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Long monochromatic paths and cycles in 2-edge-colored multipartite graphs

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We solve four similar problems: for every fixed s and large n, we describe all values of $n_1, \dots, n_s$ such that for every 2-edge-coloring of the complete s-partite graph $K_{n_1,\dots,n_s}$ there exists a monochromatic (i) cycle C_{2n} with $2n$ vertices, (ii) cycle $C_{\geq 2n}$ with at least $2n$ vertices, (iii) path P_{2n} with $2n$ vertices, and (iv) path P_{2n+1} with $2n+1$ vertices.

This implies a generalization for large n of the conjecture by Gyárfás, Ruszinkó, Sárközy and Szemerédi that for every 2-edge-coloring of the complete 3-partite graph $K_{n,n,n}$ there is a monochromatic path P_{2n+1}. An important tool is our recent stability theorem on monochromatic connected matchings.

1. Introduction

A *connected matching* in a graph G is a matching whose edges are all in the same component of G. By M_n we will always denote a connected matching with n edges and by P_n the path with n vertices. Also by C_n we denote the cycle with n vertices, and by $C_{\geq n}$ a cycle of length at least n.

For graphs $G_0, \dots, G_k$ we write $G_0 \mapsto (G_1, \dots, G_k)$ if for every k-coloring of the edges of G_0, for some $i \in [k]$ there is a copy of G_i with all edges of color i. The *Ramsey number* $R(G_1, \dots, G_k)$ is the minimum N such that $K_N \mapsto (G_1, \dots, G_k)$, and $R_k(G) = R(G_1, \dots, G_k)$, where $G_1 = \dots = G_k = G$.

Gerencsér and Gyárfás [1967] proved that the n-vertex path P_n satisfies $R_2(P_n) = \lfloor \frac{1}{2}(3n-2) \rfloor$. They actually proved a stronger result:

Theorem 1 [Gerencsér and Gyárfás 1967]. *For any two positive integers $k \geq \ell$, $R(P_k, P_\ell) = k - 1 + \lfloor \frac{1}{2}\ell \rfloor$.*

Many significant results bounding $R_k(P_n)$ for $k \geq 3$ and $R_k(C_n)$ for even n were proved in [Benevides et al. 2012; Benevides and Skokan 2009; Bondy and Erdős 1973; DeBiasio and Krueger 2018; DeBiasio et al. 2020; Faudree and Schelp 1974; Figaj and Łuczak 2007; 2018; Gyárfás et al. 2007a; Knierim and Su 2019; Łuczak 1999; Łuczak et al. 2012; Sárközy 2016]. Many proofs used the Szemerédi Regularity Lemma [1978] and a number of them used the idea of connected matchings in regular partitions due to [Łuczak 1999].

Ramsey-type problems when the host graphs are not complete but complete bipartite were studied by Gyárfás and Lehel [1973], Faudree and Schelp [1975], DeBiasio, Gyárfás, Krueger, Ruszinkó, and

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Sárközy [Gyárfás et al. 2007a], DeBiasio and Krueger [2018], Bucić, Letzter, and Sudakov [Bucić et al. 2019a; 2019b], and Zhang, Sun, and Wu [Zhang et al. 2013], and when the host graphs are complete 3-partite by Gyárfás, Ruszinkó, Sárközy, and Szemerédi [Gyárfás et al. 2007b]. The main result in [Faudree and Schelp 1975] and [Gyárfás and Lehel 1973] was:

Theorem 2 [Faudree and Schelp 1975; Gyárfás and Lehel 1973]. *For every positive integer* n, $K_{n,n} \mapsto (P_{2\lceil n/2\rceil}, P_{2\lceil n/2\rceil})$. *Furthermore,* $K_{n,n} \not\mapsto (P_{2\lceil n/2\rceil+1}, P_{2\lceil n/2\rceil+1})$.

DeBiasio and Krueger [2018] extended the result from paths $P_{2\lceil n/2\rceil}$ to cycles of length at least $2\lfloor \frac{1}{2}n \rfloor$ for large n.

The main result in [Gyárfás et al. 2007b] was:

Theorem 3 [Gyárfás et al. 2007b]. *For every positive integer* n, $K_{n,n,n} \mapsto (P_{2n-o(n)}, P_{2n-o(n)})$.

The following exact bound was also conjectured:

Conjecture 4 [Gyárfás et al. 2007b]. *For every positive integer* n, $K_{n,n,n} \mapsto (P_{2n+1}, P_{2n+1})$.

The goal of this paper is to prove for large n Conjecture 4 and similar exact bounds for paths P_{2n} (parity matters here) and cycles C_{2n}. We do it in a more general setting: for multipartite graphs with possibly different part sizes. In the next section, we discuss extremal examples, define some notions and state our main results. In Section 3, we describe our tools. In Sections 4–8, we prove the main part, namely, the result for even cycles C_{2n}. In Sections 9–11 we use the main result to derive similar results for cycles $C_{\geq 2n}$ and paths P_{2n} and P_{2n+1}.

2. Examples and results

For a graph G and disjoint sets $A, B \subset V(G)$, by $G[A]$ we denote the subgraph of G induced by A, and by $G[A, B]$ the bipartite subgraph of G with parts A and B formed by all edges of G connecting A with B.

Our edge-colorings always will be with red (color 1) and blue (color 2).

We consider necessary restrictions on $n_1 \geq n_2 \geq \cdots \geq n_s$ providing that each 2-edge-coloring of $K_{n_1,n_2,\ldots,n_s}$ contains (a) a monochromatic path P_{2n}, (b) a monochromatic path P_{2n+1}, (c) a monochromatic cycle C_{2n} and (d) a monochromatic cycle $C_{\geq 2n}$. Each condition we add is motivated by an example showing that the condition is necessary.

First, recall that each of P_{2n}, P_{2n+1}, C_{2n}, and $C_{\geq 2n}$ contains a connected matching M_n. Thus a graph with no M_n also contains neither P_{2n} nor P_{2n+1} nor $C_{\geq 2n}$.

2.1. ***Example with no monochromatic M_n: too few vertices.*** Let $G = K_{3n-2}$. Clearly, $G \supseteq K_{n_1,n_2,\ldots,n_s}$ for each $n_1, \ldots, n_s$ with $n_1 + \cdots + n_s = 3n - 2$. Partition $V(G)$ into sets U_1 and U_2 with $|U_1| = 2n - 1$ and $|U_2| = n - 1$. Color the edges of $G[U_1, U_2]$ with red and the rest of the edges with blue. Since neither K_{2n-1} nor $K_{n-1,2n-1}$ contains M_n, we conclude $G \not\mapsto (M_n, M_n)$; see Figure 1.

To rule out this example, we add the condition

$$N := n_1 + \cdots + n_s \geq 3n - 1. \tag{1}$$

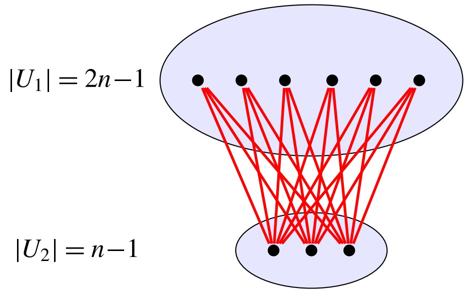

Figure 1. Section 2.1.

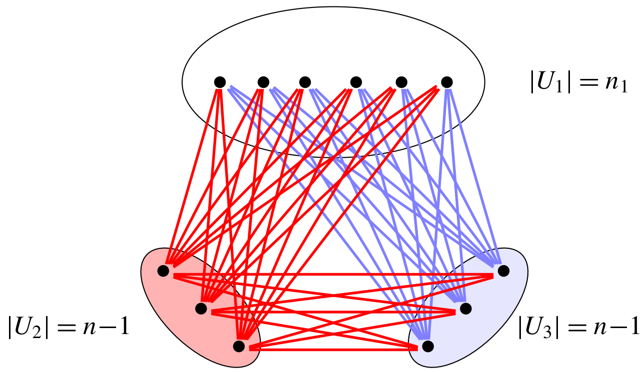

Figure 2. Section 2.2.

2.2. *Example with no monochromatic M_n: too few vertices outside V_1.* Choose any n_1 and let $N = n_1 + 2n - 2$. Let G be obtained from K_N by deleting the edges inside a vertex subset U_1 with $|U_1| = n_1$. Graph G contains every $K_{n_1,n_2,\ldots,n_s}$ with $n_2 + \cdots + n_s = 2n - 2$. Partition $V(G) - U_1$ into sets U_2 and U_3 with $|U_2| = |U_3| = n - 1$. Color all edges incident with U_2 red, and the remaining edges of G blue. Since the red and blue subgraphs of G have vertex covers of size $n - 1$ (namely, U_2 and U_3), neither of them contains M_n. Thus $G \not\mapsto (M_n, M_n)$; see Figure 2.

To rule out this example, we add the condition

$$N - n_1 = n_2 + \cdots + n_s \geq 2n - 1. \tag{2}$$

2.3. *Example with no red M_n and no blue P_{2n+1}: too few vertices.* Let $G = K_{3n-1}$. Partition $V(G)$ into sets U_1 and U_2 with $|U_1| = 2n$ and $|U_2| = n - 1$. Color the edges of $G[U_1, U_2]$ red and the rest of the edges blue. Since the red subgraph of G has vertex cover U_2 with $|U_2| = n - 1$, it does not contain M_n. Since each component of the blue subgraph of G has fewer than $2n + 1$ vertices, it does not contain P_{2n+1}.

Therefore

$$R(P_{2n}, P_{2n+1}) \geq R(M_n, P_{2n+1}) \geq 3n,$$

which yields for P_{2n+1} the following strengthening of (1):

$$\text{for } P_{2n+1}, \quad N \geq 3n. \tag{3}$$

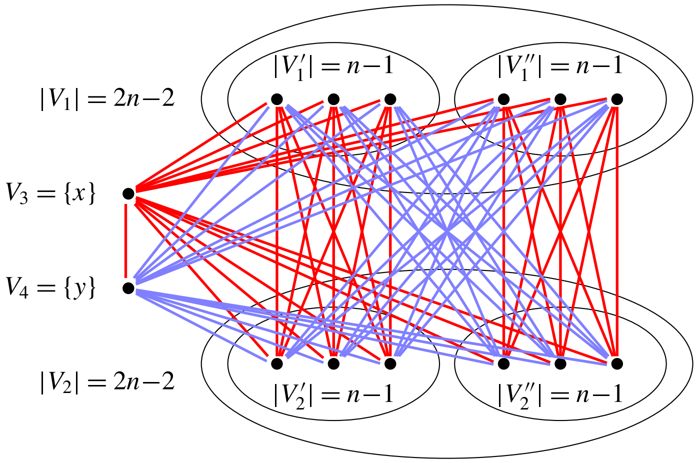

Figure 3. Section 2.4.

2.4. ***Example with no monochromatic $C_{\geq 2n}$ when $N - n_1 - n_2 \leq 2$.*** This example, and all the ones that follow, show that additional restrictions are necessary when G is bipartite or close to bipartite.

Let $G = K_{n_1,\dots,n_s}$ satisfy (1) and (2) with $N - n_1 - n_2 \leq 2$ such that $n_1 \leq 2n-2$. Then also $n_2 \leq 2n-2$, so $G \subseteq K_{2n-2,2n-2,1,1}$. Thus we assume $G = K_{2n-2,2n-2,1,1}$, with $V_1 = \{v_1, \dots, v_{2n-2}\}$, $V_2 = \{u_1, \dots, u_{2n-2}\}$, $V_3 = \{x\}$, and $V_4 = \{y\}$. Let $V_1' = \{v_1, \dots, v_{n-1}\}$, $V_1'' = V_1 - V_1'$, $V_2' = \{u_1, \dots, u_{n-1}\}$, $V_2'' = V_2 - V_2'$. Color the edges in $G[V_1', V_2']$, $G[V_1'', V_2'']$ and in $G[V_3, V_1 \cup V_2 \cup V_4]$ red, and all other edges blue. Then the red graph G_1 has cut vertex x, and the components of $G_1 - x$ have sizes $2n-2$, $2n-2$, and 1, so G_1 has no $C_{\geq 2n}$. Similarly, G_2 contains no $C_{\geq 2n}$; see Figure 3.

To rule out this example, we add the condition

$$\text{for } C_{\geq 2n}, \quad \text{if } N - n_1 - n_2 \leq 2, \quad \text{then } n_1 \geq 2n-1. \tag{4}$$

2.5. ***Example with no monochromatic $C_{\geq 2n}$ when $N - n_1 - n_2 \leq 1$.*** Let $G = K_{n_1,\dots,n_s}$ satisfy (1), (2) and (4) with $N - n_1 - n_2 \leq 1$ such that $N + n_1 \leq 6n-3$. Since by (4), $n_1 \geq 2n-1$, we get $N - n_1 \leq (6n-3) - 2(2n-1) = 2n-1$, but (2) implies $N - n_1 \geq 2n-1$; therefore both inequalities are tight and $N - n_1 = n_1 = 2n-1$. Hence $G \subseteq K_{2n-1,2n-2,1}$, which is a subgraph of the graph $K_{2n-2,2n-2,1,1}$ considered in Section 2.4.

This example is not ruled out by (4), so we add the condition

$$\text{for } C_{\geq 2n}, \quad \text{if } N - n_1 - n_2 \leq 1, \quad \text{then } n_1 + N \geq 6n-2. \tag{5}$$

2.6. ***Example with no monochromatic P_{2n+1} when G is bipartite.*** Suppose $n_3 = 0$ and $n_1 \leq 2n$. Then $n_2 \leq 2n$ as well, so $G \subseteq K_{2n,2n}$. Thus we assume $G = K_{2n,2n}$ with $V_1 = \{v_1, \dots, v_{2n}\}$ and $V_2 = \{u_1, \dots, u_{2n}\}$. Let $V_1' = \{v_1, \dots, v_n\}$, $V_1'' = V_1 - V_1'$, $V_2' = \{u_1, \dots, u_n\}$, $V_2'' = V_2 - V_2'$. Color the edges in $G(V_1', V_2')$ and $G(V_1'', V_2'')$ red, and all other edges blue. Then each component in the red graph and each component in the blue graph has $2n$ vertices and thus does not contain P_{2n+1}; see Figure 4.

To rule out this example, we add the condition

$$\text{for } P_{2n+1}, \quad \text{if } n_3 = 0, \quad \text{then } n_1 \geq 2n+1. \tag{6}$$

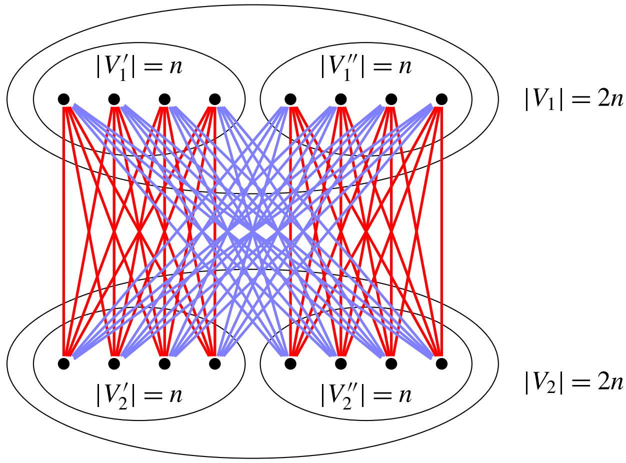

Figure 4. Section 2.5.

2.7. ***Example with no monochromatic C_{2n} when $N - n_1 - n_2 \le 2$.*** Let $G = K_{n_1,\ldots,n_s}$ satisfy (1), (2) and (4) with $N - n_1 - n_2 = 2$ such that $N \le 4n - 2$. By (4), $N - n_1 \le 2n - 1$. Now (2) implies $N - n_1 = 2n - 1 = n_1$. Hence $G \subseteq K_{2n-1,2n-3,1,1}$. Thus we assume $G = K_{2n-1,2n-3,1,1}$ with $V_1 = \{v_1, \ldots, v_{2n-1}\}$, $V_2 = \{u_1, \ldots, u_{2n-3}\}$, $V_3 = \{x\}$, and $V_4 = \{y\}$. Define $A = \{v_2, v_3, \ldots, v_n\}$, $B = \{v_{n+1}, v_{n+2}, \ldots, v_{2n-1}\}$, $C = \{u_1, u_2, \ldots, u_{n-1}\}$, and $D = \{u_n, u_{n+1}, \ldots, u_{2n-3}\}$. We assign the colors to the edges of G as follows:

(1) $G[A, C]$ and $G[B, D]$ are complete bipartite red graphs.

(2) $G[A, D]$ and $G[B, C]$ are complete bipartite blue graphs.

(3) v_1 has all blue edges to V_2.

(4) x has all red edges to $V_1 \cup V_2 \cup \{y\}$.

(5) y has all red edges to $B \cup D \cup \{x\}$ and all blue edges to $A \cup C \cup \{v_1\}$.

We claim that G has no monochromatic cycle of length $2n$. Indeed, consider first the red graph G_1. The graph $G_1 - x$ has three components: (a) $A \cup C$ of size $2n - 2$, (b) $\{v_1\}$ of size 1, and (c) $B \cup D \cup \{y\}$ of size $2n - 2$. Thus G has no red cycle of length $2n$ since the largest block of G_1 has order $2n - 1$.

Consider now the blue graph G_2. We ignore x since it is isolated. Suppose G_2 contains a $2n$-cycle F. Since v_1 is a cut vertex of $G_2 - \{y\}$ with the components of $G_2 - \{y, v_1\}$ of order $2n - 3$ and $2n - 2$, F contains y.

If we delete from G_2 all edges in $G_2[\{y\}, C]$, then both blocks in the remaining blue graph will be of order $2n - 1$; thus F contains an edge from y to C, say yz. Furthermore, if yz is the only edge in F connecting y to C, then all other edges in F belong to the bipartite graph $H = G_2[A \cup B \cup \{v_1\}, D \cup \{y\} \cup C]$. But this bipartite graph H cannot have a path of odd length $2n - 1$ between the vertices y and z in the same part.

Thus, F has to use two edges from y to C, say yz_1 and yz_2. Then the problem is reduced to finding a blue path from z_1 to z_2 of length $2n - 2$ in $G_2[C, B \cup \{v_1\}]$. However, it is impossible because $|C| = n - 1$ and the longest path from z_1 to z_2 in $G_2[C, B \cup \{v_1\}]$ has $2n - 3$ vertices.

Note that this example has cycles of length greater than $2n - 1$, but all such cycles are odd.

To rule out this example, we add the condition

$$\text{for } C_{2n}, \quad \text{if } N - n_1 - n_2 \le 2, \quad \text{then } N \ge 4n - 1. \tag{7}$$

2.8. *Results.* Our key result is that for large n, the necessary conditions (1), (2) and (7) for the presence in a 2-edge-colored $K_{n_1,\dots,n_s}$ of a monochromatic C_{2n} together are also sufficient for this.

Theorem 5. *Let $s \ge 2$ and n be sufficiently large. Let $n_1 \ge \cdots \ge n_s$ and $N = n_1 + \cdots + n_s$ satisfy (1), (2) and (7). Then for each 2-edge-coloring f of the complete s-partite graph $K_{n_1,\dots,n_s}$, there exists a monochromatic cycle C_{2n}.*

Based on Theorem 5, we derive our other results. The first of them is on cycles of length at least $2n$ (it extends a result of [DeBiasio and Krueger 2018]). Recall that (7) is not necessary for the existence of a monochromatic $C_{\ge 2n}$, but (1), (2), (4) and (5) are.

Theorem 6. *Let $s \ge 2$ and n be sufficiently large. Let $n_1 \ge \cdots \ge n_s$ and $N = n_1 + \cdots + n_s$ satisfy (1), (2), (4) and (5). Then for each 2-edge-coloring f of the complete s-partite graph $K_{n_1,\dots,n_s}$, there exists a monochromatic cycle $C_{\ge 2n}$.*

The results for paths of even and odd lengths are somewhat different. The first of them shows that for large n, the necessary conditions (1) and (2) for the presence in a 2-edge-colored $K_{n_1,\dots,n_s}$ of a monochromatic connected matching M_n together are sufficient for the presence of the monochromatic path P_{2n}.

Theorem 7. *Let $s \ge 2$ and n be sufficiently large. Let $n_1 \ge \cdots \ge n_s$ and $N = n_1 + \cdots + n_s$ satisfy (1) and (2). Then for each 2-edge-coloring f of the complete s-partite graph $K_{n_1,\dots,n_s}$, there exists a monochromatic path P_{2n}.*

Our last result implies Conjecture 4:

Theorem 8. *Let $s \ge 2$ and n be sufficiently large. Let $n_1 \ge \cdots \ge n_s$ and $N = n_1 + \cdots + n_s$ satisfy (2), (3) and (6). Then for each 2-edge-coloring f of the complete s-partite graph $K_{n_1,\dots,n_s}$, there exists a monochromatic path P_{2n+1}.*

In the next section, we describe our main tools: the Szemerédi Regularity Lemma, connected matchings, and theorems on the existence of Hamiltonian cycles in dense graphs. In Section 4 we set up and describe the structure of the proof of Theorem 5, and in the next four sections we present this proof. In the last three sections we prove Theorems 6, 7 and 8.

3. Tools

As in many recent papers on Ramsey numbers of paths (see [Benevides et al. 2012; Benevides and Skokan 2009; DeBiasio and Krueger 2018; Figaj and Łuczak 2007; Gyárfás et al. 2007a; Knierim and Su 2019; Łuczak et al. 2012; Sárközy 2016]), our proof heavily uses the Szemerédi Regularity Lemma [1978] and the idea of connected matchings in regular partitions of reduced graphs due to [Łuczak 1999].

3.1. *Regularity.* We say that a pair (V_1, V_2) of two disjoint vertex sets $V_1, V_2 \subseteq V(G)$ is (ϵ, G)-*regular* if

$$\left| \frac{|E(X,Y)|}{|X||Y|} - \frac{|E(V_1,V_2)|}{|V_1||V_2|} \right| < \epsilon$$

for all $X \subseteq V_1$ and $Y \subseteq V_2$ with $|X| > \epsilon |V_1|$ and $|Y| > \epsilon |V_2|$.

We use a 2-color version of the Regularity Lemma, following Gyárfás, Ruszinkó, Sárközy, and Szemerédi [Gyárfás et al. 2007a].

Lemma 9 (2-color version of the Szemerédi Regularity Lemma). *For every $\epsilon > 0$ and integer $m > 0$, there are positive integers M and n_0 such that for $n \geq n_0$ the following holds. For all graphs G_1 and G_2 with $V(G_1) = V(G_2) = V$, $|V| = n$, there is a partition of V into $L+1$ disjoint classes (clusters) $(V_0, V_1, V_2, \ldots, V_L)$ such that*

- $m \leq L \leq M$,
- $|V_1| = |V_2| = \cdots = |V_L|$,
- $|V_0| < \epsilon n$,
- *Apart from at most $\epsilon\binom{L}{2}$ exceptional pairs, the pairs $\{V_i, V_j\}$ are (ϵ, G_q)-regular for $q = 1$ and 2.*

Additionally, if $G_1 \cup G_2$ is a multipartite graph with partition $V = V_1^* \cup V_2^* \cup \cdots \cup V_s^*$, with $s < 6$, we can guarantee that *each of the clusters $V_1, V_2, \ldots, V_L$ is contained entirely in a single part of this partition.*

To do so, for a given $\epsilon > 0$, we begin by arbitrarily partitioning each V_i^* into parts $V_{i,1}^*, V_{i,2}^*, \ldots,$ each of size $\lfloor \frac{1}{10}\epsilon n \rfloor$, with a part $V_{i,0}^*$ of size at most $\frac{1}{10}\epsilon n$ left over. This is an equitable partition of $V - \bigcup_{i=1}^{k} V_{i,0}^*$, a set of at least $\left(1 - \frac{9}{10}\epsilon\right)n$ vertices. The Regularity Lemma allows us to refine any equitable partition into one that satisfies the conclusions of Lemma 9. Working with the subgraphs of G_1 and G_2 excluding the vertices in $\bigcup_{i=1}^{k} V_{i,0}^*$, take such a refinement with parameters $\frac{1}{9}\epsilon$ and m, then add $\bigcup_{i=1}^{k} V_{i,0}^*$ to its exceptional cluster V_0. The resulting exceptional cluster still has size at most ϵn, so we have obtained a partition satisfying the conditions of Lemma 9 in which each of $V_1, V_2, \ldots, V_L$ is entirely contained in one of $V_1^*, V_2^*, \ldots, V_k^*$.

3.2. *Connected matchings.* Let $\alpha'(G)$ denote the size of a largest matching and $\alpha'_*(G)$ denote the size of a largest connected matching in G. Let $\alpha(G)$ denote the independence number and $\beta(G)$ denote the size of a smallest vertex cover in G.

Łuczak [1999] was the first to use the fact that the existence of large connected matchings in the reduced graph of a regular partition of a large graph G implies the existence of long paths and cycles in G. A flavor of it is illustrated by the following fact.

Lemma 10 [Łuczak et al. 2012, Lemma 8; Knierim and Su 2019, Lemma 1]. *Let a real number $c > 0$ and a positive integer k be given. If for every $\epsilon > 0$ there exists a $\delta > 0$ and an n_0 such that for every even $n > n_0$ and each graph G with $v(G) > (1+\epsilon)cn$ and $e(G) \geq (1-\delta)\binom{v(G)}{2}$ each k-edge-coloring of G has a monochromatic connected matching $M_{n/2}$, then for large N, we have $R_k(C_N) \leq (c + o(1))N$ (and hence $R_k(P_N) \leq (c + o(1))N$).*

We use the following property of (ϵ, G)-regular pairs:

Lemma 11 [Gyárfás et al. 2007a, Lemma 3]. *For every $\delta > 0$ there exist $\epsilon > 0$ and t_0 such that the following holds. Let G be a bipartite graph with bipartition (V_1, V_2) such that $|V_1| = |V_2| = t \geq t_0$, and let the pair (V_1, V_2) be (ϵ, G)-regular. Moreover, assume that $\deg_G(v) > \delta t$ for all $v \in V(G)$.*

Then for every pair of vertices $v_1 \in V_1$, $v_2 \in V_2$, the graph G contains a Hamiltonian path with endpoints v_1 and v_2.

Since we are aiming at an exact bound, we need a stability version of a result similar to Lemma 10. To state it, we need some definitions.

Definition 12. For $\epsilon > 0$, an N-vertex s-partite graph G with parts $V_1, \dots, V_s$ of sizes $n_1 \geq n_2 \geq \cdots \geq n_s$, and a 2-edge-coloring $E = E_1 \cup E_2$ is (n, s, ϵ)-*suitable* if the conditions

$$N = n_1 + \cdots + n_s \geq 3n - 1, \tag{S1}$$

$$n_2 + n_3 + \cdots + n_s \geq 2n - 1 \tag{S2}$$

hold, and if $\widetilde{V}_i$ is the set of vertices in V_i of degree at most $N - \epsilon n - n_i$ and $\widetilde{V} = \bigcup_{i=1}^{s} \widetilde{V}_i$, then

$$|\widetilde{V}| = |\widetilde{V}_1| + \cdots + |\widetilde{V}_s| < \epsilon n. \tag{S3}$$

We do not require $E_1 \cap E_2 = \varnothing$; an edge can have one or both colors. We write $G_i = G[E_i]$ for $i = 1, 2$.

Our stability theorem gives a partition of the vertices of near-extremal graphs called a (λ, i, j)-*bad partition*. There are two types of bad partitions.

Definition 13. For $i \in \{1, 2\}$, $\lambda > 0$, and an (n, s, ϵ)-suitable graph G, a partition $V(G) = W_1 \cup W_2$ of $V(G)$ is $(\lambda, i, 1)$-*bad* if the following hold:

(i) $(1 - \lambda)n \leq |W_2| \leq (1 + \lambda)n_1$.

(ii) $|E(G_i[W_1, W_2])| \leq \lambda n^2$.

(iii) $|E(G_{3-i}[W_1])| \leq \lambda n^2$.

Definition 14. For $i \in \{1, 2\}$, $\lambda > 0$, and an (n, s, ϵ)-suitable graph G, a partition $V(G) = V_j \cup U_1 \cup U_2$, $j \in [s]$, of $V(G)$ is $(\lambda, i, 2)$-bad if the following hold:

(i) $|E(G_i[V_j, U_1])| \leq \lambda n^2$.

(ii) $|E(G_{3-i}[V_j, U_2])| \leq \lambda n^2$.

(iii) $n_j = |V_j| \geq (1 - \lambda)n$.

(iv) $(1 - \lambda)n \leq |U_1| \leq (1 + \lambda)n$.

(v) $(1 - \lambda)n \leq |U_2| \leq (1 + \lambda)n$.

Our stability theorem is:

Theorem 15 [Balogh et al. 2019, Theorem 9]. *Let* $0 < \epsilon < 10^{-3}\gamma < 10^{-6}$, $n \geq s \geq 2$, *and* $n > 100/\gamma$. *Let* G *be an* (n, s, ϵ)*-suitable graph. If* $\max\{\alpha'_*(G_1), \alpha'_*(G_2)\} \leq n(1 + \gamma)$, *then for some* $i \in [2]$ *and* $j \in [2]$, $V(G)$ *has a* $(68\gamma, i, j)$*-bad partition.*

3.3. *Theorems on Hamiltonian cycles in bipartite graphs.*

Theorem 16 ([Chvátal 1972]; see also [Berge 1976, Corollary 5 in Chapter 10]). *Let* H *be a* $2n$*-vertex bipartite graph with vertices* $u_1, u_2, \dots, u_n$ *on one side and* $v_1, v_2, \dots, v_n$ *on the other such that* $d(u_1) \leq \cdots \leq d(u_n)$ *and* $d(v_1) \leq \cdots \leq d(v_n)$.

If $d_H(u_i) \leq i < n$ *implies* $d_H(v_{n-i}) \geq n - i + 1$, *then* H *is Hamiltonian.*

Theorem 17 [Berge 1976]. *Let H be a $2m$-vertex bipartite graph with vertices $u_1, u_2, \dots, u_m$ on one side and $v_1, v_2, \dots, v_m$ on the other such that $d(u_1) \le \dots \le d(u_m)$ and $d(v_1) \le \dots \le d(v_m)$. Suppose that for the smallest two indices i and j such that $d(u_i) \le i+1$ and $d(v_j) \le j+1$, we have $d(u_i)+d(v_j) \ge m+2$.*

Then H is Hamiltonian biconnected: for every i and j, there is a Hamiltonian path with endpoints u_i and v_j.

Theorem 18 ([Las Vergnas 1970]; see also [Berge 1976, Theorem 11 on page 214]). *Let H be a $2n$-vertex bipartite graph with vertices $u_1, u_2, \dots, u_n$ on one side and $v_1, v_2, \dots, v_n$ on the other such that $d(u_1) \le \dots \le d(u_n)$ and $d(v_1) \le \dots \le d(v_n)$. Let q be an integer, $0 \le q \le n-1$.*

If, whenever $u_i v_j \notin E(H)$, $d(u_i) \le i+q$, and $d(v_j) \le j+q$, we have

$$d(u_i) + d(v_j) \ge n + q + 1,$$

then each set of q edges that form vertex-disjoint paths is contained in a Hamiltonian cycle of G.

3.4. ***Using the tools.*** Our strategy to prove Theorem 5 is: We first apply a 2-colored version of the Regularity Lemma to G to obtain a reduced graph G^r. If G^r has a large monochromatic connected matching then we find a long monochromatic cycle using Lemma 10. If G^r does not have a large monochromatic connected matching, then we use Theorem 15 to obtain a bad partition of G^r. We then transfer the bad partition of G^r to a bad partition of G and work with this partition. In some important cases, theorems on Hamiltonian cycles help to find a monochromatic cycle C_{2n} in G.

4. Setup of the proof of Theorem 5

Formally, we need to prove the theorem for every N-vertex complete s-partite graph G with parts $(V_1^*, V_2^*, \dots, V_s^*)$ such that the numbers $n_i = |V_i^*|$ satisfy $n_1 \ge n_2 \ge \dots \ge n_s$ and the three conditions

$$N = n_1 + \dots + n_s \ge 3n - 1, \tag{S1$'$}$$

$$N - n_1 = n_2 + \dots + n_s \ge 2n - 1, \tag{S2$'$}$$

$$\text{if } N - n_1 - n_2 \le 2, \quad \text{then } N \ge 4n - 1. \tag{S3$'$}$$

For a given large n, we consider a possible counterexample with the minimum $N + s$. In view of this, it is enough to consider the lists $(n_1, \dots, n_s)$ satisfying (S1$'$), (S2$'$) and (S3$'$) such that:

(a) For each $1 \le i \le s$, if $n_i > n_{i+1}$, then the list $(n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_s)$ does not satisfy some of (S1$'$), (S2$'$) and (S3$'$).

(b) If $s \ge 4$, then the list $(n_1, \dots, n_{s-2}, n_{s-1} + n_s)$ (possibly with the entries rearranged into a nonincreasing order) does not satisfy some of (S1$'$), (S2$'$) and (S3$'$).

<u>Case 1</u>: $N - n_1 - n_2 \ge 3$ and $N > 3n - 1$. Then (S3$'$) holds by default. If $n_1 > n_2$, then the list $(n_1 - 1, n_2, n_3, \dots, n_s)$ still satisfies the conditions (S1$'$), (S2$'$) and (S3$'$), a contradiction to (a). Hence $n_1 = n_2$. Choose the maximum i such that $n_1 = n_i$. If $N - n_1 > 2n - 1$, consider the list $(n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_s)$. In this case (S1$'$) and (S2$'$) still are satisfied for this list; so by (a), (S3$'$) fails for it. As we assumed $N - n_1 - n_2 \ge 3$, we must have $i \ge 3$ and $N - n_1 - n_2 = 3$ for (S3$'$) to fail for this list; this further implies $n_1 = n_i \le 3$, so $N = n_1 + n_2 + 3 \le 9$, a contradiction. Thus in this case $N - n_1 = 2n - 1$. Therefore, $n_1 = N - (N - n_1) \ge 3n - (2n - 1) = n + 1$ and hence $n_2 \ge n + 1$,

so $N - n_1 - n_2 \le (2n-1) - (n+1) = n-2$. Then the list $(n_1, n_1, N - 2n_1)$ satisfies (S1′)–(S3′). Summarizing, we get

$$\text{if } N - n_1 - n_2 \ge 3 \text{ and } N > 3n-1, \quad \text{then } s = 3,\ n_2 + n_3 = 2n-1, \text{ and } n_1 = n_2. \tag{8}$$

Case 2: $N - n_1 - n_2 \ge 3$ and $N = 3n-1$. Again (S3′) holds by default. By (S2′), $n_1 \le n$; hence $N - n_1 - n_2 \ge n-1$. If $s \ge 4$ and $n_{s-1} + n_s \le n$, then let L be the list obtained from $(n_1, \dots, n_s)$ by replacing the two entries n_{s-1} and n_s with $n_{s-1} + n_s$ and then possibly rearrange the entries into nonincreasing order. By construction, L satisfies (S1′)–(S3′), a contradiction to (b). Hence $n_{s-1} + n_s \ge n+1$. We also have $n_{s-1} + n_s \ge n+1$ if $s = 3$, since in this case $n_{s-1} + n_s = N - n_1 \ge 2n-1$. If $s \ge 6$, then $N \ge 3(n_{s-1} + n_s) \ge 3n+3$, contradicting $N = 3n-1$. Thus

$$\text{if } N - n_1 - n_2 \ge 3 \text{ and } N = 3n-1, \quad \text{then } n_1 \le n,\ s \le 5,\ n_{s-1} + n_s \ge n+1. \tag{9}$$

Case 3: $N - n_1 - n_2 \le 2$. Then $N \le 2n_1 + 2$, so by (S3′), $2n_1 + 2 \ge N \ge 4n-1$, implying $n_1 \ge 2n-1$. If $n_1 \ge 2n$, then (S2′) implies $G \supseteq K_{2n,2n-1}$. If $n_1 = 2n-1$, then by (S3′), $N - n_1 \ge 2n$, so again $G \supseteq K_{2n,2n-1}$. Thus we can assume that

$$\text{if } N - n_1 - n_2 \le 2, \quad \text{then } G = K_{2n,2n-1}. \tag{10}$$

As we have seen,

$$\text{in each of Cases 1, 2 and 3 we have } s \le 5. \tag{11}$$

Fix an arbitrary 2-edge-coloring $E(G) = E_1 \cup E_2$ of G. For $i \in [2]$ and $v \in V(G)$, let $G_i := (V(G), E_i)$ and $d_i(v)$ denote the degree of v in G_i.

5. Regularity

5.1. *Applying the 2-colored version of the Regularity Lemma.* We first choose parameter α so that $0 < \alpha < 10^{-10}$ and then choose ϵ such that $\epsilon < 10^{-20}$ and $0 < 10^6\epsilon < \alpha$ so that the pair $\left(\frac{1}{2}\alpha, 3\epsilon\right)$ satisfies the relation of (δ, ϵ) in Lemma 11 with $\frac{1}{2}\alpha$ playing the role of δ. Here, ϵ is the parameter for the Regularity Lemma, and α is our cutoff for the edge density at which we give an edge of the reduced graph a color.

We apply Lemma 9 to obtain a partition $(V_0, V_1, \dots, V_L)$ of $V(G)$, with each of $V_1, V_2, \dots, V_L$ contained entirely in one of $V_1^*, V_2^*, \dots, V_k^*$. Define the k-partite *reduced graph* G^r as follows:

- The vertices of G^r are v_i for $i = 1, 2, \dots, L$. A k-partition $(V_1', V_2', \dots, V_k')$ of $V(G^r)$ is induced by the k-partition of G, and reordered if necessary so that $|V_1'| \ge |V_2'| \ge \dots \ge |V_k'|$.
- There is an edge between v_i and v_j if and only if v_i and v_j are in different parts of the k-partition and the pair $\{V_i, V_j\}$ is (ϵ, G_q)-regular for both $q = 1$ and $q = 2$.
- The reduced graph G^r is missing at most $\epsilon\binom{L}{2}$ edges between distinct pairs $\{V_i', V_j'\}$.
- We give G^r a 2-edge-multicoloring: two graphs (G_1^r, G_2^r) whose union includes every edge of G^r, but are not necessarily edge-disjoint. We add edge $v_i v_j \in E(G^r)$ to G_q^r if G_q contains at least $\alpha|V_i||V_j|$ of the edges between V_i and V_j. Since $G = G_1 \cup G_2$ contains all $|V_i||V_j|$ edges between V_i and V_j, each edge of G^r is added to either G_1^r or G_2^r, and possibly to both.

Let $t = |V_1| = |V_2| = \cdots = |V_L|$, $\ell_i = |V_i'|$ for $i = 1, \ldots, k$, and $\ell := (n - \epsilon N)/t$; since $N \leq 4n - 1$, we have $\ell t \geq (1 - 5\epsilon)n$.

Because $|V_0| \leq \epsilon N$, we have $(1 - \epsilon)N \leq Lt \leq N$ and $n_i - \epsilon N \leq \ell_i t \leq n_i$. Therefore:

- $Lt \geq (1 - \epsilon)N \geq 3n - 1 - \epsilon N = 3(\ell t + \epsilon N) - 1 - \epsilon N \geq 3\ell t - 1 + 2\epsilon n$, which means $L \geq 3\ell - 1$.
- $Lt \leq N \leq 4n - 1 = 4(\ell t + \epsilon N) - 1 \leq 5\ell t$, which means $L \leq 5\ell$.
- $Lt - \ell_1 t \geq N - n_1 - \epsilon N \geq 2n - 1 - \epsilon N \geq 2(\ell t + \epsilon N) - 1 - \epsilon N \geq 2\ell t - 1 + \epsilon N$, which means $L - \ell_1 \geq 2\ell - 1$.

Recall that G^r is missing at most $\epsilon\binom{L}{2} \leq \epsilon \frac{1}{2} L^2 < 16\epsilon L^2$ edges between distinct pairs $\{V_i', V_j'\}$. Since the number of V_i's missing at least $4\sqrt{\epsilon}\ell$ edges is less than $4\sqrt{\epsilon}\ell$, we know G^r is $(\ell, k, 4\sqrt{\epsilon})$-suitable. We apply Theorem 15 to the graph G^r with γ such that $10^{-6} > \gamma > 1000\alpha$ and $\gamma > 4000\sqrt{\epsilon}$. Then we conclude that either G^r has a monochromatic connected matching of size $(1 + \gamma)\ell$, or else $V(G)$ has a $(68\gamma, i, j)$-bad partition for some $i \in [2]$ and $j \in [2]$.

5.2. ***Handling a large connected matching in the reduced graph.*** For every edge $v_i v_j \in G_1^r$, the corresponding pair (V_i, V_j) is (ϵ, G_1)-regular and contains at least αt^2 edges of G_1. Let $X_{ij} \subseteq V_i$ be the set of all vertices of V_i with fewer than $\frac{1}{2}\alpha t$ edges of G_1 to V_j, and let $Y_{ij} \subseteq V_j$ be the set of all vertices of V_j with fewer than $\frac{1}{2}\alpha t$ edges of G_1 to V_i. Note we have $Y_{ij} = X_{ji}$ but we keep using the notation Y_{ij} for emphasizing they are in different parts. Then

$$\frac{|E(X_{ij}, V_j)|}{|X_{ij}||V_j|} \leq \frac{\alpha}{2},$$

so $|X_{ij}| \leq \epsilon t$ to avoid violating (ϵ, G_1)-regularity; similarly, $|Y_{ij}| \leq \epsilon t$. Call vertices of $V_i \cup V_j$ which are not in $X_{ij} \cup Y_{ij}$ *typical* for the pair (V_i, V_j) (or for the edge $v_i v_j$ of G_1).

Let $\mathcal{M}$ be a connected matching in G_1^r of size $(1 + \gamma)\ell$. Give the edges in $\mathcal{M}$ an arbitrary cyclic ordering.

If $v_{i_1} v_{j_1}$ and $v_{i_2} v_{j_2}$ are edges of $\mathcal{M}$ which are consecutive in the ordering, we shall find a path $P(j_1, i_2)$ in G_1 joining a vertex of $V_{j_1} \setminus Y_{i_1 j_1}$ to a vertex of $V_{i_2} \setminus X_{i_2 j_2}$. To do so, we begin by finding a path P^r from v_{j_1} to v_{i_2} in G_1^r, then find a realization of that path in G_1. Pick a starting point of $P(j_1, i_2)$ typical both for the edge $v_{i_1} v_{j_1}$ and for the first edge of P^r. Next, choose the path greedily, making sure to satisfy the following conditions:

- Choose a neighbor of the previous vertex chosen which is typical for the next edge of P^r (or for $v_{i_2 j_2}$ when we reach the end of P^r).
- Choose a vertex which has not been chosen for any previous paths.

As we construct $P(j_1, i_2)$, the last vertex we have chosen is always typical for the edge of P^r we are about to realize; therefore we have at least $\frac{1}{2}\alpha t$ options for its neighbors. At most ϵt of them are eliminated because they are not typical for the next edge, and at most L^2 are eliminated because they have been chosen for previous paths. Since L is upper bounded by M which is independent of n, and $\epsilon < 10^{-6}\alpha$, we can always choose such a vertex.

Moreover, we may choose the paths such that their total length has the same parity as $|\mathcal{M}|$. If the component of G_1^r containing $\mathcal{M}$ is not bipartite, then each path can be chosen to have any parity we like. If the component of G_1^r containing $\mathcal{M}$ is bipartite, then this condition is satisfied automatically:

if we join the paths of P^r we chose by the edges of $\mathcal{M}$, we get a closed walk, which must have even length.

Once all these paths are chosen, we combine them into a long even cycle in G_1. For each edge $v_i v_j$ in the matching $\mathcal{M}$, we have vertices $x \in V_i$ and $y \in V_j$, both typical for (V_i, V_j), which are the endpoints of two paths we have constructed. We show that we can find a path from x to y using only edges of G_1 between V_i and V_j of any odd length between $t-1$ and $(1-3\epsilon)2t-1$.

To do so, we choose any $X \subseteq V_i$ with $|X| \geq \frac{1}{2}t$ that contains x and at least $\frac{1}{2}\alpha t$ neighbors of y; similarly, we choose $Y \subseteq V_j$ with $|Y| = |X|$ that contains y and at least $\frac{1}{2}\alpha t$ neighbors of x. If we want the path to have length $2Ct-1$, where $C \in \left[\frac{1}{2}, 1-3\epsilon\right]$, we begin by choosing X and Y of size $(C+3\epsilon)t$. The pair (X, Y) is $(2\epsilon, G_1)$-regular with density at least $\alpha - \epsilon$, so there are at most 2ϵ vertices in each of X and Y which have fewer than $\frac{1}{2}\alpha t$ neighbors on the other side; by our construction of X and Y, x and y are not among them.

Let $X' \subseteq X$ and $Y' \subseteq Y$ be the subsets obtained by deleting these low-degree vertices, leaving at least $(C+\epsilon)t$ vertices on each side, and then deleting enough vertices from each part to make $|X'| = |Y'| = Ct$. The pair (X', Y') is $(3\epsilon, G_1)$-regular, and all vertices have minimum degree at least $(\alpha - 3\epsilon)t$, so by Lemma 11, there is a path from x to y using all vertices of X' and Y', which has the desired length $2Ct-1$.

If we use $C = 1-3\epsilon$ for each edge $v_i v_j$ in the matching $\mathcal{M}$, then the cycle contains at least $2(1-3\epsilon)t$ vertices for each edge of $\mathcal{M}$, even ignoring the paths we constructed between them, while $|\mathcal{M}| \geq (1+10\epsilon)\ell$; therefore the total length is at least

$$2(1-3\epsilon)(1+10\epsilon)\ell t \geq 2(1-3\epsilon)(1+10\epsilon)(1-5\epsilon)n \geq (1+\epsilon)2n.$$

If we use $C = \frac{1}{2}$ each edge $v_i v_j$, then the cycle contains only t vertices for each edge of $\mathcal{M}$, giving approximately half as many edges. Up to parity, we are free to choose any length in this range, and therefore it is possible to construct a path in G_1 of length exactly $2n$.

5.3. ***Handling a bad partition of the reduced graph.*** We will show in Sections 6 and 7 how to find a long monochromatic cycle in a bad partition of G. In this subsection, we show that a bad partition of G^r corresponds to a bad partition of G.

(1) If $X \subseteq V(G^r)$ has size $C\ell$, then the corresponding set of vertices in G is $\bigcup_{v_i \in X} V_i$. It has size $C\ell t$, which is in the range $[(1-5\epsilon)Cn, Cn]$.

(2) If $|E_{G_i^r}(X)| \leq \lambda\ell^2$, then each of those $\lambda\ell^2$ edges of G_i^r corresponds to at most t^2 edges of G_i for $\lambda\ell^2 t^2 \leq \lambda n^2$ edges.

   Additionally, edges not in G_i^r may appear in G_i; across all of G_i there are at most $\alpha t^2\binom{L}{2} \leq \frac{1}{2}\alpha N^2 \leq 10\alpha n^2$ edges that occur in this way.

   Moreover, edges from at most $\epsilon\binom{L}{2}$ exceptional pairs may appear in G_i, contributing at most $10\epsilon n^2$ edges in total by the same calculation.

   To summarize, there are at most $(\lambda + 10\alpha + 10\epsilon)n^2$ edges in G_i corresponding to $E_{G_i^r}(X)$. A similar argument applies to a bound on $|E_{G_i^r}(X, Y)|$ for $X, Y \subseteq V(G^r)$.

(3) There are fewer than $\epsilon N \leq 5\epsilon n$ vertices from the exceptional part V_0, which can generally be assigned to any part of any bad partition without changing the approximate structure.

Thus, for $10^{-3} > \lambda > 1000\alpha > 10^9\epsilon > 0$, if G^r has a $(\lambda, i, 1)$-bad partition ($i \in [2]$) $V(G^r) = W_1^r \cup W_2^r$, then G has a corresponding $(2\lambda, i, 1)$-bad partition with:

(0) $W_1 := \left(\bigcup_{v_i \in W_1^r} V_i\right) \cup V_0$ and $W_2 := \bigcup_{v_i \in W_2^r} V_i$.

(i) $(1-2\lambda)n \le (1-\lambda)(1-5\epsilon)n \le (1-\lambda)\ell t \le |W_2| \le (1+\lambda)\ell_1 t \le (1+\lambda)n_1$.

(ii) $|E(G_i[W_1, W_2])| \le (\lambda + 10\alpha + 10\epsilon + 5\epsilon)n^2 \le 2\lambda n^2$.

(iii) $|E(G_{3-i}[W_1])| \le \left(\lambda + 10\alpha + 10\epsilon + 5\epsilon + \frac{25}{2}\epsilon^2\right)n^2 \le 2\lambda n^2$.

If G^r has a $(\lambda, i, 2)$-bad partition ($i \in [2]$) $V(G^r) = V_j' \cup U_1^r \cup U_2^r$ then G has a corresponding $(2\lambda, i, 2)$-bad partition with:

(0) $U_1 := \bigcup_{v_i \in U_1^r} V_i \cup (V_0 - V_j^*)$ and $U_2 := \bigcup_{v_i \in U_2^r} V_i$.

(i) $|E(G_i[V_j^*, U_1])| \le (\lambda + 10\alpha + 10\epsilon + 5\epsilon)n^2 \le 2\lambda n^2$.

(ii) $|E(G_{3-i}[V_j, U_2])| \le (\lambda + 10\alpha + 10\epsilon + 5\epsilon)n^2 \le 2\lambda n^2$.

(iii) $n_j = |V_j^*| \ge \ell_j t \ge (1-\lambda)\ell t \ge (1-\lambda)(1-5\epsilon)n \ge (1-2\lambda)n$.

(iv) $(1+2\lambda)n \ge (1+\lambda)n + 5\epsilon n \ge (1+\lambda)\ell t + 5\epsilon n \ge |U_1| \ge (1-\lambda)\ell t \ge (1-\lambda)(1-5\epsilon)n \ge (1-2\lambda)n$.

(v) $(1+\lambda)n \ge (1+\lambda)\ell t \ge |U_2| \ge (1-\lambda)\ell t \ge (1-\lambda)(1-5\epsilon)n \ge (1-2\lambda)n$.

Therefore, a $(68\gamma, i, j)$-bad partition of G^r corresponds to a $(136\gamma, i, j)$-bad partition of G for some $i \in [2]$ and $j \in [2]$. In the next three sections we show how to find a monochromatic cycle of length exactly $2n$ when G has a (λ, i, j)-bad partition for some $i \in [2]$ and $j \in [2]$, where $\lambda = 136\gamma$.

6. Dealing with $(\lambda, i, 1)$-bad partitions when $N - n_1 - n_2 \ge 3$

6.1. ***Setup.*** Without loss of generality, let $i = 1$. Recall that $d_k(v)$ is the degree of v in G_k, where $k \in [2]$. We assume that for some $\lambda < 0.01$, there is a partition $V(G) = W_1 \cup W_2$ such that

$$(1-\lambda)n \le |W_2| \le (1+\lambda)n_1, \tag{12}$$

$$|E(G_1[W_1, W_2])| \le \lambda n^2, \tag{13}$$

$$|E(G_2[W_1])| \le \lambda n^2. \tag{14}$$

If G has at least four parts then $n_1 \le n$ by (8) and (9). If G is tripartite, then we could have n_1 much larger than n, but in this section, we will assume $n_1 < \frac{5}{3}n$. The alternative, that G is tripartite and $n_1 \ge \frac{5}{3}n$, is handled in Section 6.2.

We know that $|W_1| \ge N - (1+\lambda)n_1 \ge 2n - 1 - \lambda n_1 \ge (2-5\lambda)n$ since $n_1 \le 2n$. For any vertex x, fewer than $\frac{5}{3}n$ vertices of W_1 can be in the same part V_i of G as x, so at least $\left(\frac{1}{3} - 5\lambda\right)n > \frac{1}{4}n$ are in other parts of G. In other words, we have $d(x, W_1) \ge \frac{1}{4}n$ for all $x \in V(G)$.

We call a vertex $x \in V(G)$ W_1-*typical* if $d_1(x, W_1) \ge \frac{3}{4}d(x, W_1)$, and W_2-*typical* if $d_1(x, W_1) < \frac{3}{4}d(x, W_1)$.

If x is W_1-typical, then $d_1(x, W_1) \ge \frac{3}{4} \cdot \frac{1}{4}n = \frac{3}{16}n$. Since

$$\sum_{x \in W_2} d_1(x, W_1) = |E(G_1[W_1, W_2])| \le \lambda n^2,$$

the number of W_1-typical vertices in W_2 is at most

$$\frac{\lambda n^2}{\frac{3}{16}n} < 6\lambda n.$$

Similarly, if x is W_2-typical, then $d_2(x, W_1) \geq \frac{1}{4} \cdot \frac{1}{4}n = \frac{1}{16}n$. Since

$$\sum_{x \in W_1} d_2(x, W_1) = 2|E(G_2[W_1])| \leq 2\lambda n^2,$$

the number of W_2-typical vertices in W_1 is at most

$$\frac{2\lambda n^2}{\frac{1}{16}n} = 32\lambda n.$$

Let W_1' be the set of all W_1-typical vertices and W_2' be the set of all W_2-typical vertices. The partition (W_1', W_2') is almost exactly the same as the partition (W_1, W_2): at most $40\lambda n$ vertices have been moved from one part to the other part to obtain (W_1', W_2') from (W_1, W_2). Therefore, if $x \in W_1'$, we still have $d_1(x, W_1') \geq \frac{3}{4}d(x, W_1) - 40\lambda n$, and if $x \in W_2'$, we still have $d_1(x, W_1') < \frac{3}{4}d(x, W_1) + 40\lambda n$. In either case, we still have $d(x, W_1') \geq \frac{1}{4}n - 40\lambda n$ for all x.

Moreover, W_1' and W_2' still satisfy similar conditions to W_1 and W_2:

(1) $(1 - 41\lambda)n \leq |W_2'| \leq (1 + \lambda)n_1 + 40\lambda n \leq (1 + 81\lambda)n_1$ (since $n_1 \geq \frac{1}{2}n$ in all cases).

(2) $|E(G_1[W_1', W_2'])| \leq \lambda n^2 + N \cdot (40\lambda n) \leq 161\lambda n^2$, since we move at most $40\lambda n$ vertices with degree less than N.

(3) $|E(G_2[W_1'])| \leq \lambda n^2 + N \cdot (6\lambda n) \leq 25\lambda n^2$, since we move at most $6\lambda n$ vertices with degree less than N into W_1'.

For convenience, let $\delta = 200\lambda$, which is at least as large as all multiples of λ used above.

Our goal is to find a cycle of length $2n$ in either G_1 or G_2. We decide which type of cycle we will attempt to find based on the relative sizes of W_1' and W_2'.

Suppose that $|W_1'| \geq 2n$ and, moreover, $|W_1' \setminus V_i| \geq n$ for all i. In this case, we find a cycle of length $2n$ in G_1; this is done in Section 6.3.

Otherwise, we must have $|W_2'| \geq n$: either $|W_1'| \leq 2n-1$ and $|W_2'| = N - |W_1'| \geq n$, or else $|W_1' \setminus V_i| \leq n-1$ for some i, and

$$|W_2'| \geq |W_2' \setminus V_i| = |V \setminus V_i| - |W_1' \setminus V_i| \geq (N - n_i) - (n-1) \geq (2n-1) - (n-1) = n.$$

In this case, we find a cycle of length $2n$ in G_2; this is done in Section 6.4.

We use the following lemma to pick out "well-behaved" vertices in W_1' and W_2'. For example, we commonly apply it to $G_2[W_1']$ or to $G_1[W_1', W_2']$.

Lemma 19. *Let H be an n-vertex graph with at most ϵn^2 edges for some $\epsilon > 0$ and let $S \subseteq V(H)$. If $S' \subseteq S$ is any subset that excludes the k vertices of S with the highest degree, then every $v \in S'$ satisfies $d_H(v) < 2\epsilon n^2/k$.*

Additionally, when H is bipartite, and S is entirely contained in one part of H, every $v \in S'$ satisfies $d_H(v) < \epsilon n^2/k$.

Proof. In the first case, if we have $d_H(v) \geq 2\epsilon n^2/k$ for any $v \in S'$, then we also have $d_H(v) \geq d$ for the k vertices of S with the highest degree, which we excluded from S'. The sum of degrees of these $k+1$ vertices exceeds $2\epsilon n^2$, so it is greater than twice the number of edges in H, a contradiction.

In the second case, if we have $d_H(v) \geq \epsilon n^2/k$ for any $v \in S'$, the same sum of degrees exceeds ϵn^2. But since the vertices of S are all on one side of the bipartition of H, this sum of degrees cannot be greater than the number of edges in H, which is again a contradiction. □

6.2. *The nearly bipartite subcase.* In this subsection, we assume that G is tripartite with $n_1 \geq \frac{5}{3}n$. Recall that when G is tripartite we have $n_1 = n_2$ and $n_1 + n_3 = n_2 + n_3 = 2n - 1$, and that throughout Section 6 we assume $N - n_1 - n_2 \geq 3$, or in this case that $n_3 \geq 3$.

Case 1: $|W_1 \cap V_i| \geq (1 + 10\lambda)n$ for $i = 1$ or $i = 2$. We assume $i = 1$; the proof for the case $i = 2$ is the same. In this case, let X be an n-vertex subset of $V_1 \cap W_1$ avoiding the $5\lambda n$ vertices of $V_1 \cap W_1$ with the most edges of G_2 to $W_1 \setminus V_1$ and the $5\lambda n$ vertices of $V_1 \cap W_1$ with the most edges of G_1 to $W_2 \setminus V_1$.

For any vertex $v \in X$, we have

$$d_2(v, W_1 \setminus V_1) \leq \frac{\lambda n^2}{5\lambda n} = \frac{n}{5} \quad \text{and} \quad d_1(v, W_2 \setminus V_1) \leq \frac{n}{5}$$

by Lemma 19.

We partition $V_2 \cup V_3$ into sets Y_1 and Y_2 by the following procedure:

(1) The $2\lambda n$ vertices of $W_1 \setminus V_1$ with the most edges of G_2 to X are set aside, and the remaining vertices of $W_1 \setminus V_1$ are assigned to Y_1.
 By Lemma 19, any vertex v assigned to Y_1 in this step has $d_2(v, X) \leq \frac{1}{2}n$.

(2) The $2\lambda n$ vertices of $W_2 \setminus V_1$ with the most edges of G_1 to X are set aside, and the remaining vertices of $W_2 \setminus V_1$ are assigned to Y_2.
 By Lemma 19, any vertex v assigned to Y_2 in this step has $d_1(v, X) \leq \frac{1}{2}n$.

(3) Each remaining vertex v is assigned to Y_1 if $d_1(v, X) \geq \frac{1}{2}n$ and to Y_2 otherwise (in which case $d_2(v, X) \geq \frac{1}{2}n$).

Since $|V_2 \cup V_3| = 2n - 1$, we must have $|Y_1| \geq n$ or $|Y_2| \geq n$. Let Y_j' be an n-vertex subset of Y_j, where $j \in [2]$ and $|Y_j| \geq n$. We apply Theorem 16 to find a Hamiltonian cycle in the bipartite graph $H = G_j[X, Y_j']$.

The minimum H-degree in X is $\frac{4}{5}n - 2\lambda n$, since each $v \in X$ had at most $\frac{1}{5}n$ edges to $W_j \setminus V_1$ which were not in G_j, and at most $2\lambda n$ vertices of Y_j' did not come from $W_j \setminus V_1$ originally. The minimum H-degree in Y_j' is $\frac{1}{2}n$, so the condition of Theorem 16 is satisfied: whenever $d_H(u_i) \leq i$, we have $i \geq \left(\frac{4}{5} - 2\lambda\right)n$, so $d_H(v_{n-i}) \geq \frac{1}{2}n \geq \left(\frac{1}{5} + 2\lambda\right)n + 1$.

Case 2: $|V_i \cap W_1| < (1 + 10\lambda)n$ for $i = 1$ and $i = 2$. By (12), we must have $|W_1| \geq N - (1 + \lambda)n_1 = 2n - 1 - \lambda n_1 > 2n - 3\lambda n$. Since $n_1 = n_2 \geq \frac{5}{3}n$ and $n_2 + n_3 = 2n - 1$, fewer than $\frac{1}{3}n$ vertices of W_1 are in V_3, so at least $\left(\frac{5}{3} - 3\lambda\right)n$ of them are in $V_1 \cup V_2$; therefore $|W_1 \cap V_1| > \left(\frac{2}{3} - 13\lambda\right)n$ and $|W_1 \cap V_2| > \left(\frac{2}{3} - 13\lambda\right)n$.

Because $2n > n_1 = n_2 \geq \frac{5}{3}n$, we have $\left(\frac{2}{3} - 10\lambda\right)n < |V_i \cap W_2| < \left(\frac{4}{3} + 13\lambda\right)n$ for $i = 1, 2$, as well.

Next, we choose subsets $X_{ij} \subseteq V_i \cap W_j$ with $|X_{11}| = |X_{21}| = |X_{12}| = |X_{22}| = \frac{1}{2}n + 10$. To choose X_{11} and X_{21}, avoid the $\frac{1}{20}n$ vertices with the most edges in G_1 to W_2 and the $\frac{1}{20}n$ vertices with the most edges in G_2 to W_1, so that each chosen vertex has at most $20\lambda n$ edges of each kind by Lemma 19. To choose X_{12} and X_{22}, avoid the $\frac{1}{10}n$ vertices with the most edges in G_1 to W_1, so that each chosen vertex has at most $10\lambda n$ such edges by Lemma 19.

First, we observe that if H is any of the graphs $G_1[X_{11}, X_{21}]$, $G_2[X_{12}, X_{21}]$, or $G_2[X_{11}, X_{22}]$, then given any vertices v, w in H, we can find a (v, w)-path in H on m vertices, provided that $n - 10 \le m \le n + 10$ (this is not optimal, but it is more than we need) and that the parity of m is correct.

To do so, we apply Theorem 18. If v and w are on the same side of H, add a vertex x to the other side adjacent to all vertices in the side containing v and w; if not, add an edge vw. Then take a subgraph containing $\lceil \frac{1}{2}m \rceil$ vertices from each side, making sure to include v, w and if applicable x. In this subgraph, the minimum degree is at least $\lceil \frac{1}{2}m \rceil - 20\lambda n$, so we can use Theorem 18 to find a Hamiltonian cycle in this graph containing either the edge vw or the edges vx and xw. Deleting the vertex x or the edge vw, whichever applies, creates a (v, w)-path in H of the correct length.

Suppose that $G_2[X_{12}, X_{22}]$ contains a matching $M = \{u_1u_2, v_1v_2\}$ of size 2, where $u_1, v_1 \in X_{12}$ and $u_2, v_2 \in X_{22}$. In that case, we can find a (u_1, v_1)-path P in $G_2[X_{12}, X_{21}]$ on $2\lceil \frac{1}{2}n \rceil + 1$ vertices and a (u_2, v_2)-path Q in $G_2[X_{11}, X_{22}]$ on $2\lfloor \frac{1}{2}n \rfloor - 1$ vertices by the previous observation. Joining the paths P and Q using the edges of the matching M, we find a cycle of length $2n$ in G_2.

Now we assume $G_2[X_{12}, X_{22}]$ does not contain a matching of size 2. If the size of a maximum matching in this graph is 1, then there is a vertex cover of size 1 since $G_2[X_{12}, X_{22}]$ is bipartite. We delete this vertex cover from X_{12} or X_{22} (it depends on where this vertex cover is). Having changed X_{12} and X_{22} in this way, $G_1[X_{12}, X_{22}]$ is a complete bipartite graph, so it also has the property that any two vertices in it can be joined by a path on m vertices, provided that $n - 10 \le m \le n + 10$ and that the parity of m is correct.

Note that there are at least three vertices in V_3.

We say that a vertex $v \in V_3$

- is *j-adjacent* to a set S if it has at least two edges in G_j to S,
- *S-connects* G_j if it is j-adjacent to both X_{11} and X_{12}, or if it is j-adjacent to both X_{21} and X_{22} ("S-connects" because it is j-adjacent to two sets in the *same* part of V_1 or V_2),
- *C-connects* G_1 if it is 1-adjacent to both X_{11} and X_{22}, or if it is 1-adjacent to both X_{12} and X_{21} ("C-connects" because the j-adjacency *crosses* from V_1 to V_2),
- *C-connects* G_2 if it is 2-adjacent to both X_{11} and X_{21}, or if it is 2-adjacent to both X_{12} and X_{22},
- *folds into* G_1 if it is 1-adjacent to both X_{11} and X_{21}, or if it is 1-adjacent to both X_{12} and X_{22},
- *folds into* G_2 if it is 2-adjacent to both X_{11} and X_{22}, or if it is 2-adjacent to both X_{12} and X_{21}.

Some comments on these definitions: first, a vertex that is j-adjacent to at least three of X_{11}, X_{12}, X_{21}, X_{22} is guaranteed to both S-connect and C-connect G_j. Second, a vertex that is j-adjacent to only two of $X_{11}, X_{12}, X_{21}, X_{22}$ for each value of j may S-connect both G_1 and G_2, or C-connect G_1 and fold into G_2, or C-connect G_2 and fold into G_1. In particular, each vertex either S-connects or C-connects some G_j.

If there are two vertices in V_3 that both S-connect G_j, or both C-connect G_j, then we can find a cycle of length $2n$ in G_j. The cases are all symmetric; without loss of generality, suppose $v, w \in V_3$ both S-connect G_1. We can find a path P in $G_1[X_{11}, X_{21}]$ on $2\lceil \frac{1}{2}n \rceil - 1$ vertices that starts at a G_1-neighbor of v and ends at a G_1-neighbor of w, and a path Q in $G_1[X_{12}, X_{22}]$ on $2\lfloor \frac{1}{2}n \rfloor - 1$ vertices that starts at a G_1-neighbor of v and ends at a G_1-neighbor of w. Joining P and Q via v at one endpoint and via w on the other creates a cycle of length $2n$ in G_1.

If we cannot find two vertices as in the previous paragraph, then the best we can do is to find, for some j, a vertex $v \in V_3$ that S-connects G_j and another vertex $w \in V_3$ that C-connects G_j. Since v does not C-connect G_j, it must also S-connect G_{3-j}.

There is at least one more vertex $x \in V_3$. By assumption, it does not S-connect G_{3-j} and neither S-connects nor C-connects G_j, so it must fold into G_j (and C-connect G_{3-j}).

Without loss of generality, suppose that $j = 1$ and x has a G_1-neighbor in both X_{11} and X_{21}. We add an artificial edge e_x between a pair of such neighbors of x.

As before, we can find a path P in $G_1[X_{11}, X_{21}]$ joining a neighbor of v to a different neighbor of w; we add the requirement that it uses the edge e_x, which is still possible by Theorem 18. We can also find a path Q in $G_1[X_{12}, X_{22}]$ joining a neighbor of v to a different neighbor of w. Since v S-connects G_1 and w C-connects G_1, one of these paths will have even length and the other will have odd length, but we can choose them to have $2n-3$ vertices total.

Now join the paths P and Q using the vertices v and w, then replace the artificial edge e_x by two edges to x from its endpoints. The result is a cycle of length $2n$ in G_1.

6.3. *Finding a cycle in G_1.* In this subsection, we are considering a 2-edge-colored graph G and a partition $W_1' \cup W_2'$ of $V(G)$ satisfying the following properties:

(1) G is a complete s-partite graph with parts $V_1, V_2, \dots, V_s$ of sizes $n_1, n_2, \dots, n_s$, with $s \geq 3$ and $n_1 + \cdots + n_s \leq 4n$.

(2) $(1-\delta)n \leq |W_2'| \leq (1+\delta)n_1$.

(3) $|E(G_1[W_1', W_2'])| \leq \delta n^2$ and $|E(G_2[W_1'])| \leq \delta n^2$.

(4) If $x \in W_1'$, then $d_1(x, W_1') \geq \frac{3}{4} d(x, W_1) - \delta n$.

(5) $|W_1'| \geq 2n$ and $|W_1' \setminus V_i| \geq n$ for all i. (This is the assumption that leads to this subsection as opposed to Section 6.4.)

We can deduce a further degree condition that holds for all vertices $x \in W_1'$:

(6) By properties (1) and (2), $|W_1'| = |V(G)| - |W_2'| \leq 4n - (1-\delta)n = (3+\delta)n$, so $d(x, W_1') \leq (3+\delta)n$. By property (4), we have $d_2(x, W_1) \leq \frac{1}{4}(3+\delta)n + \delta n \leq \left(\frac{3}{4} + 2\delta\right)n$.

To find a cycle of length $2n$ in G_1, we will choose two disjoint sets $X, Y \subseteq W_1'$ of size n, then apply Theorem 16 to find a Hamiltonian cycle in $H = G_1[X, Y]$.

Let $a, b \in \{1, 2, \dots, s\}$ be such that $V_a \cap W_1'$ is the largest part of $G_1[W_1']$ and $V_b \cap W_1'$ is the second-largest part of $G_1[W_1']$. To define X and Y, we begin by assigning $V_a \cap W_1'$ to X and $V_b \cap W_1'$ to Y. If either of these exceeds n vertices, we choose n of the vertices arbitrarily.

Continue by assigning the parts $V_i \cap W_1'$ to either X or Y arbitrarily for as long as this does not make $|X|$ or $|Y|$ exceed n. Once this is no longer possible, then:

- If there are still at least two parts $V_i \cap W_1'$ left unassigned, then each of them must have more than $\max\{n - |X|, n - |Y|\}$ vertices. Therefore we can add vertices from one of them to X to make $|X| = n$ (if necessary), and add vertices from the other to Y to make $|Y| = n$ (if necessary).

- If there is only one part of $G_1[W_1']$ left unassigned, call it $V_{\text{split}} \cap W_1'$. We assign $n - |X|$ vertices of $V_{\text{split}} \cap W_1'$ to X and $n - |Y|$ other vertices of $V_{\text{split}} \cap W_1'$ to Y.
- If there are no parts left unassigned, then we must have $|X| = |Y| = n$.

We must show that we do not run out of vertices in either of the last two cases. If $|V_a \cap W_1'| \le n$, then we do not run out because $|W_1'| \ge 2n$ (by property (5)) and all vertices in $W_1' \setminus V_{\text{split}}$ are assigned to either X or Y, so either $V_{\text{split}} \cap W_1'$ must contain enough vertices to fill X and Y or X and Y are already full. If $|V_a \cap W_1'| > n$, then we do not run out because $|W_1' \setminus V_a| \ge n$ (again, by property (5)), and after $V_a \cap W_1'$ is assigned, all vertices of W_1' are added to Y until it is full.

The most difficult case for us is the one in which some part $V_{\text{split}} \cap W_1'$ is divided between X and Y. To handle all cases at once, we assume this happens; if necessary, we choose some part $V_i \cap W_1'$ ($i \ne a, b$) to be a degenerate instance of V_{split} which is entirely in X or Y.

Let $n_x = |V_{\text{split}} \cap X|$ and $n_y = |V_{\text{split}} \cap Y|$. We assigned the largest part of $G[W_1']$ to X and the second-largest to Y; therefore X and Y both contain at least $n_x + n_y$ vertices not in V_{split}. Since $|X| = |Y| = n$, we must have $n_x + (n_x + n_y) \le n$ and $n_y + (n_x + n_y) \le n$; therefore $n_x + n_y \le \frac{2}{3}n$, while individually $n_x \le \frac{1}{2}n$ and $n_y \le \frac{1}{2}n$.

We first prove some bounds on $d_1(x, Y)$ for $x \in X$ (and, by symmetry, $d_1(y, X)$ for $y \in Y$). If $x \notin V_{\text{split}}$, then $d(x, Y) = n$ (since there are no vertices of Y in the same part of G as x), while $d_2(x, W_1') \le \left(\frac{3}{4} + 2\delta\right)n$ by property (6), so $d_1(x, Y) \ge \left(\frac{1}{4} - 2\delta\right)n$. If $x \in V_{\text{split}}$, then $d(x, W_1') = (n - n_x) + (n - n_y)$, since all vertices of W_1' outside V_{split} have been assigned to either X or Y, so $d_2(x, W_1') \le \frac{1}{4}(2n - n_x - n_y) + \delta n$ by property (4). This leaves $d_1(x, Y) \ge \frac{1}{2}n - \frac{3}{4}n_y - \delta n \ge \left(\frac{1}{8} - \delta\right)n$.

If we exclude the $\frac{1}{10}n$ vertices of X with the most edges to W_1' in G_2, then by Lemma 19, the remaining vertices $x \in X$ have $d_2(x, W_1') \le 20\delta n$. If $x \notin V_{\text{split}}$, this means $d_1(x, Y) \ge (1 - 20\delta)n$, and if $x \in V_{\text{split}}$, this means that $d_1(x, Y) \ge n - n_y - 20\delta n$.

Let $H = G_1[X, Y]$, let $u_1, u_2, \dots, u_n$ be the vertices of X ordered so that $d_H(u_1) \le \cdots \le d_H(u_n)$, and let $v_1, v_2, \dots, v_n$ be the vertices of Y ordered so that $d_H(v_1) \le \cdots \le d_H(v_n)$.

Suppose $u_i \in X$ satisfies $d_H(u_i) \le i < n$. We have shown $d_1(x, Y) \ge \left(\frac{1}{8} - \delta\right)n$, so among $u_1, u_2, \dots, u_i$, there must be a vertex not among the $\frac{1}{10}n$ vertices of X with the most edges to W_1' in G_2. For such a vertex, $d_1(x, Y) \ge n - n_y - 20\delta n$, so in particular $d_H(u_i) \ge n - n_y - 20\delta n$, which means $i \ge n - n_y - 20\delta n$.

If we had $d_H(v_{n-i}) \le n - i$, then by repeating this argument for vertices in Y, we would have $d_H(v_{n-i}) \ge n - n_x - 20\delta n$, which would mean $n - i \ge n - n_x - 20\delta n$. Adding this to the inequality on i, we would get $n \ge 2n - n_x - n_y - 40\delta n$, which is impossible since $n_x + n_y \le \frac{2}{3}n$. So we must have $d_H(v_{n-i}) \ge n - i + 1$, and by Theorem 16, H contains a Hamiltonian cycle. This gives a cycle of length $2n$ in G_1.

6.4. ***Finding a cycle in G_2.*** In this subsection, we are considering a 2-edge-colored graph G and a partition $W_1' \cup W_2'$ of $V(G)$ satisfying the following properties:

(1) G is a complete s-partite graph with parts $V_1, V_2, \dots, V_s$ of size $n_1, n_2, \dots, n_s$, with $s \ge 3$ and $n_1 + \cdots + n_s \le 4n$. Moreover, $\frac{5}{3}n > n_1 \ge \cdots \ge n_s$; we considered the case $n_1 \ge \frac{5}{3}n$ in Section 6.2.

(2) Either $N - n_1 > 2n - 1$ and $|V_i| \le n$ for all i, or $n_1 = n_2 \ge n$, $s = 3$, and $N - n_1 = N - n_2 = 2n - 1$.

(3) $|E(G_1[W_1', W_2'])| \le \delta n^2$ and $|E(G_2[W_1'])| \le \delta n^2$.

(4) If $x \in W_2'$, then $d(x, W_1') \ge \frac{1}{4}n - \delta n$, and $d_2(x, W_1') \ge \frac{1}{4}d(x, W_1) - \delta n$.

(5) $n \le |W_2'| \le (1+\delta)n_1$. (The lower bound is the assumption that leads to this subsection as opposed to Section 6.3.)

Let Bad consist of the $\sqrt{\delta}n$ vertices of W_2' that maximize $d_1(x, W_1')$; let Good $= W_2' \setminus$ Bad. By Lemma 19, $d_1(x, W_1') \le \sqrt{\delta}n$ for all $x \in$ Good.

Our strategy is to handle the vertices in Bad: first by finding short vertex-disjoint paths containing the vertices in Bad, then by combining them into a single path. Finally, we extend this path to a cycle of length $2n$ in $G_2[W_1', W_2']$.

6.4.1. *Constructing paths containing each vertex of* Bad. For every vertex $x \in$ Bad, we find a four-edge path $P(x)$ in G_2, which contains x, but begins and ends at a vertex of Good. We construct these paths one at a time; for each vertex x, we must keep in mind that in each of W_1' and W_2', up to $2\sqrt{\delta}n$ vertices may have been used for previously chosen paths.

This is not always possible; when it is not, we find a cycle of length $2n$ in another way.

Lemma 20. *One of the following holds*:

(1) G_2 *contains a collection* $\{P(x) : x \in \text{Bad}\}$ *of vertex-disjoint paths of length* 4 *such that, for all* $x \in$ Bad, $P(x)$ *begins and ends at a vertex of* Good, *and also contains x and two vertices in* W_1'.

(2) G_2 *contains a cycle of length* $2n$.

Proof. We attempt to find the collection of vertex-disjoint paths, one vertex of Bad at a time.

By property (4) at the beginning of this section, even if $x \in$ Bad, we have $d(x, W_1') \ge \left(\frac{1}{4} - \delta\right)n$ and $d_2(x, W_1') \ge \frac{1}{4}d(x, W_1') - \delta n$, so $d_2(x, W_1') \ge \left(\frac{1}{16} - \frac{5}{4}\delta\right)n$. There is a part V_i with $d_2(x, W_1' \cap V_i) \ge \left(\frac{1}{64} - \frac{5}{16}\delta\right)n$.

First we consider the first case of property (2). That is, suppose $N - n_1 > 2n - 1$; then we have $|V_i| = n_i \le n_1 \le n$, so $|W_2' \cap V_i| \le \left(\frac{63}{64} + \frac{5}{16}\delta\right)n$. But $|W_2'| \ge n$ in total, so there must be another part V_j with $|W_2' \cap V_j| \ge \frac{1}{4}\left(\frac{1}{64} - \frac{5}{16}\delta\right)n$. We can choose two vertices $v, w \in V_j$ to use as the endpoints of $P(x)$: ruling out the vertices of $V_j \cap$ Bad (at most $\sqrt{\delta}n$) and previously used vertices of W_2' in V_j (at most $2\sqrt{\delta}n$) we still have a number of choices linear in n.

Now we know not just the center vertex x of the path $P(x)$ but also its two endpoints v and w. To complete $P(x)$, we must find a common neighbor of v and x, and another common neighbor of w and x. This is possible, since there are at least $\left(\frac{1}{64} - \frac{5}{16}\delta\right)n$ neighbors of x in $W_1' \cap V_i$; v and w have edges in G_2 to all but at most $\sqrt{\delta}n$ of them, and we exclude at most $2\sqrt{\delta}n$ more that have been already used.

We call the method above of choosing the collection $\{P(x) : x \in \text{Bad}\}$ the *greedy strategy*. As we have seen, it always works in the first case of property (2); it remains to see when it works in the second case. Now, we assume that G is tripartite, $n_1 = n_2 \ge n$, and $N - n_1 = N - n_2 = 2n - 1$.

The greedy strategy continues to work if we can always choose the part V_j from which to pick the endpoints of $P(x)$. For this choice to always be possible, it is enough that at least two parts of G contain $3\sqrt{\delta}n$ vertices of W_2': both of them will have vertices outside Bad not previously chosen for any path, and one of them will not be the same as V_i.

If this does not occur, then one part V_a of G contains all but $6\sqrt{\delta}n$ vertices of W_2', and each of the other two parts contains fewer than $3\sqrt{\delta}n$ vertices of W_2'. If V_a contains fewer than $\frac{1}{20}n$ vertices of W_1', then the greedy strategy still works: for any $x \in$ Bad, we have $d_2(x, W_1') \ge \left(\frac{1}{16} - \frac{5}{4}\delta\right)n > |V_a \cap W_1'| + 2\sqrt{\delta}n$, so we can always choose a part of G other than V_a to play the part of V_i. In this case, it does not matter that only V_a contains many vertices of W_2', because we only need to choose the endpoints of $P(x)$ from vertices in V_a.

The greedy strategy fails in the remaining case: when V_a contains all but $6\sqrt{\delta}n$ vertices of W_2' and at least $\frac{1}{20}n$ vertices of W_1'. Then $|V_a| > n$, so without loss of generality, $V_a = V_2$. In this case, we do not try to find the paths $P(x)$ and instead find a cycle of length $2n$ in G_1 or G_2 directly.

We have a lower bound on $n_1 = n_2 = |V_2|$: it is $|V_2 \cap W_1'| + V_2 \cap W_2'| \geq \big(1 + \frac{1}{20} - 6\sqrt{\delta}\big)n$. Since $|V_1 \cap W_2'| \leq 3\sqrt{\delta}n$, we have $|V_1 \cap W_1'| \geq \big(\frac{21}{20} - 9\sqrt{\delta}\big)n > n$.

Let Y_1 be a subset of exactly n vertices of $V_1 \cap W_1'$, chosen to avoid the $\sqrt{\delta}n$ vertices of $V_1 \cap W_1'$ with largest degree in $G_1[W_1', W_2']$ and the $\sqrt{\delta}n$ vertices of $V_1 \cap W_1'$ with largest degree in $G_2[V_1 \cap W_1', W_1' \setminus V_1]$. (This is possible since $\big(\frac{21}{20} - 11\sqrt{\delta}\big)n > n$ as well.) In both cases, if a vertex $x \in Y_1$ has degree d in the corresponding graph, we get at least $\sqrt{\delta}nd$ edges in either $G_1[W_1', W_2']$ or $G_2[W_1']$ by looking at the vertices we deleted; therefore $\sqrt{\delta}nd \leq \delta n^2$ and $d \leq \sqrt{\delta}n$.

Redistribute vertices of $V_2 \cup V_3$ into two parts (X_1, X_2) as follows:

- All vertices of $W_1' \setminus V_1$, except the $\sqrt{\delta}n$ vertices v maximizing $d_2(v, Y_1)$, are put in X_1. A vertex v of this type is guaranteed to have $d_2(v, Y_1) \leq \sqrt{\delta}n$.
- All vertices of $W_2' \setminus V_1$, except the vertices in Bad, are put in X_2. A vertex v of this type is guaranteed to have $d_1(v, Y_1) \leq \sqrt{\delta}n$.
- The remaining vertices, of which there are at most $2\sqrt{\delta}n$, are assigned to X_1 or X_2 based on their edges to Y_1. If $d_1(v, Y_1) \geq \frac{1}{2}n$, then v is put into X_1; otherwise, $d_2(v, Y_1) \geq \frac{1}{2}n$, and v is put into X_2.

The sets X_1, X_2, Y_1 satisfy the following properties. For any $v \in X_1$ we have $d_1(v, Y_1) \geq \frac{1}{2}n$. For any $v \in X_2$ we have $d_2(v, Y_1) \geq \frac{1}{2}n$. For any $v \in Y_1$ we have $d_2(v, X_1) \leq 4\sqrt{\delta}n$, since $d_2(v, W_1') \leq \sqrt{\delta}n$ and X_1 contains at most $3\sqrt{\delta}n$ vertices of W_2'; similarly, for any $v \in Y_1$ we have $d_1(v, X_2) \leq 4\sqrt{\delta}n$.

Since $|X_1| + |X_2| = |V_2 \cup V_3| = 2n - 1$, either $|X_1| \geq n$ or $|X_2| \geq n$.

If $|X_1| \geq n$, then we let X_1' be a subset of exactly n vertices of X_1, and find a cycle of length $2n$ in $H = G_1[X_1', Y_1]$ by applying Theorem 16. The hypotheses of the theorem are satisfied by the minimum degree conditions above: for $u \in X_1'$ we have $d_H(u) \geq \frac{1}{2}n$, and for $v \in Y_1$ we have $d_H(v) \geq (1 - 4\sqrt{\delta})n$.

Similarly, if $|X_2| \geq n$, we let X_2' be a subset of exactly n vertices of X_2 and find a cycle of length $2n$ in $H = G_2[X_2', Y_1]$ by applying Theorem 16. The argument is the same as in the previous paragraph. □

6.4.2. *Finding a cycle using Theorem 18.* Applying Lemma 20, each of the $\sqrt{\delta}n$ vertices $x \in$ Bad is the center of a length-4 path $P(x)$. Let A be the $2\sqrt{\delta}n$ vertices of W_1' in these paths and B be the $3\sqrt{\delta}n$ vertices of W_2' in these paths (including the vertices in Bad). Additionally, let C be the set of $\sqrt{\delta}n$ vertices of $W_1' \setminus A$ with the most edges to W_2' in G_1; by Lemma 19, every $x \in W_1' \setminus (A \cup C)$ satisfies $d_1(x, W_2') \leq \sqrt{\delta}n$.

Next, we will construct a bipartite graph H by choosing subsets $W_1'' \subseteq W_1' \setminus (A \cup C)$ of size $n - 2\sqrt{\delta}n$, and $W_2'' \subseteq W_2' \setminus B$ of size $n - 3\sqrt{\delta}n$; the edges of H are the edges of $G_2[W_1'' \cup A, W_2'' \cup B]$, except that we artificially join every internal vertex of every path $P(x)$ to every vertex on the other side of H. We will apply Theorem 18 to find a Hamiltonian cycle in H containing all $q = 4\sqrt{\delta}n$ edges belonging to the paths $P(x)$, after choosing W_1'' and W_2'' to make sure that the hypotheses of this theorem hold.

In terms of our future choice of (W_1'', W_2''), let $n_{i,j} = |V_i \cap W_j''|$. If $u \in V_i \cap W_1''$, then the degree of u in H is at least $n - n_{i,2} - \sqrt{\delta}n$: u has at most $\sqrt{\delta}n$ edges to W_2'' that are in G_1, not G_2, and its degree is further reduced by the $n_{i,2}$ vertices of W_2'' that are also in V_i. Similarly, if $v \in V_i \cap W_2''$, then the degree of v in H is at least $n - n_{i,1} - \sqrt{\delta}n$.

Let $n_{*,1} \geq n_{**,1}$ be the two largest values of $n_{i,1}$ and let $n_{*,2} \geq n_{**,2}$ be the two largest values of $n_{i,2}$. As in the statement of Theorem 18 let $u_1, u_2, \dots, u_n$ be the vertices of $W_1'' \cup A$ and let $v_1, v_2, \dots, v_n$ be the vertices of $W_2'' \cup B$, ordered by degree in H.

We begin with a lemma showing that some choices of (W_1'', W_2'') are guaranteed to satisfy the conditions of Theorem 18:

Lemma 21. *Theorem 18 can be applied, letting us find a cycle of length $2n$ in H, if we can choose W_1'' and W_2'' to satisfy the following two conditions:*

(1) *For each i, either $n_{i,1} + n_{i,2} \leq n - 10\sqrt{\delta}n$, or $n_{i,1} = 0$.*

(2) *For either $j = 1$ or $j = 2$, at most one value of $n_{i,j}$ exceeds $\left(\frac{1}{2} - 10\sqrt{\delta}\right)n$.*

Proof. Suppose that $u_i \in W_1'' \cup A$ and $d(u_i) \leq i + q = i + 4\sqrt{\delta}n$. The minimum H-degree of vertices in $W_1'' \cup A$ is $n - n_{*,2} - \sqrt{\delta}n$, so we must have $i \geq n - n_{*,2} - 5\sqrt{\delta}n$. By condition (1), at most $n - n_{*,2} - 10\sqrt{\delta}n$ vertices in W_1'' are in the same part as the largest part of W_2''; at most $2\sqrt{\delta}n$ vertices are endpoints of paths $P(x)$, so together these make up at most $n - n_{*,2} - 8\sqrt{\delta}n < i$ vertices. Therefore some of the vertices $u_1, \dots, u_i$ are vertices of W_1'' in a different part, and therefore $d(u_i) \geq n - n_{**,2} - \sqrt{\delta}n$.

Similarly, suppose that $v_j \in W_2'' \cup B$ and $d(v_j) \leq j + q \leq j + 4\sqrt{\delta}n$. The minimum H-degree of vertices in $W_2'' \cup B$ is $n - n_{*,1} - \sqrt{\delta}n$, so we must have $j \geq n - n_{*,1} - 5\sqrt{\delta}n$. By condition (1), at most $n - n_{*,1} - 10\sqrt{\delta}n + |B|$ vertices in W_2'' are in the same part as the largest part of W_1'', which is fewer than j. Therefore some of the vertices $v_1, \dots, v_j$ are vertices of W_2'' in a different part, and hence $d(v_j) \geq n - n_{**,1} - \sqrt{\delta}n$.

In such a case, we have $d(u_i) + d(v_j) \geq 2n - n_{**,1} - n_{**,2} - 2\sqrt{\delta}n$. We have $n_{**,1}, n_{**,2} \leq \frac{1}{2}n$, and additionally by condition (2), $n_{**,j} \leq \frac{1}{2}n - 10\sqrt{\delta}n$ for some j. Therefore $d(u_i) + d(v_j) \geq n + 8\sqrt{\delta}n \geq n + 4\sqrt{\delta}n + 1$, and the hypothesis of Theorem 18 holds. □

It remains to choose W_1'' and W_2'' so that they satisfy the conditions of Lemma 21, or to deal separately with the cases where this is impossible.

First, we consider the case in which all parts of G have size at most $\frac{5}{4}n$. (By property (2), this automatically holds when G has more than three parts: if so, all parts of G have size at most n.) Choose W_2'' arbitrarily. W_1' must contain at least $N - (1+\delta)n_1 \geq N - n_1 - \delta n_1 \geq 2n - 1 - 2\delta n$ vertices, of which only $2\sqrt{\delta}n$ vertices have been used by paths and $\sqrt{\delta}n$ more have been thrown away as C; therefore we have at least $2n - 1 - 3\sqrt{\delta}n - 2\delta n$ choices for vertices in W_1''.

We set aside vertices of W_1' which we forbid from being in W_1''. From each part, V_i, forbid either at least $|V_i| - (1 - 10\sqrt{\delta})n$ vertices, or else all vertices of $V_i \cap W_1'$, whichever is smaller. This forbids at most $\left(\frac{1}{4} + 10\sqrt{\delta}\right)n$ vertices from each part, and at most $10\sqrt{\delta}n$ vertices in the case $n_i \leq n$. There are at most two parts with $n_i > n$, so we forbid at most $\left(\frac{1}{2} + 50\sqrt{\delta}\right)n$ vertices. Now condition (1) of Lemma 21 will be satisfied no matter what: for each part i, we will either have $n_{i,1} + n_{i,2} \leq (1 - 10\sqrt{\delta})n$, or else $n_{i,1} = 0$.

Next, we attempt to ensure that condition (2) of Lemma 21 holds. Call a part V_i of G *W_1''-rich* if, after excluding the forbidden vertices, and vertices of $A \cup C$, there are still at least $20\sqrt{\delta}n$ vertices of W_1' left in V_i; call it *W_1''-poor* otherwise.

If there are at least three W_1''-rich parts, then we can choose $20\sqrt{\delta}n$ vertices from each of them for W_1'', and complete the choice of W_1'' arbitrarily. Condition (2) of Lemma 21 must now hold for $j = 1$: if we had $n_{*,1} \geq \left(\frac{1}{2} - 10\sqrt{\delta}\right)n$ and $n_{**,1} \geq \left(\frac{1}{2} - 10\sqrt{\delta}\right)n$, then together these two parts would contain all but $20\sqrt{\delta}n$ vertices of W_1''. This is impossible, since there is a third W_1''-rich part containing at least that many vertices of W_1''.

If there are not at least three W_1''-rich parts, we give up on Lemma 21, and satisfy the conditions of Theorem 18 by a different strategy.

If V_i is W_1''-poor, it must have many vertices of W_2''. More precisely, V_i has at least $\min\{n, n_i\} - 10\sqrt{\delta}n$ vertices that we have not forbidden. Among these, there are up to $3\sqrt{\delta}n$ vertices which are in $A \cup C$, up to $3\sqrt{\delta}n$ vertices which are in B, and fewer than $20\sqrt{\delta}n$ vertices that can be added to W_1'', so the remaining $\min\{n, n_i\} - 36\sqrt{\delta}n$ vertices must be in $W_2' \setminus B$.

Moreover, when G is tripartite, $n_i \geq \frac{3}{4}n - 1$ for any part, so if a part is W_1''-poor, it contains at least $\frac{3}{4}n - 36\sqrt{\delta}n - 1$ vertices of $W_2' \setminus B$. When G has more than three parts, at least two parts must be W_1''-poor; any two parts V_i, V_j have $n_i + n_j > n$, so together, two W_1''-poor parts have at least $n - 72\sqrt{\delta}n$ vertices of $W_2' \setminus B$. In either case, there are one or two W_1''-poor parts which together contain at least $\frac{2}{3}n$ vertices of $W_2' \setminus B$.

We change our choice of W_2'', if necessary, to include at least $\frac{2}{3}n$ vertices from this W_1''-poor part or parts; otherwise, the choice is still arbitrary. Meanwhile, we choose no vertices from these parts from W_1''; this rules out at most $40\sqrt{\delta}n$ vertices in addition to our previous restrictions. Completing the choice of W_1'' arbitrarily, we are left with a pair (W_1'', W_2'') that satisfies condition (1) of Lemma 21, but possibly not condition (2).

From condition (1), we know that if $v_j \in W_2''$ satisfies $d(v_j) \leq j + q$, we have $d(v_j) \geq n - n_{**,2} - \sqrt{\delta}n \geq \frac{1}{2}n - \sqrt{\delta}n$. Additionally, we know that for any $u_i \in W_1''$, $d(u_i) \geq \frac{2}{3}n - \sqrt{\delta}n$, since there are at least $\frac{2}{3}n$ vertices of W_2'' in a different part of G. Then $d(u_i) + d(v_j) \geq \frac{7}{6}n - 2\sqrt{\delta}n \geq n + q + 1$, satisfying the hypothesis of Theorem 18.

Next, we consider the case where G has at most three parts and $n_1 > \frac{5}{4}n$. By (9), $N > 3n - 1$. Hence by (8) we know that $n_1 = n_2$ and $N - n_1 = 2n - 1$. The case of $n_1 \geq \frac{5}{3}n$ was handled in Section 6.2. Thus, we may assume $n_1 < \frac{5}{3}n$, so $n_3 = (2n - 1) - n_2 > \frac{1}{3}n - 1$.

Assume first that one of $W_1' \setminus (A \cup C)$ or $W_2' \setminus B$ intersects each part of G in at least $20\sqrt{\delta}n$ vertices, and the other has at least $30\sqrt{\delta}n$ vertices outside each part of G; we will consider departures from this assumption later. This implies that for $j = 1$ or $j = 2$, we can choose $20\sqrt{\delta}n$ vertices from each part to add to W_j'', and match these by choosing $60\sqrt{\delta}n$ vertices to add to W_{3-j}'' with no more than $30\sqrt{\delta}n$ of these from one part. (No V_i has more than $50\sqrt{\delta}n$ vertices chosen from it at this point.)

Then proceed by an iterative strategy. At each step, choose one vertex from $W_1' \setminus (A \cup C)$ not previously added to W_1'', and a vertex from $W_2' \setminus B$ not previously added to W_2'', so that these vertices are in different parts of G. Then add them to W_1'' and W_2'' respectively. This step is always possible when $|W_1'' \cup A|, |W_2'' \cup B| < n$: in this case, at least two parts still have unchosen vertices, since $|V_1|, |V_2| \geq \frac{5}{4}n$ but fewer than n vertices have been chosen. Additionally, choosing a pair of vertices, one from W_1' and one from W_2', is only impossible if $W_2' \setminus B$ has no more vertices, in which case W_2'' has reached its desired size.

Stop when $|W_2'' \cup B| = n$. When this happens, W_1'' still needs $\sqrt{\delta}n$ more vertices, and these can be chosen arbitrarily.

This process guarantees that conditions (1) and (2) of Lemma 21 hold. Before we begin iterating, we have chosen $60\sqrt{\delta}n$ vertices, but at most $50\sqrt{\delta}n$ from each part. After we begin iterating, we add at most one vertex from each part at each step. Therefore in the end, $n_{i,1} + n_{i,2} \le n - 10\sqrt{\delta}n$ for each i, satisfying condition (1). Moreover, for some j, we added at least $20\sqrt{\delta}n$ vertices from each part to W_j'', ensuring that at most one value of $n_{i,j}$ can exceed $\left(\frac{1}{2} - 10\sqrt{\delta}\right)n$ and satisfying condition (2).

Now we consider alternatives to our initial assumptions in this case. We cannot have $W_1' \setminus (A \cup C)$ have fewer than $30\sqrt{\delta}n$ vertices outside V_i for any i, since it contains at least $2n - 1 - 4\sqrt{\delta}n - 2\delta n$ vertices, and no V_i is larger than $\frac{5}{3}n$. But it is possible that one of V_1 or V_2 contains all but $30\sqrt{\delta}n$ vertices of $W_2' \setminus B$; without loss of generality, it is V_1.

In this case, if $|V_1 \cap W_2' \setminus B| > n$, then let W_2'' be any n-element subset of $V_1 \cap W_2' \setminus B$; otherwise, let W_2'' be any n-element subset of $W_2' \setminus B$ containing $V_1 \cap W_2' \setminus B$. The set $V_2 \cup V_3$ has $2n - 1$ vertices, at most $30\sqrt{\delta}n + |B| = 33\sqrt{\delta}n$ of which are in W_2', so we can pick all n vertices of W_1'' from $V_2 \cup V_3$. Choose at least $10\sqrt{\delta}n$ of them from V_3 to satisfy condition (1) of Lemma 21 for $i = 2$. Condition (1) also holds for $i = 1$ (since $n_{i,1} = 0$) and $i = 3$ $\left(\text{since } n_3 < \frac{3}{4}n\right)$; condition (2) holds for $j = 2$.

Finally, we also violate the assumptions at the beginning of this case when neither $W_1' \setminus (A \cup C)$ nor $W_2' \setminus B$ have at least $20\sqrt{\delta}n$ vertices from each part of G. It is impossible that both of them have at most $20\sqrt{\delta}n$ vertices from V_3, so one of them has at most $20\sqrt{\delta}n$ vertices from one of V_1 or V_2.

If one of them (without loss of generality, V_1) contains at most $20\sqrt{\delta}n$ vertices of $W_1' \setminus (A \cup C)$, it must have at least n vertices of $W_2' \setminus B$, since $|V_1| \ge \frac{5}{4}n$, so choose all remaining vertices out of W_2'' from there. Outside V_1, we have at least $(2n - 1 - 4\sqrt{\delta}n - 2\delta n) - 20\sqrt{\delta}n$ vertices of $W_1' \setminus (A \cup C)$, which leaves at most $24\sqrt{\delta}n + 2\delta n$ vertices we *cannot* choose for W_1''. Choose n vertices outside V_1 for W_1'', including at least $10\sqrt{\delta}n$ vertices of V_3. This satisfies condition (1) for $i = 1$ (since $n_{i,1} = 0$), $i = 2$ (since $n_{i,1} = 0$ and $n_{i,2} < n - 10\sqrt{\delta}n$), and $i = 3$ $\left(\text{since } n_3 < \frac{3}{4}n\right)$; condition (2) holds for $j = 2$.

If one of V_1 or V_2 (without loss of generality, V_1) contains at most $20\sqrt{\delta}n$ vertices of $W_2' \setminus B$, choose $n - 30\sqrt{\delta}n$ vertices of W_1'' from V_1 (satisfying condition (1) for $i = 1$ and condition (2) by taking $j = 1$). If V_3 contains at least $30\sqrt{\delta}n$ vertices of $W_1' \setminus (A \cup C)$, take the remaining vertices of W_1'' from W_3. Otherwise, V_3 contains at least $60\sqrt{\delta}n$ vertices of $W_2' \setminus B$; choosing as many vertices as possible from $V_1 \cup V_3$ to add to W_2'', and the remaining vertices of W_1'' arbitrarily, we end up choosing no more than $n - 10\sqrt{\delta}n$ vertices from V_2. So condition (1) holds for $i = 2$ either because $n_{i,1} = 0$ or because $n_{i,1} + n_{i,2} \le n - 10\sqrt{\delta}n$; condition (1) holds for $i = 3$ because $n_3 < \frac{3}{4}n$.

7. Dealing with $(\lambda, i, 2)$-bad partitions when $N - n_1 - n_2 \ge 3$

A *cherry* is a path on three vertices. The *center* of a cherry is the vertex with degree 2.

Suppose $N - n_1 - n_2 \ge 3$. By (8)–(10), we have two cases:

(1) $N > 3n - 1$, $s = 3$, $n_2 + n_3 = 2n - 1$ and $n_1 = n_2$ (i.e., (8) holds), or

(2) $N = 3n - 1$, $n_1 \le n$, $s \le 5$, and if $s \ge 4$, then $n_{s-1} + n_s \ge n + 1$ (i.e., (9) holds).

7.1. ***The case when (8) holds.*** By (8), $n_1 = n_2 > n,\ s = 3$, and $0 < n_3 = 2n - 1 - n_2 < n$.

Lemma 22. *Let $G = K_{n_1,n_2,n_3}$ with $n_1 = n_2$ and $n_2 + n_3 = 2n - 1$ be 2-edge-colored with a $(\lambda, i, 2)$-bad partition. Then G has a monochromatic cycle of length $2n$.*

In this section, we prove Lemma 22, but postpone technical details of how the monochromatic cycles are constructed in each of four cases; these details are given in Claims 23–26.

Proof of Lemma 22. Without loss of generality, let $i = 2$; we call color 1 red, color 2 blue, and use d_1 (d_2) to denote the red (blue) degree.

We begin by assuming that in the $(\lambda, 2, 2)$-bad partition (V_j, U_1, U_2), $j = 3$. Later, in Section 7.1.5, we discuss the modifications to the proof when $j \neq 3$.

Since (V_j, U_1, U_2) is a 2-bad partition, we know the following conditions hold:

(i) $|V_3| \geq (1 - \lambda)n$.

(ii) $(1 - \lambda)n \leq |U_1| \leq (1 + \lambda)n$.

(iii) $(1 - \lambda)n \leq |U_2| \leq (1 + \lambda)n$.

(iv) $E(G_2[V_3, U_1]) \leq \lambda n^2$.

(v) $E(G_1[V_3, U_2]) \leq \lambda n^2$.

If a vertex u_1 in U_1 has blue degree at least $\frac{1}{2}n_3$ to V_3 then we move u_1 to U_2. If a vertex u_2 in U_2 has red degree at least $\frac{1}{2}n_3$ to V_3 then we move u_2 to U_1. Since there are at most $3\lambda n$ vertices in U_1 with blue degree at least $\frac{1}{2}n_3$ to V_3 and there are at most $3\lambda n$ vertices in U_2 with red degree at least $\frac{1}{2}n_3$ to V_3, we moved at most $3\lambda n$ vertices out of U_1 and U_2 respectively and moved at most $3\lambda n$ vertices into U_1 and U_2 respectively. Thus, we may assume $|U_1| \geq |U_2|,\ |U_1| = n + a_1,\ |U_2| = n + a_2$, and $a_1 \geq 0$.

Note that (iv) and (v) change to:

(iv) $|E(G_2[V_3, U_1])| \leq 4\lambda n^2$.

(v) $|E(G_1[V_3, U_2])| \leq 4\lambda n^2$.

Let $|V_3| = n - a_3$, where $a_3 \leq 10\lambda n$. Let B be the set of vertices in V_3 with blue degree at least $0.9n$ to U_1 and $|B| = b$. Let R be the set of vertices in V_3 with blue degree at most $0.05n$ to U_1. By condition (iv), we know

$$|B| \leq 5\lambda n \quad \text{and} \quad |R| \geq n - a_3 - 80\lambda n.$$

Let C be a maximum collection of vertex-disjoint red cherries with center in U_2 and leaves in U_1. If there are at least $m := a_3 + b$ cherries in C, then we use them, together with the edges between U_1 and V_3, to find a red cycle of length $2n$; this is done in Claim 23.

Otherwise, we assume that $|C| \leq m - 1$: there are at most $m - 1$ red cherries from U_2 to U_1. Every vertex in $U_2 - V(C)$ has red degree at most $2m - 1$ to U_1, since otherwise we have a larger collection of red cherries.

When $|U_2| = n + a_2 \geq n - b$, we can find a blue cycle using edges between U_2 and V_3, as well as enough edges between U_1 and B to make up for the size of U_2 when $|U_2| < n$. This is done in Claim 24.

Otherwise, we assume that $|U_2| \leq n - b - 1$; in other words,

$$a_2 \leq -(b + 1). \tag{15}$$

Our goal is now to use edges within U_1 to find a monochromatic cycle. Without loss of generality, we may assume that $|U_1 \cap V_1| \geq |U_1 \cap V_2|$. We first argue that $U_1 \cap V_2$ cannot be too small.

Earlier, we defined $|U_1| = n + a_1$, $|U_2| = n + a_2$, $|V_3| = n - a_3$. Since $|V_1| + |V_3| = |V_2| + |V_3| = 2n - 1$ and $U_1 \cup U_2 = V_1 \cup V_2$, we have

$$2n + a_1 + a_2 = |V_1| + |V_2| = 4n - 2 - 2|V_3| = 2n + 2a_3 - 2$$

or

$$a_1 + a_2 = 2a_3 - 2. \tag{16}$$

Therefore

$$\begin{aligned} |U_1 \cap V_2| &\geq |U_1| - |V_1| = |U_1| - \tfrac{1}{2}(|U_1| + |U_2|) = n + a_1 - n - \tfrac{1}{2}(a_1 + a_2) \\ &= \tfrac{1}{2}(a_1 - a_2) = a_3 - a_2 - 1 = (b + a_3) + (-b - a_2) - 1. \end{aligned}$$

There are two possibilities for the vertices of $U_1 \cap V_2$:

- There are at least $m = b + a_3$ vertices in $U_1 \cap V_2$ which have red degree at least $0.1n$ to $U_1 \cap V_1$. In this case, we use Claim 25 to find a red cycle of length exactly $2n$.
- There are at least $m' := -b - a_2$ vertices in $U_1 \cap V_2$ which have blue degree at least $|U_1 \cap V_1| - 0.1n \geq 0.4n$ to $U_1 \cap V_1$. In this case, we use Claim 26 to find a blue cycle of length exactly $2n$.

One of these must hold, since $|U_1 \cap V_2| \geq m + m' - 1$, while by (15), $m' = -b - a_2 \geq 1$: therefore there are either m vertices for Claim 25 or m' vertices for Claim 26. In either case, we obtain a monochromatic cycle of length exactly $2n$, completing the proof. □

7.1.1. *The case of many cherries*: $|C| \geq m$. Recall that C is a maximum collection of vertex-disjoint red cherries with centers in U_2 and leaves in U_1; $m = b + a_3$, where $b = |B|$ and $a_3 = n - |V_3|$.

Claim 23. *If $|C| \geq m$, then we have a red cycle of length exactly $2n$.*

Proof. We do the following steps. Let $C' \subseteq C$ be a collection of m red cherries with centers in U_2 and leaves in U_1. Let $\{u_1, \dots, u_m\} = V(C') \cap U_2$ and $\{v_1, \dots, v_{2m}\} = V(C') \cap U_1$ such that each $v_{2i-1}u_i v_{2i}$ is a cherry with center u_i, where $1 \leq i \leq m$.

To find a cycle of length $2n$ in G_1 that contains the edges of C', we will apply Theorem 18 to an appropriately chosen bipartite graph.

First, create an auxiliary graph G_1' by starting with G_1 and adding every edge between $\{u_1, \dots, u_m\}$ and U_1. This will help us to satisfy the degree conditions of Theorem 18; however, these artificial edges will never be used by a cycle containing all the edges of C', since each of $\{u_1, \dots, u_m\}$ already has degree 2 in C'.

Second, let $X = (V_3 - B) \cup \{u_1, u_2, \dots, u_m\}$ (a set of n vertices total) and let $Y \subseteq U_1$ be any set of size n such that $\{v_1, \dots, v_{2m}\} \subseteq Y$. We check that the hypotheses of Theorem 18 apply to $G_1'[X, Y]$.

Order vertices in X and Y separately by their degree from smallest to largest. Since vertices in Y have red degree at least $\frac{1}{2}n_3 - b \geq 0.4n$ to X and at most $100\lambda n \ll 0.001n$ vertices in Y have blue degree at least $0.04n$ to X, the smallest index k such that $d_1(y_k) \leq k + q$ satisfies $d_1(y_k) \geq 0.95n$. Since vertices in X have blue degree at most $0.9n$ to U_1, they have red degree at least $n - 0.9n = 0.1n \gg 0.09n$ to Y. The smallest index j such that $d_1(x_j) \leq j + q$ satisfies $d_1(x_j) \geq 0.09n$. By Theorem 18 and $0.09n + 0.95n \gg n + q + 1$, we can find a Hamiltonian cycle in $G_1'[X, Y]$ of length $2n$ containing the edges of C', which is a cycle of length $2n$ in G_1. □

7.1.2. *The case of large U_2: $|U_2| \geq n-b$.* Recall that $|U_2| = n + a_2$, B is the set of vertices in V_3 with blue degree at least $0.9n$ to U_1, and $b = |B|$.

Claim 24. *If $b \geq -a_2$ (in other words, if $|U_2| = n + a_2 \geq n - b$), then we have a blue cycle of size exactly $2n$.*

Proof. Let $c := |C|$; let $V(C) \cap U_2 = \{u_1, \dots, u_c\}$ and $V(C) \cap U_1 = \{v_1, v_2, \dots, v_{2c}\}$. Let B_2 be the collection of vertices in $V_3 - B$ with red degree at most $0.1n$ to U_2. By condition (v),

$$q := |B_2| \geq n - a_3 - 40\lambda n - b.$$

Since $2n_1 = |U_1| + |U_2| = 2n + a_1 + a_2$, we know

$$|U_2 \cap V_2| = n_1 - |U_1 \cap V_2| \geq n_1 - \tfrac{1}{2}(n + a_1) = n + \tfrac{1}{2}(a_1 + a_2) - \tfrac{1}{2}n - \tfrac{1}{2}a_1 = \tfrac{1}{2}(n + a_2)$$

and thus

$$|U_2 \cap V_1| \leq n + a_2 - \tfrac{1}{2}(n + a_2) = \tfrac{1}{2}(n + a_2). \tag{17}$$

Step 1: We first find a path to include $0.8n$ vertices in V_3 and $0.8n$ vertices in U_2 (all of $U_2 \cap V_1$ and $V(C)$) by Theorem 17.

Details: Since $|B_2| \geq n - a_3 - 40\lambda n - b$, we take a set $X \subseteq B_2$ such that $|X| = 0.8n$. By (17), we can take a set $Y \subseteq U_2$ such that $U_2 \cap V_1 \subseteq Y$, $V(C) \cap U_2 \subset Y$, and $Y = 0.8n$.

Now we consider $G_2[X, Y]$ and we order vertices in X and Y separately by their degree from smallest to largest. Since vertices in Y have blue degree at least $0.8n - \frac{1}{2}n_3 > 0.2n$ to X, the smallest index k such that $d_2(y_k) \leq k + 1$ satisfies $d_2(y_k) \geq 0.2n$. Since vertices in X have red degree at most $0.1n$ to U_2, they have blue degree at least $0.8n - 0.1n = 0.7n$ to Y. The smallest index j such that $d_2(x_j) \leq j + 1$ satisfies $d_2(x_j) \geq 0.7n$. By Theorem 17 and $0.7n + 0.2n > 0.8n + 2$, we can find a Hamiltonian red path P_1' from $x \in X$ to some vertex $y \in Y - V_1 - V(C)$ in $G_2[X, Y]$ of length $1.6n - 1$.

Since $x \in X \subseteq B_2$,

$$d_2(x, U_2 - Y) \geq n + a_2 - 0.8n - 0.1n > 0.05n.$$

We extend the path P_1' to P_1 of length $1.6n$ by adding a blue edge xy' such that $y' \in U_2 - Y$.

Step 2: Use $\min\{0, -a_2\}$ vertices in B to obtain a blue path. (We can skip this step if $a_2 \geq 0$.)

Details: Assume $a_2 < 0$; since $b \geq -a_2$, let $Z := \{z_1, \dots, z_{|a_2|}\} \subseteq B$.

Since

$$|U_1 \cap V_1| \geq \tfrac{1}{2}(n + a_1) \geq |U_1 \cap V_2|,$$

each vertex in B has blue degree at least $0.9n - |U_1 \cap V_2|$ to $U_1 \cap V_1$. Therefore,

$$0.9n - |U_1 \cap V_2| \geq 0.9n - (n + a_1 - |U_1 \cap V_1|) = |U_1 \cap V_1| - a_1 - 0.1n \geq \tfrac{3}{4}|U_1 \cap V_1|.$$

We can find for each pair (z_i, z_{i+1}) a common neighbor $r_i \in U_1 \cap V_1 - V(C)$, where $1 \leq i \leq |a_2| - 1$, a blue neighbor r_0 of z_1, and a blue neighbor $r_{|a_2|}$ of $z_{|a_2|}$ such that $r_0, \dots, r_{|a_2|}$ are all distinct.

We obtain a blue path

$$P_2 = r_0 z_1 r_1 \cdots z_i r_i \cdots z_{|a_2|} r_{|a_2|}$$

of length $2|a_2|$.

Since y' has at most one red neighbor to $U_1 - V(C)$, at least one of $\{r_0, r_{|a_2|}\}$ is a blue neighbor of y'. We may assume $r_{|a_2|}y'$ is blue.

Step 3: Include the rest of vertices in U_2 to U_1.

Details: We proceed differently depending on whether $a_2 < 0$ or $a_2 \geq 0$.

- If $a_2 < 0$ then we do the following. Let $K := (U_2 - Y - \{y'\}) \cup \{y\} = \{y, f_1, \dots, f_{k-1}\}$. Note that $k = |K| = n + a_2 - 0.8n = 0.2n + a_2$ and $K \subseteq U_2 \cap V_2 - V(C)$. Since each vertex in K has at most one red neighbor to $U_1 - V_2 - V(C) - \{r_0, r_1, \dots, r_{|a_2|}\}$, we find for (y, f_1) a blue common neighbor $h_0 \in U_1 - V_2 - V(C) - \{r_0, r_1, \dots, r_{|a_2|}\}$ and each pair (f_i, f_{i+1}) a distinct blue common neighbor, h_i, in $U_1 - V_2 - V(C) - \{r_0, r_1, \dots, r_{|a_2|}\}$, where $1 \leq i \leq k-2$. We obtain a blue path

$$P_3 = yh_0 f_1 \cdots f_i h_i f_{i+1} \cdots f_{k-1}$$

of length $2k - 2 = 0.4n + 2a_2 - 2$.

We may assume $f_{k-1}r_0$ is blue since f_{k-1} has only one red neighbor to $U_1 \cap V_1 - V(C)$ and there are many choices when we choose r_0 to connect with z_1.

Finally, we connect P_2 and P_1 by adding the edge $r_{|a_2|}y'$, glue the paths P_1 and P_3 at y, then add the edge $f_{k-1}r_0$ to complete a blue cycle of length exactly

$$2|a_2| + 1 + 1.6n + 0.4n + 2a_2 - 2 + 1 = 2n.$$

- If $a_2 \geq 0$ then in the previous argument we take $K = \{y, y', f_1, \dots, f_{k-2}\}$ of size $0.2n + 1$ and find common neighbors h_0 for (y, f_1), h_i for (f_i, f_{i+1}), where $1 \leq i \leq k-3$, and h_{k-2} for (f_{k-2}, y').

In either case, we obtain a path

$$P_3 = yh_0 f_1 \cdots f_i h_i f_{i+1} \cdots f_{k-2} h_{k-2} y'$$

of length $2k - 2 = 0.4n$. We glue P_1 and P_3 at y and y' to obtain a blue cycle of length exactly $1.6n + 0.4n = 2n$. □

7.1.3. *Handling many vertices in $U_1 \cap V_2$ incident to red edges.* We will find a red cycle. Note that the size of $U_1 \cap V_2$ is at least $n + a_1 - n_1$.

Claim 25. *If there are at least $m = b + a_3$ vertices in $U_1 \cap V_2$ of red degree at least $0.1n$ to $U_1 \cap V_1$, then we have a red cycle of length exactly $2n$.*

Proof. Let B' be the collection of vertices in U_1 with blue degree at least $0.05n$ to V_3. By (iv), we have

$$|B'| \leq 80\lambda n.$$

Step 1: We first find a collection of red cherries C_3 with center in $U_1 \cap V_2$ and leaves in $U_1 \cap V_1 - B'$ of size $b + a_3 =: m$.

Details: Since there are at least m vertices in $U_1 \cap V_2$ of red degree at least $0.1n$ to $U_1 \cap V_1$ and $0.1n - 80\lambda n \gg 2m$, we can find a collection of red cherries C_3 with centers in $U_1 \cap V_2$ and leaves in $U_1 \cap V_1 - B'$ of size m. Let $V(C_3) \cap V_2 = \{u_1, \dots, u_m\}$ and $V(C_3) \cap V_1 = \{v_1, \dots, v_{2m}\}$.

Recall that $R \subseteq V_3$ is the collection of vertices in V_3 with blue degree at most $0.05n$ to U_1.

Step 2: Then by Hall's theorem we find a matching M for $V(C_3)\cap V_1$ to R and then find a common neighbor back to connect those vertices.

Details: Since $\{v_2,\dots,v_{2m}\}\cap B'=\varnothing$, each of them has red degree at least $n-a_3-0.05n-80\lambda n>0.9n$ to R. Thus, we can find a matching M for $\{v_2,\dots,v_{2m}\}$ such that $V(M)\cap V_3=\{w_2,\dots,w_{2m}\}$ and each v_iw_i is a matching edge, where $2\le i\le 2m$.

Since $V(M)\cap V_3\subseteq R$, we can find for each pair (w_{2i},w_{2i+1}) a common red neighbor $g_i\in U_1$, where $1\le i\le m-1$.

Therefore, we obtained a path

$$P_1=v_1u_1v_2w_2g_1w_3v_3u_2v_4w_4\cdots v_{2m-1}u_mv_{2m}w_{2m}$$

of length $6m-3$.

Step 3: We use Theorem 17 to get a path saturating all vertices left in $V_3-B-V(M)$.

Details: Let $X=V_3-B-\{w_2,\dots,w_{2m-1}\}$ and we know

$$|X|=n-a_3-b-(2m-2)=n-3m+2.$$

Choose $Y\subseteq U_1-\{u_1,\dots,u_m\}-\{v_2,\dots,v_{2m}\}-\{g_1,\dots,g_{m-1}\}$ such that $v_1\in Y$. By (16),

$$a_1=-a_2+2a_3-2\ge b+1+a_3+a_3-2=m+a_3-1\ge m \tag{18}$$

and thus

$$n+a_1-m-(2m-1)-(m-1)\ge n-3m+2.$$

Hence we can require $|Y|=n-3m+2$.

Now we consider $G_1[X,Y]$ and we order vertices in X and Y separately by their degree from smallest to largest. Since vertices in U_1 have red degree at least $\frac{1}{2}n_3$ to V_3, they have red degree at least $\frac{1}{2}n_3-b-(2m-2)>0.4n$ to X.

By condition (iv), there are at most $80\lambda n$ vertices in U_1 with blue degree at least $0.05n$ to V_3. Thus, at least $|Y|-80\lambda n$ vertices in Y have red degree at least $|X|-0.05n>0.94n$ to X, the smallest index k such that $d_1(y_k,X)\le k+1$ satisfies $d_1(y_k,X)\ge 0.94n-1$. Since vertices in X have blue degree at most $0.9n$ to U_1, they have red degree at least $n+a_1-m-(2m-1)-(m-1)-0.9n>0.09n$ to Y. The smallest index j such that $d_1(x_j,Y)\le j+1$ satisfies $d_1(x_j,Y)\ge 0.09n$. By Theorem 17 and $0.09n+0.94n\gg n+2$, we can find a Hamiltonian red path P_2 from v_1 to w_{2m} in $G_1[X,Y]$ of length

$$2(n-3m+2)-1=2n-6m+3.$$

We glue P_1 and P_2 at v_1 and w_{2m} to obtain a red cycle of size exactly

$$6m-3+2n-6m+3=2n. \qquad \square$$

7.1.4. *Handling many vertices in $U_1\cap V_2$ incident to blue edges.* In this case, there are many disjoint blue cherries inside U_1, and we will find a blue cycle. Recall that C is a collection of at most $m-1$ cherries with centers in U_2 and leaves in U_1, which is defined three paragraphs ahead of (15).

Claim 26. *If there are at least $-a_2-b$ vertices in $U_1\cap V_2$ of blue degree at least $|U_1\cap V_1|-0.1n\ge 0.4n$ to $U_1\cap V_1$, then we find a blue cycle of length exactly $2n$.*

Proof. Step 1: We find $m' = -a_2 - b$ blue cherries with centers in $U_1 \cap V_2$ and leaves in $U_1 \cap V_1$. Possibly avoiding bad vertices. Then find common neighbors in $U_2 \cap V_2$ to connect those cherries.

Details: Since vertices in $U_2 \cap V_2 - V(C)$ have red degree at most 1 to $U_1 \cap V_1 - V(C)$, there are at most $|U_2 \cap V_2| \leq \lambda n^2$ red edges between $U_2 \cap V_2 - V(C)$ and $U_1 \cap V_1 - V(C)$. Therefore, there are at most $20\lambda n$ vertices in $U_1 \cap V_1 - V(C)$ with red degree at least $0.05n$ to $U_2 \cap V_2 - V(C)$ and at least $|U_1 \cap V_1| - |V(C) \cap U_1| - 20\lambda n$ vertices in $U_1 \cap V_1 - V(C)$ with blue degree at least $|U_2 \cap V_2| - |V(C)| - 0.05n > \frac{3}{4}|U_2 \cap V_2|$ to $U_2 \cap V_2 - V(C)$; we call those vertices B_3.

Since there are m' vertices in $U_1 \cap V_2$ of blue degree at least $|U_1 \cap V_1| - 0.1n - |V(C)| - 20\lambda n > 0.3n$ to B_3, we find m' blue cherries, C_4, with center in $U_1 \cap V_2$ and leaves in B_3. Let $V(C_4) \cap V_2 = \{u_1, \ldots, u_{m'}\}$ and $V(C_4) \cap V_1 = \{v_1, \ldots, v_{2m'}\}$.

We can find for each pair (v_{2i}, v_{2i+1}) a common blue neighbor, w_i, in $U_2 \cap V_2 - V(C)$, where $1 \leq i \leq m' - 1$. We also find for v_1 a blue neighbor w_0 and $v_{2m'}$ a blue neighbor $w_{m'}$ distinct from $\{w_1, \ldots, w_{m'-1}\}$ and $V(C)$.

We obtain a blue path

$$P_1 = w_0 v_1 u_1 v_2 w_1 \cdots v_{2m'-1} u_{m'} v_{2m'} w_{m'}$$

of length $4m'$.

Step 2: We find for vertices in B common neighbors in $U_1 \cap V_1$, avoiding vertices already used.

Details: Since

$$|U_1 \cap V_1| \geq \tfrac{1}{2}(n + a_1) \geq |U_1 \cap V_2|, \tag{19}$$

each vertex in B has blue degree at least $0.9n - 2m' - |U_1 \cap V_2| - |V(C) \cap U_1|$ to $U_1 \cap V_1 - V(C)$. Therefore,

$$\begin{aligned} 0.9n - 2m' - |U_1 \cap V_2| - |V(C) \cap U_1| &\geq 0.9n - 2m' - (n + a_1 - |U_1 \cap V_1|) - 2(m-1) \\ &= |U_1 \cap V_1| - a_1 - 2m' - 0.1n - 2m + 2 \geq \tfrac{3}{4}|U_1 \cap V_1|. \end{aligned}$$

Let $B = \{z_1, \ldots, z_b\}$. We can find for each pair (z_i, z_{i+1}) a common neighbor r_i, where $1 \leq i \leq b - 1$, a blue neighbor r_0 of z_1, and a blue neighbor r_b of z_b such that $r_0, \ldots, r_b$ are all distinct and in $U_1 \cap V_1 - V(C)$.

We obtain a blue path

$$P_2 = r_0 z_1 r_1 \cdots z_i r_i \cdots z_b r_b$$

of length $2b$.

Step 3: Take $0.9n$ vertices in V_3 and $0.9n$ vertices in U_2 including $U_2 \cap V_1$ and $V(C)$. Use Theorem 17 to find a path.

Details: Recall that B_2 is the collection of vertices in V_3 with red degree at most $0.1n$ to U_2 and $|B_2| \geq n - a_3 - 40\lambda n - b$. Since $|B_2| \geq n - a_3 - 40\lambda n - b$, we take a set $X \subseteq B_2$ such that $|X| = 0.9n$. By (19), $|U_2 \cap V_1| \leq 0.6n$ and we can take a set $Y \subseteq U_2 - \{w_0, w_1, \ldots, w_{m'-1}\}$ such that $U_2 \cap V_1 \subseteq Y$, $V(C) \subseteq Y$, $w_{m'} \in Y$, and $Y = 0.9n$.

First we find a blue edge $v'u'$ with $v' \in X$ and $u' \in U_2 - Y$. Now we consider $G_2[X, Y]$ and we order vertices in X and Y separately by their degree from smallest to largest. Since vertices in Y have blue degree at least $0.9n - \frac{1}{2}n_3 > 0.3n$ to X, the smallest index k such that $d_2(y_k, X) \leq k + 1$ satisfies

$d_2(y_k, X) \geq 0.3n$. Since vertices in X have red degree at most $0.1n$ to U_2, they have blue degree at least $0.9n - 0.1n = 0.8n$ to Y. The smallest index j such that $d_2(x_j, Y) \leq j + 1$ satisfies $d_2(x_j, Y) \geq 0.8n$. By Theorem 17 and $0.8n + 0.3n > 0.9n + 2$, we can find a Hamiltonian blue path P_3' from $w_{m'}$ to v' in $G_2[X, Y]$ of length $1.8n - 1$. We then extend the path P_3' to P_3 by adding the edge $v'u'$. Thus, the path P_3 has length $1.8n$.

Step 4: Finally, the rest of the vertices in $U_2 \cap V_2$ have large blue degree to $U_1 \cap V_1$, and we find common neighbors to include them.

Details: Let $K := (U_2 - Y - \{w_0, w_1, \dots, w_{m'-1}\}) = \{u', f_1, \dots, f_{k-1}\}$. Note that $k = |K| = n + a_2 - 0.9n - m' = 0.1n + a_2 - m'$ and $K \subseteq U_2 \cap V_2 - V(C)$. Since each vertex in K has at most one red neighbor to $U_1 \cap V_1 - V(C) - \{v_1, \dots, v_{2m'}\} - \{r_0, \dots, r_b\}$, we find for (u', f_1) a distinct blue common neighbor h_0, and for each pair (f_i, f_{i+1}) a distinct blue common neighbor, h_i, in $U_1 \cap V_1 - V(C) - \{v_1, \dots, v_{2m'}\} - \{r_0, \dots, r_b\}$, where $1 \leq i \leq k-2$. We may assume that $r_0 f_{k-1}$ is blue (since f_{k-1} has at most one red neighbor to $U_1 \cap V_1$ and z_1 has very large blue degree to $U_1 \cap V_1$, if $r_0 f_{k-1}$ is not blue then we choose r_0 such that $r_0 f_{k-1}$ is blue).

We obtain a blue path

$$P_4 = u' h_0 f_1 \cdots f_i h_i f_{i+1} \cdots h_{k-2} f_{k-1}$$

of size $2k - 2 = 0.2n + 2a_2 - 2m' - 2$.

Finally, we add the edge $r_b w_0$ to connect P_2 and P_1, glue P_1 and P_3 at $w_{m'}$, glue P_3 and P_4 at u', and add the edge $r_0 f_{k-1}$ to complete the cycle of length

$$1 + 4m' + 2b + 1.8n + 0.2n + 2a_2 - 2m' + 1 = 2n. \qquad \square$$

7.1.5. *Changes of the proof when* $j \neq 3$. When $j \neq 3$, essentially the same proof works, with minor modifications.

Without loss of generality, we assume $j = 1$. We use the same setup as in the case when $j = 3$ but replace every place of V_3 by V_1 and n_3 by n_1.

Case 1: $n_1 \geq n + b$.

Since $n_1 \geq n + b$ and $|U_1| \geq n$, we take a set of vertices $X \subseteq V_1 - B$ of size n and a set of vertices $Y \subseteq U_1$ of size n.

Now we consider $G_1[X, Y]$ and we order vertices in X and Y separately by their degree from smallest to largest. Since vertices in Y have red degree at least $0.5n_1$ to X and there are at most $80\lambda n$ vertices with blue degree at least $0.05n$ to V_1, the smallest index k such that $d_1(y_k, X) \leq k + 1$ satisfies $d_1(y_k, X) \geq 0.95n$. Since vertices in X have blue degree at most $0.9n$ to U_1, they have red degree at least $0.1n$ to Y. The smallest index j such that $d_1(x_j, Y) \leq j + 1$ satisfies $d_1(x_j, Y) \geq 0.1n$. By Theorem 18 and $0.1n + 0.95n \gg n + 1$, there is a Hamiltonian cycle in $G_1[X, Y]$ of length $2n$.

Case 2: $n + 1 \leq n_1 \leq n + b - 1$.

We still assume $n_1 = n - a_3$ with $a_3 < 0$. It is included in Case 1 by replacing n_3 with n_1, V_3 with V_1, V_1 with V_2, and V_2 with V_3. Note that in this case we have

$$n + a_1 + n + a_2 = 2n - 1$$

and thus

$$a_1 + a_2 = -1. \tag{20}$$

Equation (17) changes to

$$|U_2 \cap V_3| = n_3 - |U_1 \cap V_3| \geq 2n - 1 - n + a_3 - \tfrac{1}{2}(n + a_1) = \tfrac{1}{2}n - 1 + a_3 - \tfrac{1}{2}a_1$$

and thus

$$|U_2 \cap V_2| \leq n + a_2 - \left(\tfrac{1}{2}n - 1 + a_3 - \tfrac{1}{2}a_1\right) = \tfrac{1}{2}n + 1 + a_2 - a_3 + \tfrac{1}{2}a_1 = \tfrac{1}{2}n - a_3 - \tfrac{1}{2}a_1.$$

Moreover, by $a_3 < 0$, the inequality $a_1 \geq m$ in (18) still holds under the assumption $a_2 \leq -b - 1$ since

$$a_1 = -1 - a_2 \geq b \geq b + a_3 = m.$$

When choosing between Claims 25 and 26, we still have by (20)

$$|U_1| - |V_2| \geq n + a_1 - n + a_3 = a_1 + a_3 = -1 - a_2 + a_3 = (b + a_3) + (-b - a_2) - 1$$

and therefore one of the two claims can still be applied.

7.2. *The case when (9) holds.*

7.2.1. *Statement and setup of the main lemma.* In this case, we have

$$n_1 + n_2 + \cdots + n_s = 3n - 1 \tag{21}$$

and

$$n_2 + \cdots + n_s \geq 2n - 1. \tag{22}$$

By (11), $s \leq 5$. Our main lemma in this subsection is:

Lemma 27. *Let $G = K_{n_1, n_2, \ldots, n_s}$ satisfying (21) and (22) be 2-edge-colored with a $(\lambda, i, 2)$-bad partition. Then G has a monochromatic cycle of length $2n$.*

Proof. Without loss of generality, let $i = 2$. By the definition of a $(\lambda, i, 2)$-bad partition, there is a $j \in [s]$ such that:

(i) $n \geq |V_j| \geq (1 - \lambda)n$.

(ii) $(1 - \lambda)n \leq |U_1| \leq (1 + \lambda)n$.

(iii) $(1 - \lambda)n \leq |U_2| \leq (1 + \lambda)n$.

(iv) $E(G_2[V_j, U_1]) \leq \lambda n^2$.

(v) $E(G_1[V_j, U_2]) \leq \lambda n^2$.

Our plan is as follows. In this and the next three subsections we handle the case $s = 4$ and renumber the parts so that $j = 1$ and $n_2 \geq n_3 \geq n_4$. Later, in Section 7.2.5, we return to the original numbering of the parts ($n_1 \geq \cdots \geq n_s$) and describe modifications to the proof for $s \neq 4$.

Since (9) holds, we have $n_i \leq n$ for all i; we also know that $n_2 \geq n_3 \geq n_4$, $n_1 = |V_j| \geq (1 - \lambda)n$, and

$$|U_1| + |U_2| = n_2 + n_3 + n_4 = 3n - 1 - n_1 \leq 2n + \lambda n - 1,$$

so $n_2 \geq \frac{1}{3}(n_2 + n_3 + n_4) \geq \frac{2}{3}n$.

We move vertices as we did in the previous section so that for each $u \in U_1$ we have $d_1(u, V_1) \geq \frac{1}{2}n_1$ and for each $v \in U_2$ we have $d_2(v, V_1) \geq \frac{1}{2}n_1$. Note that (iv) and (v) change to (iv) $|E(G_2[V_1, U_1])| \leq 4\lambda n^2$ and (v) $|E(G_1[V_1, U_2])| \leq 4\lambda n^2$.

Let $|U_1| = n + a_1$, $|U_2| = n + a_2$, and $|V_1| = n - a_3$. Let B be the set of vertices in V_1 with blue degree at least $0.9n$ to U_1, and let $b := |B|$. By condition (iv), we know $b \leq 5\lambda n$.

Let C be a maximum collection of vertex-disjoint red cherries with center in U_2 and leaves in U_1. If there are at least $m := a_3 + b$ cherries in C, then we use them, together with the edges between U_1 and V_1, to find a red cycle of length $2n$. This is done in exactly the same way as in Claim 23, except with V_1 playing the role of V_3.

Otherwise, we assume that $c := |C| \leq m - 1$, which means every vertex in $U_2 - V(C)$ has red degree at most $2m - 1$ to U_1.

When $|U_2| = n + a_2 \geq n - b$, we can find a blue cycle in almost the same way as in Claim 24; the updated proof is given in Claim 28.

Otherwise, we may assume that $|U_2| \leq n - b - 1$, in which case (15) holds.

As before, to proceed, we want to use edges within U_1. Let k be such that $|U_1 \cap V_k|$ is maximized. This intersection is still at most $|V_k| \leq n$, while $|U_1| = n + a_1$, so $|U_1 - V_k| \geq a_1$.

Since

$$(n + a_1) + (n + a_2) = |U_1| + |U_2| = 3n - 1 - |V_1| = 2n + a_3 - 1,$$

we have $a_1 + a_2 = a_3 - 1$, and therefore

$$|U_1 - V_k| \geq a_3 - a_2 - 1 = (b + a_3) + (-a_2 - b) - 1.$$

There are two possibilities:

- There are at least $m = b + a_3$ vertices in $U_1 - V_k$ of red degree at least $0.1n$ to $U_1 \cap V_k$. In this case, we will find a red cycle of length exactly $2n$ by Claim 29.
- There are at least $m' = -a_2 - b$ vertices in $U_1 - V_k$ of blue degree at least $|U_1 \cap V_k| - 0.1n \geq 0.2n$ to $U_1 \cap V_k$. In this case, we find a blue cycle of length exactly $2n$ by Claim 30.

One of these must hold, since $|U_1 - V_k| \geq m + m' - 1$, while by (15), $m' \geq 1$; therefore there are either m vertices for Claim 29 or m' vertices for Claim 30. In either case, we obtain a monochromatic cycle of length exactly $2n$, completing the proof. □

7.2.2. *The case of large* U_2: $|U_2| \geq n - b$.

Claim 28. *If* $|U_2| = n + a_2 \geq n - b$, *then we have a blue cycle of size exactly* $2n$.

Proof. Since $|U_2| = n + a_2 \geq n - 4\lambda n$, we know that the largest among $U_2 \cap V_2$, $U_2 \cap V_3$, $U_2 \cap V_4$ has size at least $0.33n$. We assume $|U_2 \cap V_p|$ is the largest and

$$|U_2 \cap V_p| \geq 0.33n. \tag{23}$$

By (23) and $|V_p| \leq n$, we have

$$|U_1 \cap V_p| \leq 0.67n$$

and there is a $q \in \{2, 3, 4\} - \{p\}$ such that

$$|U_1 \cap V_q| \geq 0.16n. \tag{24}$$

Step 1: We first find a path to include say $0.8n$ vertices in V_1 and $0.8n$ vertices in U_2 (all of $(V - V_p) \cap U_2$ and $V(C)$) by Theorem 17.

Details: The details are almost the same as in Step 2 of Claim 24 except every place of n_3 is replaced by n_1, every place of V_3 is replaced by V_1, V_1 is replaced by $(V - V_p)$.

- If $a_2 \geq 0$, then we do not need Step 2 and go to Step 3 directly.

Step 2: Use $|a_2|$ vertices in B to obtain a blue path.

Details: Since $b \geq |a_2|$, let $Z := \{z_1, \ldots, z_{|a_2|}\} \subseteq B$.

By (24) and each vertex v in B having blue degree at least $0.9n \gg \frac{1}{2}|U_1|$ to U_1, we can find for each pair (z_i, z_{i+1}) a blue common neighbor $r_i \in U_1 - V(C)$, where $1 \leq i \leq |a_2| - 1$, a blue neighbor r_0 of z_1 such that $r_0 \in V_q \cap U_1 - V(C)$, and a blue neighbor $r_{|a_2|}$ of $z_{|a_2|}$ such that $r_{|a_2|} \in V_q \cap U_1 - V(C)$ and $r_0, \ldots, r_{|a_2|}$ are all distinct.

Since y' has at most one red neighbor to $U_1 - V(C)$, we choose $r_{|a_2|}$ to be in $U_1 \cap V_q - V(C)$ and such that $r_{|a_2|}y'$ is blue.

We obtain a blue path

$$P_2 = r_0 z_1 r_1 \cdots z_i r_i \cdots z_{|a_2|} r_{|a_2|}$$

of length $2|a_2|$.

Step 3: Include the rest of vertices in U_2 to U_1 by Theorem 17.

Details: The details are almost the same as in Step 3 of Claim 24 except every place of V_2 is replaced by V_p. □

7.2.3. *Handling many vertices in $U_1 - V_k$ incident to red edges.*

Claim 29. *If there are at least $m = b + a_3$ vertices in $(V - V_k) \cap U_1$ of red degree at least $0.1n$ to $U_1 \cap V_k$, then we have a red cycle of length exactly $2n$.*

Proof. Let B' be the collection of vertices in U_1 with blue degree at least $0.05n$ to V_1. Since there are at most $4\lambda n^2$ blue edges between U_1 and V_1, we have

$$|B'| \leq 80\lambda n.$$

Step 1: We first find a collection of red cherries C_3 with center in $U_1 \cap (V - V_k)$ and leaves in $U_1 \cap V_k - B'$ of size m.

Details: The details are almost the same as in Step 1 of Claim 25 except we replace everywhere V_2 by $V - V_k$, V_1 by V_k, and V_3 by V_1.

Step 2: By Hall's theorem we find a matching M for $V(C_3) \cap V_k$ to R and then find a common neighbor back to connect those vertices.

Details: The details are almost the same as in Step 2 of Claim 25 except we replace everywhere V_3 by V_1 and n_3 by n_1.

Step 3: Use Theorem 17 to get a path saturating all vertices left in $V_1 - B - V(M)$.

Details: Let $X = V_1 - B - \{w_2, \ldots, w_{2m-1}\}$ and we know $|X| = n - a_3 - b - (2m-2) = n - 3m + 2$. We have $a_1 = a_3 - a_2 - 1 = m - a_2 - b - 1 \geq m$, and therefore

$$n + a_1 - m - (2m-1) - (m-1) = n + a_1 - 4m + 2 \geq n - 3m + 2.$$

We can take $Y \subseteq U_1 - \{u_1, \ldots, u_m\} - \{v_2, \ldots, v_{2m}\} - \{g_1, \ldots, g_{m-1}\}$ such that $v_1 \in Y$ and $|Y| = n - 3m + 2$.

The rest of details are almost the same as in Step 3 of Claim 25 except we replace everywhere V_3 by V_1 and n_3 by n_1. □

7.2.4. *Handling many vertices in $U_1 - V_k$ incident to blue edges.* In the case when many vertices in $U_1 - V_k$ are incident to blue edges, there are many disjoint blue cherries inside U_1, and we find a blue cycle.

Claim 30. *If there are at least $m' = -a_2 - b$ vertices in $U_1 - V_k$ of blue degree at least $|U_1 \cap V_k| - 0.1n$ to $U_1 \cap V_k$, then we have a blue cycle of length exactly $2n$.*

Proof. Since $U_1 \cap V_k$ is the largest among $U_1 \cap V_2$, $V_3 \cap U_1$, and $V_4 \cap U_1$, we know

$$|U_1 \cap V_k| \geq 0.33n, \quad |U_2 \cap V_k| \leq 0.67n, \quad \text{and} \quad |U_2 - V_k| \geq 0.32n. \tag{25}$$

Step 1: We find m' blue cherries from $U_1 \cap (V - V_k)$ to $U_1 \cap V_k$, possibly avoiding bad vertices. Then we find common neighbors in U_2 to connect those cherries.

Details: The details are almost the same as in Step 1 of Claim 26 until the following sentence except that we replace everywhere V_2 by $V - V_k$ and V_1 by V_k.

For all pairs (v_{2i}, v_{2i+1}) we can find distinct common blue neighbors, w_i, in $(V - V_k) \cap U_2 - V(C)$, where $1 \leq i \leq m' - 1$.

By (25), there is an $\ell \in \{2, 3, 4\} - \{k\}$ such that

$$|V_\ell \cap U_2| \geq 0.16n. \tag{26}$$

We also find for v_1 a blue neighbor $w_0 \in V_\ell \cap U_2$ and $v_{2m'}$ a blue neighbor $w_{m'} \in V_\ell \cap U_2$ distinct from $\{w_1, \ldots, w_{m'-1}\}$ and $V(C)$.

We obtain a blue path

$$P_1 = w_0 v_1 u_1 v_2 w_1 \cdots v_{2m'-1} u_{m'} v_{2m'} w_{m'}$$

of length $4m'$.

Step 2: We find for vertices in B common neighbors in $U_1 \cap V_k$, avoiding vertices already used.

Details: By (25) and each vertex v in B having red degree at most $0.1n + a_1$ to U_1, v has at least

$$|U_1 \cap V_k| - 2m' - 0.1n - a_1 > 0.6|U_1 \cap V_k - V(C)| \tag{27}$$

edges to $U_1 \cap V_k - V(C)$. We can find for each pair (z_i, z_{i+1}) a common neighbor r_i, where $1 \leq i \leq b-1$, a blue neighbor r_0 of z_1, and a blue neighbor r_b of z_b such that $\{r_0, \ldots, r_b\} \subseteq U_1 \cap V_k - V(C)$ are all distinct and $w_0 r_b$ is blue.

We obtain a blue path

$$P_2 = r_0 z_1 r_1 \cdots z_i r_i \cdots z_b r_b$$

of length $2b$.

Step 3: Take $0.9n$ vertices in V_1 and $0.9n$ vertices in U_2 including $(V - V_\ell) \cap U_2$ and $V(C)$. Use Theorem 17 to find a path.

Details: The details are almost the same as in Step 3 of Claim 26 except we replace everywhere V_1 by $V - V_\ell$, V_3 by V_1, and n_3 by n_1.

Step 4: Finally, the rest of vertices in $U_2 \cap V_\ell$ have large blue degree to $(V - V_\ell) \cap U_1$, and we find common neighbors to include them.

Details: The details are almost the same as in Step 4 of Claim 26 except we replace everywhere V_1 by $V - V_\ell$, V_2 by V_ℓ, V_3 by V_1, and n_3 by n_1. □

7.2.5. *Changes in the proof when $s \neq 4$.* When $s \neq 4$, essentially the proof for $s = 4$ works, with minor modifications.

Case 1: $s = 3$. Then $n_2 + n_3 \geq 2n - 1$ implies $n_1 \geq n_2 \geq n$ and therefore

$$n_1 = n_2 = n \quad \text{and} \quad n_3 = n - 1.$$

This case is addressed in Lemma 22.

Case 2: $s = 5$. If $j = 2$, then since $n_4 + n_5 > n$, $n_1 \geq n_2 \geq (1 - \lambda)n$, and $n_3 > \frac{1}{2}n$, we have

$$N = n_1 + n_2 + n_3 + n_4 + n_5 \geq 2(1 - \lambda)n + \tfrac{3}{2}n > 3n,$$

which is not the case. By a similar argument, $j \notin \{3, 4, 5\}$. Thus, we may assume $j = 1$.

The argument is almost the same as for $s = 4$. We only mention differences.

In our case, $n_4 + n_5 > n$ implies

$$n_1 \geq n_2 \geq n_3 \geq n_4 > \tfrac{1}{2}n; \tag{28}$$

thus

$$n_2 + n_3 = 3n - 1 - n_1 - n_4 - n_5 < n + \lambda n - 1. \tag{29}$$

By (28) and (29), we have

$$\tfrac{1}{2}n - \lambda n \leq n_5 \leq n_4 \leq n_3 \leq n_2 \leq \tfrac{1}{2}n + \lambda n. \tag{30}$$

In Section 7.2.2, in (23) we now can only guarantee $|U_2 \cap V_p| \geq 0.24n$ instead of $0.33n$. By (30), we can find a $q \in \{2, 3, 4, 5\} - \{p\}$ such that $|U_1 \cap V_q| \geq 0.16n$.

In Section 7.2.4, in (25) we can now only guarantee the largest $|U_1 \cap V_k| \geq 0.24n$. Equation (26) still holds with $\ell \in \{2, 3, 4, 5\} - \{k\}$. Everything else is the same.

8. Completion of the proof of Theorem 5

In the previous three sections, we proved Theorem 5 in the cases when $N - n_1 - n_2 \geq 3$. By (10), in the case $N - n_1 - n_2 \leq 2$, it is sufficient to show that for every 2-edge-coloring of $K_{2n,2n-1}$, there is a monochromatic cycle of length exactly $2n$. Thus, the next lemma completes the proof of Theorem 5.

Lemma 31. *If n is sufficiently large, then for every 2-edge-coloring of $K_{2n,2n-1}$, there is a monochromatic cycle of length exactly $2n$.*

Proof. Let $G = K_{2n,2n-1}$. From Section 5, we know that if the reduced graph G^r has a connected matching of size at least $(1 + \gamma)n$, then we can find a monochromatic cycle of length exactly $2n$. Suppose G^r has no connected matching of size $(1 + \gamma)n$ and thus, by Section 5 again, G has a (λ, i, j)-bad partition for some $i \in [2]$ and $j \in [2]$.

Without loss of generality, we assume $i = 1$ and discuss separately cases $j = 1$ and $j = 2$.

Case 1: G has a $(\lambda, 1, 1)$-bad partition. By the setup in Section 6, we have a partition $W_1 \cup W_2$ of $V(G)$ such that

(i) $(1-\lambda)n \le |W_2| \le (1+\lambda)n_1 = (1+\lambda)\cdot 2n$,

(ii) $|E(G_1[W_1, W_2])| \le \lambda n^2$,

(iii) $|E(G_2[W_1])| \le \lambda n^2$.

We know $|W_1| = N - |W_2| = 4n - 1 - |W_2|$, so by condition (i),

$$(2-3\lambda)n \le |W_1| \le (3+\lambda)n. \tag{31}$$

For simplicity, let $A := W_1 \cap V_1,\ B := W_2 \cap V_1,\ C := W_1 \cap V_2$, and $D := W_2 \cap V_2$. Let A^* be the collection of vertices in A with less than $0.6|C|$ red edges to C, B^* be the collection of vertices in B with at least $0.6|C|$ red edges to C, C^* be the collection of vertices in C with less than $0.6|A|$ red edges to A, and D^* be the collection of vertices in D with at least $0.6|A|$ red edges to A. Let $A = (A - A^*) \cup B^*$, $B = (B - B^*) \cup A^*,\ C = (C - C^*) \cup D^*$, and $D = (D - D^*) \cup C^*$. By conditions (ii) and (iii),

$$|A^*| \le \frac{5}{2|C|}\lambda n^2, \quad |B^*| \le \frac{5}{3|C|}\lambda n^2, \quad |C^*| \le \frac{5}{2|A|}\lambda n^2, \quad \text{and} \quad |D^*| \le \frac{5}{3|A|}\lambda n^2.$$

Let $\lambda' = 10\lambda$, $W_1 = A \cup C$, and $W_2 = B \cup D$.

Remark 32. Conditions (i)–(iii) still hold with λ' replacing λ and every vertex in A has red degree at least $0.59|C|$ to C, every vertex in B has blue degree at least $0.39|C|$ to C, every vertex in C has red degree at least $0.59|A|$ to A, and every vertex in D has red degree at least $0.39|A|$ to A.

Case 1.1: $|A| \ge n$ and $|C| \ge n$. Let $X \subseteq A$ and $Y \subseteq C$ such that $|X| = |Y| = n$. For each $x \in X$ and $y \in Y$, by $|A|, |C| \le 2n$ and Remark 32,

$$d_1(x, Y) \ge |Y| - 0.41|C| \ge n - 0.82n = 0.18n \quad \text{and similarly} \quad d_1(y, X) \ge |X| - 0.41|A| \ge 0.18n.$$

By condition (iii), we know that the number of vertices in X with at least $0.95n$ edges to Y in G_1 is at least $n - 20\lambda' n$ and the number of vertices in Y with at least $0.95n$ edges to X in G_1 is at least $n - 20\lambda' n$. Therefore, if we order vertices in X by their degrees in nondecreasing order, say the ordering follows from $d(x_1) \le \cdots \le d(x_n)$, then the smallest index i such that $d(x_i) \le i+1$ has the property that $d(x_i) \ge 0.95n$. Similarly, if we order vertices in Y by their degree in nondecreasing order, say the ordering follows from $d(y_1) \le \cdots \le d(y_n)$, then the smallest index j such that $d(y_j) \le j+1$ has the property that $d(y_j) \ge 0.95n$. Since $d(x_i) + d(y_j) \gg n+2$, by Theorem 17, we know $G_1[X, Y]$ is Hamiltonian biconnected and we can find a cycle in G_1 of length exactly $2n$.

Remark 33. The same proof shows that there is a red cycle of length exactly $\min\{|A|, |C|\}$.

Case 1.2: $|A| \le (1 - 30\lambda')n$. By (31) and $|V_1| = 2n$,

$$|C| \ge (1 + 27\lambda')n \quad \text{and} \quad |B| \ge (1 + 30\lambda')n. \tag{32}$$

By condition (ii), there are at most $20\lambda' n$ vertices in C with red degree at least $0.05n$ to B. Let C' be the $20\lambda' n$ vertices in C of largest red degree to B. Let Y be a subset of $C - C'$ with size n. Similarly, let B' be the $20\lambda' n$ vertices in B of largest red degree to C and we define $X \subseteq B - B'$ of size n. We show there is a blue cycle of length exactly $2n$ in $G_2[X, Y]$.

By the definitions of X and Y, we know that $d_2(x, Y) \geq 0.95n$ for $x \in X$ and $d_2(y, X) \geq 0.95n$ for $y \in Y$. By an argument similar to the last paragraph of Case 1.1, we can find a blue cycle of length exactly $2n$ in $G_2[X, Y]$.

Case 1.3: $|C| \leq (1 - 30\lambda')n$. We find a blue cycle by an argument similar to Case 1.2.

Case 1.4: $|A| \geq (1 + 30\lambda')n$ and $|D| \geq n$. By condition (iii), there are at most $20\lambda' n$ vertices in A of red degree at least $0.05n$ to D. Let X' be the $20\lambda' n$ vertices in A of largest red degree to D.

By condition (ii), there are at most $20\lambda' n$ vertices in D of red degree at least $0.05n$ to A. Let R be the $20\lambda' n$ vertices in D of largest red degree to A. Since $d_2(v, A) \geq 0.39|A| > 0.39n$ for each $v \in R$ and $|R| = 20\lambda' n =: m$, we can order vertices in R so that $R = \{r_1, \ldots, r_m\}$ and find for R a distinct collection of blue cherries to $A - X'$. We may assume the other ends of the cherries are $S = \{s_1, \ldots, s_{2m}\}$ so that each $s_{2i-1} r_i s_{2i}$ is a cherry. Since $S \subseteq A - X'$, each s_i has blue degree at least $|D| - 0.05n$ to D and we can find for each (s_{2i}, s_{2i+1}) a distinct common blue neighbor f_i in $D - R$, where $1 \leq i \leq m - 1$, and thus form a blue path

$$P_1 = s_1 r_1 s_2 f_1 s_3 \cdots s_{2m}$$

from s_1 to s_{2m}. We then extend the path P_1 by finding a blue neighbor r_0 of s_1 in $D - R$ distinct from each vertex chosen in P_1. Note now P_1 has length $4m - 1$ from r_0 to s_{2m}.

Let $X \subseteq (A - X' - V(P_1)) \cup \{s_{2m}\}$ such that $s_{2m} \in X$ and $|X| = n - 2m + 1$. Let $Y \subseteq (D - R - V(P_1)) \cup \{r_0\}$ such that $|Y| = n - 2m + 1$. Since $d_2(y, X) \geq 0.9n$ for $y \in Y$ and $d_2(x, Y) \geq 0.9n$ for $x \in X$, we claim that $G_2[X, Y]$ is Hamiltonian biconnected by an argument similar to the last paragraph of Case 1.2. Therefore, we can find a blue path P_2 of length $2n - 4m + 1$ from r_0 to s_{2m}.

Finally, we glue P_1 and P_2 at r_0 and s_{2m} to complete a blue cycle of length exactly $2n$.

Case 1.5: $|C| \geq (1 + 30\lambda')n$ and $|B| \geq n$. It is similar to Case 1.4.

Case 1.6: $|B| \geq n$ and $|D| \geq n$.

• If there is no blue edge in $G[B, D]$, then $G_1[B, D]$ is a complete bipartite graph and thus we can find a red cycle of length exactly $2n$.

• If there is a blue matching of size 2 in $G_2[B, D]$, say the two matching edges are $v_1 v_2$ and $u_1 u_2$, where $v_1, u_1 \in V_1$ and $v_2, u_2 \in V_2$, then by Cases 1.2 and 1.3, we know $|A| \geq (1 - 30\lambda')n$ and $|C| \geq (1 - 30\lambda')n$. By condition (ii), there are at most $20\lambda' n$ vertices in A such that the red degree to D is at least $0.05n$ and there are at most $20\lambda' n$ vertices in D such that the red degree to A is at least $0.05n$. Similarly, there are at most $20\lambda' n$ vertices in C such that the red degree to B is at least $0.05n$ and there are at most $20\lambda' n$ vertices in B such that the red degree to C is at least $0.05n$.

Let $A' \subseteq A$ be the $|A| - 20\lambda' n$ vertices with the largest blue degree to D, $D' \subseteq D$ be the $|D| - 20\lambda' n$ vertices with the largest blue degree to A, $C' \subseteq C$ be the $|C| - 20\lambda' n$ vertices with the largest blue degree to B, and $B' \subseteq B$ be the $|B| - 20\lambda' n$ vertices with largest blue degree to C.

By condition (i) and $|W_2| = |B| + |D| \geq 2n$, we know $|A| \geq n - 2\lambda' n$. Thus, by Remark 32,

$$d_2(u_2, A) \geq 0.39|A| \geq 0.38n.$$

We find a blue neighbor $w_1 \in A'$ of u_2. Let $A'' \subseteq A$ such that $w_1 \in A''$ and $|A''| = \lfloor \frac{1}{2} n \rfloor$. Let $D'' \subseteq D'$ such that $v_2 \in D''$ and $|D''| = \lfloor \frac{1}{2} n \rfloor$. By $A'' \subset A'$ and $D'' \subseteq D'$, $d_2(v, A'') \geq 0.4n$ for every $v \in D''$ and $d_2(v, D'') \geq 0.4n$ for every $v \in A''$. Since $0.4n + 0.4n > 0.5n + 1$, we can use Theorem 17 to find a blue path P_1 of length $2\left(\lfloor \frac{1}{2}n \rfloor - 1\right)$ from v_2 to w_1 and then extend P_1 by adding w_1u_2. Similarly, we can find a blue path P_2 with vertices in $B \cup C$ from v_1 to u_1 of length exactly $2\left(\lceil \frac{1}{2}n \rceil - 1\right)$.

Finally, we connect P_1 and P_2 by adding the edge v_1v_2 and u_1u_2 to form a blue cycle of length exactly $2n$.

Remark 34. The argument also works whenever all of A, B, C, D are of size in $[n - 100\lambda', n + 100\lambda' n]$.

• If the size of a maximum matching in $G_2[B, D]$ is exactly 1, then let v_1v_2 be a blue edge, and let $\{v_2\} \subseteq D$ be a smallest vertex cover in $G_2[B, D]$ (the case $\{v_1\}$ is a smallest vertex cover has a similar proof and is simpler). If we delete v_2, then the remaining graph is a complete bipartite graph in G_1. If $|D| \geq n + 1$ then we can find a red cycle of length $2n$ in $G_1[B, D - \{v_2\}]$. Thus, we may assume $|D| = n$ and $|C| = n - 1$.

Let $B'' \subseteq B$ such that $|B''| = n$. We find a blue cycle in $G_2[B'', C \cup \{v_2\}]$. By condition (i) and $|W_2| = |B| + |D| \geq 2n$, we know $|C| \geq n - 2\lambda' n$. Thus, by Remark 32, for each $v \in B''$ we have

$$d_2(v, C) \geq 0.39|C| \geq 0.38n.$$

We also know that each vertex v_c in $C \cup \{v_2\}$ can have red degree at most 1 to B (so it has blue degree at least $n - 1$ to B'') since otherwise with vertices in $D - \{v_2\}$ we can find a red cycle of length $2n$. Since $n - 1 + 0.19n > n + 1$, we can use Theorem 17 to find a blue cycle of length exactly $2n$.

Case 1.7: $n + 1 \leq |A| \leq (n + 30\lambda' n)$ and $n \leq |D| \leq n + 30\lambda' n$. By Remark 34, the size of a maximum matching in $G_2[B, D]$ is at most 1. Let $v_1v_2 \in G_2$ such that $v_1 \in B$ and $v_2 \in D$. We may also assume that $\{v_2\}$ is a minimum vertex cover of $G_2[B, D]$ (the case $\{v_1\}$ is a smallest vertex cover has a similar proof and is simpler). Let $R \subseteq A$ be the set of vertices with red degree at least $0.8n$ to D. By condition (ii), we know $|R| \leq 2\lambda' n$.

We first show that $|D| = n$. Assume not, i.e., $|D| \geq n + 1$. Then $|D - \{v_2\}| \geq n$.

If $|A - R| \geq n$, then we find a blue cycle of length $2n$ in $G_2[A - R, D]$. To do so, take a subset $A' \subseteq A - R$ of size n and $D' \subseteq D - \{v_2\}$ of size n. By Remark 32, for every $v \in D$ we have

$$d_2(v, C) \geq 0.39|C| = 0.39(2n - |D|) \geq 0.38n.$$

Thus, $d_2(v, A') \geq$ for $v \in D'$. By the definition of A', we know $d_2(v, D') \geq 0.2n$ for $v \in A'$. By condition (ii), we also know there are at most $20\lambda' n$ vertices in A' of red degree at least $0.05n$ to D and thus if we order vertices in A' and D' in nondecreasing order respectively, say $A' = \{u_1, \ldots, u_n\}$ and $D' = \{w_1, \ldots, w_n\}$, then the smallest index such that $d_2(u_i) \leq i + 1$ has $d_2(u_i) \geq 0.95n$ and the smallest index such that $d_2(w_j) \leq j + 1$ has $d_2(u_j) \geq 0.19n$. Since $0.95n + 0.19n > n + 1$, we can use Theorem 17 to find a blue cycle of length exactly $2n$ in $G_2[A', D']$.

If $|A - R| \leq n - 1$, then we find a red cycle of length exactly $2n$ in $G_1[B \cup R, D - \{v_2\}]$. To do so, note that (1) $|B \cup R| = 2n - |A - R| \geq n + 1$, (2) $G_1[B, D - \{v_2\}]$ is a red complete bipartite graph, and (3) each vertex in R has degree at least $0.8n$ to $D - \{v_2\}$. We can use Theorem 17 to find a red cycle of length exactly $2n$, since this red graph is very dense and has both parts large enough.

Remark 35. The proof also shows we can find a monochromatic cycle when $|A| \in [n - 100\lambda' n, n + 100\lambda' n]$ and $n + 1 \leq |D| \leq (1 + 100\lambda')n$.

We assume $|D| = n$ from now on. Since each vertex in R has red degree at least $0.8n$ to D, if there are at least two vertices in R, say r_1 and r_2, then we find a red common neighbor $w \in D$ for r_1 and r_2. Note that by Remark 33, $G_1[A, C]$ is Hamiltonian-biconnected. Therefore, we can find a red cycle of length exactly $2n$ from a path P_1 from r_1 to r_2 of length $2n-2$ glued with the path $P_2 = r_1wr_2$. The only case remaining is $|R| \leq 1$. Then we have $|A - R| \geq n$ and we find a blue cycle of length $2n$ by the same argument as in two paragraphs ahead of this paragraph.

Remark 36. Note that the last sentence of the previous paragraph shows why we need $|A| \geq n+1$.

The only uncovered case is:

Case 1.8: $n \leq |C| \leq (1+30\lambda')n$ and $(1-30\lambda')n \leq |A| \leq n-1$. We define R to be vertices in C with red degree at least $0.8n$ to B. By Remark 34, we may assume that the size of a maximum matching in $G_2[B, D]$ is at most 1.

If $|C - R| \geq n$, then we find a blue cycle of length exactly $2n$ in $G_2[B, C-R]$. Thus, we may assume

$$|C - R| \leq n - 1. \tag{33}$$

• If there is no edge in $G_2[B, D]$, then $G_1[B, D]$ is a complete bipartite graph and we are done if $|D \cup R| \geq n$. Thus, we may assume that $|D \cup R| \leq n-1$. Since $|C-R| + |R| + |D| = 2n-1$, $|C-R| \geq n$ and we have a contradiction.

• If the size of a maximum matching in $G_2[B, D]$ is exactly 1, say v_1v_2 is such a matching with $v_1 \in B$ and $v_2 \in D$, then one of $\{v_1\}$ or $\{v_2\}$ is a minimum vertex cover of $G_2[B, D]$. We may assume that $\{v_2\}$ is a minimum vertex cover of $G_2[B, D]$, and the case when $\{v_1\}$ is a minimum vertex cover has a similar proof and is simpler.

Since $G_1[B, D - \{v_2\}]$ is a complete bipartite graph, we are done if $|D| \geq n+1$. Thus, we may assume $|D| \leq n$. Moreover, if $|D \cup R - \{v_2\}| \geq n$ then we can find a red cycle of length $2n$ in $G_1[D \cup R - \{v_2\}, B]$; hence we may assume

$$|D| + |R| - 1 \leq n - 1.$$

But we also know that $|D| + |R| + |C - R| = 2n - 1$. Thus,

$$|C - R| \geq n - 1,$$

and by (33) we know

$$|C - R| = n - 1 \quad \text{and} \quad |D \cup R| = n.$$

If v_2 has at least two red edges to B then we can find a red cycle in $G_1[B, D \cup R]$ by first considering the two edges incident with v_2. Thus, v_2 has at most one red edge to B and thus has at least $|B| - 1$ blue edges to B. We can find a blue cycle in $G_2[(C - R) \cup \{v_2\}, B]$.

Case 2: G has a $(\lambda, 1, 2)$-bad partition. This case is covered in Case 1 in Section 7.1.5 (with the same proof). □

9. Proof of Theorem 6 on monochromatic $C_{\geq 2n}$

For large n, we need to prove the theorem for every N-vertex complete s-partite graph G with parts $(V_1^*, V_2^*, \dots, V_s^*)$ such that the numbers $n_i = |V_i^*|$ satisfy $n_1 \geq n_2 \geq \cdots \geq n_s$ and conditions (1), (2), (4) and (5).

Consider a possible counterexample G with 2-edge-coloring f and minimum $N+s$. If $N-n_1-n_2 \geq 3$, then restriction (7) does not apply, so by Theorem 5, G has a monochromatic C_{2n}, a contradiction. If $N-n_1-n_2 \leq 2$ and (7) holds, then again by Theorem 5, G has a monochromatic C_{2n}. Hence we need to consider only the case that $N-n_1-n_2 \leq 2$, all (1), (2), (4) and (5) hold, but (7) does not hold. In particular, $n_1 \geq 2n-1$, but $N \leq 4n-2$. This means $N-n_1 \leq (4n-2)-(2n-1) = 2n-1$, so by (2), $N = 4n-2$ and $n_1 = 2n-1$. If $N-n_1-n_2 \leq 1$, this does not satisfy (5). Thus $N-n_1-n_2 = 2$, and hence $G \supseteq K_{2n-1,2n-3,2}$. Therefore, the following lemma implies Theorem 6.

Lemma 37. *If n is sufficiently large, then for every 2-edge-coloring of $K_{2n-1,2n-3,2}$, there is a monochromatic cycle of length at least $2n$.*

Proof. The set-up of the proof is similar to the proof of Lemma 31. We only show the differences.

Let $V_3 = \{u_1, u_2\}$. Define $V_1' = V_1$ and $V_2' = V_2 \cup V_3$. We first consider $G[V_1', V_2']$ and then use the fact that $V_2' = V_2 \cup V_3$. Note that we have $|V_1'| = |V_2'| = 2n-1$.

By the proof in Lemma 31, we narrow the uncovered cases to (1) $|A| = n-1$ and $n \leq |C| \leq (1+30\lambda')n$ and (2) $n \leq |A| \leq (1+30\lambda')n$ and $|C| = n-1$.

<u>Case 1</u>: $|A| = n-1$ and $n \leq |C| \leq (1+30\lambda')n$.

Then we know $|B| = n$ and $(1-30\lambda')n - 1 \leq |D| \leq n-1$. By Remark 34, we know the size of a maximum matching, α', in $G_2[B, D]$ is at most 1. Let R be the set of vertices in C with at least $0.8n$ red neighbors in B. By condition (ii), $|R| \leq 2\lambda' n$.

Claim 38. *If $|C - R| \geq n$ then we find a blue cycle of length $2n$ in $G_2[B, C-R]$.*

Proof. We pick $C' \subseteq C - R$ of size n. We know:

(1) By Remark 32 and the definition of R, each vertex in B has blue degree at least $0.38n$ to C' and each vertex in C' has blue degree at least $0.2n$ to B.

(2) By condition (ii), all but at most $20\lambda' n$ vertices in B have red degree at most $0.05n$ to C' and all but at most $20\lambda' n$ vertices in C have red degree at most $0.05n$ to B.

(3) If we order vertices in C' and B in nondecreasing order by their degree in $G_2[C', B]$ respectively, then the smallest index with $d(x_i) \leq i+1$ and the smallest index with $d(y_j) \leq j+1$ satisfy $d(x_i) \geq 0.95n$ and $d(y_j) \geq 0.95n$.

Since $0.95n + 0.95n > n+1$, we can use Theorem 17 to show $G_2[C', B]$ is Hamiltonian biconnected and thus we can find a cycle by fixing an edge e first and then find a Hamiltonian path in $G_2[C', B]$ without e, which is still Hamiltonian biconnected. □

Remark 39. Similarly to Claim 38, we can show:

(1) For any two vertices $c_1 \in C$, $a_1 \in A$, graph $G_1[A, C]$ has a red path of length $2n-3$ from c_1 to a_1.

(2) For any two vertices $c_1, c_2 \in C$, graph $G_1[A, C]$ has a red path of length $2n-2$ from c_1 to c_2.

(3) For any two vertices $b_1, b_2 \in B$, graph $G_2[B, C-R]$ has a blue path of length $2n-2$ from b_1 to b_2.

(4) For any two vertices $c_1 \in C - R$, $b_1 \in B$, graph $G_2[B, C - R]$ has a blue path of length $2n - 3$ from c_1 to b_1.

Therefore, we may assume

$$|C - R| \leq n - 1 \quad \text{and thus} \quad |D \cup R| \geq n. \tag{34}$$

If $|R| \geq 2$, say $r_1, r_2 \in R$, then we find a common neighbor $r_b \in B$ for them. By Remark 39, we can find a red path P_1 of length $2n - 2$ in $G_1[C, A]$ and then extend P_1 to a red cycle of length $2n$ by adding $r_1 r_b r_2$. Thus, we may assume

$$|C - R| = n - 1, \quad |R| = 1 \quad \text{and} \quad |D| = n - 1. \tag{35}$$

Let $R = \{r\}$. If $\alpha' = 0$, then $G_1[B, D]$ is a complete bipartite graph. We can find a red cycle of length $2n$ in $G_1[B, D \cup R]$ by first fixing two neighbors in B for r.

If $\alpha' = 1$, say $v_1 v_2$ is a maximum matching in $G_2[B, D]$, where $v_1 \in B$ and $v_2 \in D$. If $\{v_2\}$ is a minimum vertex cover, then v_2 has at most one red edge to B since otherwise we find a red cycle by (35) in $G_1[D \cup R, B]$ by first fixing two neighbors in B for v_2. Thus, we may assume v_2 has at least $|B| - 1$ blue edges to B and thus we can find a blue cycle in $G_2[(C - R) \cup \{v_2\}, B]$ by Remark 39.

We may assume $\{v_1\}$ is a minimum vertex cover. Note that v_1 has at most one red edge to D since otherwise we find a red cycle in $G_1[B, D \cup R]$ by first fixing two red neighbors for v_1. For the same reason, each vertex in A has at most one red edge to D. We use vertices in V_3 to find a monochromatic cycle.

If there is a red edge from D to $C - R$, say $u_1 y_1$ with $u_1 \in D$ and $y_1 \in C$, then we find a red cycle of length at least $2n$. To do so, by Remark 39, we first find a red path P_1 from y_1 to r of length $2n - 2$ in $G_1[A, C]$. Since r has at least $0.8n$ red neighbors in B and $G_1[B - \{v_1\}, D]$ is complete bipartite, we find for r and u_1 a red common neighbor in $B - \{v_1\}$, say r_b. Finally, we extend P_1 to a red cycle of length $2n + 1$ by adding the red path $r r_b u_1 y_1$. Since at least one of u_1 and u_2 are not in R, say $u_1 \notin R$, we may assume there is a blue edge $u_1 y_1$ from $C - R$ to D with $u_1 \in C - R$ and $y_1 \in D$.

We find a blue cycle of length at least $2n$ by using u_1. To do so, by Remark 32, each vertex in D has blue degree at least $0.38n$ to $A \cup \{v_1\}$ and each vertex in $C - R$ has blue degree at least $0.2n - 1$ to B. We first fix a blue neighbor z_1 of y_1 with $z_1 \in A$ and then find a common blue neighbor, say $y_2 \in D - \{y_1\}$, for v_1 and z_1. We can find a blue path P_1 of length $2n - 3$ from u_1 to v_1 in $G_2[C - R, B]$ by Remark 39 and then extend P_1 by adding the path $v_1 y_2 z_1 y_1 u_1$ to obtain a blue cycle of length $2n + 1$.

Case 2: $n \leq |A| \leq (1 + 30\lambda')n$ and $|C| = n - 1$. It is symmetric to Case 1 until we use vertices in V_3. Thus, we may assume the maximum size of a matching in $G_2[B, D]$ is 1, $v_1 v_2$ is one maximum matching and $\{v_2\}$ is a minimum vertex cover and every vertex in $C \cup \{v_2\}$ has blue degree at least $|B| - 1$ to B. Moreover, we may define $R \subseteq A$ similarly to Case 1; i.e., R is the collection of vertices in A with at least $0.8n$ red degrees to D, and assume

$$|A - R| = n - 1, \quad |R| = 1 \quad \text{and} \quad |B| = n - 1. \tag{36}$$

Let $R = \{r\}$. If there is a red edge from C to $D - \{v_2\}$, say $u_1 y_1$ with $u_1 \in C$ and $y_1 \in D$, then we can find a red cycle of length at least $2n$. To do so, we first find a red path P_1 of length $2n - 3$ from u_1 to r

by Remark 39. Then we find a red neighbor r_d of r in $D-\{v_2, y_1\}$ and a common red neighbor r_b of r_d and y_1 in B. We extend the path P_1 to a red cycle of length $2n+1$ by adding the red path $rr_dr_by_1u_1$ to P_1.

Then we may assume there is a blue edge from C to $D-\{v_2\}$, say u_1y_1 with $u_1 \in C$ and $y_1 \in D-\{v_2\}$. We first find a blue path of length $2n-2$ from y_1 to v_2 in $G_2[A-R, D]$ by Remark 39 and then find a common blue neighbor $y \in B$ for v_2 and u_1. Finally, we add the path $y_1u_1yv_2$ to P_1 to obtain a blue cycle of length $2n+1$. □

10. Proof of Theorem 7 on monochromatic P_{2n}

10.1. ***A useful lemma.*** If G contains a monochromatic C_{2n}, then it certainly contains a monochromatic P_{2n}. So suppose $G = K_{n_1,\dots,n_s}$ does not have a monochromatic C_{2n}. The lemma below is very helpful here and in the next section.

Lemma 40. *Let $s \geq 3$ and n be sufficiently large. Let $n_1 \geq \cdots \geq n_s$ and $N = n_1 + \cdots + n_s$ satisfy (1) and (2). Suppose that for some 2-edge-coloring f of the complete s-partite graph $G = K_{n_1,\dots,n_s}$, there are no monochromatic cycles C_{2n}. Then G contains a monochromatic P_{2n+1}.*

Proof. By Theorem 5, if (1) and (2) hold but G does not have a monochromatic C_{2n}, then (7) fails. In particular, $N - n_1 - n_2 \leq 2$. Since $s \geq 3$, $N - n_1 - n_2 \geq 1$. We may assume $s = 3$: if $s > 3$, then $N - n_1 - n_2 \leq 2$ yields $s = 4$ and $n_3 = n_4 = 1$. In this case, deleting the edges between V_3 and V_4 and combining them into one part (of size 2) only makes the case harder.

We use condition (7) to find a monochromatic C_{2n} only in the nearly-bipartite subcase of Section 6: in Section 6.2. Therefore, if there is no monochromatic C_{2n}, but (1) and (2) hold, we have a graph G that falls under this subcase.

In this case, we have found disjoint subsets $X_{11}, X_{12} \subseteq V_1$ and $X_{21}, X_{22} \subseteq V_2$ with $|X_{11}| = |X_{21}| = |X_{12}| = |X_{22}| = \frac{1}{2}n + 10$ satisfying the following property: if H is any of the graphs $G_1[X_{11}, X_{21}]$, $G_1[X_{12}, X_{22}]$, $G_2[X_{12}, X_{21}]$, or $G_2[X_{11}, X_{22}]$, then given any vertices v, w in H, we can find a (v, w)-path in H on m vertices, provided that $n - 10 \leq m \leq n + 10$ and that the parity of m is correct.

Now let $x \in V_3$ be an arbitrary vertex (since we know that $1 \leq n_3 \leq 2$). Without loss of generality, we may assume that x has an edge in G_1 to X_{11}. If x also has an edge in G_1 to $X_{12} \cup X_{22}$, then we obtain a long path in G_1 as follows:

- Let P_1 be a path in $G_1[X_{11}, X_{21}]$ of length at least n starting from a neighbor of x in X_{11}.
- Let P_2 be a path in $G_1[X_{12}, X_{22}]$ of length at least n starting from a neighbor of x.
- Use x to join P_1 and P_2 into a path.

Otherwise, all edges of x to $X_{12} \cup X_{22}$ are in G_2; in particular, x has a neighbor in G_2 in both X_{12} and X_{22}. We obtain a long path in G_2 in a similar way:

- Let P_1 be a path in $G_2[X_{12}, X_{21}]$ of length at least n starting from a neighbor of x in X_{12}.
- Let P_2 be a path in $G_2[X_{11}, X_{22}]$ of length at least n starting from a neighbor of x in X_{22}.
- Use x to join P_1 and P_2 into a path.

In either case, G contains a monochromatic P_{2n+1}. □

10.2. ***Completion of the proof of Theorem 7.*** As observed above, if G has a monochromatic C_{2n}, then we are done. Otherwise, by Theorem 5 and Lemma 40, G is bipartite. In this case, (2) yields $n_2 \geq 2n-1$. Hence $n_1 \geq 2n-1$, and $G \supseteq K_{2n-1,2n-1}$. In this case, Theorem 2 yields the result. □

11. Proof of Theorem 8 on monochromatic P_{2n+1}

11.1. ***Setup of the proof.*** For large n, we need to prove the theorem for each complete s-partite graph $G = K_{n_1,\dots,n_s}$ such that the numbers n_i satisfy $n_1 \geq n_2 \geq \cdots \geq n_s$ and the three conditions

$$N = n_1 + \cdots + n_s \geq 3n, \tag{T1$'$}$$

$$N - n_1 = n_2 + \cdots + n_s \geq 2n-1, \tag{T2$'$}$$

$$\text{if } s = 2, \quad \text{then } n_1 \geq 2n+1. \tag{T3$'$}$$

For a given large n, we consider a possible counterexample with the minimum $N+s$. In view of this, it is enough to consider the lists $(n_1, \cdots, n_s)$ satisfying (T1$'$), (T2$'$) and (T3$'$) such that:

(a) For each $1 \leq j \leq s$, if $n_i > n_{i+1}$, then the list $(n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_s)$ does not satisfy some of (T1$'$), (T2$'$) and (T3$'$).

(b) If $s \geq 4$, then the list $(n_1, \dots, n_{s-2}, n_{s-1} + n_s)$ (possibly with the entries rearranged into a nonincreasing order) does not satisfy some of (T1$'$), (T2$'$) and (T3$'$).

<u>Case 1</u>: $s \geq 3$ and $N > 3n$. Then (T3$'$) holds by default. If $n_1 > n_2$, then the list $(n_1 - 1, n_2, n_3, \dots, n_s)$ still satisfies the conditions (T1$'$), (T2$'$) and (T3$'$), a contradiction to (a). Hence $n_1 = n_2$. Choose the maximum i such that $n_1 = n_i$. If $N - n_1 > 2n-1$, consider the list $(n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_s)$. In this case (T1$'$) and (T2$'$) still are satisfied; so by (a), (T3$'$) fails. But this means $s = 3$ and $n_1 = n_i = 1$, so $N \leq 3$, a contradiction. Thus in this case $N - n_1 = 2n-1$. Therefore, $n_1 = N - (N - n_1) \geq 3n + 1 - (2n-1) = n+2$ and hence $n_2 \geq n+2$, so $N - n_1 - n_2 \leq (2n-1) - (n+2) = n-3$. Then the list $(n_1, n_1, N - 2n_1)$ satisfies (T1$'$)–(T3$'$). Summarizing, we get

$$\text{if } s \geq 3 \text{ and } N > 3n, \quad \text{then } s = 3,\ n_2 + n_3 = 2n-1 \text{ and } n_1 = n_2 \geq n+2. \tag{37}$$

<u>Case 2</u>: $s \geq 3$ and $N = 3n$. Again (T3$'$) holds by default. By (T2$'$), $n_1 \leq n+1$; hence $N - n_1 - n_2 \geq n-2$. If $s \geq 4$ and $n_{s-1} + n_s \leq n+1$, then let L be the list obtained from $(n_1, \dots, n_s)$ by replacing the two entries n_{s-1} and n_s with $n_{s-1} + n_s$ and then possibly rearrange the entries into nonincreasing order. By construction, L satisfies (T1$'$)–(T3$'$), a contradiction to (b). Hence $n_{s-1} + n_s \geq n+2$. If $s \geq 6$, then $N \geq 3(n_{s-1} + n_s) \geq 3n + 6$, contradicting $N = 3n$. Thus

$$\text{if } s \geq 3 \text{ and } N = 3n, \text{ then } s \leq 5 \quad \text{and} \quad \text{if } s \geq 4, \text{ then } n_{s-1} + n_s \geq n+2. \tag{38}$$

<u>Case 3</u>: $s = 2$. Then by (T3$'$), $n_1 \geq 2n+1$ and by (T2$'$), $n_2 \geq 2n-1$. Thus $G \supseteq K_{2n+1,2n-1}$, and we can assume that

$$\text{if } s = 2, \text{ then } G = K_{2n+1,2n-1}. \tag{39}$$

As we have seen, always $s \leq 5$.

11.2. ***Completion of the proof.*** Suppose G satisfies (37)–(39), and f is a 2-edge-coloring G such that there is no monochromatic P_{2n+1}.

If G has no monochromatic C_{2n}, then by Lemma 40, G is bipartite. So by (39), $G = K_{2n+1,2n-1}$. But by Lemma 31, $K_{2n,2n-1} \mapsto (C_{2n}, C_{2n})$. Therefore, below we assume that the 2-edge-coloring f of G is such that G contains a red cycle C with $2n$ vertices (i.e., G_1 contains C).

Let $V' = V(C)$ and $V'' = V(G) - V'$. Similarly, for $j = 1, \dots, s$, let $V'_j = V_j \cap C$ and $V''_j = V_j - V'_j$. If some red edge e connects V' with V'', then $C + e$ contains a red P_{2n+1}, so below we assume that

$$\text{all the edges in } G[V', V''] \text{ are blue,} \quad \text{i.e., } G_2[V', V''] = G[V', V'']. \tag{40}$$

Case 1: $s = 2$. Then $|V'_1| = |V'_2| = n$. By (39), $|V''_1| = n + 1$. By (40), $G_2[V''_1, V'_2] = K_{n+1,n}$, but $K_{n+1,n}$ contains P_{2n+1}.

Case 2: $s \geq 3$ and $n_1 \geq n$. If $V_1 \supseteq V''$, then (since $|V''| \geq n$ by (38))

$$G_2[V'', V(G) - V_1] = G[V'', V(G) - V_1] = K_{n,N-n_1} \supseteq K_{n,2n-1} \supseteq P_{2n+1}.$$

Because C is a cycle of length $2n$ and V'_1 is an independent set, $|V'_1| \leq n$. In particular, since $s \geq 3$,

$$\text{there are distinct } 2 \leq j_1, j_2 \leq s \text{ such that there are vertices } v_1 \in V'_{j_1} \text{ and } v_2 \in V''_{j_2}.$$

If $|V''_1| \geq n$, then $G_2[V''_1, V' - V'_1]$ is a complete bipartite graph with parts of size at least n, so it contains a path P with $2n$ vertices, starting from v_1. Adding to it edge v_1v_2, we get a blue P_{2n+1}.

Suppose now $|V''_1| \leq n - 1$. Then the complete bipartite graph $G_2[V''_1, V' - V'_1]$ has a path Q_1 with $2|V''_1| + 1$ vertices starting from v_1 and ending in $V' - V_1$. Also since $n_1 \geq n$ and $|V''| \geq n$, the complete bipartite graph $G[V'_1, V'' - V_1]$ contains $K_{n-|V''_1|,n-|V''_1|}$ and hence contains a path Q_2 with $2(n - |V''_1|)$ vertices starting from v_2. Then connecting Q_1 with Q_2 by the edge v_1v_2 we create a P_{2n+1}.

Case 3: $s \geq 3$ and $n_1 \leq n - 1$. In this case, $N/n_1 > 3$, so $s \geq 4$. Then (37)–(39) imply that $N = 3n$ and $4 \leq s \leq 5$. In particular,

$$N - n_i \geq 3n - (n - 1) = 2n + 1 \quad \text{for every } 1 \leq i \leq s. \tag{41}$$

Relabel the V_i's so that $|V''_1| \geq \dots \geq |V''_s|$. Let s' be the largest i such that $V''_i \neq \varnothing$. We construct a path Q with $2n + 1$ vertices greedily in two stages.

Stage 1: For $i = 1, \dots, s' - 1$, find a vertex $w_i \in V' - V_i - V_{i+1}$ so that all $s' - 1$ of them are distinct. We can do it because V''_i and V''_{i+1} are nonempty, so

$$|V'_i \cup V'_{i+1}| \leq (n_i - 1) + (n_{i+1} - 1) \leq 2n - 4 = |V'| - 4.$$

At least four choices for each of the $s' - 1 \leq 4$ vertices w_i allow us to choose them all distinct. Then we choose $w_0 \in V' - V_1$ and $w_{s'} \in V' - V_{s'}$ so that all $w_0, \dots, w_{s'}$ are distinct.

Stage 2: For $i = 0, \dots, s' - 1$ we find a (w_i, w_{i+1})-path Q_i such that (i) $V(Q_i) \cap V'' = V''_{i+1}$, and (ii) all paths $Q_0, \dots, Q_{s'-1}$ are internally disjoint.

If we succeed, then $\bigcup_{i=0}^{s'-1} Q_i$ is a path that we are seeking.

Suppose we are constructing Q_i and $V''_{i+1} = \{u_1, \dots, u_q\}$. We start Q_i by the edge w_iu_1. Then on Step j for $j = 1, \dots, q$, do as follows.

If $j = q$, then add edge $u_q w_{i+1}$ and finish Q_i. Otherwise, find a vertex $z_j \in V' - V_{i+1}$ not yet used in any $Q_{i'}$, then add to Q_i edges $u_j z_j$ and $z_j u_{j+1}$, and then go to Step $j+1$. We can find this z_j because by (41), $|V - V_i| \geq 2n+1$, at most $n-2$ of these vertices are in V'', and at most n vertices of all paths $Q_{i'}$ are already chosen in V'. Since we always can choose z_j, our greedy procedure constructs Q_i, and all Q_i together form the promised path Q. □

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