Moscow Journal of
Combinatorics and
Number Theory

2019
vol. 8 no. 4

Correction to the article
Intersection theorems for $(0, \pm 1)$-vectors and s-cross-intersecting families

Peter Frankl and Andrey Kupavskii

msp

Moscow Journal of Combinatorics and Number Theory
Vol. 8, No. 4, 2019
dx.doi.org/10.2140/moscow.2019.8.389

Correction to the article Intersection theorems for $(0, \pm 1)$-vectors and s-cross-intersecting families

Peter Frankl and Andrey Kupavskii

Volume 7:2 (2017), 91–109

We modify the statement and proof of Theorem 1 in "Intersection theorems for $\{0, \pm 1\}$-vectors and s-cross-intersecting families". A version of the paper that incorporates the errata is uploaded on the arXiv: https://arxiv.org/abs/1603.00938. We thank Danila Cherkashin and Sergei Kiselev for pointing out the error.

Part 2 of Theorem 1 in [Frankl and Kupavskii 2017] is incorrect. To give a corrected version, let us introduce some notation: $\mathcal{V}(n, m_1, m_2) \subset \{0, \pm 1\}^n$ is the collection of all vectors with exactly m_1 ones and m_2 minus ones, and

$$g(n, m_1, m_2) := \max\{|\mathcal{V}| : \mathcal{V} \subset \mathcal{V}(n, m_1, m_2) \text{ and } \langle \boldsymbol{v}, \boldsymbol{w} \rangle \geq -2m_2 + 1 \text{ for any } \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}\}.$$

Theorem 1 (part 2). *For $n \geq n_0(k)$ and $0 \leq l \leq k$ we have*

$$F(n, k, -l) = \begin{cases} \sum_{i=0}^{l/2} \binom{k}{i}\binom{n}{k} & \text{for even } l, \\ g\left(n, k - \frac{l+1}{2}, \frac{l+1}{2}\right) + \sum_{i=0}^{(l-1)/2} \binom{k}{i}\binom{n}{k} & \text{for odd } l. \end{cases}$$

Thus, the statement is the same for even l and is different for odd l. The value of $g(n, m_1, m_2)$ seems to be very difficult to determine in general. We have studied this quantity in [Frankl and Kupavskii 2018a; 2018b]. In the former one, we determined the value of $g(n, m_1, 1)$ for any n, m_1. This allows us to determine exactly the value of $F(n, k, -1)$. In the latter one, we obtained the bounds

$$\binom{n}{k}\binom{k-1}{\frac{l-1}{2}} \leq g\left(n, k - \tfrac{l+1}{2}, \tfrac{l+1}{2}\right) \leq \binom{n}{k}\binom{k-1}{\frac{l-1}{2}} + \binom{n}{l+1}\binom{l+1}{\frac{l+1}{2}}\binom{n-l-2}{k-l-2}.$$

We go on to the proof. The proof is correct until (and including) Claim 3. Let us give the corrected version of the remainder of the proof. We first deal with the case of even l.

Claim 4. *Let l be even. If $I \subset \binom{[n]}{l+1}$ is bad, then for at least $\binom{n-k}{k-l-1}$ sets $S \supset I$, $S \in \binom{[n]}{k}$, we have $|\mathcal{V}_g(S)| \leq f(k, l) - 1$.*

Proof. Consider the family $\mathcal{A} \subset 2^S$ of subsets of S defined as $\mathcal{A} = \{N(\boldsymbol{w}) \cap S : \boldsymbol{w} \in \mathcal{V}_g(S)\}$. In view of the uniqueness part of Katona's theorem, it is sufficient to show that $\mathcal{A}$ does not contain one of the sets from the extremal family $\mathcal{U}^l$ for at least $\binom{n-k}{k-l-1}$ choices of S.

MSC2010: 05D05.
Keywords: families of vectors, intersecting family.

If I is bad then there exists a vector $\boldsymbol{v}$ of length $l+1$ such that both $\mathcal{V}(I,\boldsymbol{v})$ and $\mathcal{V}(I,\bar{\boldsymbol{v}})$ are nonempty. Assume without loss of generality that $|N(\boldsymbol{v})| \le l/2$ and take a vector $\boldsymbol{w} \in \mathcal{V}$ such that $\boldsymbol{w}|_I = \bar{\boldsymbol{v}}$. Then for any S such that $S \cap S(\boldsymbol{w}) = I$ the set $N(\boldsymbol{v})$ is missing from $\mathcal{A}$ (and, consequently, $|\mathcal{V}_g(S)| \le f(k,l)-1$). There are exactly $\binom{n-k}{k-l-1}$ such sets. □

Assume now that there are t bad sets $I \subset \binom{[n]}{l+1}$. Then the number of sets $S \subset \binom{[n]}{k}$ such that $|\mathcal{V}_g(S)| \le f(k,\ell)-1$ is at least $t\binom{n-k}{k-l-1}/\binom{k}{l+1}$. Therefore, by the original Claim 3 and the corrected Claim 4, we have

$$|\mathcal{V}| - f(k,l)\binom{n}{k} \le -t\frac{\binom{n-k}{k-l-1}}{\binom{k}{l+1}} + \sum_{\text{bad } I}\frac{1}{2}\sum_{v\in\{\pm1\}^{l+1}}(|\mathcal{V}(I,\boldsymbol{v})| + |\mathcal{V}(I,\bar{\boldsymbol{v}})|)$$
$$\le -t\left(\frac{\binom{n-k}{k-l-1}}{\binom{k}{l+1}} - 2^k(k-l)\binom{n-l-2}{k-l-2}\right) < 0,$$

provided $n > 2^k k^2\binom{k}{l+1}$. We note that taking $n > 4^k k^2$ makes the choice of n for which the proof works independent of l.

The case of odd l turns out to be harder. We shall need the following variant of Katona's theorem.

Theorem I. *Assume that $\mathcal{F} \subset 2^{[n]} \setminus \binom{[n]}{m+1}$ and for any $F, G \in \mathcal{F}$ we have $|F \cup G| \le 2m+1$. Then $|\mathcal{F}| \le \sum_{i=0}^{m}\binom{n}{i}$; moreover, for $n > 2m+2$ the only example attaining the bound is $\bigcup_{i=0}^{m}\binom{[n]}{i}$.*

Proof. Without loss of generality, we may assume that $\mathcal{F}$ is shifted (we discuss the effect of this assumption on the uniqueness at the end of the proof). The proof is by induction. The statement is clear for $m=0$; moreover, the extremal family is unique. For $n = 2m+2$, it is easy to see that $2^{[n]} \setminus \binom{[n]}{m+1}$ splits into pairs of complementary sets, which implies the statement.

Assume that the statement holds for $(n-1,m)$ and $(n-1,m-1)$, and let us prove it for (n,m), $n > 2m+2$. We have $|\mathcal{F}| = |\mathcal{F}(n)| + |\mathcal{F}(\bar{n})|$. By induction, we have

$$|\mathcal{F}(\bar{n})| \le \sum_{i=0}^{m}\binom{n-1}{i}.$$

Moreover, by shiftedness it follows that $|F \cup G| \le 2m-1$ for any $F, G \in \mathcal{F}(n)$, and thus $|\mathcal{F}(n)| \le \sum_{i=0}^{m-1}\binom{n-1}{i}$. Since $n-1 > 2(m-1)+1$, this inequality is sharp unless $\mathcal{F}(n) = \bigcup_{i=0}^{m-1}\binom{[n-1]}{i}$. In the case of equality, from here it should be clear that $\mathcal{F}(\bar{n}) \subset \bigcup_{i=0}^{m}\binom{[n-1]}{i}$ and thus

$$\mathcal{F}(\bar{n}) = \bigcup_{i=0}^{m}\binom{[n-1]}{i}.$$

We remark that if $\mathcal{F}$ was not shifted initially, then it could not shift into $\bigcup_{i=0}^{m}\binom{[n]}{i}$; thus the uniqueness part holds for nonshifted families as well. □

Let us return to the case of odd l. Consider the subfamily $\mathcal{V}' \subset \mathcal{V}$ of all vectors from $\mathcal{V}$ that have exactly $\frac{l+1}{2}$ minus ones, and put $\mathcal{V}'' := \mathcal{V} \setminus \mathcal{V}'$. Arguing as in the case of even l, but applying Theorem I, we get

$$|\mathcal{V}''| \le \sum_{i=0}^{(l-1)/2}\binom{k}{i}\binom{n}{k},$$

and the inequality is sharp if $\mathcal{V}'' \neq \mathcal{U}$, where $\mathcal{U}$ consists of all $\{-1, 0, 1\}$-vectors with k nonzero coordinates and at most $\frac{l-1}{2}$ minus ones.

Note also that any vector with $\frac{l+1}{2}$ minus ones has scalar product at least $-l$ with any vector from $\mathcal{U}$. It is clear that $\mathcal{V}'$ must avoid scalar product $-l-1$. Moreover, it is sufficient for $\mathcal{V}' \cup \mathcal{U}$ to have all scalar products at least $-l$. Therefore, $|\mathcal{V}'| \le g\left(n, k - \frac{l+1}{2}, \frac{l+1}{2}\right)$ and the largest $\mathcal{V}$ satisfying the requirements has size

$$g\left(n, k - \tfrac{l+1}{2}, \tfrac{l+1}{2}\right) + \sum_{i=0}^{(l-1)/2} \binom{k}{i}\binom{n}{k}.$$

References

[Frankl and Kupavskii 2017] P. Frankl and A. Kupavskii, "Intersection theorems for $\{0, \pm 1\}$-vectors and s-cross-intersecting families", *Mosc. J. Comb. Number Theory* **7**:2 (2017), 91–109. MR Zbl

[Frankl and Kupavskii 2018a] P. Frankl and A. Kupavskii, "Erdős–Ko–Rado theorem for $\{0, \pm 1\}$-vectors", *J. Combin. Theory Ser. A* **155** (2018), 157–179. MR Zbl

[Frankl and Kupavskii 2018b] P. Frankl and A. Kupavskii, "Families of vectors without antipodal pairs", *Studia Sci. Math. Hungar.* **55**:2 (2018), 231–237. MR Zbl

Received 29 May 2019. Revised 16 Jul 2019.

PETER FRANKL:
peter.frankl@gmail.com
Rényi Institute, Hungarian Academy of Sciences, Budapest, Hungary

ANDREY KUPAVSKII:
kupavskii@ya.ru
Moscow Institute of Physics and Technology, Moscow, Russia
and
University of Oxford, Oxford, United Kingdom

Moscow Journal of Combinatorics and Number Theory

msp.org/moscow

See inside back cover or msp.org/moscow for submission instructions.

The subscription price for 2019 is US $310/year for the electronic version, and $365/year (+$20, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Moscow Journal of Combinatorics and Number Theory (ISSN 2640-7361 electronic, 2220-5438 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

MJCNT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Moscow Journal of Combinatorics and Number Theory

vol. 8 no. 4 2019