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Convex sequences may have thin additive bases

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For a fixed $c > 0$ we construct an arbitrarily large set B of size n such that its sum set $B + B$ contains a convex sequence of size cn^2, answering a question of Hegarty.

Notation

The following notation is used throughout the paper. The expressions $X \gg Y$, $Y \ll X$, $Y = O(X)$, $X = \Omega(Y)$ all have the same meaning in that there is an absolute constant c such that $|Y| \leq c|X|$.

If X is a set then $|X|$ denotes its cardinality.

For sets of numbers A and B the *sumset* $A + B$ is the set of all pairwise sums

$$\{a + b : a \in A, b \in B\}.$$

1. Introduction

Let $A = \{a_i\}$, $i = 1, \dots, n$, be a set of real numbers ordered in a way that $a_1 \leq a_2 \leq \dots \leq a_n$. (We also refer to A a sequence, if we wish to emphasize the ordering.) Recall that A is called *convex* if the gaps between consecutive elements of A are strictly increasing, that is

$$a_2 - a_1 < a_3 - a_2 < \dots < a_n - a_{n-1}.$$

Studies of convex sets were initiated by Erdős, who conjectured that any convex set must grow with respect to addition, so that the size of the set of sums $A + A := \{a_1 + a_2 : a_1, a_2 \in A\}$ is significantly larger than the size of A.

The first nontrivial bound confirming the conjecture of Erdős was obtained by Hegyvári [1986]. The state of the art bound for the size of $A + A$ of a convex sequence A is due to Shkredov [2015]:

$$|A + A| \gg |A|^{58/37} \log^{-20/37} |A|.$$

The best bound for the size of the difference set $A - A$ is due to Schoen and Shkredov [2011], who proved that

$$|A - A| \gg |A|^{8/5} \log^{-2/5} |A|$$

if A is arbitrary convex sequence. It is conjectured that in fact

$$|A + A| \geq C(\epsilon)|A|^{2-\epsilon}$$

holds for any $\epsilon > 0$ and some $C > 0$ which depends only on ϵ.

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In general, it is believed that convex sets cannot be additively structured. In particular, Hegarty [2012] asked whether there is a constant $c > 0$ with the property that there is a set B of arbitrarily large size n such that $B + B$ contains a convex set of size cn^2.

Recall that B is a *basis* (of order two) for a set A if $A \subset B + B$. In other words, Hegarty asked if a convex set of size n can have a thin additive basis (of order two) of size as small as $O(n^{1/2})$, which is clearly the smallest possible size up to a constant.

Perhaps contrary to the intuition that convex sets lack additive structure, we present a construction which answers Hegarty's question in the affirmative. Our main result is as follows.

Theorem 1. *There is $c > 0$ such that for any m there is a set B of size $n > m$ such that $B + B$ contains a convex set of size cn^2.*

2. Construction

Assume n is fixed and large. We will construct a set B of size $O(n)$ such that $B + B$ contains a convex set of size $\Omega(n^2)$. Theorem 1 will clearly follow.

The following constants (we assume n is fixed) will be used throughout the proof:

$$\alpha := \frac{1}{n^2}, \quad \gamma := \frac{1}{1000n^3}, \quad \epsilon := 0.1.$$

Define

$$x_i = i + (\alpha + \gamma)i^2, \quad y_j = j - \alpha j^2.$$

Next, we define

$$B_k = \{x_i + y_j : i + j = k\},$$

where i and j are allowed to be negative.

Let $k \in [0.999n, n]$ so that $\alpha k^2 \in [0.99, 1]$. For such an integer k writing $j = k - i$ we have that the i-th element of B_k is given by

$$b_i^{(k)} = k + (\alpha + \gamma)i^2 - \alpha(k - i)^2 = (k - \alpha k^2) + \gamma i^2 + 2ik\alpha. \tag{1}$$

Now assume that i ranges in $[-n, 2n]$. The consecutive differences $b_{i+1}^{(k)} - b_i^{(k)}$ are then given by

$$\Delta_i^{(k)} := \gamma(2i + 1) + 2k\alpha.$$

Observe that $\Delta_i^{(k)}$ are positive and increasing, thus the block $B_k := \{b_i^{(k)}\}_{-n}^{2n}$ is convex. Further, by (1) for sufficiently large n we have

$$b_{-n}^{(k)} = k - \alpha k^2 + \gamma n^2 - 2nk\alpha \in [k - 2.9, k - 3], \tag{2}$$

$$b_{2n}^{(k)} = k - \alpha k^2 + \gamma(2n)^2 + 4nk\alpha \in [k + 2.9, k + 3.1], \tag{3}$$

so $B_k \subset [k - 3, k + 3] + [-\epsilon, \epsilon]$.

Now we are going to build a large convex sequence out of blocks B_k with $4 \mid k$. Since each B_k is already convex, it remains to show how to glue together B_k and B_{k+4} so that the resulting set is again convex. We proceed with the following simple lemma.

Lemma 2. *Let* $X = \{x_i\}_{i=0}^{N}$ *and* $Y = \{y_j\}_{j=0}^{M}$ *be two convex sequences and there are indices* u *and* v *such that*

$$[x_u, x_{u+1}] \subset [y_v, y_{v+1}].$$

Then

$$Z := \{x_i\}_{i=0}^{u} \cup \{y_j\}_{j=v+1}^{M}$$

is a convex sequence.

Proof. Since $[x_u, x_{u+1}] \subset [y_v, y_{v+1}]$ we have that

$$x_u - x_{u-1} < x_{u+1} - x_u < y_{v+1} - x_u.$$

On the other hand,

$$y_{v+1} - x_u < y_{v+1} - y_v < y_{v+2} - y_{v+1}. \qquad \square$$

By Lemma 2, in order to merge B_k and B_{k+4} it suffices to find two consecutive elements $b_i^{(k)}, b_{i+1}^{(k)} \in B_k$ in between two consecutive elements $b_j^{(k+4)}, b_{j+1}^{(k+4)} \in B_{k+4}$. Define

$$\delta := \max_{i\in[-n,2n]} \Delta_i^{(k)}, \qquad \Delta := \min_{i\in[-n,2n]} \Delta_i^{(k+4)}.$$

We have

$$\delta < 4n\gamma + 2k\alpha < \frac{2.1}{n}, \tag{4}$$

$$\Delta - \delta > 8\alpha - 10n\gamma > \frac{6}{n^2}. \tag{5}$$

Let $b_v^{(k)}$ be the least element in B_k greater than $b_{-n}^{(k+4)}$ (such an element exists by (2) and (3)). We claim that with $m := \lceil n/2\rceil + 1$ holds $b_{-n+m}^{(k+4)} > b_{v+m}^{(k)}$, which in turn by the pigeonhole principle guarantees the arrangement of elements required by Lemma 2.

Indeed, by our choice of v,

$$0 \le d := b_v^{(k)} - b_{-n}^{(k+4)} \le \delta. \tag{6}$$

But by (4) and (5),

$$b_{-n+m}^{(k+4)} - b_{v+m}^{(k)} > -d + m(\Delta - \delta) > \frac{3}{n} - \delta > 0, \tag{7}$$

so the claim follows.

It remains to note that by (3),

$$b_{v+m}^{(k)} < b_{-n}^{(k+4)} + m\Delta < (k+1+\epsilon) + \frac{2n^2\alpha}{2} + 4\gamma nm < k + 2.2,$$

and thus $v + m < 2n$ again by (3). This verifies that $b_v^{(k)}, b_{v+m}^{(k)} \in B_k$.

3. Putting everything together

Applying the procedure described in the previous section, we can glue together consecutive blocks B_{4l} with $4l := k \in [0.999n, n]$. Let A be the resulting convex sequence. First, observe there are $\Omega(n)$ blocks being merged. Moreover, each interval $[4l - 1 + \epsilon, 4l + 1 - \epsilon]$ is covered only by the block B_{4l} and by (2), (3), and (4) it contains $\Omega(n)$ elements from B_{4l}, so $|A| = \Omega(n^2)$. On the other hand, by our construction, A is contained in the sumset $B + B$ of $B := \{x_i\}_{-2n}^{2n} \cup \{y_j\}_{-2n}^{2n}$ of size $O(n)$.

Remark 3. It follows from our construction that there are arbitrarily large convex sets A such that the equation

$$a_1 - a_2 = x : a_1, a_2 \in A$$

has $\Omega(|A|^{1/2})$ solutions (a_1, a_2) for at least $\Omega(|A|^{1/2})$ values of x.

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