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Generalized simultaneous approximation to m linearly dependent reals

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In order to analyse the simultaneous approximation properties of m reals, the parametric geometry of numbers studies the joint behaviour of the successive minima functions with respect to a one-parameter family of convex bodies and a lattice defined in terms of the m given reals. For simultaneous approximation in the sense of Dirichlet, the linear independence over $\mathbb{Q}$ of these reals together with 1 is equivalent to a certain nice intersection property that any two consecutive minima functions enjoy. This paper focusses on a slightly generalized version of simultaneous approximation where this equivalence is no longer in place and investigates conditions for that intersection property in the case of linearly dependent irrationals.

1. Introduction

In Diophantine approximation the simultaneous approximation to $m := n-1$ real numbers $\xi_1, \dots, \xi_m$ has a long tradition, starting with Dirichlet who proved the existence of nontrivial solutions $(x, y_1, \dots, y_m) \in \mathbb{Z}^n$ to the system

$$\begin{aligned} |x| &\le e^q, \\ |\xi_1 x - y_1| &\le e^{-q/m}, \\ &\vdots \\ |\xi_m x - y_m| &\le e^{-q/m} \end{aligned} \tag{$\star$}$$

for any parameter $q > 0$. In other words, if $\mathcal{B}(q)$ consists of points $(p_0, p_1, \dots, p_m)$ with $|p_0| \le e^q$, $|p_i| \le e^{-q/m}$ for $1 \le i \le m$, and $\Lambda = \Lambda(\boldsymbol{\xi})$ the lattice of points $(x, \xi_1 x - y_1, \dots, \xi_m x - y_m)$ with $(x, y_1, \dots, y_m) \in \mathbb{Z}^n$, Dirichlet's theorem asserts that there is a nonzero lattice point in $\mathcal{B}(q)$, i.e., that the first minimum $\lambda_1(q)$ with respect to $\mathcal{B}(q)$ and Λ is at most 1.

Lately, the successive minima functions $\lambda_1(q), \dots, \lambda_n(q)$ have been intensively studied within the framework of parametric geometry of numbers, culminating in a fundamental paper of D. Roy [2015] in which he reduces the problem of describing the joint spectrum of a family of exponents of Diophantine approximation relative to ($\star$) to combinatorial analysis. A main tool for the investigation of the successive minima functions is the following result from [Schmidt and Summerer 2009]:

Proposition 1.1. *Suppose $1, \xi_1, \dots, \xi_m$ are linearly independent over $\mathbb{Q}$ and let $\lambda_i(q)$ denote the successive minima with respect to $\Lambda(\boldsymbol{\xi})$ and $\mathcal{B}(q)$. Then for every $s < n$ there exist arbitrarily large values of q for which $\lambda_s(q) = \lambda_{s+1}(q)$.*

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An analogous result holds in the more general situation where a system of exponents $(\nu_0, -\nu_1, \dots, -\nu_m)$ with $\nu_i > 0$ for $1 \le i \le m$ and $\nu_0 - \nu_1 - \cdots - \nu_m = 0$ is considered (see [Schmidt and Summerer 2009], page 72, Corollary 2.2). Here we normalize to the case $\nu_0 = 1$ so that $\nu_1 + \cdots + \nu_m = 1$ and denote by $\mathcal{B}^\nu(q)$ the box of points $(p_0, p_1, \dots, p_m)$ defined by $|p_0| \le e^q$, $|p_i| \le e^{-\nu_i q}$ for $1 \le i \le m$. This modifies the initial system to

$$\begin{aligned} |x| &\le e^q, \\ |\xi_1 x - y_1| &\le e^{-\nu_1 q}, \\ &\vdots \\ |\xi_m x - y_m| &\le e^{-\nu_m q}. \end{aligned} \tag{$\star\star$}$$

When $A = \{i_1 < \cdots < i_s\} \subseteq \{1, \dots, m\}$, let $\pi_A : \mathbb{R}^n \to \mathbb{R}^s$ be the map with

$$\pi_A((p_0, p_1, \dots, p_m)) = (p_{i_1}, \dots, p_{i_s}) \in \mathbb{R}^s.$$

Proposition 1.1 and its generalization to successive minima with respect to $\Lambda(\boldsymbol{\xi})$ and $\mathcal{B}^\nu(q)$ were proved in [Schmidt and Summerer 2009] by showing that the assumption of Theorem 1.1, page 69 of that paper is fulfilled for $\Lambda(\xi)$ and $\mathcal{B}^\nu(q)$ if $1, \xi_1, \dots, \xi_m$ are linearly independent over $\mathbb{Q}$. For the convenience of the reader we state this result here in the present notation:

Theorem 1.2. *Suppose for every s-dimensional space S spanned by lattice points (i.e., points of Λ), there is some $A \subseteq \{1, \dots, m\}$ of cardinality s with $\pi_A(S) = \mathbb{R}^s$. Then there are arbitrarily large values of q with $\lambda_s(q) = \lambda_{s+1}(q)$.*

The question of whether the condition in Theorem 1.2 and the condition of linear independence of $1, \xi_1, \dots, \xi_m$ in Proposition 1.1 are also necessary to guarantee that for given s we have arbitrarily large values of q with $\lambda_s(q) = \lambda_{s+1}(q)$ (in the cases ($\star$) and ($\star\star$)) was the major motivation for the subsequent investigations. Regarding the set of exponents, we will without loss of generality suppose that

$$0 < \nu_1 \le \nu_2 \le \cdots \le \nu_m \tag{1-0}$$

in addition to $\nu_1 + \cdots + \nu_m = 1$.

It will follow from our exposition that in the standard simultaneous approximation case ($\star$) where $\nu_i = 1/m$ we have $\lambda_{n-1}(q) = \lambda_n(q)$ for some arbitrarily large q if and only if the linear independence condition is satisfied, in particular:

Corollary 1.3. *Suppose $\xi_1, \xi_2, \dots, \xi_m$ are real numbers with $\xi_k = \xi_{k+1}$ for some $1 \le k \le m$, and $\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_m$ together with 1 are linearly independent over $\mathbb{Q}$, and let $\lambda_i(q)$, $1 \le i \le n$, denote the successive minima with respect to $\Lambda(\boldsymbol{\xi})$ and $\mathcal{B}(q)$. Then $\lambda_{n-1}(q) < \lambda_n(q)$ for all sufficiently large q.*

On the other hand, if $\xi_k = \xi_{k+1}$ and $\mathcal{B}(q)$ is replaced by $\mathcal{B}^\nu(q)$ with ν_m sufficiently large compared to ν_{m-1}, the situation may be different. In fact, for $\xi_1 = \xi_2$ in the three-dimensional case (i.e., $m = 2$) we will give a bound for ν_2 that guarantees $\lambda_2(q) = \lambda_3(q)$ for some arbitrarily large q in Section 4. All these particular cases of simultaneous approximation to linearly dependent reals fit in the general situation where for some k and r the real numbers $\xi_k, \xi_{k+1}, \dots, \xi_{k+r-1}$ are linear combinations of $1, \xi_{k+r}, \dots, \xi_m$ with rational coefficients. For this setting we will state conditions that guarantee that $\lambda_{n-r}(q) = \lambda_{n-r+1}(q)$ in Section 2. The proof of this result will be given in Section 3.

2. Basic notation and statement of the main result

We fix some exponents $(1, -\nu_1, \dots, -\nu_m)$ with $\nu_1 + \dots + \nu_m = 1$ satisfying (1-0) in ($\star\star$) and write $\mathcal{B}(q)$ briefly for the body introduced as $\mathcal{B}^\nu(q)$ in the Introduction. Moreover we choose $r \in \{1, \dots, m-1\}$ and $k \in \{1, \dots, m-1-r\}$, set $s := n - r$ and define the sets $B := \{k, \dots, k+r-1\}$, $C := \{0, 1, \dots, m\} \setminus B$, $D := \{0, k+r, \dots, m\}$ with cardinalities

$$|B| = r, \quad |C| = s, \quad |D| = s - k + 1,$$

as well as $C' := C \setminus \{0\}$, $D' := D \setminus \{0\}$. Also let

$$\nu_B := \sum_{i \in B} \nu_i, \quad \nu_{C'} := \sum_{i \in C'} \nu_i,$$

so that $\nu_B + \nu_{C'} = 1$.

We will now consider the case of linearly dependent components ξ_i, more precisely the case where

$$\xi_j = \mathcal{L}_j(1, \xi_1, \dots, \xi_m) \quad \text{for } j \in B, \tag{2-0}$$

with r linear forms

$$\mathcal{L}_j(p_0, p_1, \dots, p_m) = \sum_{i \in D} c_i^{(j)} p_i$$

with rational coefficients $c_i^{(j)}$ so that $\xi_j = c_0^{(j)} + \sum_{i \in D'} c_i^{(j)} \xi_i$. Further put

$$c^{(j)} := \sum_{i \in D} |c_i^{(j)}| \quad \text{as well as} \quad c := \max\bigl(1, \max_{j \in B} c^{(j)}\bigr),$$

and let d be the least common denominator of the $c_i^{(j)}$ with $j \in B$, $i \in D$. Note that d as well as c depend only on the coefficients of the system (2-0).

To any m-tuple $(\xi_1, \dots, \xi_m)$ we had already associated the lattice $\Lambda = \Lambda(\boldsymbol{\xi})$ of points $p(\boldsymbol{x}) := (x, \xi_1 x - y_1, \dots, \xi_m x - y_m)$, with $\boldsymbol{x} := (x, y_1, \dots, y_m) \in \mathbb{Z}^n$, and the successive minima $\lambda_1(q), \dots, \lambda_n(q)$ with respect to $\mathcal{B}(q)$. We will write $L_i(q) = \log(\lambda_i(q))$ for $i = 1, \dots, n$ so that by Minkowski's second theorem

$$L_1(q) + \dots + L_n(q) \le 0. \tag{2-1}$$

Now let S be the s-dimensional subspace of $\mathbb{R}^n$ spanned by the lattice points with $y_j = \mathcal{L}_j(x, y_1, \dots, y_m)$ for $j \in B$. Further we write S^C for the s-dimensional space of points with coordinates η_i, where $i \in C$, and let $\Lambda^C \subseteq S^C$ denote the s-dimensional lattice $\pi_C(\Lambda)$ consisting of points

$$(x, \xi_1 x - y_1, \dots, \xi_{k-1} x - y_{k-1}, \xi_{k+r} x - y_{k+r}, \dots, \xi_m x - y_m),$$

with $(x, y_1, \dots, y_{k-1}, y_{k+r}, \dots, y_m) \in \mathbb{Z}^s$. Let $\mathcal{B}^C(q) \subseteq S^C$ be the box with

$$|\eta_0| \le e^q, \qquad |\eta_i| \le e^{-\nu_i q} \quad (i \in C').$$

This box has volume $2^s e^{q - \nu_C q} = 2^s e^{\nu_B q}$. We will also need the successive minima $\lambda_j^C(q)$ as well as their logarithms $L_j^C(q)$, $1 \le j \le s$, that are defined in terms of $\mathcal{B}^C(q)$ and Λ^C. Minkowski's second theorem then implies

$$-\nu_B q - n \log n < L_1^C(q) + \dots + L_s^C(q). \tag{2-2}$$

Note that in the present situation the condition of Theorem 1.2 is not fulfilled for the s-dimensional subspace S defined above. In fact, for any $A \subset \{1, \dots, m\}$ of cardinality s we have $|A^c| = r$ and A^c contains 0. Now S is the span of lattice points with $y_j = \mathcal{L}_j(x, y_1, \dots, y_m)$ for $j \in B$ and in view of (2-0) these lattice points have

$$\xi_j x - y_j = \mathcal{L}_j(0, x\xi_1 - y_1, \dots, x\xi_m - y_m), \quad j \in B.$$

This may be interpreted as a system of r linear equations among the $p_i = x\xi_i - y_i$, with $i \in B \cup D'$. As $0 \notin B \cup D'$, at most $r-1$ of these indices are not in A. It follows that the p_i with $i \in (B \cup D') \cap A$ satisfy at least $r - (r-1)$ linear relations; hence the projection $\pi_A : S \to \mathbb{R}^s$ is not surjective.

However it will turn out that the condition is not necessary for the conclusion $\lambda_s(q) = \lambda_{s+1}(q)$ for arbitrarily large q. More precisely we will show:

Theorem 2.1. *Let $\xi_1, \xi_2, \dots, \xi_m$ be real numbers satisfying (2-0) and $s = n - r$ as already defined.*

(a) *The relation*

$$L_s^C(q) \le \nu_k q - \log c - 2\log d - 1 \tag{2-3}$$

implies $L_s(q) < L_{s+1}(q)$. If (2-3) holds for every large q, and $\{\xi_i : i \in C'\}$ together with 1 are linearly independent over $\mathbb{Q}$, then for each $j < s$ there are arbitrarily large values of q with $L_j(q) = L_{j+1}(q)$.

(b) *Assume that (2-3) is fulfilled for certain arbitrarily large q and that for some (other) arbitrarily large q we have*

$$L_s^C(q) \ge \nu_B q + n^2. \tag{2-4}$$

Then there exist arbitrarily large q with $L_s(q) = L_{s+1}(q)$.

In the special case where (2-0) is reduced to

$$\xi_k = \cdots = \xi_{k+r}, \tag{2-5}$$

we have $\mathcal{L}_j(1, \xi_1, \dots, \xi_m) = \xi_{k+r}$ so that $c_{k+r}^{(j)} = 1$ for $j = k, \dots, k+r-1$ and all other coefficients are zero so that obviously $c = d = 1$. As (2-5) clearly implies $\xi_{k+l} = \cdots = \xi_{k+r}$ for any $l \in \{1, \dots, r\}$, we may as well apply the above results with $\widetilde{B} := \{k+l, \dots, k+r-1\}$ and $\widetilde{C} := \{0, 1, \dots, m\} \setminus \widetilde{B}$. In this way we see that the relation

$$L_{s+l}^{\widetilde{C}}(q) \le \nu_{k+l} q - 1 \tag{2-6}$$

implies $L_{s+l}(q) < L_{s+l+1}(q)$ and that the fact (2-6) is fulfilled for certain arbitrarily large q together with

$$L_s^{\widetilde{C}}(q) \ge \nu_{\widetilde{B}} q + n^2 \tag{2-7}$$

for some other arbitrarily large q guarantees that there exist arbitrarily large q with $L_{s+l}(q) = L_{s+l+1}(q)$.

These results highlight the interest of considering parametric geometry of numbers in a more general context than the classical simultaneous approximation problem as initiated in [Schmidt and Summerer 2009] and investigated in much more detail in [Schmidt $\ge$ 2019].

3. Deduction of Theorem 2.1

Assume that (2-0) holds for $\xi_1, \xi_2, \dots, \xi_m$ and keep all notation as introduced in Section 2. For points $p(\boldsymbol{x})$ in $\Lambda \cap S$ with $\pi_C(p(\boldsymbol{x})) \in \mathcal{B}^C(q)$ we get for $j \in B$

$$\begin{aligned}
|\xi_j x - y_j| &= |\mathcal{L}_j(1, \xi_1, \dots, \xi_m)x - \mathcal{L}_j(x, y_1, \dots, y_m)| \\
&\le |c^{(j)}_{k+r}||\xi_{k+r}x - y_{k+r}| + \dots + |c^{(j)}_m||\xi_m x - y_m| \\
&\le |c^{(j)}_{k+r}|e^{-\nu_{k+r}q} + \dots + |c^{(j)}_m|e^{-\nu_m q} \\
&\le c^{(j)}e^{-\nu_{k+r}q} \\
&\le c^{(j)}e^{-\nu_j q}
\end{aligned} \tag{3-0}$$

for large q in view of (1-0). Hence by the definition of c we have $p(\boldsymbol{x}) \in c\mathcal{B}(q)$. So if $\lambda\mathcal{B}^C(q)$ contains s linearly independent points of Λ^C, then $c\lambda\mathcal{B}(q)$ contains s linearly independent points $p(\boldsymbol{x})$ where $\boldsymbol{x} \in d^{-1}\mathbb{Z}^n$ and thus $dc\mathcal{B}(q)$ contains s linearly independent points $p(\boldsymbol{x})$ of $\Lambda \cap S$. It follows that $\lambda_s(q) \le dc\lambda^C_s(q)$ and consequently

$$L_s(q) \le L^C_s(q) + \log c + \log d. \tag{3-1}$$

In combination with (2-3) that we assume in (a), (3-1) yields

$$L_s(q) \le \nu_k q - \log d - 1. \tag{3-2}$$

On the other hand, points in Λ outside S have $y_{j_0} \neq \mathcal{L}_j(x, y_1, \dots, y_m)$ for at least one $j_0 \in B$, so that $|\mathcal{L}_j(x, y_1, \dots, y_m) - y_{j_0}| \ge d^{-1}$. This implies

$$\begin{aligned}
|\xi_{j_0}x - y_{j_0}| &= |\mathcal{L}_{j_0}(1, \xi_1, \dots, \xi_m)x - y_{j_0}| \\
&= |\mathcal{L}_{j_0}(1, \xi_1, \dots, \xi_m)x - \mathcal{L}_{j_0}(x, y_1, \dots, y_m) + \mathcal{L}_{j_0}(x, y_1, \dots, y_m) - y_{j_0}| \\
&\ge |\mathcal{L}_{j_0}(x, y_1, \dots, y_m) - y_{j_0}| - |\mathcal{L}_{j_0}(1, \xi_1, \dots, \xi_m)x - \mathcal{L}_{j_0}(x, y_1, \dots, y_m)| \\
&\ge d^{-1} - c^{(j_0)}e^{-\nu_{j_0}q}
\end{aligned}$$

and hence $|\xi_{j_0}x - y_{j_0}| \ge d^{-1} - ce^{-\nu_k q}$ by the definition of c and (1-0). Denoting by $\lambda_{\boldsymbol{x}}(q)$ the least $\lambda > 0$ with $p(\boldsymbol{x}) \in \lambda\mathcal{B}(q)$ and writing $L_{\boldsymbol{x}}(q) = \log \lambda_{\boldsymbol{x}}(q)$, we thus have

$$\lambda_{\boldsymbol{x}}(q) = \inf_{\boldsymbol{x} \in \lambda\mathcal{B}(q)} \lambda \ge d^{-1}e^{\nu_k q} - c$$

for $p(\boldsymbol{x}) \in \Lambda \setminus S$, so that any lattice point outside S has

$$L_{\boldsymbol{x}}(q) > \nu_k q - \log d - 1 \tag{3-3}$$

for sufficiently large q, so that certainly

$$L_{s+1}(q) > \nu_k q - \log d - 1. \tag{3-4}$$

Together (3-2) and (3-4) imply $L_s(q) < L_{s+1}(q)$, i.e., the first assertion of (a).

To prove the second assertion of (a) and part (b) we introduce the function

$$G(q) := \min_{\boldsymbol{x}\in\Lambda\setminus S} L_{\boldsymbol{x}}(q),$$

which by (3-3) satisfies

$$G(q) > \nu_k q - \log d - 1, \tag{3-5}$$

and is continuous and piecewise linear. In particular, for those q for which (2-3) holds we have $L_s^C(q) < G(q)$ and thus

$$L_j(q) = L_j^C(q) \tag{3-6}$$

for all $j \le s$.

Now assume that (2-3), hence (3-6), holds for all large q. If $\{\xi_i : i \in C'\}$ together with 1 are linearly independent over $\mathbb{Q}$ then Proposition 1.1 applied to simultaneous approximation of $\{\xi_i : i \in C'\}$, i.e., successive minima defined with respect to Λ^C and $\mathcal{B}^C(q)$, implies the existence of arbitrarily large q with $L_j^C(q) = L_{j+1}^C(q)$ for any $j < s$. In combination with (3-6) the second assertion of (a) follows.

In general, given any q, at least one of $L_1(q), \dots, L_{s+1}(q)$ will stem from a point $p(\boldsymbol{x})$ outside S, say $L_l(q) = L_{\boldsymbol{x}}(q)$ with $p(\boldsymbol{x}) \notin S$, where l is chosen minimal subject to this property. Note that the definition of l implies that (3-6) now holds for $i = 1, \dots, l-1$.

If $l = s+1$, it follows from (2-2) that

$$L_1^C(q) + \cdots + L_s^C(q) > -\nu_B q - n^2 \tag{3-7}$$

and by the definition of G combined with (2-4)

$$L_{s+1}(q) = G(q) > L_s^C(q) > \nu_B q + n^2 \tag{3-8}$$

holds for certain arbitrarily large $q = q_0$. Together (3-6)–(3-8) would imply

$$L_1(q_0) + \cdots + L_{s+1}(q_0) > 0,$$

and as $0 < L_{s+1}(q_0) \le L_{s+2}(q_0) + \cdots + L_n(q_0)$ this would contradict (2-1).

If $l \le s$ then (2-1) yields

$$L_1(q) + \cdots + L_{l-1}(q) + (n-l+1)G(q) \le 0,$$

which can be rephrased as

$$\begin{aligned}
(n-l+1)G(q) &\le -L_1(q) - \cdots - L_{l-1}(q) \\
&= -L_1^C(q) - \cdots - L_{l-1}^C(q) && \text{(by (3-6))} \\
&< L_l^C(q) + \cdots + L_s^C(q) + \nu_B q + n^2 && \text{(by (2-2))} \\
&\le (s+1-l)L_s^C(q) + \nu_B q + n^2.
\end{aligned}$$

For $q = q_0$ with (2-4) this yields $(n-l+1)G(q) \le (s-l+2)L_s^C(q_0)$; therefore

$$G(q_0) < \frac{s-l+2}{n-l+1} L_s^C(q_0) \le L_s^C(q_0)$$

for some arbitrarily large q_0 since $s \le n-1$ by definition. By assumption there are also arbitrarily large q_1 with (2-3) for which we have $L_s^C(q_1) < G(q_1)$, as already noticed. Since L_s^C as well as G are continuous, there will be some q in (q_0, q_1) with

$$L_s^C(q) = G(q). \tag{3-9}$$

Since S has dimension s, we have $L_{s+1}(q) \ge G(q)$ for every q. There are s linearly independent lattice points $p(\boldsymbol{x})$ in S with $L_{\boldsymbol{x}}(q) \le L_s^C(q)$, as well as a lattice point $\boldsymbol{x} \notin S$ with $L_{\boldsymbol{x}}(q) = G(q)$, so that by (3-9) we have $L_{s+1}(q) \le G(q)$; hence $L_{s+1}(q) = G(q)$. Also there are fewer than s independent lattice points $p(\boldsymbol{x})$ with $L_{\boldsymbol{x}}(q) < L_s^C(q)$ so that $L_s(q) = L_s^C(q)$. Therefore $L_s(q) = L_{s+1}(q)$; hence (b) is proved.

4. Another version of Theorem 2.1

In order to apply Theorem 2.1 it is essential to be able to check whether the conditions (2-3) and (2-4) are fulfilled for the given ξ_i and the given exponents. For this purpose, let us first replace the functions $L_s^C(q)$ defined with respect to $\mathcal{B}^C(q)$ by functions $\hat{L}_s^C(q)$ defined with respect to a set $\hat{\mathcal{B}}^C(q)$ of volume 2^s.

Define ρ and σ by

$$\rho(s-\nu_B) = s \quad \text{and} \quad \sigma = \rho - 1. \tag{4-0}$$

For $i \in C$ set $\mu_i := \rho\nu_i + \sigma$ so that

$$\begin{aligned}\sum_{i\in C}\mu_i &= \rho\nu_C + (s-1)\sigma \\ &= \rho(1-\nu_B+s-1)+1-s = \rho(s-\nu_B)-s+1 = 1\end{aligned}$$

by (4-0). The box $\hat{\mathcal{B}}^C(q)$ is now defined by

$$|\eta_0| \le e^q, \qquad |\eta_i| \le e^{-\mu_i q} \quad (i \in C'),$$

which may also be written as

$$|\eta_0| \le e^{-\sigma q + \rho q}, \qquad |\eta_i| \le e^{-\sigma q - \rho\nu_i q} \quad (i \in C').$$

Thus $\hat{\mathcal{B}}^C(q)$ is $e^{-\sigma q}\mathcal{B}^C(\rho q)$. The corresponding quantities $\hat{L}_j^C(q)$ for $1 \le j \le s$ have

$$\hat{L}_j^C(q) = \sigma q + L_j^C(\rho q).$$

Therefore (2-3) becomes

$$\begin{aligned}\hat{L}_s^C(q) &\le \sigma q + \rho\nu_k q - \log c - 2\log d - 1 \\ &= (\rho(1+\nu_k)-1)q - \log c - 2\log d - 1 \\ &= \frac{s\nu_k+\nu_B}{s-\nu_B}q - \log c - 2\log d - 1.\end{aligned}$$

Moreover (2-4) becomes

$$\hat{L}_s^C(q) \ge \sigma q + \rho\nu_B q + n^2 = (\rho(1+\nu_B)-1)q + n^2 = \frac{(s+1)\nu_B}{s-\nu_B}q + n^2.$$

We may thus rewrite Theorem 2.1 as:

Corollary 4.1. *Let $\xi_1, \xi_2, \dots, \xi_m$ be real numbers satisfying (2-0).*

(a) *The relation*

$$\hat{L}_s^C(q) \leq \frac{s\nu_k + \nu_B}{s - \nu_B} q - \log c - 2\log d - 1 \tag{4-1}$$

implies $L_s(q) < L_{s+1}(q)$. If (4-1) holds for every large q, and $\{\xi_i : i \in C'\}$ together with 1 are linearly independent over $\mathbb{Q}$, then for each $j < s$ there are arbitrarily large values of q with $L_j(q) = L_{j+1}(q)$.

(b) *Assume that (4-1) is fulfilled for certain arbitrarily large q and that for some (other) arbitrarily large q we have*

$$\hat{L}_s^C(q) \geq \frac{(s+1)\nu_B}{s - \nu_B} q + n^2. \tag{4-2}$$

Then there exist arbitrarily large q with $L_s(q) = L_{s+1}(q)$.

In this reformulation of the main result, the conditions to check, i.e., (4-1) and (4-2), are concerned with the functions $\hat{L}_i^C(q)$, whose behaviour is rather well understood in the case where they stem from a classical simultaneous approximation problem in lower dimension, hence when all $\mu_i,\ i \in C$ are equal, which amounts to all $\nu_i,\ i \in C$, are equal.

In particular, when all ν_i are equal this leads to the deduction Corollary 1.3: (2-0) reduces to the equation $\xi_k = \xi_{k+1}$, which is of the form (2-5) and we have $B = \{k\}$; hence $C' = \{1, \dots, k-1, k+1, \dots, m\}$ and thus $s = n - 1 = m$. Moreover in the case of classical simultaneous approximation one has $\nu_i = 1/m$ for $i = 1, \dots, m$ so that relation (4-1) reads

$$\hat{L}_m^C(q) \leq \frac{1 + 1/m}{m - 1/m} q - 1 = \frac{1}{m-1} q - 1. \tag{4-3}$$

We claim that this relation holds for all sufficiently large q, so that assertion (a) of Corollary 4.1 yields $L_m(q) = L_{n-1}(q) < L_n(q)$ for all large q. Indeed for the simultaneous approximation of $m-1$ linearly independent reals, here these are $\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_m$, one always has $\hat{L}_m^C(q) < q/(m-1) - g(q)$ for some function g tending to infinity (see [Schmidt and Summerer 2009], page 77, equation (4.9)), which implies (4-3).

Our next example deals with a case where not all the ν_i are identical and shows the existence of $\xi_1, \dots, \xi_m$ and exponents $\nu_1, \dots, \nu_m$ for which the intersection properties of the successive minima functions with respect to $\mathcal{B}^\nu(q)$ differ from those with respect to $\mathcal{B}(q)$.

We consider the case $m = 2$ of simultaneous approximation to (ξ, ξ), where ξ is an irrational number with $\omega(\xi) > 1$. Here $\omega(\xi)$ is the supremum of all η such that there are arbitrarily large values of Q for which $|\xi x - y| \leq Q^{-\eta}$ has a nontrivial integer solution (x, y) with $|x| \leq Q$. Then the (single) approximation constant

$$\bar{\varphi}_2(\xi) = \frac{\omega - 1}{\omega + 1}$$

(as defined in [Schmidt and Summerer 2013], page 3) has $\bar{\varphi}_2(\xi) > 0$. By Corollary 1.3 applied in the case $\xi_1 = \xi_2 = \xi$, i.e., for classical simultaneous approximation to (ξ, ξ), we have $\lambda_1(q) = \lambda_2(q)$ for some arbitrarily large q since ξ is irrational, whereas $\lambda_2(q) < \lambda_3(q)$ for all sufficiently large q.

We claim that this will not be the case for approximation relative to exponents (ν_1, ν_2) provided ν_2 is sufficiently large.

Corollary 4.2. *Let ξ be an irrational number with $\bar{\varphi}_2(\xi) > 0$ and let (ν_1, ν_2) be a system of exponents with*

$$\nu_2 > \frac{3 - \bar{\varphi}_2(\xi)}{3 + \bar{\varphi}_2(\xi)}.$$

Then for $s \in \{1, 2\}$ there exist arbitrarily large $q = q(s)$ with $L_s(q) = L_{s+1}(q)$.

Proof. For $s = 1$ this is clear by the irrationality of ξ. So let $s = 2$ and apply Corollary 4.1 with $B = \{1\}$ and $C = \{2\}$ so that $s = 2$ and $\nu_B = 1 - \nu_2$. Note that by the definition of $\bar{\varphi}_2(\xi)$ and $\hat{\mathcal{B}}^C(q)$ we have $\limsup_{q\to\infty} \hat{L}_2^C(q)/q = \bar{\varphi}_2(\xi)$.

Moreover $c = d = 1$ so that (4-1) reads

$$\hat{L}_2^C(q) \le \frac{3 - 3\nu_2}{1 + \nu_2} q - 1,$$

which is certainly fulfilled for some arbitrarily large q as $3 - 3\nu_2 > 0$ and $\liminf_{q\to\infty} \hat{L}_2^C(q)/q = 0$ for single approximation.

On the other hand (4-2) becomes

$$\hat{L}_2^C(q) \ge \frac{3 - 3\nu_2}{1 + \nu_2} q + n^2,$$

which is fulfilled for certain arbitrarily large q provided

$$\frac{3 - 3\nu_2}{1 - +nu_2} < \bar{\varphi}_2(\xi) \quad \Longleftrightarrow \quad \nu_2 > \frac{3 - \bar{\varphi}_2(\xi)}{3 + \bar{\varphi}_2(\xi)}.$$

So part (b) of Corollary 4.1 implies $L_2(q) = L_3(q)$ for some arbitrarily large q as desired. □

It remains to say a few words on the case where the ν_i, $i \in C$, are distinct. Then the μ_i will be as well and it is not clear how to check conditions (4-1) and (4-2) when the functions $\hat{L}_s^C(q)$ do not stem from classical simultaneous approximation. However in [Schmidt $\ge$ 2019] a very precise description of the possible behaviour of the successive minima functions defined with respect to $\Lambda(\xi)$ and $\mathcal{B}^\nu(q)$ is sketched. In order to show the existence of real numbers for which those successive minima functions follow a prescribed behaviour, an appropriate analogue of Roy's results [2015, Theorem 1.3, Corollary 1.4] for generalized systems of exponents would be needed. This would considerably broaden the range of applications of the results in this paper.

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