Moscow Journal of Combinatorics and Number Theory

2019
vol. 8 no. 2

A simple proof of the Hilton–Milner theorem

Peter Frankl

msp

Moscow Journal of Combinatorics and Number Theory
Vol. 8, No. 2, 2019
dx.doi.org/10.2140/moscow.2019.8.97

A simple proof of the Hilton–Milner theorem

Peter Frankl

Let $n \geq 2k \geq 4$ be integers and $\mathcal{F}$ a family of k-subsets of $\{1, 2, \ldots, n\}$. We call $\mathcal{F}$ *intersecting* if $F \cap F' \neq \varnothing$ for all $F, F' \in \mathcal{F}$, and we call $\mathcal{F}$ *nontrivial* if $\bigcap_{F \in \mathcal{F}} F = \varnothing$. Strengthening the famous Erdős–Ko–Rado theorem, Hilton and Milner proved that $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ if $\mathcal{F}$ is nontrivial and intersecting. We provide a proof by injection of this result.

1. Introduction

The proof of the Hilton–Milner theorem that we are going to present is very short but it is based on the very useful operation of *shifting* and two old results of the author. We are going to review these in this section.

Let $[n] = \{1, \ldots, n\}$ be the standard n-element set and $2^{[n]}$ its power set. Subsets $\mathcal{F} \subset 2^{[n]}$ are called families. For $i \in [n]$ we use the standard notation $\mathcal{F}(i) = \{F \setminus \{i\} : i \in F \in \mathcal{F}\}$ and $\mathcal{F}(\bar{i}) = \{F : i \notin F \in \mathcal{F}\}$. Note that

$$|\mathcal{F}| = |\mathcal{F}(i)| + |\mathcal{F}(\bar{i})|.$$

For a positive integer t the family $\mathcal{F}$ is said to be *t-intersecting* if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$. For $t = 1$ we use the term *intersecting*.

Let us recall the definition of the $S_{i,j}$ shift, an important operation on families, discovered by Erdős, Ko and Rado [Erdős et al. 1961].

Definition 1.1. For $1 \leq i < j \leq n$ and a family $\mathcal{F} \subset 2^{[n]}$, one defines $S_{i,j}(\mathcal{F}) = \{S_{i,j}(F) : F \in \mathcal{F}\}$, where

$$S_{i,j}(F) = \begin{cases} F' := (F \setminus \{j\}) \cup \{i\} & \text{if } j \in F,\ i \notin F \text{ and } F' \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

From the definition, $|S_{i,j}(\mathcal{F})| = |\mathcal{F}|$ and $|S_{i,j}(F)| = |F|$ should be obvious. More importantly, if $\mathcal{F}$ is t-intersecting then $S_{i,j}(\mathcal{F})$ is t-intersecting as well.

If $S_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$ then $\mathcal{F}$ is called *shifted.*

Let us use the notation $(a_1, a_2, \ldots, a_r)$ to denote the set $\{a_1, a_2, \ldots, a_r\}$, where $a_1 < a_2 < \cdots < a_r$. For two subsets $F = (a_1, \ldots, a_r)$ and $G = (b_1, \ldots, b_r)$ we say that F is smaller than G if $a_i \leq b_i$ for all $1 \leq i \leq r$. We denote this by $F \prec G$.

It is not hard to see that $\mathcal{F}$ is shifted if and only if for all pairs of F, G with $F \prec G$, we have $G \in \mathcal{F}$ implies $F \in \mathcal{F}$. For the proof of this and many other useful properties of shifting see [Frankl 1987b].

We shall need the following simple result.

MSC2010: 05D05.
Keywords: finite sets, intersection, hypergraphs.

Proposition 1.2 [Frankl 1978]. *Let $\mathcal{F} \subset 2^{[n]}$ be a shifted t-intersecting family. Then the following hold*:

(i) *For every $F \in \mathcal{F}$ there exists an integer $\ell \geq t$ such that*
$$|F \cap [2\ell - t]| \geq \ell.$$

(ii) *For all $F, G \in \mathcal{F}$ there exists an integer $h \geq t$ such that*
$$|F \cap [h]| + |G \cap [h]| \geq h + t. \tag{1-1}$$

Note that (1-1) *implies $|F \cap G \cap [h]| \geq t$.*

For $F \in \mathcal{F}$ define $\ell(F) = \{\max \ell, t \leq \ell \leq \frac{n}{2} : |F \cap [2\ell]| \geq \ell\}$. Note that if $2|F| \leq n$ then the maximality of $\ell(F)$ implies
$$|F \cap [2\ell(F)]| = \ell(F). \tag{1-2}$$

Let $k \geq s \geq 2$ be integers. Let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n]$.

Example 1.3. Define
$$\mathcal{E}(n, k, s) = \left\{E \in \binom{[n]}{k} : 1 \in E,\ E \cap [2, s+1] \neq \varnothing\right\} \cup \left\{F \subset \binom{[2n]}{k} : [2, s+1] \subset F\right\}.$$

Note that $\mathcal{E}(n, k, s)$ is intersecting, $E \cap [2, s+1] \neq \varnothing$ for all $E \in \mathcal{E}(n, k, s)$ and
$$|\mathcal{E}(n, k, s)| = \binom{n-1}{k-1} - \binom{n-s-1}{k-1} + \binom{n-s-1}{k-s}.$$

Theorem 1.4. *Let $n \geq 2k \geq 2s \geq 4$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is a shifted intersecting family satisfying $F \cap [2, s+1] \neq \varnothing$ for all $F \in \mathcal{F}$. Then*
$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-s-1}{k-1} + \binom{n-s-1}{k-s}. \tag{1-3}$$

This result is somewhat technical but its proof is rather special. We are going to prove it through an explicit injection from $\mathcal{F}$ into $\mathcal{E}(n, k, s)$.

For sets A, B let $A \triangle B$ denote their symmetric difference. Let us define the map $\alpha : \mathcal{F} \to \mathcal{E}(n, k, s)$ by
$$\alpha(F) = \begin{cases} F & \text{if } 1 \in F \text{ or if } [2, s+1] \subset F, \\ F \triangle [2\ell(F)] & \text{otherwise.} \end{cases}$$

To prove (1-3) it is sufficient to prove the following.

Proposition 1.5. *The map α is an injection into $\mathcal{E}(n, k, s)$.*

Let us recall two important results concerning intersecting families of k-sets.

Erdős–Ko–Rado theorem [Erdős et al. 1961]. *Suppose that $n \geq 2k > 0$ and $\mathcal{F} \subset \binom{[n]}{k}$ is an intersecting family. Then*
$$|\mathcal{F}| \leq \binom{n-1}{k-1}. \tag{1-4}$$

Taking all k-sets containing a fixed element shows that (1-4) is the best possible bound.

An intersecting family is called *nontrivial* if there is no element common to all its members. For $k = 1$ there is no nontrivial k-intersecting family. For $k = 2$ the only such family is the triangle: $\binom{[3]}{2}$.

Hilton–Milner theorem [1967]. *Suppose that $n \geq 2k \geq 4$ and $\mathcal{F} \subset \binom{[n]}{k}$ is a nontrivial intersecting family. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1. \tag{1-5}$$

Recently Hurlbert and Kamat [2018] gave an injective proof for (1-4). We extend their work by providing an injective proof for (1-5). For this we need the following proposition.

Proposition 1.6 [Frankl 1987b]. *Suppose that $n \geq 2k \geq 4$ and $\mathcal{F} \subset \binom{[n]}{k}$ is a nontrivial intersecting family of maximal size. Then there exists a nontrivial intersecting family $\widetilde{\mathcal{F}} \subset \binom{[n]}{k}$ such that $|\widetilde{\mathcal{F}}| = |\mathcal{F}|$ and $\widetilde{\mathcal{F}}$ is shifted.*

Once one has Proposition 1.6, to establish (1-5) is easy. One only needs to apply the case $s = k$ of Theorem 1.4 to the family $\widetilde{\mathcal{F}}$. Indeed, since $\widetilde{\mathcal{F}}$ is nontrivial and shifted, $[2, k+1] \in \widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}$ being intersecting imply that $F \cap [2, k+1] \neq \varnothing$ holds for all $F \in \widetilde{\mathcal{F}}$.

Since the proof of Proposition 1.6 is quite short and somewhat hidden in [Frankl 1987b], we reproduce it in Section 2.

Let us mention that there are several other, known proofs of the Hilton–Milner theorem: [Frankl and Füredi 1986; Frankl and Tokushige 1992; Mörs 1985; Kupavskii and Zakharov 2018].

We should also mention that in [Hilton and Milner 1967] the essentially unique families attaining equality are determined as well. This can be done via the present proof as well. However, it is rather technical and very similar to the corresponding part of previous proofs. Therefore we prefer to omit it.

2. The proofs of Propositions 1.5 and 1.6

We divide the proof of Proposition 1.5 into two lemmas. The first shows that for $F \in \mathcal{F} \setminus \mathcal{E}(n, k, s)$ the image $\alpha(F)$ is in $\mathcal{E}(n, k, s) \setminus \mathcal{F}$.

The second shows that α is an injection.

Lemma 2.1. *Suppose that $F \in \mathcal{F}(\bar{1})$ and $[2, s+1] \not\subset F$. Then the following hold:*

(i) $1 \in \alpha(F)$.

(ii) $\alpha(F) \notin \mathcal{F}$.

(iii) $\alpha(F) \cap [2, s+1] \neq \varnothing$.

Proof. (i) Recall that $\alpha(F) = F \bigtriangleup [2\ell(F)]$. As $1 \notin F$ implies $1 \in \alpha(F)$, (i) is true.

(ii) Suppose for contradiction that $\alpha(F) \in \mathcal{F}$. Apply Proposition 1.2 to F and $\alpha(F)$. By (1-2), $F \cap [2\ell(F)]$ and $\alpha(F) \cap [2\ell(F)]$ are complementary ℓ-element subsets of $[2\ell(F)]$. Consequently $h > 2\ell(F)$.

However, for $h \geq 2\ell$, we have $|F \cap [h]| = |\alpha(F) \cap [h]|$. Thus $2|F \cap [h]| \geq h+1$ implies

$$|F \cap [h]| \geq \tfrac{1}{2}(h+1). \tag{2-1}$$

Thus

$$|F \cap [h+1]| \geq \tfrac{1}{2}(h+1)$$

as well, and we get a contradiction with the maximality of $\ell(F)$.

(iii) Define $i(F) = \min\{i : 2 \le i \le n,\ i \notin F\}$. As $\ell(F) \ge 2$, (1-2) implies $i(F) \le 2\ell(F)$. Also, $[2, s+1] \not\subset F$ implies $i(F) \le s+1$. Consequently $i(F) \in [2\ell(F)]$ and $i(F) \in [2, s+1]$ hold. Therefore $i(F) \in \alpha(F) \cap [2, s+1]$. □

Lemma 2.2. *For distinct $F, F' \in \mathcal{F} \setminus \mathcal{E}(n, k, s)$, it holds that $\alpha(F) \ne \alpha(F')$.*

Proof. Since $F, F' \notin \mathcal{E}(n, k, s)$, we have $\alpha(F) = F \triangle [2\ell(F)]$ and $\alpha(F') = F' \triangle [2\ell(F')]$. If $\ell(F) = \ell(F')$ then $\alpha(F) \ne \alpha(F')$ is evident from $F \ne F'$.

By symmetry suppose $\ell(F) < \ell(F')$. The maximality of $\ell(F)$ implies $|F \cap [2\ell(F')]| < \ell(F')$. Using $|F \cap [2\ell(F)]| = \ell(F) = |\alpha(F) \cap [2\ell(F)]|$, it follows that $|\alpha(F) \cap [2\ell(F')]| < \ell(F') = |\alpha(F') \cap [2\ell(F')]|$. This proves $\alpha(F) \ne \alpha(F')$. □

Since $\alpha(F) = F$ for $F \in \mathcal{F} \cap \mathcal{E}(n, k, s)$, Lemmas 2.1 and 2.2 prove that α is an injection into $\mathcal{E}(n, k, s)$.

The proof of Proposition 1.6. Starting with a nontrivial intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ of maximal size, we can keep on applying the S_{ij} shift for various pairs until we run into trouble. The possible trouble is that $S_{ij}(\mathcal{F})$ ceases to be nontrivial, i.e., all its members contain the element i. Then $\{i, j\} \cap F \ne \varnothing$ must hold for all $F \in \mathcal{F}$. By symmetry let $i = 1$, $j = 2$.

The maximality of $|\mathcal{F}|$ implies that *all* k-sets G with $\{1, 2\} \subset G \subset [n]$ are in $\mathcal{F}$. Therefore continuing with the $S_{a,b}$ shift for $3 \le a < b \le n$ will never produce a trivial intersecting family. Eventually we obtain a nontrivial intersecting family $\mathcal{H}$, with $|\mathcal{H}| = |\mathcal{F}|$, such that $S_{a,b}(\mathcal{H}) = \mathcal{H}$ for all $3 \le a < b \le n$.

Consequently, both $\{1, 3, 4, \dots, k+1\}$ and $\{2, 3, 4, \dots, k+1\}$ are in $\mathcal{H}$. Since all $G \in \binom{[n]}{k}$ with $\{1, 2\} \subset G \subset [n]$ are unchanged under the shift $S_{a,b}$ for $3 \le a < b \le n$, we infer that $\binom{[k+1]}{k} \subset \mathcal{H}$.

Noting that $\binom{[k+1]}{k}$ is not affected by $S_{i,j}$ for $1 \le i < j \le n$, we can continue shifting and eventually obtain a shifted, nontrivial intersecting family of the same size. □

3. Concluding remarks

For a family $\mathcal{F} \subset 2^{[n]}$, let $\triangle(\mathcal{F})$ be its *maximum degree*, that is, $\max_i |\mathcal{F}(i)|$. Then $\gamma(\mathcal{F}) = |\mathcal{F}| - \triangle(\mathcal{F})$ is called the *diversity* of $\mathcal{F}$. With this terminology, for intersecting families $\mathcal{F}$, with $\mathcal{F} \subset \binom{[n]}{k}$, $n \ge 2k$, the Hilton–Milner theorem shows that $\gamma(\mathcal{F}) \ge 1$ implies

$$|\mathcal{F}| \le |\mathcal{E}(n, k, k)| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

In [Frankl 1987a] the author proved that $\gamma(\mathcal{F}) \ge \binom{n-s-1}{k-s}$, $3 \le s \le k$, implies $|\mathcal{F}| \le |\mathcal{E}(n, k, s)|$. Kupavskii and Zakharov [2018] gave a new proof for a stronger version of this result. It would be desirable to have a proof by injection. Let us note that for $\mathcal{F} \subset \mathcal{G}$ necessarily $\gamma(\mathcal{F}) \le \gamma(\mathcal{G})$ holds.

In the case of Theorem 1.4, we may replace $\mathcal{F}$ by another family $\mathcal{G}$, with $\mathcal{F} \subset \mathcal{G} \subset \binom{[n]}{k}$ where $\mathcal{G}$ is shifted, intersecting and all $G \in \binom{[n]}{k}$ with $[2, s+1] \subset G$ are members of $\mathcal{G}$. For such a special case Theorem 1.4 provides an injective proof. However the general case seems to be harder.

The proofs in [Frankl 1987a; Kupavskii and Zakharov 2018] rely heavily on the Kruskal–Katona theorem; see [Kruskal 1963; Katona 1968]. Therefore we feel that it would be desirable to have a proof by injection for this important result as well.

Note in proof

Hurlbert and Kamat [2018] independently gave a very similar proof in the new version of their paper.

References

[Erdős et al. 1961] P. Erdős, C. Ko, and R. Rado, "Intersection theorems for systems of finite sets", *Quart. J. Math. Oxford Ser.* (2) **12** (1961), 313–320. MR Zbl

[Frankl 1978] P. Frankl, "The Erdős–Ko–Rado theorem is true for $n = ckt$", pp. 365–375 in *Combinatorics, I: Proc. Fifth Hungarian Colloq.* (Keszthely, 1976), edited by A. Hajnal and V. T. Sós, Colloq. Math. Soc. János Bolyai **18**, North-Holland, Amsterdam, 1978. MR Zbl

[Frankl 1987a] P. Frankl, "Erdős–Ko–Rado theorem with conditions on the maximal degree", *J. Combin. Theory Ser. A* **46**:2 (1987), 252–263. MR Zbl

[Frankl 1987b] P. Frankl, "The shifting technique in extremal set theory", pp. 81–110 in *Surveys in combinatorics 1987* (New Cross, 1987), edited by C. Whitehead, London Math. Soc. Lecture Note Ser. **123**, Cambridge Univ. Press, 1987. MR Zbl

[Frankl and Füredi 1986] P. Frankl and Z. Füredi, "Nontrivial intersecting families", *J. Combin. Theory Ser. A* **41**:1 (1986), 150–153. MR Zbl

[Frankl and Tokushige 1992] P. Frankl and N. Tokushige, "Some best possible inequalities concerning cross-intersecting families", *J. Combin. Theory Ser. A* **61**:1 (1992), 87–97. MR Zbl

[Hilton and Milner 1967] A. J. W. Hilton and E. C. Milner, "Some intersection theorems for systems of finite sets", *Quart. J. Math. Oxford Ser.* (2) **18** (1967), 369–384. MR Zbl

[Hurlbert and Kamat 2018] G. Hurlbert and V. Kamat, "New injective proofs of the Erdős–Ko–Rado and Hilton–Milner theorems", *Discrete Math.* **341**:6 (2018), 1749–1754. MR Zbl

[Katona 1968] G. Katona, "A theorem of finite sets", pp. 187–207 in *Theory of graphs* (Tihany, 1966), edited by P. Erős and G. Katona, Academic, New York, 1968. MR Zbl

[Kruskal 1963] J. B. Kruskal, "The number of simplices in a complex", pp. 251–278 in *Mathematical optimization techniques*, edited by R. Bellman, Univ. of California Press, Berkeley, CA, 1963. MR Zbl

[Kupavskii and Zakharov 2018] A. Kupavskii and D. Zakharov, "Regular bipartite graphs and intersecting families", *J. Combin. Theory Ser. A* **155** (2018), 180–189. MR Zbl

[Mörs 1985] M. Mörs, "A generalization of a theorem of Kruskal", *Graphs Combin.* **1**:2 (1985), 167–183. MR Zbl

Received 9 Oct 2017.

PETER FRANKL:
peter.frankl@gmail.com
Rényi Institute, Budapest, Hungary

Moscow Journal of Combinatorics and Number Theory

msp.org/moscow

See inside back cover or msp.org/moscow for submission instructions.

The subscription price for 2019 is US $310/year for the electronic version, and $365/year (+$20, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Moscow Journal of Combinatorics and Number Theory (ISSN 2640-7361 electronic, 2220-5438 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

MJCNT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

Moscow Journal of Combinatorics and Number Theory

vol. 8 no. 2 2019