Moscow Journal of Combinatorics and Number Theory

2019
vol. 8 no. 1

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Moscow Journal of Combinatorics and Number Theory
Vol. 8, No. 1, 2019
dx.doi.org/10.2140/moscow.2019.8.71

Algebraic results for the values $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ of the Jacobi theta-constant

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Let $\vartheta_3(\tau) = 1 + 2\sum_{\nu=1}^{\infty} e^{\pi i \nu^2 \tau}$ denote the classical Jacobi theta-constant. We prove that the two values $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$ for any τ in the upper half-plane such that $q = e^{\pi i \tau}$ is an algebraic number, where $m, n \geq 2$ are distinct integers.

1. Introduction and statement of the results

Throughout this paper, let τ be a complex variable in the upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$. The three classical theta functions

$$\vartheta_2(\tau) = 2\sum_{\nu=0}^{\infty} q^{(\nu+1/2)^2}, \quad \vartheta_3(\tau) = 1 + 2\sum_{\nu=1}^{\infty} q^{\nu^2}, \quad \vartheta_4(\tau) = 1 + 2\sum_{\nu=1}^{\infty} (-1)^\nu q^{\nu^2}$$

are known as theta-constants or Thetanullwerte, where $q := e^{\pi i \tau}$. These theta-constants are holomorphic in $\mathbb{H}$ and never vanish for any $\tau \in \mathbb{H}$. In particular, the function $\vartheta_3(\tau)$ is called a Jacobi theta-constant or Thetanullwert of the Jacobi theta function $\vartheta(z \mid \tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i \nu^2 \tau + 2\pi i \nu z}$. For an extensive discussion of the Jacobi theta function and theta-constants we refer the reader to [Stein and Shakarchi 2003, Chapter 10]. Y. V. Nesterenko [2006] has improved upon a result from [Grinspan 2001] and obtained some identities for the theta-constants.

Theorem A [Nesterenko 2006, Theorem 1]. *For any odd integer $n \geq 3$ there exists an integer polynomial $P_n(X, Y)$ with $\deg_X P_n(X, Y) = \psi(n)$ such that*

$$P_n\left(n^2 \frac{\vartheta_3^4(n\tau)}{\vartheta_3^4(\tau)}, 16\frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}\right) = 0$$

holds for any $\tau \in \mathbb{H}$, where

$$\psi(n) := n \prod_{p \mid n} \left(1 + \frac{1}{p}\right).$$

For example, the first polynomials P_3 and P_5 are given in [Nesterenko 2006] by

$$P_3 = 9 - (28 - 16Y + Y^2)X + 30X^2 - 12X^3 + X^4,$$

$$P_5 = 25 - (126 - 832Y + 308Y^2 - 32Y^3 + Y^4)X + (255 + 1920Y - 120Y^2)X^2 + (-260 + 320Y - 20Y^2)X^3 + 135X^4 - 30X^5 + X^6$$

MSC2010: primary 11J85; secondary 11J91, 11F27.
Keywords: algebraic independence, Jacobi theta-constants, modular functions.

and the polynomials P_7, P_9, and P_{11} are listed in the appendix of [Elsner 2015]. Recently one of us (Elsner) constructed similar integer polynomials in two variables X and Y, which vanish identically at certain rational functions of theta-constants including the function $\vartheta_3(n\tau)$ for $n = 2^m$. He applied this result and Theorem A to settle the algebraic independence problem of the two values $\vartheta_3(\tau)$ and $\vartheta_3(n\tau)$ for integers $n \geq 2$, and obtained the following Theorem B.

Theorem B [Elsner 2015, Theorem 1.1]. *Let $\tau \in \mathbb{H}$ such that $e^{\pi i \tau}$ is an algebraic number. Then the two values $\vartheta_3(\tau)$ and $\vartheta_3(2^m\tau)$ are algebraically independent over $\mathbb{Q}$ for each integer $m \geq 1$. Furthermore, the same holds for the two values $\vartheta_3(\tau)$ and $\vartheta_3(n\tau)$ if $n = 3, 5, 6, 7, 9, 10, 11, 12$.*

The proof of Theorem B is based on an algebraic independence criterion, see [Elsner et al. 2011, Lemma 3.1], which requires a nonvanishing of a Jacobian determinant. In particular, to prove the latter assertion in Theorem B, he needed the explicit forms of the polynomials P_3, P_5, P_7, P_9 and P_{11} stated above. In [Elsner and Tachiya 2017], two of us obtained the following Theorem C by studying the specific properties of the polynomials P_n.

Theorem C [Elsner and Tachiya 2017, Theorem 1.2]. *Let $n \geq 2$ be an integer and $j \in \{2, 3, 4\}$. Then for any $\tau \in \mathbb{H}$ at least three of the numbers $e^{\pi i \tau}$, $\vartheta_3(\tau)$, $\vartheta_3(n\tau)$, and $D\vartheta_j(\tau)$ are algebraically independent over $\mathbb{Q}$, where $D := (\pi i)^{-1} d/d\tau$ denotes a differential operator.*

An application of Theorem C gives an improvement of Theorem B as follows:

Theorem D. *Let $\tau \in \mathbb{H}$ be such that $e^{\pi i \tau}$ is an algebraic number. Then the two numbers $\vartheta_3(\tau)$ and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$ for each integer $n \geq 2$.*

On the other hand, the algebraic dependence result is also obtained in [Elsner and Tachiya 2017] through the properties of the polynomials P_n.

Theorem E [Elsner and Tachiya 2017, Theorem 1.4]. *Let $\ell, m, n \geq 1$ be integers and $\tau \in \mathbb{H}$ be any complex number. Then the three values $\vartheta_3(\ell\tau)$, $\vartheta_3(m\tau)$, and $\vartheta_3(n\tau)$ are algebraically dependent over $\mathbb{Q}$.*

In this paper, we fill the gap between Theorems D and E. Our main result is the following.

Theorem 1. *Let $m, n \geq 1$ be distinct integers and $\tau \in \mathbb{H}$. Then at least two of the numbers $e^{\pi i \tau}$, $\vartheta_3(m\tau)$, and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$. In particular, the two numbers $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$ for any $\tau \in \mathbb{H}$ such that $e^{\pi i \tau}$ is an algebraic number.*

Of course the two numbers $\vartheta_3(m\tau)$ and $\vartheta_3(n\tau)$ can be algebraically dependent over $\mathbb{Q}$ without an algebraic condition on $e^{\pi i \tau}$. Indeed, for $\tau = i \in \mathbb{H}$ the two numbers $\vartheta_3(i)$ and $\vartheta_3(2i)$ are algebraically dependent over $\mathbb{Q}$, since the nontrivial relation

$$4\vartheta_3^2(2i) - (\sqrt{2}+2)\vartheta_3^2(i) = 0 \tag{1}$$

exists; see [Berndt 1998, p. 325]. Note that the number $e^{\pi} = i^{-2i}$ was shown to be transcendental for the first time by A. O. Gelfond (1929) and, a few years later, this property of e^{π} was corroborated by the Gelfond–Schneider theorem (1934). Conversely, the above identity (1) and Theorem 1 imply the transcendence of e^{π} as well.

2. Some properties of $P_n(X, Y)$

We now discuss some properties of $P_n(X, Y)$ given in Theorem A. We start with a short description of the construction of $P_n(X, Y)$; for details, see [Nesterenko 2006]. Let $\Gamma(2)$ be the principal congruence subgroup of level 2 in $\mathrm{SL}(2, \mathbb{Z})$; that is,

$$\Gamma(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2 \right\}.$$

Then for each odd integer $n \geq 3$ the set of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2, \qquad (a, b, c, d) = 1, \;\; ad - bc = n,$$

is a union of $\psi(n)$ equivalence classes with respect to the left–multiplication on the elements of $\Gamma(2)$, and the class representatives are given by

$$\alpha_\nu := \begin{pmatrix} u & 2v \\ 0 & w \end{pmatrix}, \qquad (u, v, w) = 1, \;\; uw = n, \;\; 0 \leq v < w. \tag{2}$$

For these $\psi(n)$ matrices $\alpha_1, \ldots, \alpha_{\psi(n)}$ in (2), we define the polynomial

$$\prod_{\nu=1}^{\psi(n)} (X - x_\nu(\tau)) =: X^{\psi(n)} + a_1(\tau) X^{\psi(n)-1} + \cdots + a_{\psi(n)-1}(\tau) X + a_{\psi(n)}(\tau),$$

where

$$x_\nu(\tau) := u^2 \frac{\vartheta_3^4((u\tau + 2v)/w)}{\vartheta_3^4(\tau)} \quad \text{with} \quad \begin{pmatrix} u & 2v \\ 0 & w \end{pmatrix} = \alpha_\nu, \;\; \nu = 1, \ldots, \psi(n). \tag{3}$$

Then, using the modular method as well as Galois considerations, one finds that there exist polynomials $R_j(Y) \in \mathbb{Z}[Y]$, $j = 1, \ldots, \psi(n)$, such that

$$a_j(\tau) = R_j(16\lambda(\tau)), \qquad \lambda(\tau) := \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}. \tag{4}$$

Thus, the integer polynomial

$$P_n(X, Y) := X^{\psi(n)} + R_1(Y) X^{\psi(n)-1} + \cdots + R_{\psi(n)-1}(Y) X + R_{\psi(n)}(Y) \tag{5}$$

vanishes identically at $X = n^2 \vartheta_3^4(n\tau)/\vartheta_3^4(\tau)$ and $Y = 16\lambda(\tau)$.

Lemma 2. *For each odd integer $n \geq 3$, the polynomial $P_n(X, 16\lambda(\tau))$ is irreducible over the field $\mathbb{C}(\lambda(\tau))$.*

Proof. The group $\Gamma(2)$ fixes the function $\lambda(\tau) = \vartheta_2^4(\tau)/\vartheta_3^4(\tau)$, since the functions $\vartheta_3^4(\tau)$ and $\vartheta_4^4(\tau)$ are modular forms of weight 2 with respect of the subgroup $\Gamma(2)$. Moreover, we have the transformation formula

$$x_\nu\left(\frac{a\tau + b}{c\tau + d}\right) = x_\mu(\tau) \tag{6}$$

for a proper matrix $\beta := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$ and subscripts ν, μ such that a proper matrix $\gamma \in \Gamma(2)$ satisfies $\alpha_\nu \beta = \gamma \alpha_\mu$; see formulae (6) and (7) in [Nesterenko 2006]. This may be regarded as an equivalence relation over the matrices $\alpha_1, \alpha_2, \ldots, \alpha_{\psi(n)}$ from (2). One can show that any two matrices α_ν and α_μ, $1 \le \nu, \mu \le \psi(n)$, are equivalent. Together with (6) it turns out that the group $\Gamma(2)$ permutes the $\psi(n)$ distinct functions $x_1(\tau), \ldots, x_{\psi(n)}(\tau)$ transitively. This implies that $P_n(X, 16\lambda(\tau))$ is a minimal polynomial of $x_1(\tau)$ over the field $\mathbb{C}(\lambda(\tau))$. □

Remark 3. There is no complex number α such that $P_n(\alpha, Y)$ is identically zero. If such an α existed, the polynomial $P_n(X, Y)$ would be divisible by $(X-\alpha)$, which is impossible by Lemma 2. This fact can also be checked directly from the definition of $x_\nu(\tau)$; see [Elsner and Tachiya 2017, Lemma 2.1]. In particular, $P_n(X, Y)$ has positive degree in Y.

Lemma 4. *We have*

$$P_n(X, 0) = \prod_{u \mid n,\ u \ge 1} (X - u^2)^{w(u, n/u)},$$

where

$$w(a, b) := \sum_{\substack{(a,b,k)=1 \\ 0 \le k < b}} 1.$$

Proof. This follows immediately from the relation

$$P_n(X, 16\lambda(\tau)) = \prod_{\nu=1}^{\psi(n)} (X - x_\nu(\tau))$$

as $\tau \to i\infty$, since we have $\lambda(\tau) \to 0$ and $x_\nu(\tau) \to u^2$ for each $\nu = 1, \ldots, \psi(n)$ in (3), respectively. □

Example 5. For the polynomial P_3 given in Section 1, we have

$$P_3(X, 0) = 9 - 28X + 30X^2 - 12X^3 + X^4 = (X-1)^3(X-3^2).$$

Here, $\psi(3) = 4$ and the four triples (u, v, w) in (2) are given by

$$(3, 0, 1), \quad (1, 0, 3), \quad (1, 1, 3), \quad (1, 2, 3).$$

More generally, $P_p(X, 0) = (X-1)^p(X-p^2)$ for any odd prime $p \ge 3$.

3. Lemmas

Let $\tau \in \mathbb{H}$. We prove in Lemmas 7 and 8 below that the number $\vartheta_3(\tau)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(u\tau), \vartheta_3(v\tau))$ for certain positive integers u and v. To see this, we need the following Lemma 6. Note that $P_n(0, Y)$ is a *nonzero integer* for the polynomial $P_n(X, Y)$ in Theorem A; see [Elsner and Tachiya 2017, Lemma 2.3].

Lemma 6 [Elsner and Tachiya 2017, Lemma 2.5]. *Let $n = 2^\alpha m$ be an integer with $\alpha \ge 1$ and odd integer $m \ge 3$. Then there exists a polynomial $Q_n(X, Y) \in \mathbb{Z}[X, Y]$ such that*

$$Q_n\left(\frac{\vartheta_3^4(n\tau)}{\vartheta_3^4(\tau)}, \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}\right) = 0$$

for any $\tau \in \mathbb{H}$. Furthermore, the polynomial $Q_n(X, Y)$ is of the form

$$Q_n(X, Y) = c^{2^\alpha} Y^{2^\alpha \psi(m)} + \sum_{j=0}^{2^\alpha \psi(m)-1} R_{n,j}(X) Y^j, \tag{7}$$

with

$$Q_n(0, Y) = c^{2^\alpha} Y^{2^\alpha \psi(m)},$$

where c is equal to the nonzero integer $P_m(0, Y)$.

First we consider the case where u and v have different parity.

Lemma 7. *Let $u \geq 1$ be an odd integer and $v \geq 2$ be an even integer which is not a power of 2. Then for any $\tau \in \mathbb{H}$ the number $\vartheta_3(\tau)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(u\tau), \vartheta_3(v\tau))$.*

Proof. The assertion is clear if $u = 1$. Let $u \geq 3$ be an odd integer and $P_u(X, Y)$ be as in Theorem A. Then

$$P_u\left(u^2 \frac{\vartheta_3^4(u\tau)}{\vartheta_3^4(\tau)}, 16\frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}\right) = 0 \tag{8}$$

for any $\tau \in \mathbb{H}$. Noting that $P_u(X, Y)$ has positive degree in Y and $P_u(0, Y)$ is a nonzero integer, we have the form

$$P_u(X, Y) = \sum_{j=0}^{d_u} S_{u,j}(X) Y^j, \qquad S_{u,d_u}(X) \not\equiv 0,$$

with

$$c_u := S_{u,0}(0) = P_u(0, 0) \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad S_{u,j}(0) = 0 \quad (1 \leq j \leq d_u). \tag{9}$$

On the other hand, since v is not a power of 2, Lemma 6 shows that there exists a nonzero polynomial $Q_v(X, Y) \in \mathbb{Z}[X, Y]$ such that

$$Q_v\left(\frac{\vartheta_3^4(v\tau)}{\vartheta_3^4(\tau)}, \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}\right) = 0 \tag{10}$$

for any $\tau \in \mathbb{H}$, where $Q_v(X, Y)$ is of the form (7) with

$$Q_v(0, Y) := c_v Y^{d_v}, \quad c_v \in \mathbb{Z} \setminus \{0\}. \tag{11}$$

Let $\tau \in \mathbb{H}$ be a fixed complex number. Then, by (8) and (10), the polynomials $P_u(u^2\vartheta_3^4(u\tau)/\vartheta_3^4(\tau), 16Y)$ and $Q_v(\vartheta_3^4(v\tau)/\vartheta_3^4(\tau), Y)$ have the same common root $Y_0 = \vartheta_2^4(\tau)/\vartheta_3^4(\tau)$. Hence, the resultant

$$R_1(X, Z) := \mathrm{Res}_Y(P_u(X, 16Y), Q_v(Z, Y))$$

is given by the determinant D_Y of the square $(d_u + d_v)$ Sylvester matrix depending on the coefficients of $P_u(X, 16Y)$ and $Q_v(Z, Y)$ with respect to Y. Then, $R_1(X, Z)$ (and thus D_Y) vanishes at $X := u^2\vartheta_3^4(u\tau)/\vartheta_3^4(\tau)$ and $Z := \vartheta_3^4(v\tau)/\vartheta_3^4(\tau)$, so that the polynomial

$$R_2(W) := R_1(u^2\vartheta_3^4(u\tau)W, \vartheta_3^4(v\tau)W)$$

has a root $W_0 = \vartheta_3^{-4}(\tau)$ over the field $K := \mathbb{Q}(\vartheta_3(u\tau), \vartheta_3(v\tau))$. Note that $R_2(W)$ is not identically zero, since by (9) and (11) the determinant D_Y takes the form

$$R_2(0) = R_1(0, 0) = \det \begin{pmatrix} & & & c_u & 0 & 0 \\ & & & & \ddots & 0 \\ & & & & & c_u \\ c_v & & & & & \\ 0 & \ddots & & & & \\ 0 & 0 & c_v & & & \end{pmatrix} = \pm c_u^{d_v} c_v^{d_u} \neq 0.$$

Therefore the number $\vartheta_3(\tau)$ is algebraic over K and the proof of Lemma 7 is completed. □

Next we treat the case where both u and v are odd.

Lemma 8. *Let $u, v \geq 1$ be distinct odd integers. Then for any $\tau \in \mathbb{H}$ the number $\vartheta_3(\tau)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(u\tau), \vartheta_3(v\tau))$.*

Proof. We may assume $u, v \geq 3$. Similarly to the proof of Lemma 7, we consider the resultant

$$R_1(X, Z) := \operatorname{Res}_Y(P_u(X, Y), P_v(Z, Y)), \tag{12}$$

and the polynomial

$$R_2(W) := R_1(u^2\vartheta_3^4(u\tau)W, v^2\vartheta_3^4(v\tau)W), \tag{13}$$

which has a root $W_0 = \vartheta_3^{-4}(\tau)$. Suppose to the contrary that the above polynomial $R_2(W)$ is identically zero for some $\tau_0 \in \mathbb{H}$. Then, putting $\alpha := u^2\vartheta_3^4(u\tau_0)$ and $\beta := v^2\vartheta_3^4(v\tau_0)$, we have by (12) and (13)

$$\operatorname{Res}_Y(P_u(\alpha W, Y), P_v(\beta W, Y)) = R_1(\alpha W, \beta W) = R_2(W) \equiv 0,$$

and so there exists a common factor $H(W, Y) \in \mathbb{C}[W, Y]$ with positive degree in Y of the two polynomials $P_u(\alpha W, Y)$ and $P_v(\beta W, Y)$. Let

$$P_u(\alpha W, Y) = H(W, Y)\, G(W, Y).$$

Substituting the function $\lambda(\tau)$ defined by (4) into Y in the above, we have

$$P_u(\alpha W, 16\lambda(\tau)) = H(W, 16\lambda(\tau))\, G(W, 16\lambda(\tau)). \tag{14}$$

In what follows, we denote by $\deg H(W, Y)$, $\deg G(W, Y)$, and $\deg P_u(\alpha W, Y)$ the total degrees of the polynomials $H(W, Y)$, $G(W, Y)$, and $P_u(\alpha W, Y)$ with respect to W and Y, respectively. Then

$$\deg_W H(W, 16\lambda(\tau)) \leq \deg H(W, Y), \quad \deg_W G(W, 16\lambda(\tau)) \leq \deg G(W, Y),$$

so that

$$\begin{aligned} \deg_W P_u(\alpha W, 16\lambda(\tau)) &= \deg_W H(W, 16\lambda(\tau)) + \deg_W G(W, 16\lambda(\tau)) \\ &\leq \deg H(W, Y) + \deg G(W, Y) \\ &= \deg P_u(\alpha W, Y). \end{aligned}$$

On the other hand, it is clear that

$$\deg_W P_u(\alpha W, 16\lambda(\tau)) = \deg P_u(\alpha W, Y),$$

since by [Nesterenko 2006, Corollary 4] the inequalities

$$\deg_Y R_k(Y) \le k \cdot \frac{n-1}{n}, \quad 1 \le k \le \psi(n),$$

hold in (5). Thus, we get

$$\deg_W H(W, 16\lambda(\tau)) = \deg H(W, Y) \ge \deg_Y H(W, Y) \ge 1. \tag{15}$$

Hence by Lemma 2 together with (14) and (15), we obtain

$$P_u(\alpha W, 16\lambda(\tau)) = c_1 H(W, 16\lambda(\tau))$$

for some nonzero complex numbers c_1. Similarly there exists a nonzero complex number c_2 such that

$$P_v(\beta W, 16\lambda(\tau)) = c_2 H(W, 16\lambda(\tau)),$$

and hence

$$P_u(\alpha W, 16\lambda(\tau)) = c P_v(\beta W, 16\lambda(\tau)), \quad c := c_1/c_2.$$

Taking $\tau \to i\infty$ in the above equality, we have by Lemma 4

$$\prod_{d \mid u,\ d \ge 1} (\alpha W - d^2)^{w(d,u/d)} = c \prod_{d \mid v,\ d \ge 1} (\beta W - d^2)^{w(d,v/d)}.$$

Assume, without loss of generality, that $u > v$. Then, comparing the multiplicity of the zeros of these polynomials at $1/\alpha$, we obtain

$$u = w(1, u) \le \max_d w(d, v/d) \le v,$$

which is a contradiction. Hence, the polynomial $R_2(W)$ is not identically zero for any $\tau \in \mathbb{H}$, and the proof of Lemma 8 is completed by $R_2(\vartheta_3^{-4}(\tau)) = 0$. □

4. Proof of Theorem 1

Proof of Theorem 1. Let m and n be distinct positive integers. Define $m_1 := m/d$ and $n_1 := n/d$, where $d := \gcd(m, n)$. Without loss of generality, we may assume that m_1 is odd. In what follows, we distinguish two cases based on the parity of n_1. We first suppose that n_1 is even. Let $\tau \in \mathbb{H}$. Then, by Lemma 7 with $u := 3m_1 \ge 3$, $v := 3n_1 \ne 2^\alpha$ $(\alpha \ge 0)$, and $\tau_0 := d\tau/3 \in \mathbb{H}$, the number $\vartheta_3(\tau_0)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(u\tau_0), \vartheta_3(v\tau_0))$. Hence, we obtain

$$\begin{aligned}
\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{\pi i\tau}, \vartheta_3(m\tau), \vartheta_3(n\tau)) &= \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{\pi i\tau_0}, \vartheta_3(u\tau_0), \vartheta_3(v\tau_0)) \\
&= \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{\pi i\tau_0}, \vartheta_3(\tau_0), \vartheta_3(u\tau_0), \vartheta_3(v\tau_0)) \\
&\ge \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{\pi i\tau_0}, \vartheta_3(\tau_0), \vartheta_3(u\tau_0)) \\
&\ge 2,
\end{aligned}$$

where for the last inequality we used the fact that $u > 2$ and that at least two of the numbers $e^{\pi i\tau_0}$, $\vartheta_3(\tau_0)$ and $\vartheta_3(u\tau_0)$ are algebraically independent over $\mathbb{Q}$; see [Elsner and Tachiya 2017, Theorem 1.2]. In the case where n_1 is odd, we can deduce the same inequality as above by applying Lemma 8 with the same quantities u, v, τ_0 as above.

Therefore, at least two of the numbers $e^{\pi i\tau}$, $\vartheta_3(m\tau)$, and $\vartheta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$, and the proof of Theorem 1 is complete. □

In the case where $m > n$ with two odd integers m, n, we obtain a stronger result based on [Elsner and Tachiya 2017, Theorem 1.2] and on Lemma 8.

Theorem 9. *Let $m > n \geq 1$ be odd integers, $j \in \{2, 3, 4\}$ and $\tau \in \mathbb{H}$. Then we have*

$$\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{i\pi\tau}, \vartheta_3(m\tau), \vartheta_3(n\tau), D\vartheta_j(\tau)) \geq 3.$$

Proof. We apply Lemma 8 with $u = m$ and $v = n$. Therefore, we know that $\vartheta_3(\tau)$ is algebraic over the field $\mathbb{Q}(\vartheta_3(m\tau), \vartheta_3(n\tau))$. Hence we obtain with Theorem C,

$$\begin{aligned}\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{i\pi\tau}, \vartheta_3(m\tau), \vartheta_3(n\tau), D\vartheta_j(\tau)) &= \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{\pi i\tau}, \vartheta_3(\tau), \vartheta_3(m\tau), \vartheta_3(n\tau), D\vartheta_j(\tau)) \\ &\geq \operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(e^{i\pi\tau}, \vartheta_3(\tau), \vartheta_3(m\tau), D\vartheta_j(\tau)) \\ &\geq 3,\end{aligned}$$

as desired. This proves the theorem. □

Acknowledgments

The authors would like to express their sincere gratitude to Professor Hirofumi Tsumura, who kindly made us realize algebraic relations for the theta values as represented by (1). The authors are also grateful to the referee for careful reading the manuscript and for giving useful comments.

This work started during a very enjoyable visit of Luca at the University of Hirosaki in January 2017 and ended during a visit of Luca to the Max Planck Institute for Mathematics in Bonn from January to July 2017. He thanks those institutions for hospitality and support. In addition, Luca was supported by grant no. CPRR160325161141 and an A-rated scientist award, both from the NRF of South Africa, and by grant no. 17-02804S of the Czech Granting Agency. Tachiya was supported by JSPS, Grant-in-Aid for Young Scientists (B), 15K17504.

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Received 10 Jan 2018. Revised 18 Jun 2018.

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Moscow Journal of Combinatorics and Number Theory

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Moscow Journal of Combinatorics and Number Theory (ISSN 2220-5438 electronic, 2220-5438 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

MJCNT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

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Moscow Journal of Combinatorics and Number Theory

vol. 8 no. 1 2019