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A family of four-variable expanders with quadratic growth

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We prove that if $g(x, y)$ is a polynomial of degree d that is not a polynomial of only y, then for any finite set $A \subset \mathbb{R}$

$$|X| \gg_d |A|^2, \quad \text{where } X := \left\{ \frac{g(a_1, b_1) - g(a_2, b_2)}{b_2 - b_1} : a_1, a_2, b_1, b_2 \in A \right\}.$$

We will see this bound is also tight for some polynomial $g(x, y)$.

1. Introduction

Throughout this paper, when we write $X \gg Y$, this means that $X \geq cY$ for some absolute constant $c > 0$.

The sum set of a subset $A \subset \mathbb{R}$ is defined as $A + A := \{a + b : a, b \in A\}$. The product set is defined in a similar way, $AA := \{ab : a, b \in A\}$.

The Erdős–Szemerédi conjecture [1983] states that, for all $\epsilon > 0$ and for any finite set $A \subset \mathbb{N}$,

$$\max\{|A + A|, |AA|\} \geq c(\epsilon)|A|^{2-\epsilon}.$$

It is natural to extend this conjecture to other settings (such as $\mathbb{R}$), and also to change the polynomials $F(x, y) = x + y$ and $F(x, y) = xy$ defining the sum and product sets to other polynomials or rational functions. In recent years much research has been done in this direction.

For many such functions, the images of sets are known to always grow. For example, the authors of [Murphy et al. 2015] have studied several multivariable polynomials, including the function

$$G(x_1, x_2, x_3, x_4, x_5) = x_1(x_2 + x_3 + x_4 + x_5).$$

More precisely they showed that, for any finite set $A \subset \mathbb{R}$,

$$|A(A + A + A + A)| \gg \frac{|A|^2}{\log |A|},$$

where $A(A + A + A + A) := \{x_1(x_2 + x_3 + x_4 + x_5) : x_i \in A\}$.

In [Murphy et al. 2017], the authors studied a more complicated function, namely

$$H(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2 + x_3 + x_4)^2 + \log x_5.$$

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They showed that, for any finite $A \subset \mathbb{R}$,

$$|\{(a_1+a_2+a_3+a_4)^2+\log a_5 : a_i \in A\}| \gg \frac{|A|^2}{\log |A|}.$$

In the same circle of ideas, Balog and Roche-Newton [2015] investigated the rational function

$$F(x_1, x_2, x_3, x_4) = \frac{x_1+x_2}{x_3+x_4},$$

showing that for any finite set $A \subset \mathbb{R}$, we have

$$|F(A, A, A, A)| \geq 2|A|^2 - 1.$$

Our result is a generalization of the method of [Murphy et al. 2015, Corollary 3.1], where they used the Szemerédi–Trotter theorem to prove that for any finite set $A \subset \mathbb{R}$

$$\left|\frac{A-A}{A-A}\right| \gg |A|^2.$$

A stronger version of this result, with a multiplicative constant 1, follows from an earlier geometric result of Ungar [1982].

In this article we consider a certain class of rational functions of four variables. Suppose that $g(x, y)$ is a polynomial of two variables of degree d. Let

$$F(x_1, x_2, y_1, y_2) = \frac{g(x_1, y_1) - g(x_2, y_2)}{y_2 - y_1}$$

be a four-variable rational function in terms of x_1, x_2, y_1, y_2. The main theorem of this paper is the following result concerning the growth of F.

Theorem 1.1. *Suppose that $g(x, y)$ is a polynomial of degree d, that it is not a polynomial of only y, and that $A \subset \mathbb{R}$ is a finite set. Then*

$$|X| \gg_d |A|^2, \quad \text{where } X := \left\{\frac{g(a_1, b_1) - g(a_2, b_2)}{b_2 - b_1} : a_1, a_2, b_1, b_2 \in A\right\}.$$

Notice that the following example shows that the condition that $g(x, y)$ cannot be a polynomial of only y is necessary.

Example 1.2. Suppose that $g(x, y) = y^2$ and $A = \{1, 2, \dots, n\}$. Then

$$X = \left\{\frac{b_1^2 - b_2^2}{b_2 - b_1} : b_1, b_2 \in A\right\}$$

equals $-\{b_2 + b_1 : b_i \in A\}$ and has cardinality $O(n)$.

On the other hand, it is known that for some polynomials g, the result of Theorem 1.1 is tight. For example, if we define $g(x_1, y_1) = x_1$ then Theorem 1.1 recovers the result of [Murphy et al. 2015; Ungar

1982]. This is known to be tight, since for the set $A = \{1, \dots, N\}$,

$$\left|\frac{A-A}{A-A}\right| = O(N^2).$$

However, we are not aware of any other polynomials g for which the bound in Theorem 1.1 is tight, and whether or not the bound can be improved for some particular g is an interesting question.

Our main result has some similarities with a result of Raz, Sharir and Solymosi [Raz et al. 2015] concerning the growth of two-variable polynomials. Their result states that, if F is a two-variable polynomial with bounded degree, then for any $A, B \subset \mathbb{R}$ with $|A| = |B| = n$,

$$|F(A, B)| \gg_d n^{4/3},$$

provided that F satisfies a nondegeneracy condition. This condition states that F cannot be of one of the following forms:

(1) $F(u, v) = f(g(u) + h(v))$.

(2) $F(u, v) = f(g(u) \cdot h(v))$.

This result gave an improvement upon an earlier result of Elekes and Ronyai [2000].

The Szemerédi–Trotter theorem. The essential ingredient used to prove our result is a corollary of the Szemerédi–Trotter theorem [1983], which gives a bound for the number of lines in the plane containing at least a fixed number of points k from a given finite set, that is, the number of *k-rich* lines.

Theorem 1.3. *Let P be a finite set of points and let L be a finite set of lines. Then the number of incidences $I(P, L) := \{(p, \ell) \in P \times L : p \in \ell\}$ has the upper bound*

$$I(P, L) \ll |P|^{2/3}|L|^{2/3} + |P| + |L|.$$

More precisely,

$$I(P, L) \leq 4|P|^{2/3}|L|^{2/3} + 4|P| + |L|.$$

If each line in L appears at most d times for some constant d, then a generalization of the Szemerédi–Trotter theorem states that

$$I(P, L) \leq 4d|P|^{2/3}|L|^{2/3} + 4d|P| + d|L|.$$

The main idea of the following corollary is known in literature; we present here a slightly improved version which we could not find in the literature in the form we need.

Corollary 1.4. *Let $k, n \geq 2$ be natural numbers and fix $d \in \mathbb{N}$ such that $4d + 1 \leq k \leq d\sqrt{n}$. Let $\mathcal{L}$ be a set of n lines in the plane, and let $t_{\geq k}$ denote the number of points in the plane contained in at least k lines of $\mathcal{L}$, where each line appears with multiplicity at most d. Then*

$$t_{\geq k} = O_d\left(\frac{n^2}{k^3}\right).$$

Notice that if $\mathcal{L}$ is a set of n lines in the plane such that each line appears at most d times for some constant d, then for computing t_k, k should be greater than or equal to $4d + 1$. To see this, suppose that

P_k is the set of k-rich points. Then we have $k|P_k| \le 4d|P_k|^{2/3}|L|^{2/3} + 4d|P_k| + d|L|$. This implies

$$(k-4d)|P_k| \le 4d|P_k|^{2/3}|L|^{2/3} + d|L|.$$

Hence we may assume $k \ge 4d+1$, otherwise the inequality gives nothing.

Proof of Corollary 1.4. Let P_k be the set of k-rich points. Since each line appears at most d times we have

$$\frac{k|P_k|}{d} \ll |P_k|^{2/3}|L|^{2/3} + |L|,$$

so $k^3|P_k| \ll d^3|L|^2$ or otherwise $|P_k| \ll d|L|/k$. Plugging these bounds back into the Szemerédi–Trotter theorem gives

$$I(P_k, L) \ll |L|^{2/3}\left(\frac{d^3|L|^2}{k^3}\right)^{2/3} + |L|^{2/3}\left(\frac{d|L|}{k}\right)^{2/3} + |L| + \frac{d^3|L|^2}{k^3} + \frac{d|L|}{k}.$$

Since $k > 4d$ we can ignore last two summands and we obtain

$$I(P_k, L) \ll \frac{d^2|L|^2}{k^2} + \left(\frac{d}{k}\right)^{2/3}|L|^{4/3} + |L|.$$

Note that we have

$$\frac{d^2|L|^2}{k^2} \ge \left(\frac{d}{k}\right)^{2/3}|L|^{4/3}$$

if $k \le d\sqrt{n}$. □

2. Main results

Suppose that $A, B \subset \mathbb{R}$ are finite, and $g(x_1, y_1)$ is a polynomial of degree d. We associate an element of $A \times B$ with a line via

$$A \times B \ni (a, b) \longleftrightarrow l_{a,b} : y = bx - g(a, b).$$

Consider $\mathcal{L} = \{\ell_{a,b} : a, b \in A \times B\}$ as a multiset. Then $\mathcal{L}$ is a set of $|A||B|$ lines, such that each line appears at most d times. We also define the quantity

$$n(x, y) = |\{(a, b) \in A \times B : (x, y) \in l_{a,b}\}|,$$

which is interpreted geometrically as the number of lines of $\mathcal{L}$ that pass through (x, y).

Lemma 2.1. *Suppose that $d \in \mathbb{N}$ is fixed. Suppose that $A, B, X \subset \mathbb{R}$ are finite and satisfy*

$$|X| \le \frac{|A||B|}{4d^2},$$

with $0 \notin X$. Then

$$\sum_{x\in X}\sum_{y} n^2(x, y) \ll |A|^{3/2}|B|^{3/2}|X|^{1/2}. \tag{1}$$

Proof. The set of t-rich points is given by

$$R_t := \{(x, y) \in \mathbb{R}^2 : n(x, y) \ge t\}.$$

We first show that

$$|R_t| \ll \frac{|A|^2|B|^2}{t^3}.$$

We begin by bounding $n(x, y)$ for a given point (x, y). For fixed $b_0 \in B$ we obtain a line with slope b_0 passing through (x, y) and a one-variable polynomial equation $g(a, b_0)$. Since each line is determined uniquely, by its slope and one point on it (for fixed b_0 and (x, y) the equation $g(a, b_0) = 0$ has at most d distinct solutions), we have

$$n(x, y) \leq d|B|.$$

With a similar argument for fixed $a \in A$ we obtain a univariate polynomial equation. Since each line is determined uniquely by its y-intercept and one point on it we have

$$n(x, y) \leq d|A|.$$

These together imply

$$n(x, y) \leq d(\min\{|A|, |B|\}) \leq (d|A|d|B|)^{1/2} = d|\mathcal{L}|^{1/2}.$$

This implies there are no points incident to more than $d\sqrt{|\mathcal{L}|}$ lines in $\mathcal{L}$, and by applying Corollary 1.4 we get

$$|R_t| \ll \frac{|\mathcal{L}|^2}{t^3} \leq \frac{|A|^2|B|^2}{t^3}.$$

Let $\Delta > 2d$ be an integer to be specified later. We have

$$\sum_{x\in X}\sum_{y} n^2(x, y) \leq \sum_{x\in X}\sum_{n(x,y)\leq\Delta} n^2(x, y) + \sum_{\substack{(x,y)\\ n(x,y)>\Delta}} n^2(x, y). \tag{2}$$

The first term is bounded by $\Delta|A||B||X|$; in fact

$$\sum_{x\in X}\sum_{n(x,y)\leq\Delta} n^2(x, y) \leq \Delta\sum_{x\in X}\sum_{y} n(x, y) = \Delta|A||B|\sum_{x\in X} 1 = \Delta|A||B||X|. \tag{3}$$

For the second term we have

$$\begin{aligned}\sum_{\substack{(x,y)\\ n(x,y)>\Delta}} n^2(x, y) &= \sum_{j\geq 1}\sum_{2^{j-1}\Delta\leq n(x,y)\leq 2^j\Delta} n^2(x, y)\\ &\ll \sum_{j\geq 1}\frac{|A|^2|B|^2}{(2^j\Delta)^3}\cdot(2^j\Delta)^2 = \frac{|A|^2|B|^2}{\Delta}\sum_{j\geq 1}\frac{1}{2^j} = \frac{|A|^2|B|^2}{\Delta}. \qquad (4)\end{aligned}$$

For an optimal choice, set

$$\Delta = \left\lceil \frac{(|A||B|)^{1/2}}{|X|^{1/2}} \right\rceil > 2d.$$

Combining the bounds from (2) and (3) and (4), it follows that

$$\sum_{x}\sum_{y} n^2(x, y) \ll |A|^{3/2}|B|^{3/2}|X|^{1/2}. \qquad \square$$

Proof of Theorem 1.1. Consider

$$\begin{aligned}|Y| &= \left|\left\{(x, a_1, a_2, b_1, b_2) : x = \frac{g(a_1, b_1) - g(a_2, b_2)}{b_1 - b_2}, a_i, b_i \in A\right\}\right| \\ &= \left|\left\{(x, a_1, a_2, b_1, b_2) : b_1 x - g(a_1, b_1) = b_2 x - g(a_2, b_2)\right\}\right| \\ &= \sum_{x \in X} \sum_{y} n^2(x, y) \ll |A|^3 |X|^{1/2}.\end{aligned}$$

On the other hand, $|Y| \geq |A|^4$. Thus we obtain

$$|A|^4 \ll |A|^3 |X|^{1/2}, \quad \text{and hence} \quad |X| \gg |A|^2. \qquad \square$$

Notice that the proof of Theorem 1.1 fails when $g(x, y)$ is a polynomial of only y. In fact if $g(x, y) = h(y)$ for some polynomial h, then $\mathcal{L} = \{l_{a,b} : a, b \in A \times B\}$ is a set of $|A||B|$ lines such that each line appears at least $|A|$ times (and at most $d|A|$ times). On the other hand the generalization of the Szemerédi–Trotter theorem and its corollary hold when each line appears at most d times, where d is independent of $|P|$ and $|L|$ in Theorem 1.3.

Corollary 2.2. *Suppose that $P = A \times A$ is a set of $|A|^2$ points. Let l be the y-axis. Suppose that $B(P)$ is the set of all bisectors determined by P. Then $|B \cap l| \gg |A|^2$.*

Proof. By a simple calculation we can see that the equation of the bisector determined by two points (x_1, y_1) and (x_2, y_2) in the (s, t)-plane is

$$s = \frac{2(x_1 - x_2)t + (x_2^2 - x_1^2) + (y_2^2 - y_1^2)}{2(y_2 - y_1)}.$$

Inserting $t = 0$, the hitting point has coordinate

$$\left(0, \frac{(x_2^2 - x_1^2) + (y_2^2 - y_1^2)}{2(y_2 - y_1)}\right).$$

Setting $g(x, y) = -\frac{1}{2}(x^2 + y^2)$, we obtain the result by Theorem 1.1. $\square$

As we mentioned, this bound is tight for some polynomials, for instance $g(x, y) = x$. However, we expect that if $F(x_1, x_2, y_1, y_2)$ is a generic rational function satisfying the condition of Theorem 1.1 we have $|X| \gg |A|^3$.

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