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On the domination number of a graph defined by containment

Peter Frankl

Let $n > k > 2$ be integers. Define a bipartite graph between all k-element and all 2-element subsets of an n-element set by drawing an edge if and only if the first one contains the second. The domination number of this graph is determined up to a factor of $1 + o(1)$. The short proof relies on some extremal results concerning hypergraphs.

1. Introduction

For a graph $\mathcal{G} = (V, \mathcal{E})$ a subset $D \subset V$ is called a *dominating set* if for every vertex $x \in V \setminus D$ there is an edge $E \in \mathcal{E}$ satisfying $x \in E$ and $E \cap D \neq \varnothing$. The *domination number* $\varrho(\mathcal{G})$ is the minimum of $|D|$ over all dominating sets.

To determine $\varrho(\mathcal{G})$ for a given graph is very difficult in general. In the present paper we address this problem for a bipartite graph defined via containments of sets.

For n and k positive integers, with $n > k$, we denote by $[n] = \{1, 2, \ldots, n\}$ the standard n-element set and by $\binom{[n]}{k}$ the collection of all k-element subsets of $[n]$. For integers $n > k > \ell \geq 2$, we define the bipartite graph $\mathcal{B} = \mathcal{B}_n(k, \ell)$ on the vertex set $\binom{[n]}{k} \cup \binom{[n]}{\ell}$ by drawing an edge between $F \in \binom{[n]}{k}$ and $G \in \binom{[n]}{\ell}$ if and only if $G \subset F$.

The problem of determining or estimating $\varrho(\mathcal{B})$ was raised in [Badakhshian et al. 2019] by Badakhshian, Katona and Tuza. They determined $\varrho(\mathcal{B}_n(3, 2))$ up to a factor $1 + o(1)$, where $o(1) \to 0$ as $n \to \infty$.

In the present paper we extend their work to all $k \geq 3$.

Theorem 1.1.
$$\varrho\big(\mathcal{B}_n(k, 2)\big) = (1 + o(1))\binom{n}{2}\frac{k+3}{k^2-1}.$$

To prove the lower bound we use a result from [Erdős et al. 1986] extending the celebrated Ruzsa–Szemerédi theorem [1978]. To obtain the matching upper bound we apply a probabilistic construction based on a result of [Frankl and Rödl 1985]. To prove similar results for $\varrho\big(\mathcal{B}_n(k, \ell)\big)$ where $\ell \geq 3$ appears to be much harder (Section 4).

2. Proof of the lower bound

Let $k \geq 3$ be fixed and $\varepsilon > 0$ be arbitrarily small. Choose $\mathcal{G} \subset \binom{[n]}{2}$ and $\mathcal{F} \subset \binom{[n]}{k}$ such that $\mathcal{F} \cup \mathcal{G}$ is a dominating set for $\mathcal{B} = \mathcal{B}_n(k, 2)$. Our aim is to prove

$$|\mathcal{F}| + |\mathcal{G}| > \binom{n}{2}\left(\frac{k+3}{k^2-1} - \varepsilon\right). \tag{1}$$

This research was done while the author was visiting Academia Sinica in Taipei.

Keywords: finite sets, graphs, hypergraphs, Turán's theorem.

Since $\frac{k+3}{k^2-1} \le \frac{3}{4}$ for $k \ge 3$, we may assume that

$$|\mathcal{F}| \le \frac{3}{4}\binom{n}{2}. \tag{2}$$

Proposition 2.1. $$|\mathcal{G}| > \frac{1-\varepsilon}{k-1}\binom{n}{2} \quad \textit{for all } n > n_0(k,\varepsilon).$$

Proof of the proposition. Let m be an integer (later qualified) and consider an m-element set $R \subset [n]$. If R contains no $F \in \mathcal{F}$, then the assumption on domination is equivalent to the fact that $\mathcal{G}_{|R} := \mathcal{G} \cap \binom{R}{2}$ has no independent set of k vertices. By Turán's theorem [1941] (or see [Bollobás 1978]), we have

$$\left|\mathcal{G} \cap \binom{R}{2}\right| > (k-1)\binom{m/(k-1)}{2} = \frac{m(m-k+1)}{2(k-1)}$$
$$> \binom{m}{2}\frac{1-\varepsilon/2}{k-1} \quad \text{for } m > 2k/\varepsilon. \tag{3}$$

We now assume m is large enough that (3) is satisfied. Let us choose the set $P \in \binom{[n]}{m}$ uniformly at random.

Claim 2.2. *Let $n > m^3/\varepsilon$. Then the probability of $\binom{P}{k} \cap \mathcal{F} \ne \varnothing$ is smaller than $\varepsilon/2$.*

Proof. Since each $F \in \mathcal{F}$ is contained in $\binom{n-k}{m-k}$ subsets $R \in \binom{[n]}{m}$, (2) implies the upper bound $\frac{3}{4}\binom{n}{2}\binom{n-k}{m-k}$ on the number of R in question. Using $k \ge 3$ we obtain the upper bound

$$\frac{3}{4}\binom{n}{2}\binom{n-3}{m-3} = \binom{n}{m}\cdot\frac{m-2}{n-2}\binom{m}{2}\cdot\frac{3}{4} < \binom{n}{m}\frac{m^3}{2n} < \frac{\varepsilon}{2}\binom{n}{m}.$$

In view of the claim, for $n > m^3/\varepsilon$ a proportion of more than $(1-\varepsilon/2)$ of $R \in \binom{[n]}{m}$ satisfy (3). Now $(1-\varepsilon/2)^2 > 1-\varepsilon$ implies the inequality in Proposition 2.1, with $n_0(k,\varepsilon) > (2k/\varepsilon)^3/\varepsilon$. □

Let $\mathcal{H} = \binom{[n]}{2} \setminus \mathcal{G}$ be the graph of those edges $H \in \binom{[n]}{2}$ that are not in $\mathcal{G}$. Since $\mathcal{F} \cup \mathcal{G}$ is a dominating set for $\mathcal{B}$, for each $H \in \mathcal{H}$ there exists some $F \in \mathcal{F}$ with $H \subset F$. From this we infer

$$|\mathcal{F}| \ge \frac{|\mathcal{H}|}{\binom{k}{2}}. \tag{4}$$

Using (4) together with Proposition 2.1 one can show that

$$|\mathcal{F}| + |\mathcal{G}| \ge \frac{1-\varepsilon}{k-1}\binom{n}{2} + \frac{k-2+\varepsilon}{(k-1)}\,\frac{\binom{n}{2}}{\binom{k}{2}}$$

which is slightly weaker than (1). To prove (1), we would need (4) with $\binom{k}{2}-1$ in the denominator.

Our strategy is relatively simple. We try and list (some of) the edges of $\mathcal{F}$: $F_1, F_2, \dots, F_q$ such that $\binom{F_1}{2} \cap \mathcal{G} \ne \varnothing$, then $\binom{F_2}{2} \cap \left(\mathcal{G} \cup \binom{F_1}{2}\right) \ne \varnothing$, etc. That is, we choose sequentially F_i, $1 \le i \le q$, so that $\binom{F_i}{2} \cap \mathcal{G} \ne \varnothing$ or $|F_j \cap F_i| \ge 2$ for some $1 \le j < i$. For each F_i let $\mathcal{E}(F_i)$ consist of those $E \in \mathcal{H}$ that $E \notin F_j$ for $1 \le j < i$. From the construction it follows that

$$\left|\mathcal{E}(\mathcal{F}_i)\right| \le \binom{k}{2} - 1 \quad \text{for all } 1 \le i \le q. \tag{5}$$

Should $\mathcal{F} = \{F_1, \ldots, F_q\}$ hold, (1) would follow. In the opposite case set $\mathcal{F}_0 = \{F_1, \ldots, F_q\}$ and $\mathcal{H}_0 = \left(\binom{F_1}{2} \cup \ldots \cup \binom{F_q}{2}\right) \setminus \mathcal{G}$.

Choosing q maximal, $\binom{F}{2} \cap \mathcal{G} = \varnothing$ and $|F \cap F_i| \leq 1$ follow for $F \in \mathcal{F} \setminus \mathcal{F}_0$, $1 \leq i \leq q$.

We define $\mathcal{F}_1 = \{F_1, \ldots, F_{q_1}\}$ similarly. We choose $F_1 \in \mathcal{F} \setminus \mathcal{F}_0$ arbitrarily and once $F_1, \ldots, F_{s-1} \in \mathcal{F} \setminus \mathcal{F}_0$ are fixed, we choose an arbitrary $F_s \in \mathcal{F} \setminus \mathcal{F}_0$ from the rest, satisfying $|F_i \cap F_s| \geq 2$ for some $1 \leq j < s$. Now let $\mathcal{F}_1$ be a maximal collection obtained in this way. This choice guarantees $|F \cap F'| \leq 1$ for all $F \in \mathcal{F} \setminus (\mathcal{F}_0 \cup \mathcal{F}_1)$, $F' \in \mathcal{F}_1$.

Set $\mathcal{H}_1 = \bigcup_{F \in \mathcal{F}_1} \binom{F}{2}$. Our procedure guarantees

$$|\mathcal{H}_1| \leq 1 + |\mathcal{F}_1| \left(\binom{k}{2} - 1\right). \tag{6}$$

We iterate this procedure. Once $\mathcal{F}_1, \ldots, \mathcal{F}_p$ and thereby $\mathcal{H}_i = \bigcup_{F \in \mathcal{F}_i} \binom{F}{2}$, $1 \leq i \leq p$ are chosen we have

$$|F \cap F'| \leq 1 \quad \text{for all } F \in \mathcal{G} \cup \mathcal{F}_0 \cup \ldots \cup \mathcal{F}_p \text{ and } F' \in \mathcal{F} \setminus (\mathcal{F}_0 \cup \ldots \cup \mathcal{F}_p).$$

As long as there are sets remaining in $\mathcal{F}$ we can define $\mathcal{F}_{p+1}$ and $\mathcal{H}_{p+1}$ in the above way.

Eventually we obtain a partition,

$$\mathcal{F} = \mathcal{F}_0 \sqcup \ldots \sqcup \mathcal{F}_t$$

such that

$$\mathcal{H}_0 \sqcup \ldots \sqcup \mathcal{H}_t = \binom{[n]}{2} \setminus \mathcal{G}$$

(here we used that $\mathcal{G} \cup \mathcal{F}$ is a dominating set). Moreover (6) holds for 1 replaced by i:

$$|\mathcal{H}_i| \leq 1 + |\mathcal{F}_i| \left(\binom{k}{2} - 1\right), \qquad 1 \leq i \leq t. \tag{7}$$

Since for $i = 0$ we do not need the extra 1, we infer

$$\binom{n}{2} - |\mathcal{G}| \leq t + |\mathcal{F}| \left(\binom{k}{2} - 1\right),$$

or equivalently

$$|\mathcal{G}| + |\mathcal{F}| \geq \frac{\binom{n}{2}}{\binom{k}{2} - 1} + |\mathcal{G}| \frac{\binom{k}{2} - 2}{\binom{k}{2} - 1} - \frac{t}{\binom{k}{2} - 1}.$$

Substituting $|\mathcal{G}| > \frac{1-\varepsilon}{k-1}\binom{n}{2}$ we obtain

$$\begin{aligned} |\mathcal{G}| + |\mathcal{F}| &> \frac{\binom{n}{2}}{\binom{k}{2} - 1} \left(1 + \frac{\binom{k}{2} - 2}{k-1} - \frac{\varepsilon}{k-1}\right) - \frac{t}{\binom{k}{2} - 1} \\ &= \binom{n}{2} \left(\frac{k+3}{k^2 - 1} - \frac{2\varepsilon}{(k^2-1)(k-2)}\right) - \frac{t}{\binom{k}{2} - 1}. \end{aligned}$$

To conclude the proof of the lower bound it is clearly more than sufficient to show that $t = o\left(\binom{n}{2}\right)$. To achieve this we will need the following extension of a celebrated result from [Ruzsa and Szemerédi 1978]:

Theorem 2.3 (Erdős, Frankl, Rödl [Erdős et al. 1986]). *Suppose that* $\mathcal{T} \subset \binom{[n]}{k}$ *satisfies* $|T \cap T'| \leq 1$ *for all distinct* $T, T' \in \mathcal{T}$, *moreover one cannot find a* k*-set* $\{x_1, \ldots, x_k\} \subset [n]$ *and* $\binom{k}{2}$ *distinct members* $T(i, j) \in \mathcal{T}$, $1 \leq i < j \leq k$, *such that* $\{x_i, x_j\} \subset T(i, j)$. *Then*

$$|\mathcal{T}| = o\left(\binom{n}{2}\right). \tag{8}$$

To apply (8) we choose $F(i)$ as an arbitrary member of $\mathcal{F}_i$ for $1 \leq i \leq t$ and define

$$\mathcal{T} = \{F(i) : 1 \leq i \leq t\}.$$

The condition $|T \cap T'| \leq 1$ is automatically satisfied. To prove the second condition we argue indirectly.

Suppose that we found $F = \{x_1, \ldots, x_k\}$ and $\binom{k}{2}$ members $T(i, j) \in \mathcal{T}$ such that $\{x_i, x_j\} \subset T(i, j)$. Since $\mathcal{F} \cup \mathcal{G}$ is a dominating set for B, either $F \in \mathcal{F}$ or $G \subset F$ for some $G \in \mathcal{G}$. In the latter case $G = \{x_i, x_j\}$ for some $1 \leq i < j \leq k$. I.e., $G \subset T(i, j)$. But this is impossible since we put all such $T(i, j)$ into $\mathcal{F}_0$. Suppose next $F \in \mathcal{F}$. Assume by symmetry $T(1, 2) \in \mathcal{F}_1$, $T(1, 3) \in \mathcal{F}_2$. From $|T(1, \ell) \cap F| \geq 2$ we infer $F \in \mathcal{F}_{\ell-1}$ for $\ell = 2, 3$. This is impossible because of $\mathcal{F}_1 \cap \mathcal{F}_2 = \varnothing$, giving the desired contradiction. □

3. The proof of the upper bound

We give a probabilistic construction based on the following old result.

Let $r \geq 2$ be an integer and consider an r-uniform hypergraph $\mathcal{H} \subset \binom{X}{r}$, where $|X| = m$. For $x \in X$ let $d(x)$ be the degree of x in $\mathcal{H}$, that is, the number of $H \in \mathcal{H}$ containing x. The *double degree* $d(x, y)$ is defined analogously.

The *covering index* $b(\mathcal{H})$ is defined as the minimal number b such that there exist b edges in $\mathcal{H}$ whose union is equal to X. Obviously, $b(\mathcal{H}) \geq m/r$.

Theorem 3.1 [Frankl and Rödl 1985]. *Let* β, ε *be positive constants*, $r \geq 2$ *fixed. There exists* $\delta = \delta(r, \beta, \varepsilon)$ *such that, for every* $\mathcal{H} \subset \binom{X}{r}$ *satisfying*

(i) $\left|d(x) - |\mathcal{H}|r/m\right| < \delta|\mathcal{H}|/m$ ***or***

(ii) $d(x, y) < |\mathcal{H}|r/m^{1+\beta}$,

one has $b(\mathcal{H}) < (1 + \varepsilon)m/r$.

Now we are ready to explain the construction of a nearly optimal dominating set for $\mathcal{B}_n(k, 2)$, $k \geq 3$. (Badakhshian et al. [2019] use the same construction for the case $k = 3$.)

Let $n = p(k-1) + q$, $0 \leq q < k - 1$ and let $[n] = X_1 \sqcup \ldots \sqcup X_{k-1}$ be a partition with $p \leq |X_i| \leq p + 1$. Let $\mathcal{G} := \bigcup_{1 \leq i < k} \binom{X_i}{2}$ be the so-called *Turán graph*. By the pigeonhole principle, $\mathcal{G}$ dominates all k-sets in $\mathcal{B}_n(k, 2)$.

Set $r = \binom{k}{2} - 1$. We define an r-uniform hypergraph $\mathcal{H}$ on the partite set $\binom{[n]}{2}$ from $\mathcal{B}_n(k, 2)$. Note that for every k-set $F \subset [n]$ satisfying $F \cap X_i \neq \varnothing$ for $1 \leq i < k$ there is exactly one $j = j(F)$ such that $|F \cap X_j| = 2$. With such an F we associate the r-set $H(F) = \binom{F}{2} \setminus \{F \cap X_j\}$. Let $\mathcal{H}$ be the r-graph formed by these $H(F)$. The actual vertex set of $\mathcal{H}$ is

$$X = \binom{[n]}{2} \setminus \left(\binom{X_1}{2} \cup \ldots \cup \binom{X_{k-1}}{2} \right);$$

that is, the number of vertices is $m \sim \frac{k-2}{k-1}\binom{n}{2}$.

If $|X_1| = \cdots = |X_{k-1}|$, then $\mathcal{H}$ is regular but even in the general case it is nearly regular. That is, (i) holds for $m > m(\delta)$.

Since $|\mathcal{H}| = (k-1+o(1))p^k/2$ and $|\mathcal{H}(x, y)| < p^{k-3}$, (ii) is satisfied with e.g. $\beta = \frac{1}{3}$ if $m > m_0(k, \beta)$.

Applying Theorem 3.1 we obtain a covering of X which is, say, formed by the edges $H(F_1), \ldots, H(F_b)$, $b < (1+\varepsilon)m/r$.

Let $\mathcal{F} = \{F_1, \ldots, F_b\}$ be the corresponding family in $\binom{[n]}{k}$. Then $\mathcal{G} \cup \mathcal{F}$ is a dominating set for $\mathcal{B}_n(k, 2)$. Substituting $m = (1+o(1))\frac{k-2}{k-1}\binom{n}{2}$, $r = \binom{k}{2} - 1$, we infer

$$|\mathcal{G} \cup \mathcal{F}| \le \binom{n}{2}\left(\frac{1}{k-1} + \frac{k-2}{k-1} \cdot \frac{1}{\binom{k}{2}-1} + \varepsilon\right) = \binom{n}{2}\left(\frac{k+3}{k^2-1} + \varepsilon\right).$$

Since $\varepsilon > 0$ was arbitrary, this concludes the proof of the upper bound in Theorem 1.1. □

4. The general problem

Let us say a few words about $\varrho\big(\mathcal{B}_n(k, \ell)\big)$ in the case $\ell \ge 3$. One would imagine that to find a small dominating set imitating the strategy used for $\ell = 2$ should be the best. However, that means that first we choose $\mathcal{G} \subset \binom{[n]}{\ell}$ covering the whole of $\binom{[n]}{k}$, that is, for every $F \in \binom{[n]}{k}$ there exists $G \in \mathcal{G}$ with $G \subset F$.

The problem is that we do not know the minimal size, $|\mathcal{G}|$ for such families. It is the famous Turán's Problem (cf. [Turán 1961]) which is still open for all pairs (k, ℓ), $k > \ell \ge 3$.

At the same time there are some plausible conjectures. For example Turán [Turán 1961] conjectured that in the case $k = 5$, $\ell = 3$ and $n > n_0(k, \ell)$ the best construction is $\mathcal{G} = \binom{X}{3} \cup \binom{Y}{3}$ where $X \cup Y = [n]$ is a partition and $|X| = \lfloor \frac{n}{2} \rfloor$. Using this $\mathcal{G}$ one can use the approach of Section 3 and show that

$$\varrho\big(\mathcal{B}_n(5, 3)\big) \le (1+o(1))\left(\frac{1}{4} + \frac{3}{4}\frac{1}{\binom{5}{3}-1}\right)\binom{n}{3} = \left(\frac{1}{3} + o(1)\right)\binom{n}{3}. \tag{9}$$

Using the results of [Frankl and Rödl 2002] one can prove the matching lower bound assuming that Turán's conjecture is true.

The situation is pretty much the same for other pairs (k, ℓ) whenever the conjectured optimal family for Turán's Problem is a "highly regular" ℓ-graph.

Let us close this paper with a conjecture.

Conjecture 4.1. $\varrho\big(\mathcal{B}_n(2\ell-1, \ell)\big) = (1+o(1))\left(\frac{1}{2^{\ell-1}} + \left(1 - \frac{1}{2^{\ell-1}}\right) \Big/ \left(\binom{2\ell-1}{\ell} - 1\right)\right)\binom{n}{3}.$

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References

[Badakhshian et al. 2019] L. Badakhshian, G. O. H. Katona, and Z. Tuza, "The domination number of the graph defined by two levels of the n-cube", *Discrete Appl. Math.* **266** (2019), 30–37. MR

[Bollobás 1978] B. Bollobás, *Extremal graph theory, with emphasis on probabilistic methods*, London Math. Soc. Monographs **11**, Academic Press, 1978. Reprinted Dover, Mineola (NY), 2004. MR

[Erdős et al. 1986] P. Erdős, P. Frankl, and V. Rödl, "The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent", *Graphs Combin.* **2**:2 (1986), 113–121. MR

[Frankl and Rödl 1985] P. Frankl and V. Rödl, "Near perfect coverings in graphs and hypergraphs", *European J. Combin.* **6**:4 (1985), 317–326. MR Zbl

[Frankl and Rödl 2002] P. Frankl and V. Rödl, "Extremal problems on set systems", *Random Structures Algorithms* **20**:2 (2002), 131–164. MR

[Ruzsa and Szemerédi 1978] I. Z. Ruzsa and E. Szemerédi, "Triple systems with no six points carrying three triangles", pp. 939–945 in *Combinatorics: Proceedings of the Fifth Hungarian Colloquium* (Keszthely, 1976), vol. 2, Colloq. Math. Soc. János Bolyai **18**, North-Holland, Amsterdam, 1978. MR Zbl

[Turán 1941] P. Turán, "On an extremal problem in graph theory", *Mat. Fiz. Lapok* **48** (1941), 436–452. In Hungarian. MR Zbl

[Turán 1961] P. Turán, "Research problems", *Publ. Math. Inst. Hungar. Acad. Sci.* **6** (1961), 417–423. MR

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