Moscow Journal of Combinatorics and Number Theory

2019
vol. 8 no. 4

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msp

Moscow Journal of Combinatorics and Number Theory
Vol. 8, No. 4, 2019
dx.doi.org/10.2140/moscow.2019.8.325

Generalized Beatty sequences and complementary triples

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A generalized Beatty sequence is a sequence V defined by $V(n) = p\lfloor n\alpha\rfloor + qn + r$, for $n = 1, 2, \ldots$, where α is a real number, and p, q, r are integers. Such sequences occur, for instance, in homomorphic embeddings of Sturmian languages in the integers.

We consider the question of characterizing pairs of integer triples (p, q, r), (s, t, u) such that the two sequences $V(n) = (p\lfloor n\alpha\rfloor + qn + r)$ and $W(n) = (s\lfloor n\alpha\rfloor + tn + u)$ are complementary (their image sets are disjoint and cover the positive integers). Most of our results are for the case that α is the golden mean, but we show how some of them generalize to arbitrary quadratic irrationals.

We also study triples of sequences $V_i = (p_i\lfloor n\alpha\rfloor + q_i n + r_i)$, $i = 1, 2, 3$ that are complementary in the same sense.

1. Introduction

A Beatty sequence is the sequence $A = (A(n))_{n\geq 1}$, with $A(n) = \lfloor n\alpha\rfloor$ for $n \geq 1$, where α is a positive real number. What Beatty observed is that when $B = (B(n))_{n\geq 1}$ is the sequence defined by $B(n) = \lfloor n\beta\rfloor$, with α and β satisfying

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \tag{1}$$

then A and B are *complementary* sequences, that is, the sets $\{A(n) : n \geq 1\}$ and $\{B(n) : n \geq 1\}$ are disjoint and their union is the set of positive integers. In particular if $\alpha = \varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, this gives that the sequences $(\lfloor n\varphi\rfloor)_{n\geq 1}$ and $(\lfloor n\varphi^2\rfloor)_{n\geq 1}$ are complementary.

Carlitz, Scoville and Hoggatt [Carlitz et al. 1972] studied the monoid generated by $A = (A(n))_{n\geq 1}$ and $B = (B(n))_{n\geq 1}$ for the composition of sequences in the case where α is the golden ratio. (The composition of two integer sequences $U = (U(n))_{n\geq 1}$ and $V = (V(n))_{n\geq 1}$ is the sequence $UV := U \circ V = (U(V(n)))_{n\geq 1}$, so that the monoid generated by A and B is composed of sequences like $A^k B^j A^\ell \ldots$, where $A^k = AA\ldots A$ is the composition of k copies of A.)

Theorem 1 [Carlitz et al. 1972, Theorem 13, p. 20]. *Let $U = (U(n))_{n\geq 1}$ be a composition of copies of $A = (\lfloor n\varphi\rfloor)_{n\geq 1}$ and $B = (\lfloor n\varphi^2\rfloor)_{n\geq 1}$, containing i occurrences of A and j occurrences of B. Then for all $n \geq 1$*

$$U(n) = F_{i+2j} A(n) + F_{i+2j-1}\, n - \lambda_U,$$

where F_k are the Fibonacci numbers ($F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$) and λ_U is a constant.

MSC2010: 11B83, 11B85, 11D09, 11J70, 68R15.
Keywords: generalized Beatty sequences, complementary pairs and triples, morphic words, return words, Kimberling transform.

Definition 1. A *generalized Beatty sequence* is any sequence V of the type $V(n) = p(\lfloor n\alpha \rfloor) + qn + r$, $n \geq 1$, where α is a real number and p, q, and r are integers.

Two examples of generalized Beatty sequences are $U = AA$ and $U = AB$, where Theorem 1 gives $AA(n) = A(n) + n - 1$, and $AB(n) = 2A(n) + n$. These two formulas directly imply the following result.

Corollary 2. *Let V be a generalized Beatty sequence given by $V(n) = p(\lfloor n\varphi \rfloor) + qn + r$, $n \geq 1$. Then VA and VB are generalized Beatty sequences with parameters $(p_{VA}, q_{VA}, r_{VA}) = (p+q,\ p,\ r-p)$ and $(p_{WA}, q_{WA}, r_{WA}) = (2p+q,\ p+q,\ r)$.*

As an extension of Beatty's observation the following natural questions can be asked.

Question A. Let α be an irrational number, and let A defined by $A(n) = \lfloor n\alpha \rfloor$ for $n \geq 1$ be the Beatty sequence of α. Let Id defined by $\mathrm{Id}(n) = n$ be the identity map on the integers. For which pairs of integer triples (p, q, r) (s, t, u) are the two sequences $V = pA + q\,\mathrm{Id} + r$ and $W = sA + t\,\mathrm{Id} + u$ complementary?

Question B. For which triples of integer triples (p_1, q_1, r_1), (p_2, q_2, r_2), (p_3, q_3, r_3) are the sequences

$$V_i = p_i A + q_i\,\mathrm{Id} + r_i,\ i = 1, 2, 3,$$

a complementary triple? That is, when do they determine disjoint sets whose union is the positive integers? (Further, when is this partition "nice", or, in the terminology of [Fraenkel 1994], a *nice integer disjoint covering system*?)

Remark 3. The theorem of Carlitz, Scoville and Hoggatt above was rediscovered by Kimberling [2008, Theorem 5, p. 3], to whom it is attributed in, e.g., [Fraenkel 2010a, p. 575; Fraenkel 2010b, p. 647; Larsson et al. 2015, p. 20–21]. This was corrected in [Ballot 2017, Theorem 2, p. 2]. Theorem 1 in [Griffiths 2015] is also a special case of the same theorem of Carlitz et al.

Remark 4. Different generalizations of Beatty sequences are considered in [Mercer 1978; Artstein-Avidan et al. 2008; Kimberling 2011; Hildebrand et al. 2018].

Remark 5. One can ask whether the monoid generated by other complementary sequences by composition can be written as a subset of the set of linear combinations of a finite number of elements. Some answers for Beatty sequences can be found in a rich paper of Fraenkel [1994] (see, e.g., p. 645). Another, possibly unexpected, example is given by the Thue–Morse sequence. Namely, calling odious (resp. evil) the integers whose binary expansion contains an odd (resp. even) number of 1's, it was proved in [Allouche et al. 2016, Corollaries 1 and 3] that the sequences $(A(n))_{n\geq 0}$ and $(B(n))_{n\geq 0}$ of odious and evil numbers satisfy for all n

$$A(n) = 2n + 1 - t(n), \qquad B(n) = 2n + t(n), \qquad A(n) - B(n) = 1 - 2t(n),$$
$$A(A(n)) = 2A(n), \qquad B(B(n)) = 2B(n), \qquad A(B(n)) = 2B(n) + 1, \qquad B(A(n)) = 2A(n) + 1,$$

where $(t(n))_{n\geq 0}$ is the Thue–Morse sequence, i.e., the characteristic function of odious integers. (This sequence can be defined by $t(0) = 0$ and for all $n \geq 0$, $t(2n) = t(n)$ and $t(2n+1) = 1 - t(n)$.) This easily implies that any finite composition of $(A(n))_{n\geq 0}$ and $(B(n))_{n\geq 0}$ can be written as $(\alpha A(n) + \beta B(n) + \gamma)_{n\geq 0}$, since $t(A(n)) = 1$ and $t(B(n)) = 0$ for all n.

2. Complementary pairs

Let α be an irrational number, and let A be the Beatty sequence of α. In this section we consider Question A of the Introduction, which we call the Complementary pair problem.

In what follows we will require that as a function $A : \mathbb{N} \to \mathbb{N}$ be injective, since we then have a one-to-one correspondence between sequences and subsets of $\mathbb{N}$. (See [Kimberling and Stolarsky 2016] for noninjective Beatty sequences.)

In the case that V and W are increasing, we will also require, without loss of generality, that $V(1) = 1$. Solutions (p, q, r, s, t, u) with $p = 0$ or $s = 0$ will be called *trivial*.

The homogeneous Sturmian sequence generated by a real number $\alpha \in (0, 1)$ is the sequence

$$c_\alpha := (\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor)_{n \geq 1}.$$

(For more about Sturmian sequences, the reader can consult, e.g., [Lothaire 2002, Chapter 2].)
A real number α is called a *Sturm number* if $\alpha \in (0, 1)$ is a quadratic irrational number with algebraic conjugate $\overline{\alpha}$ satisfying $\overline{\alpha} \notin (0, 1)$. Sturm numbers have a property that is useful to recognize their generalized Beatty sequences.

Proposition 6 [Crisp et al. 1993; Allauzen 1998]. *Let α be a Sturm number. Then there exists a morphism σ_α on the alphabet $\{0, 1\}$ such that $\sigma_\alpha(c_\alpha) = c_\alpha$.*

In the following we will consider the variants of σ_α on various other alphabets than $\{0, 1\}$, but will not indicate this in the notation. The following lemma is implied trivially by

$$V = pA + q\,\mathrm{Id} + r \quad\Rightarrow\quad V(n+1) - V(n) = p(A(n+1) - A(n)) + q = p\,c_\alpha(n) + q.$$

Lemma 7. *Let α be a Sturm number. Let $V = (V(n))_{n\geq 1}$ be the generalized Beatty sequence defined by $V(n) = p(\lfloor n\alpha \rfloor) + qn + r$, and let ΔV be the sequence of its first differences. Then ΔV is the fixed point of σ_α on the alphabet $\{q, p+q\}$.*

We remark that it can be shown that the first letters of $\sigma_\alpha(0)$ and $\sigma_\alpha(1)$ are equal (see, e.g., [Dekking 2018b]), so σ_α has a unique fixed point. It is also obvious that this fixed point starts with 0 if $\alpha \in (0, 1/2)$, and with 1 if $\alpha \in (1/2, 1)$. For general α, one replaces α with $\breve{\alpha} = \alpha - \lfloor \alpha \rfloor (= \{\alpha\})$. When α is the golden mean $\varphi = (1+\sqrt{5})/2$, the morphism generating the sequence associated to the Sturm number $\breve{\varphi} = \varphi - 1$ is $0 \mapsto 1$, $1 \mapsto 10$, so one has to exchange 0 and 1 if one wishes to compare ΔV with the classical Fibonacci morphism $0 \mapsto 01$, $1 \mapsto 0$. As a special case of Lemma 7 we therefore obtain one direction of the following lemma.

Lemma 8. *Let $V = (V(n))_{n\geq 1}$ be the generalized Beatty sequence defined by $V(n) = p(\lfloor n\varphi \rfloor) + qn + r$, and let ΔV be the sequence of its first differences. Then ΔV is the Fibonacci word on the alphabet $\{2p+q, p+q\}$. Conversely, if x_{ab} is the Fibonacci word on the alphabet $\{a, b\}$, then any V with $\Delta V = x_{ab}$ is a generalized Beatty sequence $V = ((a-b)\lfloor n\varphi \rfloor) + (2b-a)n + r)$ for some integer r.*

Another observation is that the $q\,\mathrm{Id} + r$ part in a generalized Beatty sequence generates arithmetic sequences. The following lemma, which will be useful in proving Theorem 11 below, shows that in some weak sense the Wythoff part pA of a generalized Beatty sequence is orthogonal to its arithmetic sequence part, provided that $\frac{1}{3} < \{\alpha\} < \frac{2}{3}$, where $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$. We prove this for $\frac{4}{3} < \alpha < \frac{5}{3}$.

Lemma 9. *Let α satisfy $\frac{4}{3} < \alpha < \frac{5}{3}$, and let $V = (V(n))_{n\geq 1}$ be the generalized Beatty sequence defined by $V(n) = p(\lfloor n\alpha \rfloor) + qn + r$ with $p \neq 0$, then neither $(V(1), V(2), V(3))$, nor $(V(2), V(3), V(4))$ can be an arithmetic sequence of length* 3.

Proof. When $\frac{3}{2} < \alpha < \frac{5}{3}$, we have $\lfloor \alpha \rfloor = 1$, $\lfloor 2\alpha \rfloor = 3$, and $\lfloor 3\alpha \rfloor = 4$, $\lfloor 4\alpha \rfloor = 6$, so

$$
\begin{aligned}
V(2) - V(1) &= p\lfloor 2\alpha \rfloor + 2q + r - p\lfloor \alpha \rfloor - q - r = 2p + q,\\
V(3) - V(2) &= p\lfloor 3\alpha \rfloor + 3q + r - p\lfloor 2\alpha \rfloor - 2q - r = p + q,\\
V(4) - V(3) &= p\lfloor 4\alpha \rfloor + 4q + r - p\lfloor 3\alpha \rfloor - 3q - r = 2p + q,
\end{aligned}
$$

and the result follows, since $p \neq 0$. When $\frac{4}{3} < \alpha < \frac{3}{2}$, we have $\lfloor \alpha \rfloor = 1$, $\lfloor 2\alpha \rfloor = 2$, $\lfloor 3\alpha \rfloor = 4$, and $\lfloor 4\alpha \rfloor = 5$. So this time $V(2) - V(1) = p + q$, $V(3) - V(2) = 2p + q$ and $V(4) - V(3) = p + q$, leading to the same conclusion. □

Remark 10. We note for further use that solving the equations in the proof of Lemma 9 for p and q, supplemented with an equation for r, yields in the case $\frac{3}{2} < \alpha < \frac{5}{3}$ that

$$
\begin{cases}
p = -V(1) + 2V(2) - V(3)\\
q = V(1) - 3V(2) + 2V(3)\\
r = V(1) + V(2) - V(3).
\end{cases}
$$

Let $\alpha = \varphi$ be the golden mean. Then the classical solution is $(p, q, r) = (1, 0, 0)$ and $(s, t, u) = (1, 1, 0)$, which corresponds to the Beatty pair $(\lfloor n\varphi \rfloor)$, $(\lfloor n\varphi^2 \rfloor)$. Another solution is given by

$$
(p, q, r) = (-1, 3, -1), \quad (s, t, u) = (1, 2, 0),
$$

which corresponds to the Beatty pair $\left(\left\lfloor n\left(\frac{5-\sqrt{5}}{2}\right)\right\rfloor\right)$, $\left(\left\lfloor n\left(\frac{5+\sqrt{5}}{2}\right)\right\rfloor\right)$, that is, $(\lfloor n(3-\varphi) \rfloor)$, $(\lfloor n(\varphi + 2) \rfloor)$.

Theorem 11. *Let $\alpha = \varphi$. Then there are exactly two nontrivial increasing solutions to the complementary pair problem*: $(p, q, r, s, t, u) = (1, 0, 0, 1, 1, 0)$ *and* $(p, q, r, s, t, u) = (-1, 3, -1, 1, 2, 0)$.

Proof. Recall that $V(1) = 1$. Note that $V(2) < 5$, since otherwise $(W(1), W(2), W(3)) = (2, 3, 4)$, which is not allowed by Lemma 9. There are therefore three cases to consider, according to the value of $V(2)$.

(1) The case $V(1) = 1$, $V(2) = 2$. Then by Lemma 9, $V(3) = 3$ is not possible.
   - (a) If $V(3) = 4$, then, by Remark 10, $p = -1$, $q = 3$, $r = -1$, which is one of the two solutions.
   - (b) If $V(3) = 5$, then, by Remark 10, $p = -2$, $q = 5$, $r = -2$, which implies that $V(4) = 6$, $V(5) = 7$, $V(6) = 10$. So $W(1) = 3$, $W(2) = 4$, $W(3) = 8$, which gives $s = -3$, $t = 7$, $u = -1$ (Remark 10 applied to W), implying $W(5) = 10$, which contradicts complementarity.
   - (c) If $V(3) = m$ with $m > 5$, then $W(1) = 3$, $W(2) = 4$, $W(3) = 5$, which contradicts Lemma 9.

(2) The case $V(1) = 1$, $V(2) = 3$.
   - (a) If $V(3) = 4$, then, by Remark 10, $p = 1$, $q = 0$, $r = 0$, which is one of the two solutions.
   - (b) If $V(3) = 5$, then we obtain a contradiction with Lemma 9.
   - (c) If $V(3) = 6$, then, by Remark 10, $p = -1$, $q = 4$, $r = -2$, which implies $V(5) = 10$. But we must then have $W(1) = 2$, $W(2) = 4$, $W(3) = 5$, so (Remark 10 applied to W), $s = 1$, $t = 0$, $u = 1$, which implies $W(6) = 10$, a contradiction with complementarity.

(d) If $V(3) = m$ with $m > 6$, then we obtain a contradiction with Lemma 9, since then $W(2) = 4$, $W(3) = 5$, $W(4) = 6$.

(3) The case $V(1) = 1$, $V(2) = 4$.

(a) If $V(3) = 5$, then, by Remark 10, $p = 2$, $q = -1$, $r = 0$, thus $V(4) = 8$; hence $W(1) = 2$, $W(2) = 3$, $W(3) = 6$. Hence, by Remark 10 applied to W, $s = -2$, $t = 5$, $u = -1$, so that $W(5) = 8 = V(4)$, which contradicts complementarity.

(b) If $V(3) = 6$, then $W(1) = 2$, $W(2) = 3$, $W(3) = 5$. Thus, by Remark 10 applied to W, $s = -1$, $t = 3$, $u = 0$. Hence $W(4) = 6 = V(3)$, which contradicts complementarity.

(c) If $V(3) = 7$, then we obtain a contradiction with Lemma 9.

(d) If $V(3) = m$ with $m > 7$, it follows that $V(3) = 8$, since we have $W(1) = 2$, $W(2) = 3$, $W(3) = 5$, yielding, by Remark 10 applied to W, $W = (-A(n) + 3n) = 2, 3, 5, 6, 7, 9, 10, 12, 13, 14, \ldots$ With $V(3) = 8$, one obtains (by Remark 10) that $V(n) = -A(n) + 5n - 3$, but then $V(5) = 14 = W(10)$, i.e., V and W are not complementary. □

For $\alpha = \sqrt{2}$ the classical solution to the complementary pair problem is $V = A$, $W = A + 2\,\mathrm{Id}$, i.e., the Beatty pair given by $V(n) = \lfloor n\sqrt{2} \rfloor$, and $W(n) = \lfloor n(2+\sqrt{2}) \rfloor$. As $\frac{4}{3} < \sqrt{2} < \frac{3}{2}$, we can use Lemma 9 and adapt Remark 10 to prove the following result, in the same way as Theorem 11.

Theorem 12. *Let $\alpha = \sqrt{2}$. Then there is a unique nontrivial increasing solution to the complementary pair problem*: $(p, q, r, s, t, u) = (1, 0, 0, 1, 2, 0)$.

We end this section with an example where $\{\alpha\} \notin \left(\frac{1}{3}, \frac{2}{3}\right)$.

Theorem 13. *Let $\alpha = \sqrt{8}$. Then there is a unique nontrivial increasing solution to the complementary pair problem*: $(p, q, r, s, t, u) = (1, 4, 0, -1, 4, 0)$.

Proof. Since $(4 + \sqrt{8}, 4 - \sqrt{8})$ is a Beatty pair, $(p, q, r, s, t, u) = (1, 4, 0, -1, 4, 0)$ is a solution to the complementary pair problem. To prove that it is unique is more involved. We fix $V(1) = 1$.

Let $\breve{\alpha} = \alpha - 2 = \sqrt{8} - 2$. Then $\breve{\alpha} \in (0, 1)$, and $\breve{\alpha}$ has the periodic continued fraction expansion $[0; \overline{1, 4}]$. It follows then from [Crisp et al. 1993], or from Corollary 9.1.6 in [Allouche and Shallit 2003] that the morphism $\sigma_{\breve{\alpha}}$ fixing the homogeneous Sturmian sequence $c_{\breve{\alpha}}$ is given by

$$\sigma_{\breve{\alpha}} : 0 \mapsto 11110, \quad 1 \mapsto 111101.$$

Note that

$$V(n) = p\lfloor n\sqrt{8} \rfloor + qn + r = p\lfloor n(\sqrt{8} - 2) \rfloor + (2p + q)n + r = p\lfloor n\breve{\alpha} \rfloor + (2p + q)n + r.$$

The difference sequence ΔV of V is therefore the fixed point of $\sigma_{\breve{\alpha}}$ on the alphabet $\{2p + q, 3p + q\}$. Since we require V to be increasing, both $2p + q$ and $3p + q$ have to be larger than 0. We split the possibilities according to the value of $3p + q$. The arguments below are based on the fact, following from the form of $\sigma_{\breve{\alpha}}$, that V starts with an arithmetic sequence of length 5, followed by an arithmetic sequence of length 6, both with common differences $3p + q$, and separated by a distance $2p + q$.

(1) The case $3p + q \geqslant 3$. If $3p + q \geq 3$, then $W(1) = 2$, $W(2) = 3$, so $W = sA + t\mathrm{Id} + u$ has to start with an arithmetic sequence of length 5 with common difference 1, i.e., $W(1), W(2), \ldots, W(5) = 2, 3, \ldots, 6$. Moreover, since $W(6) = 7$ is not possible (as it would imply $p = 0$), we have $V(2) = 7$,

which implies $V(3) = 13$. But then the second arithmetic sequence of W, which has length 6, does not fit in between $V(2)$ and $V(3)$.

(2) <u>The case $3p+q=2$</u>. In this case $V(1), \ldots, V(5) = 1, 3, 5, 7, 9$, so $W(1), \ldots, W(5) = 2, 4, 6, 8, 10$. Then either $V(6) = 11$, or $W(6) = 11$.

In the former case we must have $2p+q = V(6) - V(5) = 2$, which implies $p = 0$, which is trivial.

In the latter case $W(6), \ldots, W(11) = 11, 13, 15, 17, 19, 21$, and $W(12) = 22$, since $W(12) - W(11) = W(6) - W(5) = 1$. But also, $V(6), \ldots, V(11)$ equals $12, 14, \ldots, 22$. So V and W are not complementary.

(3) <u>The case $3p+q=1$</u>. In this case $V(1), \ldots, V(5) = 1, 2, 3, 4, 5$, so $W(1) = 6$, since $V(6) = 6$ would imply $p = 0$. Then either $V(6) = 7$, or $W(2) = 7$.

In the former case, $V(6), \ldots, V(11) = 7, 8, 9, 10, 11, 12$, and $W(2) = 13$. This implies that $2 = V(6) - V(5) = 2p+q$, which leads to $(p, q, r) = (-1, 4, 0)$, and $(s, t, u) = (1, 4, 0)$, which is the announced solution.

In the latter case $W(1), \ldots, W(5) = 6, 7, 8, 9, 10$, and $V(6) = 11$. So $2p+q = V(6) - V(5) = 5$. This implies $(p, q, r) = (-5, 16, -5)$, and $(s, t, u) = (-6, 19, -1)$. But then $V(12) = 22$, and $W(11) = 22$. So V and W are not complementary. □

2.1. ***Generalized Pell equations.*** If V and W are not increasing, then an analysis as in the proof of Theorem 11 is still possible, but very lengthy. We therefore consider another approach in this subsection. Considering the densities of V and W in $\mathbb{N}$, one sees that a *necessary* condition for $(pA + q\,\mathrm{Id} + r)$ and $(sA + t\,\mathrm{Id} + u)$ to be a complementary pair is that

$$\frac{1}{p\alpha + q} + \frac{1}{s\alpha + t} = 1 \tag{2}$$

In what follows we concentrate on the case $\alpha = \varphi = (1+\sqrt{5})/2$, but our arguments can be generalized to the case of arbitrary quadratic irrationals.

Proposition 14. *A necessary condition for the pair $V = pA + q\,\mathrm{Id} + r$ and $W = sA + t\,\mathrm{Id} + u$ to be a complementary pair is that $p \neq 0$ is a solution to the generalized Pell equation*

$$5p^2x^2 - 4x = y^2, \quad x, y \in \mathbb{Z}.$$

Proof. Using $\varphi^2 = 1 + \varphi$, a straightforward manipulation shows that (2) implies

$$(ps + pt + qs - p - s)\varphi = q + t - ps - qt.$$

But since φ is irrational, this can only hold if

$$ps + pt + qs - p - s = 0, \quad q + t - ps - qt = 0. \tag{3}$$

The first equation gives $pt = p - (p+q-1)s$. Eliminating pt from $p^2s + (q-1)pt - pq = 0$, we obtain $p^2s + (p - (p+q-1)s)(q-1) - pq = 0$. This gives the quadratic equation

$$s\,q^2 + (p-2)s\,q - (p^2+p-1)s + p = 0.$$

Since q is an integer, $\Delta := (p-2)^2s^2 + 4s((p^2+p-1)s - p)$ has to be an integer squared. Trivial manipulations yield that

$$\Delta = 5p^2s^2 - 4ps. \tag{4}$$

Since p divides the square Δ, $5p^2s^2 - 4ps = p^2y^2$ for some integer y, and hence p also divides s. If we put $s = px$, we obtain $5p^3x^2 - 4p^2x = p^2y^2$, which finishes the proof of the proposition. □

Actually there is a simple characterization of the integers p such that the diophantine equation above has a solution.

Proposition 15. *The generalized Pell equation*

$$5p^2x^2 - 4x = y^2, \quad x, y \in \mathbb{Z},$$

has a solution for $p > 0$ *if and only if* p *divides some Fibonacci number of odd index, i.e., if and only* p *divides some number in the set* $\{1, 2, 5, 13, 34, \dots\}$.

Proof. First suppose that there are integers $p > 0$ and $x, y \in \mathbb{Z}$ such that $5p^2x^2 - 4x = y^2$. Let $d := \gcd(x, y)$ and $x' = x/d$, $y' = y/d$, so that $\gcd(x', y') = 1$. We thus have

$$5p^2dx'^2 - 4x' = dy'^2.$$

Thus x' divides dy'^2, but it is prime to y', hence x' divides d. Since clearly d divides $4x'$, we have $d = \alpha x'$ for some α dividing 4, hence α belongs to $\{1, 2, 4\}$. This yields $\alpha(5p^2x'^2 - y'^2) = 4$. We distinguish three cases.

(1) If $\alpha = 1$, then we have $5p^2x'^2 - y'^2 = 4$. But the equation $5X^2 - 4 = Y^2$ has an integer solution if and only if X is a Fibonacci number with odd index [Lind 1968, p. 91]. Hence px' must be a Fibonacci number with odd index, thus p divides a Fibonacci number with odd index.

(2) If $\alpha = 2$, then we have $5p^2x'^2 - y'^2 = 2$. Note that x' must be odd, otherwise x' and y' would be even, which contradicts $\gcd(x', y') = 1$. Thus $5p^2x'^2 \equiv p^2 \bmod 4$, hence $p^2 - 2 \equiv y'^2 \bmod 4$. If p is even, this yields $y'^2 \equiv 2 \bmod 4$, while if p is odd, this gives $y'^2 \equiv 3 \bmod 4$. There is no such y' in both cases.

(3) If $\alpha = 4$, then we have $5p^2x'^2 - y'^2 = 1$, thus $5(2px')^2 - (2y')^2 = 4$, then $2px'$ must be a Fibonacci number with odd index, thus p divides a Fibonacci number with odd index.

Now suppose that p divides some Fibonacci number with odd index, say there exists a k with $F_{2k+1} = p\beta$. We will construct an integer solution in (x, y) to the equation $5p^2x^2 - 4x = y^2$. We know (again [Lind 1968, p. 91]) that there exists some integer γ with $5F_{2k+1}^2 - 4 = \gamma^2$ thus $5p^2\beta^2 - 4 = \gamma^2$. Let $x = \beta^2$ and $y = \beta\gamma$. Then

$$5p^2x^2 - 4x = 5p^2\beta^4 - 4\beta^2 = \beta^2(5p^2\beta^2 - 4) = \beta^2\gamma^2 = y^2. \qquad \square$$

Corollary 16. *If* -1 *is not a square modulo* p, *there are no solutions to the complementary pair problem. This is in particular the case if* p *has a prime divisor congruent to* 3 *modulo* 4.

(The sequence of such integers p, which starts with $1, 2, 5, 10, 13, 17, 25, 26, 29, 34, 37, 41, \dots$, is labeled A008784 in *The on-line encyclopedia of integer sequences* [OEIS].)

Proof. We will prove that if there are solutions to the complementary problem for p, thus if p divides an odd-indexed Fibonacci number (Propositions 14 and 15), then -1 is a square modulo p. Using again the characterization in [Lind 1968, p. 91], there exist two integers x, y with $5p^2x^2-4=y^2$. We distinguish two cases.

(i) If p is odd, we have $y^2 \equiv -4 \bmod p$ and $2^2 \equiv 4 \bmod p$. But 2 is invertible modulo p, hence, by taking the quotient of the two relations, we obtain that -1 is a square modulo p.

(ii) If p is even, remembering that $px = F_{2k+1}$ for some k, we claim that p must be congruent to 2 modulo 4 and that x must be odd. Namely the sequence of odd-indexed Fibonacci numbers, reduced modulo 4, is easily seen to be the periodic sequence $(1\ 2\ 1)^\infty$. Hence it never takes the value 0 modulo 4. The equality $5p^2x^2-4=y^2$ implies that y must be even, thus we have $5(p/2)^2x^2-1=(y/2)^2$, say $(y/2)^2=-1+z(p/2)$. Up to replacing $(y/2)$ with $(y+p)/2$, we may suppose that $(y/2)$ is even (recall that $p/2$ is odd). Thus $z(p/2)$ is even, hence z is even, say $z=2z'$. This gives $(y/2)^2=-1+z'p$; thus -1 is a square modulo p. □

Remark 17. We have just seen that if the integer p divides some odd-indexed Fibonacci number (OEIS A008784) then -1 is a square modulo p. A natural question is then whether it is true that if -1 is a square modulo p, then p must divide some odd-indexed Fibonacci number. The answer is negative, since on one hand $12^2 \equiv -1 \bmod 29$, and, on the other hand, the sequence of odd-indexed Fibonacci numbers modulo 29 is the periodic sequence $(1, 2, 5, 13, 5, 2, 1)^\infty$ which is never zero.

Let us look at examples of solutions to the diophantine equation for values of p that divide some Fibonacci number with odd index. Consider, for example, the case where $p=s$. Then equation (4) becomes $\Delta = 5p^4-4p^2$, so the diophantine equation is

$$5x^2-4=y^2, \quad x, y \in \mathbb{Z}.$$

For $p=F_1=1$ we obtain the two sequences $V=A+r$ and $W=A+\mathrm{Id}+u$. These are complementary only when $r=u=0$, and we obtain the classical Beatty pair $(A, A+\mathrm{Id})$.

For $p=F_3=2$ we obtain the two sequences $V=2A+2\,\mathrm{Id}+r$ and $W=2A-2\,\mathrm{Id}+u$. These cannot be complementary for any r and u, since for $u=0$ we have $W(n)=2\lfloor n\varphi\rfloor-2n=2\lfloor n(\varphi-1)\rfloor$, which gives all even numbers, since $\varphi-1<1$. This an example where equation (2) does not apply, since W as a function is not injective.

For $p=F_5=5$ we obtain the two sequences $V=5A+4\,\mathrm{Id}+r$ and $W=5A-7\,\mathrm{Id}+u$. To make these complementary we are forced to choose $r=u=3$, and we obtain

$$V=(12, 26, 35, 49, 63, 72, 86, 95, 109, 123, 132, 146, 160, 169, 183, 192, 206, 220, 229, 243, 252, 266, \dots),$$

$$W=(1, 4, 2, 5, 8, 6, 9, 7, 10, 13, 11, 14, 17, 15, 18, 16, 19, 22, 20, 23, 21, 24, 27, 25, 28, 31, 29, 32, 30, \dots).$$

Now a proof that V and W form a complementary pair is much harder when we let V start with $V(0)=3$, to include 3 in the union. We can perform the following trick. We split W into $(W(A(n)))_{n\ge1}$, and $(W(B(n)))_{n\ge1}$ (cf. Corollary 2). The two sequences WA and WB are increasing, and we can prove that $(V(n))_{n\ge0}$, $(W(A(n)))_{n\ge1}$, and $(W(B(n)))_{n\ge1}$ form a partition of the positive integers by proving that the three-letter sequence obtained by applying the morphism $0\mapsto 1120$, $1\mapsto 11100$ to the fixed point of the morphism g given by $g: 0\mapsto 01,\ 1\mapsto 011$, has the property that the preimages of 0, 1 and 2 are precisely these three sequences. See Theorem 25 and its proof for a similar result.

For $p = F_{2m+1} \geq 13$ it seems that we can always choose r and u for in such a way that we get almost complementary sequences: namely, e.g., for $p = 13$ we find $q = 9$ and $t = -20$. If we take $r = u = 9$, then we almost get a complementary pair. One finds $V = 9, 31, 66, 88, 123, 158, 180, 215, \ldots$ and $W = 2, 8, 1, 7, 13, 6, 12, 5, 11, 17, 10 \ldots.$ So 3 and 4 are missing. We thought we could prove, perhaps using something like the Lambek–Moser Theorem [Lambek and Moser 1954], that for all $F_{2m+1} > 5$ the two sequences are complementary excluding finitely many values, but we were not successful.

3. Complementary triples

Here we will find several complementary triples consisting of sequences $V_i = p_i A + q_i \,\mathrm{Id} + r_i,\ \ i = 1, 2, 3,$ where $A(n) = \lfloor n\alpha \rfloor$, and α is a real number.

It is interesting that the case $p_1 = p_2 = p_3 = 1$ cannot be realized. This was proved by Uspensky in 1927; see [Fraenkel 1977].

The case with different α_i was analyzed in [Tijdeman 1996] for *rational* $\alpha_i,\ 1 = 1, 2, 3.$ Also see [Tijdeman 2000] for the inhomogeneous Beatty case $(V_i(n))_n = (\lfloor n\alpha_i + \beta_i \rfloor)_n,\ i = 1, 2, 3.$

There is one triple in which we will be particularly interested (see Theorem 25):

$$(p_1, q_1, r_1) = (2, -1, 0), \quad (p_2, q_2, r_2) = (4, 3, 2), \quad (p_3, q_3, r_3) = (2, -1, 2).$$

We allow the sequences (V_i) to be each indexed either by $\{0, 1, 2, \ldots\}$ or by $\{1, 2, \ldots\}$.

3.1. ***Two classical triples.*** In this subsection α is always the golden mean φ. Let once more $A(n) = \lfloor n\varphi \rfloor$ for $n \geq 1$ be the terms of the lower Wythoff sequence, and let B be given by $B(n) = \lfloor n\varphi^2 \rfloor$ for $n \geq 1$, the upper Wythoff sequence. Then we have the disjoint union

$$A(\mathbb{N}) \cup B(\mathbb{N}) = \mathbb{N}. \tag{5}$$

Since $B = A + \mathrm{Id}$, this is the classical complementary pair $((1, 0, 0), (1, 1, 0))$.

Here is a way to create complementary triples from complementary pairs.

Proposition 18. *Let (V, W) be a golden mean complementary pair $V = pA + q\,\mathrm{Id} + r$ and $W = sA + t\,\mathrm{Id} + u$. Then (V_1, V_2, V_3) is a complementary triple, where the three parameters of V_1 are $(p + q, p, r - p)$, those of V_2 are $(2p + q, p + q, r)$, and $V_3 = W$.*

Proof. Substituting (5) in $V(\mathbb{N}) \cup W(\mathbb{N}) = \mathbb{N}$ we obtain the disjoint union

$$V(A(\mathbb{N})) \cup V(B(\mathbb{N})) \cup W(\mathbb{N}) = \mathbb{N}.$$

Then Corollary 2 implies the statement of the proposition. □

Remark 19. Actually Proposition 18 and Corollary 2 can be generalized to cover an infinite family of quadratic irrationals, but their statements will not be true for all quadratic irrationals. We hope to revisit this point in a future article.

Applying Proposition 18 to the basic complementary pair $((1, 0, 0), (1, 1, 0))$ gives that

$$((1, 1, -1), (2, 1, 0), (1, 1, 0)) \quad \text{and} \quad ((1, 0, 0), (2, 1, -1), (3, 2, 0))$$

are complementary triples, which we will call classical triples. (The corresponding sequences are indexed in OEIS as (A003623, A003622, A001950) and (A000201, A035336, A101864). The first classical triple is given at the end of [Skolem 1957].)

Let $w = 1231212312312\ldots$ be the fixed point of the morphism

$$1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 12.$$

Then $w^{-1}(1) = AA,\ w^{-1}(2) = B$ and $w^{-1}(3) = AB$ give the three sequences V_1, V_3 and V_2 of the first classical triple (see [Dekking 2016]).

The question arises: is there also a morphism generating the second triple? The answer is positive.

Proposition 20. *Let* $(V_1, V_2, V_3) = (A,\ 2A + \mathrm{Id} - 1,\ 3A + 2\mathrm{Id}) = (A,\ BA,\ BB)$. *Then* (V_1, V_2, V_3) *is a complementary triple. Let* μ *be the morphism on* $\{1, 2, 3\}$ *given by*

$$\mu : 1 \mapsto 121,\ 2 \mapsto 13,\ 3 \mapsto 13,$$

with fixed point z. *Then* $z^{-1}(1) = V_1,\ z^{-1}(2) = V_2$ *and* $z^{-1}(3) = V_3$.

Proof. The four words of length 3 occurring in the infinite Fibonacci word x_{F} are 010, 100, 001, 101. Coding these with the alphabet $\{1, 2, 3, 4\}$ in the given order, they generate the 3-block morphism $\hat{f}_3$ that describes the successive occurrences of the words of length 3 in x_{F} (cf. [Dekking 2016]). It is given by

$$\hat{f}_3(1) = 12, \quad \hat{f}_3(2) = 3, \quad \hat{f}_3(3) = 14, \quad \hat{f}_3(4) = 3.$$

It has just one fixed point, which is

$$z' := 1, 2, 3, 1, 4, 1, 2, 3, 1, 2, 3, 1, 4, 1, 2, 3, \ldots.$$

We claim that

$$z'^{-1}(1) = AA,\ z'^{-1}(2) = BA, \quad z'^{-1}(3) = AB, \quad z'^{-1}(4) = BB.$$

To see this, note that the 3-block 010 in x_{F} uniquely decomposes as $010 = f(0)0$. It follows that the m^{th} occurrence of 010 in x_{F} corresponds exactly to the m^{th} occurrence of 0 in $f^{-1}(x_{\mathrm{F}}) = x_{\mathrm{F}}$. This implies that the positions of the occurrences of 010 are of the form $A(A(n))$, and also that the occurrences of 001 are of the form $A(B(n))$, since $B(\mathbb{N})$ is the complement of $A(\mathbb{N})$.

For the 3-block 100, we note that it always occurs in x_{F} as factor of 0100, which uniquely decomposes in x_{F} as $0100 = f^2(0)0$. It follows that the m^{th} occurrence of 100 in x_{F} corresponds exactly to the m^{th} occurrence of 0 in $f^{-2}(x_{\mathrm{F}}) = x_{\mathrm{F}}$. This implies that the positions of the occurrences of 100 are of the form $B(A(n))$, and also that the occurrences of 101 are of the form $B(B(n))$.

Since the 0's in x_{F} occur either as prefix of 001, or of 010, we see that we have to merge the letters 1 and 3 to obtain the sequence A. This is not possible with $\hat{f}_3$. However, the square of this 3-block morphism is given by

$$1 \mapsto 123,\ 2 \mapsto 14,\ 3 \mapsto 123,\ 4 \mapsto 14,$$

and now we *can* consistently merge 1 and 3 to the single letter 1, obtaining the morphism μ, after mapping 4 to 3. Under this projection the sequence z' maps to z. □

3.2. ***Nonclassical triples.*** Let $\mathcal{L}$ be a language, i.e., a sub-semigroup of the free semigroup generated by a finite alphabet under the concatenation operation. A homomorphism of $\mathcal{L}$ into the natural numbers is a map $\mathrm{S} : \mathcal{L} \to \mathbb{N}$ satisfying $\mathrm{S}(vw) = \mathrm{S}(v) + \mathrm{S}(w)$, for all $v, w \in \mathcal{L}$.

Let $\mathcal{L}_\mathrm{F}$ be the Fibonacci language, i.e., the set of all words occurring in the Fibonacci word x_F, the iterative fixed point of the morphism f defined on $\{0, 1\}^*$ by $f : 0 \mapsto 01,\ 1 \mapsto 0$.

Theorem 21 [Dekking 2018a]. *Let* $\mathrm{S} : \mathcal{L}_\mathrm{F} \to \mathbb{N}$ *be a homomorphism. Define* $a = \mathrm{S}(0)$, $b = \mathrm{S}(1)$. *Then* $\mathrm{S}(\mathcal{L}_\mathrm{F})$ *is the union of the two generalized Beatty sequences*

$$\big((a-b)\lfloor n\varphi\rfloor + (2b-a)n\big) \quad \textit{and} \quad \big((a-b)\lfloor n\varphi\rfloor + (2b-a)n + a - b\big).$$

For a few choices of a and b, the two sequences in $\mathrm{S}(\mathcal{L}_\mathrm{F})$ and the sequence $\mathbb{N} \setminus \mathrm{S}(\mathcal{L})$ form a complementary triple of generalized Beatty sequences. The goal of this section is to prove this for $a = 3,\ b = 1$. It turns out that the three sequences

$$(2\lfloor n\varphi\rfloor - n)_{n\ge1},\ (2\lfloor n\varphi\rfloor - n + 2)_{n\ge1},\ (4\lfloor n\varphi\rfloor + 3n + 2)_{n\ge0},$$

form a complementary triple.

Remark 22. Note that the indices for $(4\lfloor n\varphi\rfloor) + 3n + 2)_{n\ge0}$ are $(n \ge 0)$, not $(n \ge 1)$.

It is easy to see that the Fibonacci word x_F can be obtained as an infinite concatenation of two kinds of blocks, namely 01 and 001 (part (i) of Lemma 23 below). Kimberling introduced in the OEIS the sequence A284749 obtained by replacing every block 001 in this concatenation by 2. We let $x_\mathrm{K} = A284749$ denote this sequence.

Lemma 23. *Let* f, g, h, k *be the morphisms defined on* $\{0, 1\}^*$ *by*

$$f : 0 \mapsto 01,\ 1 \mapsto 0; \quad g : 0 \mapsto 01,\ 1 \mapsto 011; \quad h : 0 \mapsto 01,\ 1 \mapsto 001; \quad k : 0 \mapsto 01,\ 1 \mapsto 2.$$

The following equalities hold:

(i) $x_\mathrm{F} = f^\infty(0) = h(g^\infty(0))$.

(ii) $x_\mathrm{K} = k(g^\infty(0))$.

Proof. (i) An easy induction proves that for all $n \ge 0$ one has $hg^k = f^{2k}h$. (Note that it suffices to prove that the values of both sides are equal when applied to 0 and to 1.) By letting n tend to infinity this implies $hg^\infty(0) = f^\infty(0)$.

(ii) Assertion (i) gives that x_F is an infinite concatenation of blocks $h(0) = 01$ and $h(1) = 001$, obtained as image under h from $g^\infty(0)$. So substituting 2 for 001 in x_F is the same as substituting 01 for 0 and 2 for 1 in $g^\infty(0)$. □

It is interesting that x_K is fixed point of a morphism i, given by $i : 0 \mapsto 01,\ 1 \mapsto 2,\ 2 \mapsto 0122$. This follows from the relation $kg^n = i^{n+1}$ for all n, which is easily proved by induction.

Lemma 24. *Define the morphism* ℓ *from* $\{0, 1\}^*$ *to* $\{0, 1, 2\}^*$ *by* $\ell : 0 \mapsto 012,\ 1 \mapsto 0022$. *Then the sequence* $v = (v_n)_{n\ge1} = \ell(g^\infty(0))$ *is obtained from* x_K *by replacing* 1 *by* 0 *in all blocks* 0122 (*but not in* 0120).

Proof. Note that $kg : 0 \mapsto 012,\ 1 \mapsto 0122$. Lemma 24 then follows from $x_\mathrm{K} = k(g^\infty(0)) = kg(g^\infty(0))$. □

Theorem 25. *Let v be the sequence defined above, i.e.,* $v = \ell(g^\infty(0))$, *where g and ℓ are the morphisms defined by $g : 0 \mapsto 01$, $1 \mapsto 011$ and $\ell : 0 \mapsto 012$, $1 \mapsto 0022$. Then the increasing sequences of integers defined by $v^{-1}(0)$, $v^{-1}(1)$, $v^{-1}(2)$ form a partition of the set of positive integers $\mathbb{N}^*$. Furthermore*:

- $v^{-1}(0) = \{1, 4, 5, 8, 11, 12, 15, 16, 19, 22, \ldots\}$ *is equal to the sequence of integers* $(2\lfloor n\varphi\rfloor - n)_{n\ge1}$, *where φ is the golden ratio* $\frac{1+\sqrt{5}}{2}$ (OEIS A050140).
- $v^{-1}(1) = \{2, 9, 20, 27, \ldots\}$ *is equal to the sequence of integers* $(4\lfloor n\varphi\rfloor + 3n + 2)_{n\ge0}$.
- $v^{-1}(2) = \{3, 6, 7, 10, 13, 14, 17, 18, 21, 24, \ldots\}$ *is equal to the sequence* $(2\lfloor n\varphi\rfloor - n + 2)_{n\ge1}$ (that is, $2 +$ A050140).

Proof. We see from Lemma 24 that the positions of 2 in v are the same as the positions of 2 in $x_{\rm K}$. In Section 4 it is proved that $x_{\rm K}^{-1}(2) = (2\lfloor n\varphi\rfloor - n + 2)$, see Example 30, and so the third assertion of the theorem follows.

Inspection of the occurrences of 0 and 2 in $\ell(0)$ and $\ell(1)$ then shows the first assertion to be true too.

For the proof of the second assertion consider $\ell(01) = 0\mathbf{1}20022$, $\ell(011) = 0\mathbf{1}200220022$. Since $g^\infty(0)$ is a concatenation of the words $g(0) = 01$ and $g(1) = 011$, we see from this that the differences of indices of the positions where 1's in v occur are 7 or 11, and moreover, that a 0 in $g^\infty(0)$ generates a difference 7, a 1 in $g^\infty(0)$ generates a difference 11.

It is well known and easy to prove that $g^\infty(0)$ equals the binary complement of the Fibonacci word prefixed with the letter 1. From this it follows that $\Delta v^{-1}(1)$ is the Fibonacci word on the alphabet $\{11, 7\}$. Now Lemma 8 gives the generalized Beatty sequence $V = (4\lfloor n\varphi\rfloor + 3n + 2)$. The first element 2 in $v^{-1}(1)$ is obtained by letting V start at $n = 0$ instead of $n = 1$. □

Remark 26. Some of the sequences above are images of Sturmian sequences by a morphism. Namely $v = \ell(g^\infty(0))$, $x_{\rm K} = k(g^\infty(0))$. Such sequences are examples of sequences called *quasi-Sturmian* in [Cassaigne 1998]. Their block complexity is of the form $n + C$ for n large enough ($C = 1$ for Sturmian sequences). These sequences were studied, e.g., in [Paul 1974/75; Coven 1974/75; Cassaigne 1998].

4. Generalized Beatty sequences and return words

In this section we show that generalized Beatty sequences are closely related to return words.

Theorem 27. *Let $x_{\rm F}$ be the Fibonacci word, and let w be any word in the Fibonacci language $\mathcal{L}_{\rm F}$. Let Y be the sequence of positions of the occurrences of w in $x_{\rm F}$. Then Y is a generalized Beatty sequence, i.e., for all $n \ge 0$, $Y(n+1) = p\lfloor n\varphi\rfloor + qn + r$ with parameters p, q, r, which can be explicitly computed.*

Proof. Let $x_{\rm F} = r_0(w)r_1(w)r_2(w)r_3(w)\ldots$, written as a concatenation of return words of the word w (cf. [Huang and Wen 2015], Lemma 1.2). According to Theorem 2.11 in [Huang and Wen 2015], if we skip $r_0(w)$, then the return words occur as the Fibonacci word on the alphabet $\{r_1(w), r_2(w)\}$. Thus the distances between occurrences of w in $x_{\rm F}$ are equal to $l_1 := |r_1(w)|$ and $l_2 := |r_2(w)|$. We can apply Lemma 8, which yields $p = l_1 - l_2$ and $q = 2l_2 - l_1$. Inserting $n = 0$, we find that $r = |r_0(w)| + 1$, as the first occurrence of w is at the beginning of $r_1(w)$. □

4.1. ***The Kimberling transform.*** Here we will obtain nonclassical triples appearing in another way, namely as the three indicator functions $x^{-1}(0)$, $x^{-1}(1)$ and $x^{-1}(2)$, of a sequence x on an alphabet

$\{0, 1, 2\}$ of three symbols. In our examples the sequence x is a transform $\mathcal{T}(x_{\rm F})$ of the Fibonacci word $x_{\rm F} = 01001010010010100\ldots$. These transforms have been introduced by Kimberling. Our main example is $\mathcal{T} : [001 \mapsto 2]$. Replacing each 001 in $x_{\rm F} = 01001010010010100\ldots$ by 2 gives $x_{\rm K} = 01201220120\ldots$.

For the transform method $\mathcal{T}$ we can derive a general result similar to Theorem 27. However, since Kimberling applies the StringReplace procedure from Mathematica, which replaces occurrences of w consecutively from left to right, we do not obtain a sequence of return words in the case that w has overlaps in $x_{\rm F}$. This restricts the number of words w to which Theorem 29 below applies.

Definition 28 [Wen and Wen 1994, pp. 593–594]. Let w be a factor of the Fibonacci word $x_{\rm F}$. We say that w has an *overlap* in $x_{\rm F}$ if there exist nonempty words x, y and z such that $w = xy = yz$, and the word xyz is a factor of the Fibonacci word.

Theorem 29. *Let $x_{\rm F}$ be the Fibonacci word, and let w be a factor of $x_{\rm F}$ that has no overlap in $x_{\rm F}$. Consider the transform $\mathcal{T}(x_{\rm F})$, which replaces every occurrence of the word w in $x_{\rm F}$ by the letter 2. Let Y be the sequence $(\mathcal{T}(x_{\rm F}))^{-1}(2)$, i.e., the positions of 2's in $\mathcal{T}(x_{\rm F})$. Then Y is a generalized Beatty sequence, i.e., for all $n \geq 1$, $Y(n) = p\lfloor n\varphi\rfloor + qn + r$, with parameters p, q, r, which can be explicitly computed.*

Proof. As in the proof of Theorem 27, let $x_{\rm F} = r_0(w)r_1(w)r_2(w)\ldots$, written as a concatenation of return words of the word w. Now the distances between 2's in $\mathcal{T}(x_{\rm F})$ are equal to $l_1 := |r_1(w)| - |w| + 1$ and $l_2 := |r_2(w)| - |w| + 1$. We can apply Lemma 8, which gives $p = l_1 - l_2$, $q = 2l_2 - l_1$. Inserting $n = 1$, we find that $r = |r_0(w)| - l_2 + 1$. □

Example 30. We take $\mathcal{T} : [001 \mapsto 2]$, with image $\mathcal{T}(x_{\rm F}) = 01201220120\ldots$, so $Y = (3, 6, 7, 10, \ldots)$. Here $r_0(w) = 01$, $r_1(w) = 00101$, $r_2(w) = 001$. This gives $l_1 = 3$, $l_2 = 1$, implying $p = 2$, $q = -1$ and $r = 2$. So Y is the generalized Beatty sequence $(Y_n)_{n\geq 1} = (2\lfloor n\varphi\rfloor - n + 2)_{n\geq 1}$.

The question arises whether not only $\mathcal{T}(x_{\rm F})^{-1}(2)$, but also $\mathcal{T}(x_{\rm F})^{-1}(0)$ and $\mathcal{T}(x_{\rm F})^{-1}(1)$ are generalized Beatty sequences. In general this is not true. However, this holds for $\mathcal{T} : [001 \mapsto 2]$.

Theorem 31. *Let $\mathcal{T} : [001 \mapsto 2]$, and let $x_{\rm K} := \mathcal{T}(x_{\rm F})$. Then the three sequences $x_{\rm K}^{-1}(0)$, $x_{\rm K}^{-1}(1)$, $x_{\rm K}^{-1}(2)$ form a complementary triple of generalized Beatty sequences.*

Proof. According to Example 30 we have that $x_{\rm K}^{-1}(2) = (2\lfloor n\varphi\rfloor - n + 2)_{n\geq 1}$. Since clearly $x_{\rm K}^{-1}(0) = x_{\rm K}^{-1}(1) - 1$, our remaining task is to prove that $(Z(n))_{n\geq 0} := x_{\rm K}^{-1}(1)$ is a generalized Beatty sequence. The return word structure of the word $w = 001$ in $x_{\rm F}$ is given by

$$r_0(w) = 01, \quad r_1(w) = 00101, \quad r_2(w) = 001.$$

Note that $Z(0) = 2$, the 1 coming from $r_0(w)$. This is exactly the reason why it is convenient to start Z from index 0: the other 1's are coming from the $r_1(w)$'s — note that $r_2(w)$ is mapped to 2.

The differences between the indices of occurrences of 2 in $x_{\rm K}$ are given by the Fibonacci word $3133131331\ldots$, which codes the appearance of the words $r_1(w)$ and $r_2(w)$. Therefore, to obtain the differences between the indices of occurrences of 1 in $x_{\rm K}$, we have to map the word $w' = 13$ to 4, obtaining the word $u = 343443\ldots$. To obtain a description of u, we apply Theorem 27 a second time with $w' = 13$. We have $r_0(w') = 3$, $r_1(w') = 133$, $r_2(w') = 13$. So $l_1 = |r_1(w')| - |w'| + 1 = 2$, and $l_2 = |r_2(w')| - |w'| + 1 = 1$, which gives $p = l_1 - l_2 = 1$, $q = 2l_2 - l_1 = 0$. The conclusion is that positions of 4 in u are given by the generalized Beatty sequence $(\lfloor n\varphi\rfloor+1)_{n\geq 1}$. This forces that u is nothing else than the Fibonacci word on {4, 3}, preceded by 3. But then Z is a generalized Beatty sequence with parameters $p=1$, $q=2$. Since $Z(1)=5$, we must have $r=2$, which happens to fit perfectly with the value $Z(0)=2$. □

Here is an example where $\mathcal{T}(x_{\mathrm{F}})^{-1}(0)$ and $\mathcal{T}(x_{\mathrm{F}})^{-1}(1)$ are *not* generalized Beatty sequences.

Example 32. We take $\mathcal{T}:[00100\mapsto 2]$, with image $\mathcal{T}(x_{\mathrm{F}})=010010121010010121012\ldots$, so $Y=(8,17,21\ldots)$. Here $r_0(w)=0100101$, $r_1(w)=0010010100101$, $r_2(w)=00100101$. This gives $l_1=9$ and $l_2=4$, so $p=5$ and $q=-1$ and $r=4$. Therefore Y is the generalized Beatty sequence $(Y_n)_{n\geq 1}=(5\lfloor n\varphi\rfloor-n+4)_{n\geq 1}$. The positions of 0 are given by $(\mathcal{T}(x_{\mathrm{F}}))^{-1}(0)=1,3,4,6,10,12,13,\ldots$, with difference sequence $2,1,2,4,2,1,\ldots$, so by Lemma 8 this sequence is not a generalized Beatty sequence. However, it can be shown that $(\mathcal{T}(x_{\mathrm{F}}))^{-1}(0)$ is a union of four generalized Beatty sequences, and the same holds for $(\mathcal{T}(x_{\mathrm{F}}))^{-1}(1)$.

Here is the general result.

Theorem 33. *For a nonoverlapping word w from the Fibonacci language let $\mathcal{T}:[w\mapsto 2]$, and let $Y:=\mathcal{T}(x_{\mathrm{F}})$. Suppose w satisfies*

$$|r_0(w)|\leq|r_1(w)|-|w|. \tag{SR0}$$

Then the three sequences $Y^{-1}(0)$, $Y^{-1}(1)$, $Y^{-1}(2)$ are finite unions of generalized Beatty sequences.

Note that we already know by Theorem 29 that $Y^{-1}(2)$ is a single generalized Beatty sequence. Condition (SR0) states that $r_0(w)$ is short relative to $r_1(w)$.

For the proof of Theorem 33 one needs the following proposition.

Proposition 34. *Let w be a word from the Fibonacci language, and let $r_0(w)r_1(w)r_2(w)\ldots$ be the return sequence of w in the Fibonacci word x_{F}. Then*:

(i) *$r_0(w)$ is a suffix of $r_1(w)$.*

(ii) *If $r_2(w)=wt_2(w)$, then $t_2(w)$ is a suffix of $r_1(w)$.*

Proof. Let $s_0=1$, $s_1=00$, $s_2=101$, $s_3=00100,\ldots$ be the singular words introduced in [Wen and Wen 1994]. According to [Huang and Wen 2015, Theorem 1.9] there is a unique largest singular word s_k occurring in w, so we can write $w=\mu_1 s_k\mu_2$, for two words μ_1,μ_2 from the Fibonacci language. It is known — see [Wen and Wen 1994] and the remarks after [Huang and Wen 2015, Proposition 1.6] — that the two return words of the singular word s_k are

$$r_1(s_k)=s_ks_{k+1},\quad r_2(s_k)=s_ks_{k-1}.$$

According to [Huang and Wen 2015, Lemma 3.1], the two return words of w are given by

$$r_1(w)=\mu_1r_1(s_k)\mu_1^{-1},\quad r_2(w)=\mu_1r_2(s_k)\mu_1^{-1}.$$

Substituting the first equation in the second, we obtain the key equation

$$r_1(w)=\mu_1s_ks_{k+1}\mu_1^{-1},\quad r_2(w)=\mu_1s_ks_{k-1}\mu_1^{-1}. \tag{6}$$

Proof of (i): We compare the return word decompositions of x_{F} by s_k and by w:

$$r_0(s_k)r_1(s_k)r_2(s_k)r_1(s_k)\ldots = r_0(w)r_1(w)r_2(w)r_1(w)\ldots$$
$$= r_0(w)\mu_1 r_1(s_k)\mu_1^{-1}\mu_1 r_2(s_k)\mu_1^{-1}\mu_1 r_1(s_k)\mu_1^{-1}\ldots.$$

It follows that $r_0(s_k) = r_0(w)\mu_1$, and so $r_0(w) = r_0(s_k)\mu_1^{-1}$. By [Huang and Wen 2015, Lemma 2.3], $r_0(s_k)$ equals s_{k+1}, with the first letter deleted. Thus we obtain from (6) that $r_0(w)$ is a suffix of $r_1(w)$.

Proof of (ii): Since $s_{k+1} = s_{k-1}s_{k-3}s_{k-1}$, by [Wen and Wen 1994, Property 2], we can make the following computation, starting from (6):

$$r_1(w) = \mu_1 s_k s_{k+1}\mu_1^{-1} = w\mu_2^{-1}s_{k+1}\mu_1^{-1} = w\mu_2^{-1}s_{k-1}s_{k-3}s_{k-1}\mu_1^{-1} = w\mu_2^{-1}s_{k-1}s_{k-3}\mu_2\mu_2^{-1}s_{k-1}\mu_1^{-1}.$$

For $r_2(w)$ we have

$$r_2(w) = \mu_1 s_k s_{k-1}\mu_1^{-1} = w\mu_2^{-1}s_{k-1}\mu_1^{-1}.$$

Now note that in this concatenation μ_2^{-1} cancels against a suffix of w. We claim that it also cancels against a prefix of s_{k-1}. This follows, since by [Huang and Wen 2015, Proposition 2.5] *any* occurrence of s_k in x_{F} is directly followed by a $s_{k+1} = s_{k-1}s_{k-3}s_{k-1}$ with the last letter deleted. It now follows that $t_2(w) = \mu_2^{-1}s_{k-1}\mu_1^{-1}$, and we see that this word is a suffix of $r_1(w)$. □

Proof of Theorem 33. In view of property (ii) in Proposition 34, the return words of w can be written as

$$r_1(w) = w\, m_1(w)\, t_2(w), \quad r_2(w) = w\, t_2(w),$$

for some words $m_1(w)$ and $t_2(w)$. Let $Z := Y^{-1}(2)$ be the positions of the letter 2. If $t_2(w)$ is nonempty, then any letter in $t_2(w)$ occurs in $Y := \mathcal{T}(x_{\mathrm{F}})$ in positions which are just a shift $-|t_2(w)|, \ldots, -1$ of Z, so each letter occurs according to a generalized Beatty sequence. The word $m_1(w)$ is never empty, and any letter in $m_1(w)$ occurs in $Y = \mathcal{T}(x_{\mathrm{F}})$ in positions which are a shift of a subsequence of Z (except, possibly, for the first occurrence, which then is in $r_0(w)$). This subsequence is obtained by replacing the distances ℓ_1 and ℓ_2 of ΔZ by $\ell_1+\ell_2$ and ℓ_1. Moreover, these distances occur as the Fibonacci word x_{F} on the alphabet $\{\ell_1+\ell_2, \ell_1\}$, because x_{F} is invariant under $0 \mapsto 01$, $1 \mapsto 0$. Thus each letter in $m_1(w)$ occurs according to a generalized Beatty sequence. All these $|t_2(w)|+|m_1(w)|$ sequences start at index 1. If we let the last $|r_0(w)|$ of these sequences start at index 0, then we have taken into account *all* elements of Y. This works, because of property (i) in Proposition 34. □

Here is an example where (SR0) is not satisfied.

Example 35. We take $\mathcal{T} : [10100 \mapsto 2]$, with image $\mathcal{T}(x_{\mathrm{F}}) = 01002100221002\ldots$, so $Y = (5, 9, 10\ldots)$. Here $r_0(w) = 0100$, $r_1(w) = 10100100$, $r_2(w) = 10100$. The positions of 0 are given by the sequence $(\mathcal{T}(x_{\mathrm{F}}))^{-1}(0) = 1, 3, 4, 7, 8\ldots$, which can be written as a union of two generalized Beatty sequences, *except* that the position 1 from the first 0 in $\mathcal{T}(x_{\mathrm{F}})$ will not be in this union.

With (6) we can deduce an equivalent simple formulation of condition (SR0). If $w = \mu_1 s_k \mu_2$, then $r_0(w)$ equals $s_{k+1}\mu_1^{-1}$ with the first letter removed, and $r_1(w) = \mu_1 s_k s_{k+1}\mu_1^{-1}$, so

$$|w| = |\mu_1| + F_k + |\mu_2|, \quad |r_0(w)| = F_{k+1} - |\mu_1| - 1, \quad |r_1(w)| = F_{k+1} + F_k.$$

Filling this into condition (SR0) we obtain

$$|\mu_2| \leq 1. \tag{SR0$'$}$$

Using this condition, together with Theorem 6 in [Wen and Wen 1994], one can show that Theorem 33 does apply to at most 3 words w of length m, for all $m \geq 2$ (in fact, only 2, if m is not a Fibonacci number).

Acknowledgments

We thank the referee for useful comments. We also thank S. Levy, from the publisher's editorial staff, for several improvements to our text.

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Received 28 Dec 2018. Revised 22 May 2019.

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Moscow Journal of Combinatorics and Number Theory (ISSN 2640-7361 electronic, 2220-5438 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

MJCNT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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nonprofit scientific publishing

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Moscow Journal of Combinatorics and Number Theory
vol. 8 no. 4 2019