \mathbb{R}^2 \]
and
\[ \mathcal{P} = \{(x_1, x_2) \in \mathbb{R}^2 : (1, x_1, x_2) \in \pi \} \subset \mathbb{R}^2, \]
where \( \pi \) was defined at the end of the previous section. Let
\[ B_1(\theta) = \{ \xi \in \mathbb{R}^2 : \text{dist}(\xi, \theta) < 1 \}. \]
We will show that for any \( \eta = (\eta_1, \eta_2) \in \mathcal{P} \cap B_1(\theta) \) there exists infinitely many solutions of the inequality
\[ \max_{i=1,2} \| \theta_i x - \eta_i \| < \frac{\omega(x)}{x} \]
in integers \( x \). Denote by \( \eta_v = (\eta_{1v}, \eta_{2v}) \) the orthogonal projection of \( \eta \) onto \( \mathcal{P}_v \). From (20) we see that
\[ |\eta - \eta_v| \ll \frac{b_v}{a_v}. \]  \hspace{1cm} (21)
\(^2\)A similar procedure was explained in [Moshchevitin 2012]. There, the author provides the linear independence of coordinates of the limit vector by “going away from all rational subspaces” (the beginning of the proof of Theorem 1 in the case \( k = 1 \), p. 132 and the beginning of §5, p. 146).
For any \( \eta_v = (\eta_{1v}, \eta_{2v}) \in P_v \) the planar domain \( \bar{\eta}_v + \Pi_v, \bar{\eta}_v = (1, -\eta_{1v}, -\eta_{2v}) \) contains an integer point \( y_v = (x_v, y_{1v}, y_{2v}) \in \Lambda_v \). It is clear that

\[
|x_v| \ll q_v \tag{22}
\]

and

\[
\max_{i=1,2} |\theta_{iv} x_v - \eta_{iv} - y_{iv}| \ll \frac{d_v}{q_v}. \tag{23}
\]

By (19), (21), (22), and (23) we have

\[
\max_{i=1,2} \|\theta_i x_v - \eta_i\| \leq |x_v| \max_{i=1,2} |\theta_i - \theta_{iv}| + \max_{i=1,2} \|\theta_{iv} x_v - \eta_{iv}\| + \max_{i=1,2} |\eta_i - \eta_{iv}| \ll a_v + \frac{d_v}{q_v} + \frac{b_v}{a_v} \ll \frac{d_v}{q_v}.
\]

In the last inequality we use (9). By (16) we have

\[
\max_{i=1,2} \|\theta_i x_v - \eta_i\| \leq \frac{\omega(q_v)}{q_v}
\]

for large \( v \). As \( \bar{\eta} \in \pi \) and \( y_v \in \pi_v \), \( \max_{i=1,2} \|\theta_i x_v - \eta_i\| \neq 0 \) infinitely often (in fact for all large \( v \)).

References


SETS OF INHOMOGENEOUS LINEAR FORMS CAN BE NOT ISOTROPICALLY WINNING


Received 1 Dec 2017.

NATALIA DYAKOVA:
natalia.stepanova.msu@gmail.com

Department of Mathematics and Mechanics, Moscow State University, Moscow, Russia
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>To the reader</td>
<td>1</td>
</tr>
<tr>
<td>Nikolay Moshchevitin and Andrei Raigorodskii</td>
<td></td>
</tr>
<tr>
<td>Sets of inhomogeneous linear forms can be not isotropically winning</td>
<td>3</td>
</tr>
<tr>
<td>Natalia Dyakova</td>
<td></td>
</tr>
<tr>
<td>Some remarks on the asymmetric sum-product phenomenon</td>
<td>15</td>
</tr>
<tr>
<td>Ilya D. Shkredov</td>
<td></td>
</tr>
<tr>
<td>Convex sequences may have thin additive bases</td>
<td>43</td>
</tr>
<tr>
<td>Imre Z. Ruzsa and Dmitrii Zhelezov</td>
<td></td>
</tr>
<tr>
<td>Admissible endpoints of gaps in the Lagrange spectrum</td>
<td>47</td>
</tr>
<tr>
<td>Dmitry Gayfulin</td>
<td></td>
</tr>
<tr>
<td>Transcendence of numbers related with Cahen’s constant</td>
<td>57</td>
</tr>
<tr>
<td>Daniel Duverney, Takeshi Kurosawa and Iekata Shiokawa</td>
<td></td>
</tr>
<tr>
<td>Algebraic results for the values $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ of the Jacobi theta-constant</td>
<td>71</td>
</tr>
<tr>
<td>Carsten Elsner, Florian Luca and Yohei Tachiya</td>
<td></td>
</tr>
<tr>
<td>Linear independence of $1, \text{Li}_1$ and $\text{Li}_2$</td>
<td>81</td>
</tr>
<tr>
<td>Georges Rhin and Carlo Viola</td>
<td></td>
</tr>
</tbody>
</table>