Moscow Journal of Combinatorics and Number Theory

2019
vol. 8 no. 4

A new explicit formula for Bernoulli numbers involving the Euler number

Sumit Kumar Jha

msp

Moscow Journal of Combinatorics and Number Theory
Vol. 8, No. 4, 2019
dx.doi.org/10.2140/moscow.2019.8.385

A new explicit formula for Bernoulli numbers involving the Euler number

Sumit Kumar Jha

We derive a new explicit formula for Bernoulli numbers in terms of the Stirling numbers of the second kind and the Euler numbers. As a corollary of our result, we obtain an explicit formula for the even Euler numbers in terms of the Stirling numbers of the second kind.

Definition 1. The *Bernoulli numbers* B_n can be defined by the generating function

$$\frac{t}{e^t-1}=\sum_{n\geq 0}\frac{B_n t^n}{n!},$$

where $|t|<2\pi$.

Definition 2. A *Stirling number of the second kind*, denoted by $S(n,m)$, is the number of ways of partitioning a set of n elements into m nonempty sets.

There are many known explicit formulas known for the Bernoulli numbers [Gould 1972; Jha 2019]. The following formulas express the Bernoulli numbers explicitly in terms of the Stirling numbers of the second kind:

$$B_r=\sum_{k=1}^{r}(-1)^k\cdot k!\,\frac{S(r,k)}{k+1},$$

$$(-1)^{r-1}B_r=\sum_{k=1}^{r}(-1)^k\frac{S(r,k)}{k+1}\cdot(k-1)!\,,$$

$$B_{r+1}=\frac{(-1)^r\cdot(r+1)\cdot 2^r}{2^{r+1}-1}\sum_{k=1}^{r}\frac{S(r,k)}{k+1}(-1)^k 2^{-2k}\frac{(2k-1)!}{(k-1)!}.$$

Definition 3. The *Euler numbers* are a sequence of integers, denoted by E_n, which can be defined by the Taylor series expansion

$$\frac{1}{\cosh t}=\frac{2}{e^t+e^{-t}}=\sum_{n=0}^{\infty}\frac{E_n}{n!}\cdot t^n,$$

where $\cosh t$ is the hyperbolic cosine.

We prove the following.

MSC2010: 11B68.
Keywords: Bernoulli numbers, Stirling numbers of the second kind, Euler numbers, polylogarithm function.

Theorem 4. *We have*

$$B_{r+1} = -\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))}\left(\sum_{k=1}^{r}(-1)^k \cdot \frac{S(r,k)}{k+1} \cdot \left(\tfrac{3}{4}\right)^{(k)} + 4^{-r} E_r\right), \tag{1}$$

where $S(r,k)$ denotes the Stirling numbers of the second kind, $x^{(n)} = (x)(x+1)\cdots(x+n-1)$ denotes the rising factorial, and E_r denotes the Euler number.

Proof. We begin with the result

$$\frac{\sin n\pi}{\pi}\int_0^\infty x^{n-1}\frac{\mathrm{Li}_s(-x)}{1+x}\,dx = \zeta(s) - \zeta(s, 1-n),$$

where $\mathrm{Li}_s(-x)$ denotes the polylogarithm function, $\zeta(s)$ is the Riemann zeta function, and $\zeta(s, 1-n)$ is the Hurwitz zeta function. The integral above is valid for all $s \in \mathbb{C} \setminus \{1\}$ and $0 < n < 1$. This integral can be obtained from formula 3.2.1.6 in [Brychkov et al. 2019].

Plugging $n = \frac{3}{4}$ and $s = -r$, a negative integer, into the integral above we get

$$\int_0^\infty x^{-1/4}\frac{\mathrm{Li}_{-r}(-x)}{1+x}\,dx = \sqrt{2}\pi\left(\frac{B_{r+1}\left(\frac{1}{4}\right) - B_{r+1}}{r+1}\right).$$

Now, we use the representation from [Landsburg 2009]

$$\mathrm{Li}_{-r}(-x) = \sum_{k=1}^{r} k!\, S(r,k)\left(\frac{1}{1+x}\right)^{k+1}(-x)^k,$$

which can be easily proved using induction on r.

As a result, we have

$$\begin{aligned}
\int_0^\infty x^{-1/4}\frac{\mathrm{Li}_{-r}(-x)}{1+x}\,dx &= \sum_{k=1}^{r}(-1)^k \cdot k!\, S(r,k)\int_0^\infty \frac{x^{k-1/4}}{(1+x)^{k+2}}\,dx \\
&= \sum_{k=1}^{r}(-1)^k \cdot k!\, S(r,k) \cdot \frac{\Gamma\left(k+\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(k+2)} \\
&= \sum_{k=1}^{r}(-1)^k \cdot \frac{S(r,k)}{k+1} \cdot \Gamma\left(k+\tfrac{3}{4}\right)\Gamma\left(\tfrac{5}{4}\right) \\
&= \sum_{k=1}^{r}(-1)^k \cdot \frac{S(r,k)}{k+1} \cdot \left(\tfrac{3}{4}\right)\cdot\left(\tfrac{3}{4}+1\right)\cdots\left(\tfrac{3}{4}+k-1\right)\Gamma\left(\tfrac{3}{4}\right)\Gamma\left(\tfrac{5}{4}\right) \\
&= \sum_{k=1}^{r}(-1)^k \cdot \frac{S(r,k)}{k+1} \cdot \left(\tfrac{3}{4}\right)\cdot\left(\tfrac{3}{4}+1\right)\cdots\left(\tfrac{3}{4}+k-1\right)\tfrac{1}{2\sqrt{2}}\pi \\
&= \sum_{k=1}^{r}(-1)^k \cdot \frac{S(r,k)}{k+1} \cdot \left(\tfrac{3}{4}\right)^{(k)}\tfrac{1}{2\sqrt{2}}\pi,
\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function.

But, from [Weisstein], we have

$$\frac{B_{r+1}\left(\frac{1}{4}\right)-B_{r+1}}{r+1}=\frac{(-2^{-(r+1)}(1-2^{-r})B_{r+1}-4^{-(r+1)}(r+1)E_r-B_{r+1})}{r+1}.$$

Thus, we have

$$B_{r+1}=-\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))}\left(\sum_{k=1}^{r}(-1)^k\cdot\frac{S(r,k)}{k+1}\cdot\left(\tfrac{3}{4}\right)^{(k)}+4^{-r}E_r\right).$$

□

If we let $r=2l$, an even integer, in (1) we immediately obtain:

Corollary 5.
$$E_{2l}=-4^{2l}\sum_{k=1}^{2l}(-1)^k\cdot\frac{S(2l,k)}{k+1}\cdot\left(\tfrac{3}{4}\right)^{(k)}.$$

References

[Brychkov et al. 2019] Y. A. Brychkov, O. I. Marichev, and N. V. Savischenko, *Handbook of Mellin transforms*, CRC Press, Boca Raton, FL, 2019. MR Zbl

[Gould 1972] H. W. Gould, "Explicit formulas for Bernoulli numbers", *Amer. Math. Monthly* **79** (1972), 44–51. MR Zbl

[Jha 2019] S. K. Jha, "Two new explicit formulas for the Bernoulli numbers", preprint, 2019. arXiv

[Landsburg 2009] S. E. Landsburg, "Stirling numbers and polylogarithms", unpublished note, 2009, http://www.landsburg.com/query.pdf.

[Weisstein] E. W. Weisstein, "Bernoulli polynomial", http://mathworld.wolfram.com/BernoulliPolynomial.html. From MathWorld.

Received 17 Jun 2019. Revised 24 Jul 2019.

SUMIT KUMAR JHA:
kumarjha.sumit@research.iiit.ac.in
International Institute of Information Technology, Hyderabad, India

Moscow Journal of Combinatorics and Number Theory

msp.org/moscow

See inside back cover or msp.org/moscow for submission instructions.

The subscription price for 2019 is US $310/year for the electronic version, and $365/year (+$20, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Moscow Journal of Combinatorics and Number Theory (ISSN 2640-7361 electronic, 2220-5438 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

MJCNT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

Moscow Journal of Combinatorics and Number Theory

vol. 8 no. 4 2019