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# Intersection theorems for $(0, \pm 1)$-vectors and $s$-cross-intersecting families

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We modify the statement and proof of Theorem 1 in "Intersection theorems for $\{0, \pm 1\}$-vectors and $s$-cross-intersecting families". A version of the paper that incorporates the errata is uploaded on the arXiv: https://arxiv.org/abs/1603.00938. We thank Danila Cherkashin and Sergei Kiselev for pointing out the error.

Part 2 of Theorem 1 in [Frankl and Kupavskii 2017] is incorrect. To give a corrected version, let us introduce some notation: $\mathcal{V}(n, m_1, m_2) \subset \{0, \pm 1\}^n$ is the collection of all vectors with exactly $m_1$ ones and $m_2$ minus ones, and

$$g(n, m_1, m_2) := \max\{|\mathcal{V}| : \mathcal{V} \subset \mathcal{V}(n, m_1, m_2) \text{ and } \langle \boldsymbol{v}, \boldsymbol{w} \rangle \geq -2m_2 + 1 \text{ for any } \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}\}.$$

**Theorem 1 (part 2).** *For $n \geq n_0(k)$ and $0 \leq l \leq k$ we have*

$$F(n, k, -l) = \begin{cases} \sum_{i=0}^{l/2} \binom{k}{i}\binom{n}{k} & \text{for even } l, \\ g\left(n, k - \frac{l+1}{2}, \frac{l+1}{2}\right) + \sum_{i=0}^{(l-1)/2} \binom{k}{i}\binom{n}{k} & \text{for odd } l. \end{cases}$$

Thus, the statement is the same for even $l$ and is different for odd $l$. The value of $g(n, m_1, m_2)$ seems to be very difficult to determine in general. We have studied this quantity in [Frankl and Kupavskii 2018a; 2018b]. In the former one, we determined the value of $g(n, m_1, 1)$ for any $n, m_1$. This allows us to determine exactly the value of $F(n, k, -1)$. In the latter one, we obtained the bounds

$$\binom{n}{k}\binom{k-1}{\frac{l-1}{2}} \leq g\left(n, k - \frac{l+1}{2}, \frac{l+1}{2}\right) \leq \binom{n}{k}\binom{k-1}{\frac{l-1}{2}} + \binom{n}{l+1}\binom{l+1}{\frac{l+1}{2}}\binom{n-l-2}{k-l-2}.$$

We go on to the proof. The proof is correct until (and including) Claim 3. Let us give the corrected version of the remainder of the proof. We first deal with the case of even $l$.

**Claim 4.** *Let $l$ be even. If $I \subset \binom{[n]}{l+1}$ is bad, then for at least $\binom{n-k}{k-l-1}$ sets $S \supset I$, $S \in \binom{[n]}{k}$, we have* $|\mathcal{V}_g(S)| \leq f(k, l) - 1$.

*Proof.* Consider the family $\mathcal{A} \subset 2^S$ of subsets of $S$ defined as $\mathcal{A} = \{N(\boldsymbol{w}) \cap S : \boldsymbol{w} \in \mathcal{V}_g(S)\}$. In view of the uniqueness part of Katona's theorem, it is sufficient to show that $\mathcal{A}$ does not contain one of the sets from the extremal family $\mathcal{U}^l$ for at least $\binom{n-k}{k-l-1}$ choices of $S$.

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If $I$ is bad then there exists a vector $\boldsymbol{v}$ of length $l+1$ such that both $\mathcal{V}(I, \boldsymbol{v})$ and $\mathcal{V}(I, \bar{\boldsymbol{v}})$ are nonempty. Assume without loss of generality that $|N(\boldsymbol{v})| \leq l/2$ and take a vector $\boldsymbol{w} \in \mathcal{V}$ such that $\boldsymbol{w}|_I = \bar{\boldsymbol{v}}$. Then for any $S$ such that $S \cap S(\boldsymbol{w}) = I$ the set $N(\boldsymbol{v})$ is missing from $\mathcal{A}$ (and, consequently, $|\mathcal{V}_g(S)| \leq f(k,l)-1$). There are exactly $\binom{n-k}{k-l-1}$ such sets. □

Assume now that there are $t$ bad sets $I \subset \binom{[n]}{l+1}$. Then the number of sets $S \subset \binom{[n]}{k}$ such that $|\mathcal{V}_g(S)| \leq f(k,\ell)-1$ is at least $t\binom{n-k}{k-l-1}/\binom{k}{l+1}$. Therefore, by the original Claim 3 and the corrected Claim 4, we have

$$|\mathcal{V}| - f(k,l)\binom{n}{k} \leq -t\frac{\binom{n-k}{k-l-1}}{\binom{k}{l+1}} + \sum_{\text{bad } I} \frac{1}{2} \sum_{v \in \{\pm 1\}^{l+1}} (|\mathcal{V}(I, \boldsymbol{v})| + |\mathcal{V}(I, \bar{\boldsymbol{v}})|)$$

$$\leq -t\left(\frac{\binom{n-k}{k-l-1}}{\binom{k}{l+1}} - 2^k(k-l)\binom{n-l-2}{k-l-2}\right) < 0,$$

provided $n > 2^k k^2 \binom{k}{l+1}$. We note that taking $n > 4^k k^2$ makes the choice of $n$ for which the proof works independent of $l$.

The case of odd $l$ turns out to be harder. We shall need the following variant of Katona's theorem.

**Theorem I.** *Assume that $\mathcal{F} \subset 2^{[n]} \setminus \binom{[n]}{m+1}$ and for any $F, G \in \mathcal{F}$ we have $|F \cup G| \leq 2m+1$. Then $|\mathcal{F}| \leq \sum_{i=0}^{m} \binom{n}{i}$; moreover, for $n > 2m+2$ the only example attaining the bound is $\bigcup_{i=0}^{m} \binom{[n]}{i}$.*

*Proof.* Without loss of generality, we may assume that $\mathcal{F}$ is shifted (we discuss the effect of this assumption on the uniqueness at the end of the proof). The proof is by induction. The statement is clear for $m = 0$; moreover, the extremal family is unique. For $n = 2m+2$, it is easy to see that $2^{[n]} \setminus \binom{[n]}{m+1}$ splits into pairs of complementary sets, which implies the statement.

Assume that the statement holds for $(n-1, m)$ and $(n-1, m-1)$, and let us prove it for $(n, m)$, $n > 2m+2$. We have $|\mathcal{F}| = |\mathcal{F}(n)| + |\mathcal{F}(\bar{n})|$. By induction, we have

$$|\mathcal{F}(\bar{n})| \leq \sum_{i=0}^{m} \binom{n-1}{i}.$$

Moreover, by shiftedness it follows that $|F \cup G| \leq 2m-1$ for any $F, G \in \mathcal{F}(n)$, and thus $|\mathcal{F}(n)| \leq \sum_{i=0}^{m-1} \binom{n-1}{i}$. Since $n-1 > 2(m-1)+1$, this inequality is sharp unless $\mathcal{F}(n) = \bigcup_{i=0}^{m-1} \binom{[n-1]}{i}$. In the case of equality, from here it should be clear that $\mathcal{F}(\bar{n}) \subset \bigcup_{i=0}^{m} \binom{[n-1]}{i}$ and thus

$$\mathcal{F}(\bar{n}) = \bigcup_{i=0}^{m} \binom{[n-1]}{i}.$$

We remark that if $\mathcal{F}$ was not shifted initially, then it could not shift into $\bigcup_{i=0}^{m} \binom{[n]}{i}$; thus the uniqueness part holds for nonshifted families as well. □

Let us return to the case of odd $l$. Consider the subfamily $\mathcal{V}' \subset \mathcal{V}$ of all vectors from $\mathcal{V}$ that have exactly $\frac{l+1}{2}$ minus ones, and put $\mathcal{V}'' := \mathcal{V} \setminus \mathcal{V}'$. Arguing as in the case of even $l$, but applying Theorem I, we get

$$|\mathcal{V}''| \leq \sum_{i=0}^{(l-1)/2} \binom{k}{i}\binom{n}{k},$$

and the inequality is sharp if $\mathcal{V}'' \neq \mathcal{U}$, where $\mathcal{U}$ consists of all $\{-1, 0, 1\}$-vectors with $k$ nonzero coordinates and at most $\frac{l-1}{2}$ minus ones.

Note also that any vector with $\frac{l+1}{2}$ minus ones has scalar product at least $-l$ with any vector from $\mathcal{U}$. It is clear that $\mathcal{V}'$ must avoid scalar product $-l-1$. Moreover, it is sufficient for $\mathcal{V}' \cup \mathcal{U}$ to have all scalar products at least $-l$. Therefore, $|\mathcal{V}'| \leq g\big(n, k - \frac{l+1}{2}, \frac{l+1}{2}\big)$ and the largest $\mathcal{V}$ satisfying the requirements has size

$$g\big(n, k - \tfrac{l+1}{2}, \tfrac{l+1}{2}\big) + \sum_{i=0}^{(l-1)/2} \binom{k}{i}\binom{n}{k}.$$

## References

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