Algebraic results for the values \( \vartheta_3(m\tau) \) and \( \vartheta_3(n\tau) \) of the Jacobi theta-constant

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Let $\vartheta_3(\tau) = 1 + 2 \sum_{\nu=1}^{\infty} e^{\pi i \nu^2 \tau}$ denote the classical Jacobi theta-constant. We prove that the two values $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$ for any $\tau$ in the upper half-plane such that $q = e^{\pi i \tau}$ is an algebraic number, where $m, n \geq 2$ are distinct integers.

1. Introduction and statement of the results

Throughout this paper, let $\tau$ be a complex variable in the upper half-plane $\mathbb{H} := \{ \tau \in \mathbb{C} | \Im(\tau) > 0 \}$. The three classical theta functions

$$
\vartheta_2(\tau) = 2 \sum_{\nu=0}^{\infty} q^{(\nu+1/2)^2}, \quad \vartheta_3(\tau) = 1 + 2 \sum_{\nu=1}^{\infty} q^{\nu^2}, \quad \vartheta_4(\tau) = 1 + 2 \sum_{\nu=1}^{\infty} (-1)^\nu q^{\nu^2}
$$

are known as theta-constants or Thetanullwerte, where $q := e^{\pi i \tau}$. These theta-constants are holomorphic in $\mathbb{H}$ and never vanish for any $\tau \in \mathbb{H}$. In particular, the function $\vartheta_3(\tau)$ is called a Jacobi theta-constant or Thetanullwert of the Jacobi theta function $\vartheta(z | \tau) = \sum_{\nu=\infty}^{\infty} e^{\pi i \nu^2 \tau + 2\pi i \nu z}$. For an extensive discussion of the Jacobi theta function and theta-constants we refer the reader to [Stein and Shakarchi 2003, Chapter 10]. Y. V. Nesterenko [2006] has improved upon a result from [Grinspan 2001] and obtained some identities for the theta-constants.

**Theorem A** [Nesterenko 2006, Theorem 1]. *For any odd integer $n \geq 3$ there exists an integer polynomial $P_n(X, Y)$ with $\deg_X P_n(X, Y) = \psi(n)$ such that*

$$
P_n(n^2 \frac{\vartheta_4^2(n\tau)}{\vartheta_3^2(\tau)}, 16 \frac{\vartheta_4^2(\tau)}{\vartheta_3^2(\tau)}) = 0
$$

*holds for any $\tau \in \mathbb{H}$, where*

$$
\psi(n) := n \prod_{p | n} \left( 1 + \frac{1}{p} \right).
$$

For example, the first polynomials $P_3$ and $P_5$ are given in [Nesterenko 2006] by

\begin{align*}
P_3 &= 9 - (28 - 16Y + Y^2)X + 30X^2 - 12X^3 + X^4, \\
P_5 &= 25 - (126 - 832Y + 308Y^2 - 32Y^3 + Y^4)X + (255 + 1920Y - 120Y^2)X^2 \\
&\quad + (-260 + 320Y - 20Y^2)X^3 + 135X^4 - 30X^5 + X^6
\end{align*}

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and the polynomials $P_7$, $P_9$, and $P_{11}$ are listed in the appendix of [Elsner 2015]. Recently one of us (Elsner) constructed similar integer polynomials in two variables $X$ and $Y$, which vanish identically at certain rational functions of theta-constants including the function $\vartheta_3(n\tau)$ for $n = 2^m$. He applied this result and Theorem A to settle the algebraic independence problem of the two values $\vartheta_3(\tau)$ and $\vartheta_3(n\tau)$ for integers $n \geq 2$, and obtained the following Theorem B.

**Theorem B** [Elsner 2015, Theorem 1.1]. Let $\tau \in \mathbb{H}$ such that $e^{\pi i \tau}$ is an algebraic number. Then the two values $\vartheta_3(\tau)$ and $\vartheta_3(2^n \tau)$ are algebraically independent over $\mathbb{Q}$ for each integer $m \geq 1$. Furthermore, the same holds for the two values $\vartheta_3(\tau)$ and $\vartheta_3(n\tau)$ if $n = 3, 5, 6, 7, 9, 10, 11, 12$.

The proof of Theorem B is based on an algebraic independence criterion, see [Elsner et al. 2011, Lemma 3.1], which requires a nonvanishing of a Jacobian determinant. In particular, to prove the latter assertion in Theorem B, he needed the explicit forms of the polynomials $P_3$, $P_5$, $P_7$, $P_9$ and $P_{11}$ stated above. In [Elsner and Tachiya 2017], two of us obtained the following Theorem C by studying the specific properties of the polynomials $P_n$.

**Theorem C** [Elsner and Tachiya 2017, Theorem 1.2]. Let $n \geq 2$ be an integer and $j \in \{2, 3, 4\}$. Then for any $\tau \in \mathbb{H}$ at least three of the numbers $e^{\pi i \tau}$, $\vartheta_3(\tau)$, $\vartheta_3(n\tau)$, and $D\vartheta_j(\tau)$ are algebraically independent over $\mathbb{Q}$, where $D := (\pi i)^{-1}d/d\tau$ denotes a differential operator.

An application of Theorem C gives an improvement of Theorem B as follows:

**Theorem D.** Let $\tau \in \mathbb{H}$ be such that $e^{\pi i \tau}$ is an algebraic number. Then the two numbers $\vartheta_3(\tau)$ and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$ for each integer $n \geq 2$.

On the other hand, the algebraic dependence result is also obtained in [Elsner and Tachiya 2017] through the properties of the polynomials $P_n$.

**Theorem E** [Elsner and Tachiya 2017, Theorem 1.4]. Let $\ell, m, n \geq 1$ be integers and $\tau \in \mathbb{H}$ be any complex number. Then the three values $\vartheta_3(\ell \tau)$, $\vartheta_3(m\tau)$, and $\vartheta_3(n\tau)$ are algebraically dependent over $\mathbb{Q}$.

In this paper, we fill the gap between Theorems D and E. Our main result is the following.

**Theorem 1.** Let $m, n \geq 1$ be distinct integers and $\tau \in \mathbb{H}$. Then at least two of the numbers $e^{\pi i \tau}$, $\vartheta_3(m\tau)$, and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$. In particular, the two numbers $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$ for any $\tau \in \mathbb{H}$ such that $e^{\pi i \tau}$ is an algebraic number.

Of course the two numbers $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ can be algebraically dependent over $\mathbb{Q}$ without an algebraic condition on $e^{\pi i \tau}$. Indeed, for $\tau = i \in \mathbb{H}$ the two numbers $\vartheta_3(i)$ and $\vartheta_3(2i)$ are algebraically dependent over $\mathbb{Q}$, since the nontrivial relation

$$4\vartheta_3^2(2i) - (\sqrt{2} + 2)\vartheta_3^2(i) = 0$$  \hspace{1cm} (1)

exists; see [Berndt 1998, p. 325]. Note that the number $e^{\pi} = i^{-2i}$ was shown to be transcendental for the first time by A. O. Gelfond (1929) and, a few years later, this property of $e^{\pi}$ was corroborated by the Gelfond–Schneider theorem (1934). Conversely, the above identity (1) and Theorem 1 imply the transcendence of $e^{\pi}$ as well.
2. Some properties of $P_n(X, Y)$

We now discuss some properties of $P_n(X, Y)$ given in Theorem A. We start with a short description of the construction of $P_n(X, Y)$; for details, see [Nesterenko 2006]. Let $\Gamma(2)$ be the principal congruence subgroup of level 2 in $\text{SL}(2, \mathbb{Z})$; that is,

$$\Gamma(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2} \right\}.$$ 

Then for each odd integer $n \geq 3$ the set of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2}, \quad (a, b, c, d) = 1, \quad ad - bc = n,$$

is a union of $\psi(n)$ equivalence classes with respect to the left–multiplication on the elements of $\Gamma(2)$, and the class representatives are given by

$$\alpha_v := \begin{pmatrix} u & 2v \\ 0 & w \end{pmatrix}, \quad (u, v, w) = 1, \quad uw = n, \quad 0 \leq v < w. \quad (2)$$

For these $\psi(n)$ matrices $\alpha_1, \ldots, \alpha_{\psi(n)}$ in (2), we define the polynomial

$$\prod_{v=1}^{\psi(n)} (X - x_v(\tau)) =: X^{\psi(n)} + a_1(\tau)X^{\psi(n)-1} + \cdots + a_{\psi(n)-1}(\tau)X + a_{\psi(n)}(\tau),$$

where

$$x_v(\tau) := u^2 \frac{\vartheta_4^4((u\tau + 2v)/w)}{\vartheta_3^4(\tau)}, \quad \text{with} \quad \begin{pmatrix} u & 2v \\ 0 & w \end{pmatrix} = \alpha_v, \quad v = 1, \ldots, \psi(n). \quad (3)$$

Then, using the modular method as well as Galois considerations, one finds that there exist polynomials $R_j(Y) \in \mathbb{Z}[Y], \ j = 1, \ldots, \psi(n), \text{ such that}$

$$a_j(\tau) = R_j(16\lambda(\tau)), \quad \lambda(\tau) := \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}. \quad (4)$$

Thus, the integer polynomial

$$P_n(X, Y) := X^{\psi(n)} + R_1(Y)X^{\psi(n)-1} + \cdots + R_{\psi(n)-1}(Y)X + R_{\psi(n)}(Y) \quad (5)$$

vanishes identically at $X = n^2 \vartheta_3^4(n\tau)/\vartheta_3^4(\tau)$ and $Y = 16\lambda(\tau)$.

**Lemma 2.** For each odd integer $n \geq 3$, the polynomial $P_n(X, 16\lambda(\tau))$ is irreducible over the field $\mathbb{C}(\lambda(\tau))$.

**Proof.** The group $\Gamma(2)$ fixes the function $\lambda(\tau) = \vartheta_3^4(\tau)/\vartheta_3^4(\tau)$, since the functions $\vartheta_3^4(\tau)$ and $\vartheta_3^4(\tau)$ are modular forms of weight 2 with respect of the subgroup $\Gamma(2)$. Moreover, we have the transformation formula

$$x_v \left( \frac{a\tau + b}{c\tau + d} \right) = x_\mu(\tau) \quad (6)$$

for a proper matrix $\beta := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$ and subscripts $v, \mu$ such that a proper matrix $\gamma \in \Gamma(2)$ satisfies $\alpha_v\beta = \gamma\alpha_\mu$; see formulae (6) and (7) in [Nesterenko 2006]. This may be regarded as an equivalence relation over
the matrices $\alpha_1, \alpha_2, \ldots, \alpha_{\psi(n)}$ from (2). One can show that any two matrices $\alpha_v$ and $\alpha_{\mu}$, $1 \leq v, \mu \leq \psi(n)$, are equivalent. Together with (6) it turns out that the group $\Gamma(2)$ permutes the $\psi(n)$ distinct functions $x_1(\tau), \ldots, x_{\psi(n)}(\tau)$ transitively. This implies that $P_n(X, 16\lambda(\tau))$ is a minimal polynomial of $x_1(\tau)$ over the field $\mathbb{C}(\lambda(\tau))$.

\[\square\]

**Remark 3.** There is no complex number $\alpha$ such that $P_n(\alpha, Y)$ is identically zero. If such an $\alpha$ existed, the polynomial $P_n(X, Y)$ would be divisible by $(X - \alpha)$, which is impossible by Lemma 2. This fact can also be checked directly from the definition of $x_v(\tau)$; see [Elsner and Tachiya 2017, Lemma 2.1]. In particular, $P_n(X, Y)$ has positive degree in $Y$.

**Lemma 4.** We have

$$P_n(X, 0) = \prod_{u \mid n, u \geq 1} (X - u^2)^{w(u, n/u)},$$

where

$$w(a, b) := \sum_{(a, b, k) = 1 \atop 0 \leq k < b} 1.$$  

**Proof.** This follows immediately from the relation

$$P_n(X, 16\lambda(\tau)) = \prod_{v=1}^{\psi(n)} (X - x_v(\tau))$$

as $\tau \to i\infty$, since we have $\lambda(\tau) \to 0$ and $x_v(\tau) \to u^2$ for each $v = 1, \ldots, \psi(n)$ in (3), respectively.  

\[\square\]

**Example 5.** For the polynomial $P_3$ given in Section 1, we have

$$P_3(X, 0) = 9 - 28X + 30X^2 - 12X^3 + X^4 = (X - 1)^3(X - 3^2).$$

Here, $\psi(3) = 4$ and the four triples $(u, v, w)$ in (2) are given by

$$(3, 0, 1), \quad (1, 0, 3), \quad (1, 1, 3), \quad (1, 2, 3).$$

More generally, $P_p(X, 0) = (X - 1)^p(X - p^2)$ for any odd prime $p \geq 3$.

### 3. Lemmas

Let $\tau \in \mathbb{H}$. We prove in Lemmas 7 and 8 below that the number $\vartheta_3(\tau)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(\alpha u \tau), \vartheta_3(\alpha v \tau))$ for certain positive integers $u$ and $v$. To see this, we need the following Lemma 6. Note that $P_n(0, Y)$ is a **nonzero integer** for the polynomial $P_n(X, Y)$ in Theorem A; see [Elsner and Tachiya 2017, Lemma 2.3].

**Lemma 6 [Elsner and Tachiya 2017, Lemma 2.5].** Let $n = 2^m m$ be an integer with $\alpha \geq 1$ and odd integer $m \geq 3$. Then there exists a polynomial $Q_n(X, Y) \in \mathbb{Z}[X, Y]$ such that

$$Q_n \left( \frac{\vartheta_3^4(n \tau)}{\vartheta_3^4(\tau)}, \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)} \right) = 0.$$
for any \( \tau \in \mathbb{H} \). Furthermore, the polynomial \( Q_n(X, Y) \) is of the form

\[
Q_n(X, Y) = c^{2w} Y^{2^{w}(m-1)} + \sum_{j=0}^{2^w(m-1)} R_{n,j}(X)Y^j,
\]

(7)

with

\[
Q_n(0, Y) = c^{2w} Y^{2^{w}(m)}.
\]

where \( c \) is equal to the nonzero integer \( P_m(0, Y) \).

First we consider the case where \( u \) and \( v \) have different parity.

**Lemma 7.** Let \( u \geq 1 \) be an odd integer and \( v \geq 2 \) be an even integer which is not a power of 2. Then for any \( \tau \in \mathbb{H} \) the number \( \vartheta_3(\tau) \) is algebraic over the field \( \mathbb{Q}(\vartheta_3(u\tau), \vartheta_3(v\tau)) \).

**Proof.** The assertion is clear if \( u = 1 \). Let \( u \geq 3 \) be an odd integer and \( P_u(X, Y) \) be as in Theorem A. Then

\[
P_u\left(u^2 \vartheta_3^4(u\tau) \frac{\vartheta_3^4(\tau)}{\vartheta_3^4(\tau)} 16 \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)} \right) = 0
\]

(8)

for any \( \tau \in \mathbb{H} \). Noting that \( P_u(X, Y) \) has positive degree in \( Y \) and \( P_u(0, Y) \) is a nonzero integer, we have the form

\[
P_u(X, Y) = \sum_{j=0}^{d_u} S_{u,j}(X)Y^j, \quad S_{u,d_u}(X) \neq 0,
\]

with

\[
c_u := S_{u,0}(0) = P_u(0, 0) \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad S_{u,j}(0) = 0 \quad (1 \leq j \leq d_u).
\]

(9)

On the other hand, since \( v \) is not a power of 2, **Lemma 6** shows that there exists a nonzero polynomial \( Q_v(X, Y) \in \mathbb{Z}[X, Y] \) such that

\[
Q_v\left(\vartheta_3^4(v\tau) \frac{\vartheta_3^4(\tau)}{\vartheta_3^4(\tau)} \right) = 0
\]

(10)

for any \( \tau \in \mathbb{H} \), where \( Q_v(X, Y) \) is of the form (7) with

\[
Q_v(0, Y) := c_vY^{d_v}, \quad c_v \in \mathbb{Z} \setminus \{0\}.
\]

(11)

Let \( \tau \in \mathbb{H} \) be a fixed complex number. Then, by (8) and (10), the polynomials \( P_u(u^2\vartheta_3^4(u\tau)/\vartheta_3^4(\tau), 16Y) \) and \( Q_v(\vartheta_3^4(v\tau)/\vartheta_3^4(\tau), Y) \) have the same common root \( Y_0 = \vartheta_2^4(\tau)/\vartheta_3^4(\tau) \). Hence, the resultant

\[
R_1(X, Z) := \text{Res}_Y(P_u(X, 16Y), Q_v(Z, Y))
\]

is given by the determinant \( D_Y \) of the square \( (d_u + d_v) \) Sylvester matrix depending on the coefficients of \( P_u(X, 16Y) \) and \( Q_v(Z, Y) \) with respect to \( Y \). Then, \( R_1(X, Z) \) (and thus \( D_Y \)) vanishes at \( X := u^2\vartheta_3^4(u\tau)/\vartheta_3^4(\tau) \) and \( Z := \vartheta_3^4(v\tau)/\vartheta_3^4(\tau) \), so that the polynomial

\[
R_2(W) := R_1(u^2\vartheta_3^4(u\tau)W, \vartheta_3^4(v\tau)W)
\]
has a root $W_0 = \vartheta_3^{-4}(\tau)$ over the field $K := \mathbb{Q}(\vartheta_3(u\tau), \vartheta_3(v\tau))$. Note that $R_2(W)$ is not identically zero, since by (9) and (11) the determinant $D_Y$ takes the form

$$R_2(0) = R_1(0, 0) = \det \begin{pmatrix} c_u & 0 & 0 \\ 0 & \ddots & 0 \\ c_v & \ddots & c_u \\ 0 & \cdots & 0 \\ 0 & 0 & c_v \end{pmatrix} = \pm c_u^{d_u} c_v^{d_v} \neq 0.$$ 

Therefore the number $\vartheta_3(\tau)$ is algebraic over $K$ and the proof of Lemma 7 is completed. \(\square\)

Next we treat the case where both $u$ and $v$ are odd.

**Lemma 8.** Let $u, v \geq 1$ be distinct odd integers. Then for any $\tau \in \mathbb{H}$ the number $\vartheta_3(\tau)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(u\tau), \vartheta_3(v\tau))$.

**Proof.** We may assume $u, v \geq 3$. Similarly to the proof of Lemma 7, we consider the resultant

$$R_1(X, Z) := \text{Res}_Y(P_u(X, Y), P_v(Z, Y)), \quad (12)$$

and the polynomial

$$R_2(W) := R_1(u^2 \vartheta_3^4(u\tau)W, v^2 \vartheta_3^4(v\tau)W), \quad (13)$$

which has a root $W_0 = \vartheta_3^{-4}(\tau)$. Suppose to the contrary that the above polynomial $R_2(W)$ is identically zero for some $\tau_0 \in \mathbb{H}$. Then, putting $\alpha := u^2 \vartheta_3^4(u\tau_0)$ and $\beta := v^2 \vartheta_3^4(v\tau_0)$, we have by (12) and (13)

$$\text{Res}_Y(P_u(\alpha W, Y), P_v(\beta W, Y)) = R_1(\alpha W, \beta W) = R_2(W) \equiv 0,$$

and so there exists a common factor $H(W, Y) \in \mathbb{C}[W, Y]$ with positive degree in $Y$ of the two polynomials $P_u(\alpha W, Y)$ and $P_v(\beta W, Y)$. Let

$$P_u(\alpha W, Y) = H(W, Y) G(W, Y).$$

Substituting the function $\lambda(\tau)$ defined by (4) into $Y$ in the above, we have

$$P_u(\alpha W, 16\lambda(\tau)) = H(W, 16\lambda(\tau)) G(W, 16\lambda(\tau)). \quad (14)$$

In what follows, we denote by $\deg H(W, Y)$, $\deg G(W, Y)$, and $\deg P_u(\alpha W, Y)$ the total degrees of the polynomials $H(W, Y)$, $G(W, Y)$, and $P_u(\alpha W, Y)$ with respect to $W$ and $Y$, respectively. Then

$$\deg W H(W, 16\lambda(\tau)) \leq \deg H(W, Y), \quad \deg W G(W, 16\lambda(\tau)) \leq \deg G(W, Y),$$

so that

$$\deg W P_u(\alpha W, 16\lambda(\tau)) = \deg W H(W, 16\lambda(\tau)) + \deg W G(W, 16\lambda(\tau)) \leq \deg H(W, Y) + \deg G(W, Y) = \deg P_u(\alpha W, Y).$$

On the other hand, it is clear that

$$\deg W P_u(\alpha W, 16\lambda(\tau)) = \deg P_u(\alpha W, Y),$$
since by [Nesterenko 2006, Corollary 4] the inequalities
\[ \deg_{Y} R_k(Y) \leq k \cdot \frac{n-1}{n}, \quad 1 \leq k \leq \psi(n), \]
hold in (5). Thus, we get
\[ \deg_{W} H(W, 16\lambda(\tau)) = \deg H(W, Y) \geq \deg_{Y} H(W, Y) \geq 1. \] (15)
Hence by Lemma 2 together with (14) and (15), we obtain
\[ P_u(\alpha W, 16\lambda(\tau)) = c_1 H(W, 16\lambda(\tau)) \]
for some nonzero complex numbers \(c_1\). Similarly there exists a nonzero complex number \(c_2\) such that
\[ P_v(\beta W, 16\lambda(\tau)) = c_2 H(W, 16\lambda(\tau)), \]
and hence
\[ P_u(\alpha W, 16\lambda(\tau)) = c P_v(\beta W, 16\lambda(\tau)), \quad c := c_1/c_2. \]
Taking \(\tau \to i\infty\) in the above equality, we have by Lemma 4
\[ \prod_{d | u, d \geq 1} (\alpha W - d^2)^{w(d,u/d)} = c \prod_{d | v, d \geq 1} (\beta W - d^2)^{w(d,v/d)}. \]
Assume, without loss of generality, that \(u > v\). Then, comparing the multiplicity of the zeros of these polynomials at \(1/\alpha\), we obtain
\[ u = w(1, u) \leq \max_d w(d, v/d) \leq v, \]
which is a contradiction. Hence, the polynomial \(R_2(W)\) is not identically zero for any \(\tau \in \mathbb{H}\), and the proof of Lemma 8 is completed by \(R_2(\vartheta_3^{-4}(\tau)) = 0\). \(\square\)

4. Proof of Theorem 1

Proof of Theorem 1. Let \(m\) and \(n\) be distinct positive integers. Define \(m_1 := m/d\) and \(n_1 := n/d\), where \(d := \gcd(m, n)\). Without loss of generality, we may assume that \(m_1\) is odd. In what follows, we distinguish two cases based on the parity of \(n_1\). We first suppose that \(n_1\) is even. Let \(\tau \in \mathbb{H}\). Then, by Lemma 7 with \(u := 3m_1 \geq 3, \quad v := 3n_1 \neq 2^\alpha (\alpha \geq 0),\) and \(\tau_0 := d\tau/3 \in \mathbb{H}\), the number \(\vartheta_3(\tau_0)\) is algebraic over the field \(\mathbb{Q}(\vartheta_3(u\tau_0), \vartheta_3(v\tau_0))\). Hence, we obtain
\[
\text{trans.} \deg_{\mathbb{Q}} \mathbb{Q}(e^{\pi i \tau}, \vartheta_3(m\tau), \vartheta_3(n\tau)) = \text{trans.} \deg_{\mathbb{Q}} \mathbb{Q}(e^{\pi i \tau_0}, \vartheta_3(u\tau_0), \vartheta_3(v\tau_0))
\]
\[
= \text{trans.} \deg_{\mathbb{Q}} \mathbb{Q}(e^{\pi i \tau_0}, \vartheta_3(\tau_0), \vartheta_3(u\tau_0), \vartheta_3(v\tau_0))
\]
\[
\geq \text{trans.} \deg_{\mathbb{Q}} \mathbb{Q}(e^{\pi i \tau_0}, \vartheta_3(\tau_0), \vartheta_3(u\tau_0))
\]
\[
\geq 2,
\]
where for the last inequality we used the fact that \(u > 2\) and that at least two of the numbers \(e^{\pi i \tau_0}, \vartheta_3(\tau_0)\) and \(\vartheta_3(u\tau_0)\) are algebraically independent over \(\mathbb{Q}\); see [Elsner and Tachiya 2017, Theorem 1.2]. In the case where \(n_1\) is odd, we can deduce the same inequality as above by applying Lemma 8 with the same quantities \(u, v, \tau_0\) as above.
Therefore, at least two of the numbers $e^{\pi i \tau}$, $\vartheta_3(m \tau)$, and $\vartheta_3(n \tau)$ are algebraically independent over $\mathbb{Q}$, and the proof of Theorem 1 is complete.

In the case where $m > n$ with two odd integers $m, n$, we obtain a stronger result based on [Elsner and Tachiya 2017, Theorem 1.2] and on Lemma 8.

**Theorem 9.** Let $m > n \geq 1$ be odd integers, $j \in \{2, 3, 4\}$ and $\tau \in \mathbb{H}$. Then we have

$$\text{trans. deg}_{\mathbb{Q}} \mathbb{Q}(e^{i\pi \tau}, \vartheta_3(m \tau), \vartheta_3(n \tau), D\vartheta_j(\tau)) \geq 3.$$  

**Proof.** We apply Lemma 8 with $u = m$ and $v = n$. Therefore, we know that $\vartheta_3(\tau)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(m \tau), \vartheta_3(n \tau))$. Hence we obtain with Theorem C,

$$\text{trans. deg}_{\mathbb{Q}} \mathbb{Q}(e^{i\pi \tau}, \vartheta_3(m \tau), \vartheta_3(n \tau), D\vartheta_j(\tau)) = \text{trans. deg}_{\mathbb{Q}} \mathbb{Q}(e^{i\pi \tau}, \vartheta_3(\tau), \vartheta_3(m \tau), \vartheta_3(n \tau), D\vartheta_j(\tau)) \geq \text{trans. deg}_{\mathbb{Q}} \mathbb{Q}(e^{i\pi \tau}, \vartheta_3(\tau), \vartheta_3(m \tau), D\vartheta_j(\tau)) \geq 3,$$

as desired. This proves the theorem. □

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**References**


ALGEBRAIC RESULTS FOR THE VALUES $\vartheta_3(m\tau)$ AND $\vartheta_3(n\tau)$ OF THE JACOBI THETA-CONSTANT

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