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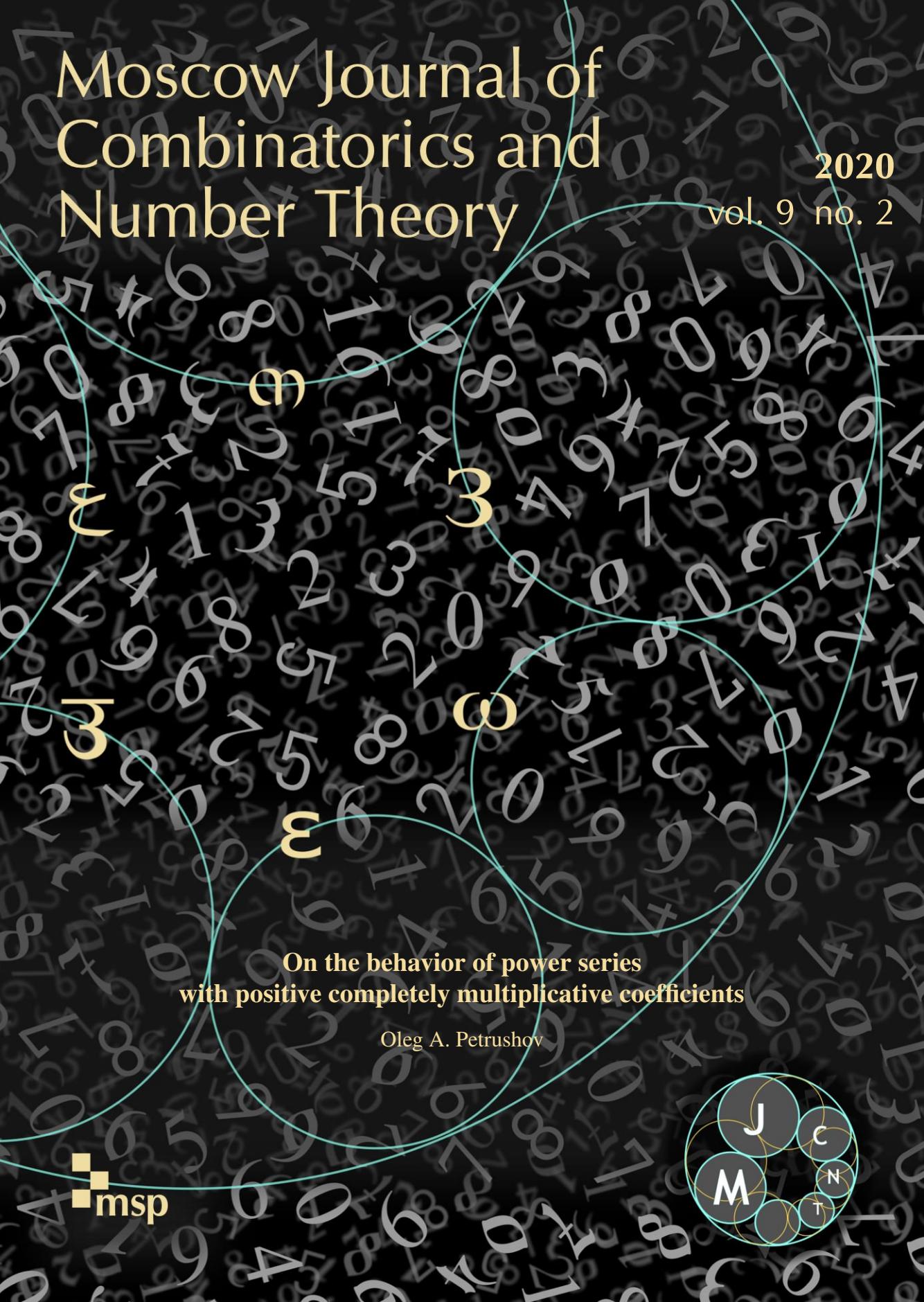

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We consider power series with positive completely multiplicative coefficients. We obtain a large family of power series that have the unit circle as natural boundary, as well as new Ω-theorems for power series with positive completely multiplicative coefficients when the argument tends to roots of unity. Also Ω-estimates for some partial sums of completely multiplicative functions are given.

1. Introduction

In this paper we study power series with completely multiplicative coefficients. Power series with coefficients that have some arithmetical structure possess interesting properties. Most of the power series with arithmetical coefficients converge on the unit disc but have no continuation beyond the unit circle. Moreover they usually have interesting properties when z tends to the unit circle along a radius.

The results of [Petrushov 2014; 2015a] are about two specific and important series $\sum_n \mu(n)z^n$ and $\sum_n \mu^2(n)z^n$. For example in [Petrushov 2015a] we exposed a connection between the asymptotic behavior of the series $\sum_{n>0} \mu^2(n)z^n$ as z tends to $e^{2\pi i\beta}$ along its radius and the Diophantine properties of β, namely its irrationality exponent when β is irrational and its denominator when β is rational. A similar study for the series $\sum_{n>0} \mu(n)z^n$ was performed in [Petrushov 2014] with less-striking conclusions.

The specific problem of determining the analytic behavior of power series with multiplicative coefficients was posed by W. Schwarz in the Oberwolfach Meeting on Number Theory. L. G. Lucht [1981] proved that for an extensive set of multiplicative functions $\alpha(n)$ the unit circle is the natural boundary of the series $\sum_{n=1}^{\infty} \alpha(n)z^n$. The set is defined by some complicated conditions. In particular it requires the existence of a complex number s with nonnegative real part, a slowly oscillating function $l(x)$ and a nonzero sequence of coefficients c_q such that for any principal character χ_0 modulo q

$$\sum_{n<x} \alpha(n)\chi_0(n) = (c_q + o(1))x^s l(x), \quad x \to +\infty,$$

and for any nonprincipal character χ

$$\sum_{n<x} \alpha(n)\chi(n) = o(x^s|l(x)|), \quad x \to +\infty.$$

In the simpler case of positive multiplicative coefficients, results of Wirsing may be applied to obtain the desired asymptotic behavior in some cases (see [Lucht 1981, Corollary 3]) but even then, the simple

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multiplicative function defined by $\alpha_0(2^a m) = 3^a$, where m is an odd number and a is a nonnegative integer, cannot satisfy the conditions of Lucht's class: since

$$\sum_{n\le 2^m-1} \alpha_0(n) = 3^m - 2^m \quad \text{and} \quad \sum_{n\le 2^m} \alpha_0(n) = 2(3^m) - 2^m,$$

the function $x^{-\log 3/\log 2}\sum_{n\le x}\alpha_0(n)$ is not slowly oscillating.

The scope of this article is restricted to nonnegative, completely multiplicative functions.

An arithmetical function $\alpha(n)$ is called completely multiplicative if

$$\alpha(mn) = \alpha(m)\alpha(n)$$

for each m and n.

For example n^z is a completely multiplicative function. A Dirichlet character is also a completely multiplicative function.

Denote $e^{2\pi i\beta}$ by $e(\beta)$. Denote by $\mathfrak{A}(z)$, where $z \in \mathbb{C}$, the power series

$$\sum_{n=1}^{\infty} \alpha(n) z^n.$$

Denote by $A(x, \beta)$, where $x \in \mathbb{R}^+,\ \beta \in \mathbb{R}$, the sum

$$\sum_{n<x} \alpha(n) e(n\beta).$$

Throughout the paper the letter p always denotes a generic prime number and σ the real part of the complex number s. Let $g(x) > 0$. The equality $f(x) = \Omega(g(x))$ when $x \to a$ means that there is an infinite sequence $t_k \to a$ such that $|f(t_k)| > \delta g(t_k)$ for some $\delta > 0$. The relation $f(x) = \Omega(g(x))$ when $x \to a$ is also equivalent to $\overline{\lim}_{x\to a} |f(x)/g(x)| > 0$.

In [Petrushov 2018] we proved Ω-estimates of power series with positive completely multiplicative coefficients.

To be specific, for a completely multiplicative function $\alpha(n)$ such that $\alpha(p) \le p$ and $0 < A \le \alpha(p) \le B < 2A$ for any prime p, we proved that there is a computable constant $C > 0$ such that for any $l \in \mathbb{Z}$ and any prime q with $\alpha(q) \ne 1$, we have

$$\mathfrak{A}\left(e\left(\frac{l}{q}\right)r\right) = \Omega\left(\frac{|\ln(1-r)|^{C-1}}{1-r}\right)$$

as $r \to 1-$.

This implies that if $\alpha(p) \ne 1$ for infinitely many primes p, the natural boundary of $\mathfrak{A}(z)$ is the unit circle.

In the present article we substantially enlarge the set of completely multiplicative functions for which the associated power series has the unit circle as natural boundary.

Theorem 1. *Let $\alpha(n)$ be a completely multiplicative function satisfying*:

(1) $\sum_p \alpha(p)(1 - \Re(\chi(p)))/p$ *diverges for any nonprincipal* χ.

(2) *The series*

$$\sum_p \frac{\alpha(p)}{p^\sigma}$$

converges for $\Re s > 1$.

If the series $\sum_{n=1}^{\infty} \alpha(n) z^n$ *has a nonsingular point on the unit circle, then* $\alpha(n) \equiv 1$.

Any completely multiplicative function $\alpha(n)$ satisfying $\alpha(p) \geq A > 0$ for every prime p satisfies condition (1) of Theorem 1, and any completely multiplicative function satisfying $|\alpha(p)| \leq B$ for every prime p satisfies condition (2). Therefore this theorem improves results obtained in [Petrushov 2018] and covers functions which are not in Lucht's class, such as our example $\alpha_0(n)$.

Theorem 1 is easily derived from Ω-estimates for the power series $\mathfrak{A}(z)$ along every radius $[0, e(l/q))$ where every prime factor p of q satisfies $\alpha(p) \neq 1$.

Theorem 2. *Let* $\alpha(n)$ *be a positive completely multiplicative function satisfying conditions* (1) *and* (2) *of Theorem 1. Let* q *be a positive integer whose prime factors all satisfy* $\alpha(p) \neq 1$. *Let* $\beta = l/q$ *with* $(l, q) = 1$.

Set $\delta = \sup_p(\log \alpha(p)/\log p) - 1$ *if there are primes* p *with* $\alpha(p) > p$, *and* $\delta = 0$ *otherwise. Choose* $m \geq 0$ *such that there are at least* m *distinct primes satisfying* $\alpha(p) = p^{1+\delta}$. *If* $\delta = 0$, *we set* $\epsilon \geq 0$ *such that there is* $c \in \mathbb{R}$ *with*

$$\sum_p \frac{\alpha(p)}{p^\sigma} \geq \epsilon |\ln(\sigma - 1)| + c \quad \text{as } \sigma \to 1+.$$

Assume first that

(3.1) $\delta = 0$.

Then for all $b < 0$

$$\mathfrak{A}(e(\beta)r) = \Omega\left(\frac{1}{(1-r)|\ln(1-r)|^{1-b}}\right) \quad \text{as } r \to 1-,$$

$$A(x, \beta) = \Omega\left(\frac{x}{(\ln x)^{1-b}}\right) \quad \text{as } x \to +\infty.$$

If moreover

(3.2) $\delta = 0$ *and* $\epsilon + m > 0$,

then one can replace b *by* $\epsilon + m$ *in the previous formulas*; *that is*,

$$\mathfrak{A}(e(\beta)r) = \Omega\left(\frac{|\ln(1-r)|^{\epsilon+m-1}}{1-r}\right) \quad \text{as } r \to 1-,$$

$$A(x, \beta) = \Omega\left(x(\ln x)^{\epsilon+m-1}\right) \quad \text{as } x \to +\infty.$$

Now, *if*

(3.3) $\delta > 0$ *and* $\alpha(q) = q^{1+\delta}$,

then

$$\mathfrak{A}(e(\beta)r) = \Omega\left(\frac{|\ln(1-r)|^{m-1}}{(1-r)^{1+\delta}}\right) \quad \text{as } r \to 1-,$$

$$A(x, \beta) = \Omega\left(x^{1+\delta}(\ln x)^{m-1}\right) \quad \text{as } x \to +\infty.$$

To get these estimates, we follow a method that we developed in [Petrushov 2015b; 2017] to study power series with additive coefficients and to prove similar Ω-estimates.

We use the Mellin transforms of $\mathfrak{A}(e(\beta)r)$ and $A(x,\beta)$, which both turn out be easily expressed in terms of the twisted Dirichlet series

$$F[\beta](s)=\sum_n e(\beta n)\frac{\alpha(n)}{n^s}.$$

When $\beta=l/q$ with $(l,q)=1$, $F[\beta](s)$ can be decomposed into a linear combination of

$$F(s)=\sum_n \frac{\alpha(n)}{n^s} \quad \text{and} \quad F(s,\chi)=\sum_n \frac{\alpha(n)\chi(n)}{n^s},$$

where χ runs among nonprincipal characters mod q. This decomposition gives a nice description of the meromorphic extension of $F[\beta](s)$ on the half-plane $\Re s>1$. Finally Tauberian arguments allow us to deduce the different Ω-estimates of Theorem 2 from the corresponding analytic properties of $F[\beta](s)$.

In Section 2 we prove that the decomposition of $F[\beta](s)$ into a linear combination of $L(s,\chi)$ is possible, study some sums with characters, and study the Mellin transform. In Section 3 we prove some general Ω-estimates. In Section 4 we prove the theorems and prove the generalization of Theorem 2.

2. Preliminary results

In this section we decompose $F[\beta](s)$ into a linear combination of $F(s,\chi)$ and prove a useful integral equality.

Let q be a natural number, $q>1$ and let $q=\prod_{i=1}^k p_i^{l_i}$ be its decomposition into prime factors throughout this section. Let $K(q)=\left\{n\in\mathbb{N}: n=\prod_{i=1}^k p_i^{m_i}\right\}$ and in this definition m_i are arbitrary nonnegative integers. From the fundamental theorem of arithmetic it easily follows that each $n\in\mathbb{N}$ has a unique representation

$$n=km, \tag{1}$$

where $k\in K(q)$, $(m,q)=1$.

For a Dirichlet character χ modulo q, we denote by $\tau(\chi,l)$ the Gauss sum

$$\sum_{n=1}^{q}\chi(n)e\left(\frac{nl}{q}\right).$$

We define $C_\chi(s)$ as the Dirichlet series

$$\sum_{k\in K(q)}\frac{\alpha(k)}{k^s}\tau(\bar{\chi},lk),$$

where χ is the conjugate character of χ. Throughout the paper $C_\chi(s)$ depends on l and q.

Lemma 3. *Let $\alpha(n)$ be a multiplicative function and let the Dirichlet series*

$$F(s)=\sum_{n=1}^{\infty}\frac{\alpha(n)}{n^s}$$

be absolutely convergent in the domain $\{\Re s > \sigma_1\}$. *Then for* $\Re s > \sigma_1$ *the following identity holds*:

$$F[\beta](s) = \frac{1}{\phi(q)} \sum_{\chi \pmod q} C_\chi(s) F(s, \chi) \tag{2}$$

(see [Petrushov 2015a, p. 20]).

Let $c_q(n)$ be Ramanujan sum,

$$c_q(n) = \sum_{\substack{0 \le l < q \\ (l,q)=1}} e\left(\frac{nl}{q}\right).$$

Recall standard properties of the Ramanujan sum:

(1) If $(q_1, q_2) = 1$ then $c_{q_1 q_2}(n) = c_{q_1}(n) c_{q_2}(n)$.

(2) $c_q(k)$ depends only on (k, q).

(3) $c_k(q) = \tau(\chi_0, k)$, where χ_0 is the principal character modulo q.

Lemma 4. *Let* f *be a completely multiplicative function with* $|f(p)| < 1$ *for* $p \mid q$. *We have*

$$\left(\sum_{n \in K(q)} f(n) c_q(n)\right) \prod_{p \mid q} (1 - f(p)) = \prod_{p^m \,||\, q} ((pf(p))^m - (pf(p))^{m-1}), \tag{3}$$

where the notation $p^m \,||\, q$ *means that the multiplicity of* p *in the prime decomposition of* q *is* m.

Proof. We see

$$\begin{aligned}
\left(\sum_{n \in K(q)} f(n) c_q(n)\right) \prod_{p \mid q} (1 - f(p)) &= \prod_{p \mid q} \left(\sum_{n \in K(p)} f(n) c_q(n)\right) (1 - f(p)) \\
&= \prod_{p^m \,||\, q} \left(\sum_{j=0}^{\infty} f^j(p) c_{p^j}(p^m)\right) (1 - f(p)) \\
&= \prod_{p^m \,||\, q} \left(-p^{m-1} f^{m-1}(p) + p^{m-1}(p-1) \frac{f^m(p)}{1 - f(p)}\right) (1 - f(p)) \\
&= \prod_{p^m \,||\, q} (-p^{m-1} f^{m-1}(p)(1 - f(p) + p^{m-1}(p-1) f^m(p)) \\
&= \prod_{p^m \,||\, q} (-p^{m-1} f^{m-1}(p) + p^{m-1} f^m(p) + p^m f^m(p) - p^{m-1} f^m(p)) \\
&= \prod_{p^m \,||\, q} ((pf(p))^m - (pf(p))^{m-1}).
\end{aligned}$$

□

Lemma 5. *Let* $\alpha(n)$ *be a completely multiplicative function, and let* σ_1 *be as in Lemma 3. The following formula holds*:

$$C_{\chi_0}(s) F(s, \chi_0) = \prod_{i=1}^{r} \left(\frac{p_i^{m_i - 1} \alpha^{m_i - 1}(p_i)}{p_i^{(m_i - 1)s}} (\alpha(p_i) p_i^{1-s} - 1)\right) F(s). \tag{4}$$

Proof. If the series $\sum_n |\alpha(n)|/n^\sigma$ converges for $\sigma > \sigma_1$, then $|\alpha(p)| \le p^{\sigma_1}$ for all prime p. The completely multiplicative function defined by $f(p) = \alpha(p)/p^s$ satisfies the condition of Lemma 4 if $\Re s > \sigma_1$. Along with the facts that $\tau(\chi_0, l) = c_q(l)$ and $F(s, \chi_0) = \prod_{p \mid q}(1 - f(p)/p^s)F(s)$, this lemma follows from Lemma 4 with $f(p) = \alpha(p)/p^s$. □

Lemma 6. *Each character modulo q can be expressed in the form*

$$\chi = \chi_1 \chi_2,$$

where χ_1 is the principal character modulo q_1, χ_2 is a character induced by a primitive character modulo $q_2^{'}$, $q_2^{'} \mid q_2$, $q = q_1 q_2$, $(q_1, q_2) = 1$, and the prime divisors of q_2 and $q_2^{'}$ are the same.

Lemma 6 follows from [Apostol 1976, Theorem 8.18, p. 171].

Lemma 7. *Let $q = q_1 q_2$, $(q_1, q_2) = 1$. Let $\chi = \chi_1 \chi_2$, where χ_1 and χ_2 are characters modulo q_1 and q_2 respectively. Then for each $l \in \mathbb{Z}$ we have $\tau(\chi, l) = \chi_2(q_1)\chi_1(q_2)\tau(\chi_1, l)\tau(\chi_2, l)$.*

Lemma 7 follows from [Montgomery and Vaughan 2007, Theorem 9.6, p. 287].

Lemma 8. *Let χ be a character modulo $q = \prod_i p_i^{l_i}$ induced by a primitive character modulo $q_2^{'}$, and suppose $q_2^{'} \mid q_2$, $q = q_1 q_2$, $(q_1, q_2) = 1$, and the prime divisors of q_2 and $q_2^{'}$ are the same. If there is an i such that $p_i^{l_i}$ divides m then $\tau(\chi, m) = 0$.*

Proof. Let $m = \prod_{i=1}^r p_i^{n_i}$. Let $\chi = \prod_{i=1}^r \chi_i$, where χ_i are characters induced by characters modulo $p_i^{r_i}$. Then by Lemma 7 and that fact that $\tau(\chi, al) = \bar{\chi}(a)\tau(\chi, l)$ (see [Montgomery and Vaughan 2007, Theorem 9.5, p. 287]), we obtain

$$\tau(\chi, lk) = C \prod_{i=1}^{r} \tau(\chi_i, p_i^{n_i}),$$

where $|C| = 1$. Since $p_i^{l_i} \mid m$ we have $n_i \ge l_i \ge r_i$. Thus $\tau(\chi_i, p_i^{n_i}) = 0$. □

Lemma 9. *Let $\Re s > \sigma_1$, where σ_1 is as defined in Lemma 3. Let*

$$\chi = \chi_1 \chi_2,$$

where χ_1 is the principal character modulo q_1, χ_2 is a character induced by a primitive character modulo $q_2^{'}$, $q_2^{'} \mid q_2$, $q = q_1 q_2$, $(q_1, q_2) = 1$, and the prime divisors of q_2 and $q_2^{'}$ are the same. Then

$$C_\chi(s) = \bar{\chi}_2(q_1)\bar{\chi}_2(l) \sum_{k_1 \in K(q_1)} \frac{\tau(\chi_1, k_1)}{k_1^s} \chi_2(k_1)\alpha(k_1) \sum_{k_2 \mid q_2} \frac{\tau(\chi_2, k_2)}{k_2^s} \alpha(k_2). \tag{5}$$

Proof. Using Lemmas 6 and 7 and the fact that $\tau(\chi, al) = \bar{\chi}(a)\tau(\chi, l)$ when $\chi(a) \ne 0$ we derive

$$C_\chi(s) = \sum_{\substack{k_1 \in K(q_1) \\ k_2 \in K(q_2)}} \frac{\tau(\chi_1 \bar{\chi}_2, lk_1k_2)}{k_1^s k_2^s} \alpha(k_1)\alpha(k_2) = \sum_{\substack{k_1 \in K(q_1) \\ k_2 \in K(q_2)}} \frac{\bar{\chi}_2(q_1)\tau(\chi_1, lk_1k_2)\tau(\bar{\chi}_2, lk_1k_2)\alpha(k_1)\alpha(k_2)}{k_1^s k_2^s}$$

$$= \bar{\chi}_2(q_1)\bar{\chi}_2(l) \sum_{k_1 \in K(q_1)} \frac{\tau(\chi_1, k_1)}{k_1^s} \chi_2(k_1)\alpha(k_1) \sum_{k_2 \in K(q_2)} \frac{\tau(\chi_2, k_2)}{k_2^s} \alpha(k_2). \tag{6}$$

By Lemma 8 for $k_2 \in K(q_2)$ we have $\tau(\chi_2, k_2) = 0$ unless $k_2 \mid q_2$. Hence the second sum may be written as a Dirichlet polynomial

$$\sum_{k_2 \mid q_2} \frac{\tau(\chi_2, k_2)}{k_2^s} \alpha(k_2). \qquad \square$$

Lemma 10. *Let*

$$\chi = \chi_1 \chi_2,$$

where χ_1 is the principal character modulo q_1, χ_2 is character induced by a primitive character modulo q_2', $q_2' \mid q_2$, $q = q_1 q_2$, $(q_1, q_2) = 1$, and the prime divisors of q_2 and q_2' are the same. Then

$$C_\chi(s) F(s, \chi) = A_\chi(s) F(s, \chi_2),$$

where $A_\chi(s)$ is an entire function.

Proof. Let $q = \prod_{i=1}^r p_i^{l_i}$. From Lemma 9 we get

$$C_\chi(s) F(s, \chi) = B_\chi(s) \sum_{k_1 \in K(q_1)} \frac{\tau(\chi_1, k_1)}{k_1^s} \chi_2(k_1) \alpha(k_1) \prod_{i=1}^r \left(1 - \frac{\chi_2(p_i)\alpha(p_i)}{p_i^s}\right) F(s, \chi_2),$$

where $B_\chi(s)$ is a Dirichlet polynomial. Hence

$$C_\chi(s) F(s, \chi) = B_\chi(s) \sum_{k_1 \in K(q_1)} \frac{\alpha(k_1)}{k_1^s} \chi_2(k_1) c_{q_1}(k_1) \prod_{i=1}^r \left(1 - \frac{\chi_2(p_i)}{p_i^s} \alpha(p_i)\right) F(s, \chi_2). \tag{7}$$

Using Lemma 4 with $f(p) = \alpha(p)\chi_2(p)/p^s$, we obtain

$$C_\chi(s) F(s, \chi) = B_\chi(s) \prod_{i=1}^r \left(\left(p \frac{\chi_2(p)\alpha(p)}{p^s}\right)^{l_i} - \left(p \frac{\chi_2(p)\alpha(p)}{p^s}\right)^{l_i - 1}\right) F(s, \chi_2) = A_\chi(s) F(s, \chi_2),$$

where $A_\chi(s)$ is a Dirichlet polynomial. $\square$

Lemma 11. *The following equality holds*:

$$F[\beta](s) = \frac{1}{\phi(q)} \sum_{\chi \in X} D_\chi(s) F(s, \chi) + \frac{1}{\phi(q)} D(s) F(s), \tag{8}$$

where X is the set of primitive characters of modulus q_1 with $q_1 \mid q$,

$$D(s) = \prod_{i=1}^r \left(\frac{p_i^{l_i - 1} \alpha^{l_i - 1}(p_i)}{p_i^{(l_i - 1)s}} (\alpha(p_i) p_i^{1-s} - 1)\right), \tag{9}$$

and $D_\chi(s)$ are Dirichlet polynomials just like $D(s)$.

Proof. The proof follows from Lemmas 3, 5, 6 and 10. $\square$

Note that $D(1) = 0$ if and only if $\alpha(p) = 1$ for some $p \mid q$.

The following proposition relates $F[l/q](s)$ to $\mathfrak{A}(e(l/q)r)$ and $A(x, \beta)$.

Proposition 12. *Let $\alpha(n)$ be a sequence such that $|\alpha(n)| \le n^{1+\delta}$. Then for any $\beta \in \mathbb{R}$, for $\Re s > 2+\delta$*

$$\Gamma(s)F[\beta](s) = \int_0^\infty t^{s-1}\mathfrak{A}(e(\beta)e^{-t})\,dt,$$

$$\frac{1}{s}F[\beta](s) = \int_0^1 t^{s-1}A(t^{-1},\beta)\,dt.$$

Proof. From inequality $|\alpha(n)| \le n^{1+\delta}$ we derive $A(x,0) \ll x^{2+\delta}$ and

$$|\mathfrak{A}(e^{-t})| \le \sum_{n=1}^{+\infty} n^{1+\delta}e^{-nt} \le e^t \int_0^{+\infty} u^{1+\delta}e^{-ut}\,dt \le e^t\Gamma(2+\delta)t^{-2-\delta}.$$

Using the Lebesgue dominated convergence theorem with $|\mathfrak{A}(e(\beta)e^{-t})| \le |\mathfrak{A}(e^{-t})|$ we obtain

$$\int_0^\infty t^{s-1}\mathfrak{A}(e(\beta)e^{-t})\,dt = \sum_{n=1}^\infty \alpha(n)e(\beta n)\int_0^\infty t^{s-1}e^{-nt}\,dt = \sum_{n=1}^\infty \alpha(n)e(\beta n)\Gamma(s)n^{-s}.$$

Let $B_m(x) = 1$ if $x < 1/m$ and $B_m(x) = 0$ if $x \ge 1/m$. Then $\sum_m \alpha(m)B_m(x) = A(1/x,0) \ll x^{-2-\delta}$. Using the Lebesgue dominated convergence theorem with $|A(1/t,\beta)| \le |A(1/t,0)|$ we obtain

$$\int_0^1 x^{s-1}\sum_{m=1}^\infty \alpha(m)e(m\beta)B_m(x)\,dx = \sum_{m=1}^\infty \alpha(m)e(m\beta)\int_0^1 x^{s-1}B_m(x)\,dx = \sum_{m=1}^\infty \alpha(m)e(m\beta)\int_0^{1/m} x^{s-1}\,dx$$

$$= \frac{1}{s}\sum_{m=1}^\infty \alpha(m)e(m\beta)m^{-s} = \frac{1}{s}F[\beta](s). \qquad \square$$

3. Growth of some functions

In this section we study growth of some Euler products as $s \to 1$.

Let

$$G(s) = \prod_{\alpha(p)\le p}\left(1-\frac{\alpha(p)}{p^s}\right)^{-1} \quad \text{and} \quad H(s) = \prod_{\alpha(p)>p}\left(1-\frac{\alpha(p)}{p^s}\right)^{-1}.$$

Throughout this section, we assume that condition (2) of Theorem 1 is satisfied, that is, we assume the convergence of the series $\sum_p \alpha(p)/p^\sigma$ for any $\sigma > 1$. It follows that $G(s)$ is an analytic function on $\{\Re s > 1\}$. It also follows that for any $\epsilon > 0$ there are at most finitely many p such that $\alpha(p) > p^{1+\epsilon}$.

Lemma 13. *Let χ be a nonprincipal character modulo q. If the series*

$$\sum_p \alpha(p)\frac{1-\Re\chi(p)}{p^{1+x}}$$

diverges, then the following relation holds:

$$G(1+x,\chi) = o(G(1+x)) \quad \text{as } x \to 0+. \tag{10}$$

Proof. The condition (10) is equivalent to

$$\ln|G(1+x)| - \ln|G(1+x,\chi)| \to +\infty$$

as $x \to 0+$. Let $x > 0$. We have

$$\ln|G(1+x)| - \ln|G(1+x,\chi)| = -\sum_{\alpha(p)\le p} \ln(1-\alpha(p)p^{-1-x}) + \sum_{\alpha(p)\le p} \ln|1-\alpha(p)\chi(p)p^{-1-x}|$$
$$= \sum_{\alpha(p)\le p} -\ln(1-\alpha(p)p^{-1-x}) + \sum_{\alpha(p)\le p} \Re\ln(1-\alpha(p)\chi(p)p^{-1-x}),$$

where ln is the principal value of the logarithm.

Using the power series expression of $-\ln(1-z)$ on the unit disk we have

$$\ln|G(1+x)| = -\sum_{\alpha(p)\le p} \ln(1-\alpha(p)p^{-1-x}) = \sum_{\alpha(p)\le p}\sum_{k\ge 1} \frac{1}{k}\alpha(p)^k p^{-k(1+x)}.$$

Notice that the summands are nonnegative and the double sum converges. Similarly

$$\ln|G(1+x,\chi)| = -\sum_{\alpha(p)\le p} \Re\ln(1-\alpha(p)p^{-1-x}\chi(p)) = \sum_{\alpha(p)\le p}\sum_{k\ge 1} \Re\frac{1}{k}(\chi^k(p)\alpha(p)^k p^{-k(1+x)}),$$

where the double sum is absolutely convergent (using the previous double sum). Therefore

$$\ln|G(1+x)| - \ln|G(1+x,\chi)| = \sum_{\alpha(p)\le p}\sum_{k>1} \frac{1}{k}(1-\Re\chi^k(p))\alpha(p)^k p^{-k(1+x)} \ge \sum_{\alpha(p)\le p} (1-\Re\chi(p))\alpha(p)p^{-(1+x)}$$

since all summands are nonnegative. If $\sum_p \alpha(p)(1-\Re\chi(p))/p^{1+x}$ is bounded as x tends to 0 then it has a limit and by the Tauberian theorem (see [Hardy and Littlewood 1914, Theorem 17]) the sum $\sum_p \alpha(p)(1-\Re\chi(p))/p$ is convergent, which contradicts our assumption. Therefore we have

$$\Re\sum_p \frac{\alpha(p)}{p^{1+x}}(1-\chi(p)) \to +\infty$$

for $x \to 0+$. □

Lemma 14. *Let assumption* (2) *of Theorem 1 hold. Then*

$$|F(1+x,\chi)| \le \prod_{p\mid q} |1-\alpha(p)p^{-1-x}|\,|F(1+x)|,$$

where q is the modulus of χ.

Proof. For any $t \in [0,+\infty)$ and $z \in \mathbb{C}$ with $|z| = 1$, we have $|t-1| \le |t-z|$(since 1 is the point of the unit circle closest to any given point of $[0,\infty)$) or again $|t-1| \le |tz-1|$. For p with $\alpha(p) \ge 0$ and $\chi(p) \ne 0$ we have $|1-\alpha(p)\chi(p)p^{-1-x}|^{-1} \le |1-\alpha(p)p^{-1-x}|^{-1}$. Hence we deduce

$$|F(1+x,\chi)| = \prod_{p\nmid q} |1-\alpha(p)\chi(p)p^{-1-x}|^{-1}$$
$$\le \prod_{p\nmid q} |1-\alpha(p)p^{-1-x}|^{-1} = \prod_{p\mid q} |1-\alpha(p)p^{-1-x}|\,|F(1+x)|.$$ □

Let f be locally integrable on $(0, +\infty)$. The Mellin transform of f is defined by the integral $f^*(s) = \int_0^\infty x^{s-1} f(x)\,dx$. The fundamental strip is the largest open strip on which it is defined. In particular, if f satisfies the asymptotic conditions $f(x) = O(x^{-\sigma})$ as $x \to 0+$ for some $\sigma \in \mathbb{R}$ and $f(x) = o(x^{-N})$ as $x \to +\infty$ for any $N > 0$, then the integral defining $f^*(s)$ is convergent for any $s \in \mathbb{C}$ such that $\Re s > \sigma$, and the Mellin transform f^* is an analytic function over $\{\Re s > \sigma\}$.

Proposition 15. *Let f be a locally integrable function on $(0, +\infty)$ such that its Mellin transform f^* is analytic on $\{\Re s > \sigma_0\}$ with $\sigma_0 > 0$. Let $t_0 \in \mathbb{R}$. If $\overline{\lim}_{\sigma \to \sigma_0+} |f^*(\sigma + it_0)| = +\infty$, then for any $b > 0$, we have*

$$\overline{\lim_{x \to 0+}} \frac{|f(x)|}{x^{-\sigma_0} |\ln x|^{b-1}} = +\infty.$$

Moreover, if there are $b > 0$ and $c > 0$ such that

$$\overline{\lim_{\sigma \to \sigma_0+}} \frac{|f^*(\sigma + it_0)|}{(\sigma - \sigma_0)^b} \geq c,$$

then

$$\overline{\lim_{x \to 0+}} \frac{|f(x)|}{x^{-\sigma_0} |\ln x|^{b-1}} \geq \frac{c}{\Gamma(b)}.$$

Proof. Since f admits a Mellin transform, there exists $\sigma_1 > \sigma_0$ such that $\int_0^{+\infty} u^{\sigma_1 - 1} |f(u)|\,du$ is convergent. Therefore, for any s such that $\Re s < \sigma_1$, we have

$$\left| \int_1^{+\infty} u^{s-1} f(u)\,du \right| \leq \int_1^{+\infty} u^{\sigma_1 - 1} |f(u)|\,du \leq |f|^*(\sigma_1).$$

Assume now there are constants $b \in \mathbb{R}$, $c > 0$, and $u_0 \in (0, 1)$ such that $|f(u)| \leq c' u^{-\sigma_0} |\ln u|^{b-1}$ for any $u \leq u_0$. In that case, for any s satisfying $\Re s > \sigma_0$, the function $u^{s-1} f(u)$ is integrable on $(0, 1)$ and

$$\left| \int_0^1 u^{s-1} f(u)\,du \right| \leq c' \int_0^{u_0} u^{\sigma - \sigma_0 - 1} |\ln u|^{b-1}\,du + \int_{u_0}^1 u^{\sigma_0 - 1} |f(u)|\,du.$$

We deduce that for any s such that $\sigma_0 < \Re s \leq \sigma_1$, we have

$$|f^*(s)| \leq c' \int_0^{u_0} u^{\sigma - \sigma_0} |\ln u|^{b-1}\,du + C,$$

where C is some constant independent of s.

If we can choose $b < 0$, then the integral $\int_0^{u_0} u^{\sigma - \sigma_0 - 1} |\ln u|^{b-1}\,du$ is bounded by the convergent integral $\int_0^{u_0} u^{-1} |\ln u|^{b-1}\,du$ which does not depend on s. We conclude that $\overline{\lim}_{\sigma \to \sigma_0+} |f^*(\sigma + it_0)| < +\infty$ for any t_0. This proves the first case.

If $b > 0$, then

$$\int_0^1 u^{(\sigma - \sigma_0) - 1} |\ln u|^{b-1}\,du = \int_0^{+\infty} e^{-(\sigma - \sigma_0)u} u^{b-1}\,du = \Gamma(b)(\sigma - \sigma_0)^b.$$

We conclude that $|f^*(s)| \leq c' \Gamma(b)(\sigma - \sigma_0)^b + C$ and consequently

$$\overline{\lim_{\sigma \to \sigma_0+}} \frac{|f^*(\sigma + it_0)|}{(\sigma - \sigma_0)^{-b}} \leq c' \Gamma(b).$$

Hence, if this limit superior is larger than c, it contradicts the previous conclusion for any $c' < c/\Gamma(b)$ and the assumption $|f(u)| \le c' u^{\sigma_0} |\ln u|^{b-1}$ has to be contradicted in any neighborhood of 0 and for any $c' < c/\Gamma(b)$. This implies

$$\varlimsup_{x\to 0+} \frac{|f(x)|}{x^{-\sigma_0}|\ln x|^{b-1}} \ge \frac{c}{\Gamma(b)}. \qquad \square$$

Proposition 16. *Let $\alpha(n)$ be a sequence satisfying $|\alpha(n)| \le n^{1+\delta}$. Let $\beta \in \mathbb{R}$ and $\sigma_0 > 0$ such that the Dirichlet series $F[\beta](s) = \sum_n e(n\beta)\alpha(n)n^{-s}$ is analytic on $\{\Re s > \sigma_0\}$. If*

$$\varlimsup_{\sigma\to\sigma_0+} |F[\beta](\sigma + it_0)| = +\infty,$$

then for any $b > 0$

$$\mathfrak{A}(e(\beta)r) = \Omega((1-r)^{-\sigma_0}|\ln(1-r)|^{b-1}),$$
$$A(x,\beta) = \Omega(x^{\sigma_0}(\ln x)^{b-1}).$$

Let

$$\varlimsup_{\sigma\to\sigma_0+} \frac{|F[l/q](\sigma + it_0)|}{(\sigma-\sigma_0)^{-b}} \ge C.$$

Then

$$\mathfrak{A}(e(\beta)r) = \Omega((1-r)^{-\sigma_0}|\ln(1-r)|^{b-1}),$$
$$A(x,\beta) = \Omega(x^{\sigma_0}(\ln x)^{b-1}).$$

Proof. The assumptions of Proposition 12 are satisfied. Therefore $\Gamma(s)F[\beta](s)$ and $(1/s)F[\beta](s)$ are the Mellin transforms of $\mathfrak{A}(e(\beta)e^{-t})$ and $A(t^{-1},\beta)\mathbf{1}_{(0,1)}(t)$ and they satisfy the assumptions of Proposition 15.

By change of variable $r = e^{-u}$ we have

$$\varlimsup_{u\to 0+} \frac{\mathfrak{A}(e(\beta)e^{-u})}{u^{-\sigma_0}|\ln u|^{b-1}} = \varlimsup_{r\to 1-} \frac{\mathfrak{A}(e(\beta)r)}{(1-r)^{-\sigma_0}|\ln(1-r)|^{b-1}},$$

and by change of variable $y = x^{-1}$ we have

$$\varlimsup_{x\to 0+} \frac{|A(x^{-1},\beta)|}{x^{-\sigma_0}|\ln x|^{b-1}} = \varlimsup_{y\to+\infty} \frac{|A(y,\beta)|}{y^{\sigma_0}(\ln y)^{b-1}}.$$

Consider the first case. We see by Proposition 12

$$f^*(s) = \Gamma(s)F\left[\frac{l}{q}\right](s) = \int_0^\infty t^{s-1}\mathfrak{A}\left(e\left(\frac{l}{q}\right)e^{-t}\right)dt,$$
$$\varlimsup_{\sigma\to\sigma_0+} |f^*(\sigma + it_0)| = +\infty.$$

Thus by Proposition 15

$$\varlimsup_{u\to 0+} \frac{\mathfrak{A}(e(l/q)e^{-u})}{u^{-\delta}|\ln u|^{b-1}} = +\infty$$

for each $b > 0$.

Similarly

$$f^*(s) = \int_0^\infty x^{-s-1} A(x^{-1}, \beta)\, dx = \frac{1}{s} F\left[\frac{l}{q}\right](s),$$
$$\overline{\lim_{\sigma \to \sigma_0+}} |f^*(\sigma + it_0)| = +\infty.$$

Thus by Proposition 15

$$\overline{\lim_{x \to 0+}} \frac{|A(x^{-1}, \beta)|}{x^{-\sigma_0} |\ln x|^{b-1}} = +\infty \quad \text{and} \quad \overline{\lim_{y \to +\infty}} \frac{|A(y, \beta)|}{y^{\sigma_0} (\ln y)^{b-1}} = +\infty$$

for each $b > 0$.

Consider the second case. Let

$$\overline{\lim_{\sigma \to \sigma_0+}} \frac{|F[l/q](\sigma + t_0)|}{(\sigma - \sigma_0)^{-b}} \geq c.$$

Thus by Proposition 15

$$\overline{\lim_{u \to 0+}} \frac{|\mathfrak{A}(e(l/q)e^{-u})|}{u^{-\delta} |\ln u|^{-1+b}} \geq \frac{c}{\Gamma(b)}.$$

Further

$$\overline{\lim_{\sigma \to \sigma_0+}} \frac{(\sigma + it_0) F(\sigma + it_0)}{(\sigma - \sigma_0)^{-b}} \geq c \frac{1}{|\sigma_0 + it_0|}.$$

Hence by Proposition 15

$$\overline{\lim_{x \to 0+}} \frac{A(x^{-1}, \beta)}{x^{-\sigma_0} |\ln x|^{-1+b}} \geq \frac{c}{|\sigma_0 + it_0| \Gamma(b)} \quad \text{and} \quad \overline{\lim_{y \to +\infty}} \frac{|A(y, \beta|)}{y^{\sigma_0} (\ln y)^{-1+b}} \geq \frac{c}{|\sigma_0 + it_0| \Gamma(b)}. \qquad \square$$

By Lemma 11

$$F[\beta](s) = \frac{1}{\phi(q)} \sum_{\chi \in X} D_\chi(s) F(s, \chi) + \frac{1}{\phi(q)} D(s) F(s),$$

where X is the set of primitive characters of modulus q_1 with $q_1 \mid q$, $D_\chi(s)$ are entire functions, $D(s)$ is defined in (9).

Proposition 17. *Let $\alpha(n)$ be a positive completely multiplicative function satisfying conditions* (1) *and* (2) *of Theorem 1. Let $\beta = l/q$ with $(l, q) = 1$.*

If $\alpha(p) \leq p$ for all primes p, and $\alpha(p) \neq 1$ for all primes p dividing q, then

$$F[\beta](1 + x) \sim \frac{D(1)}{\phi(q)} F(1 + x) \quad \text{as } x \to 0+.$$

If there exists $\delta > 0$ such that $\alpha(p) = p^{1+\delta}$ for every prime factor p of q, then

$$F[\beta](1 + x) \sim F(1 + x) \quad \text{as } x \to \delta.$$

Proof. Consider the first case. Since $D_\chi(s)$ is entire we see

$$|D_\chi(1 + x) F(1 + x, \chi)| \ll |F(1 + x, \chi)|$$

as $x \to 0+$. Further by Lemma 13 we obtain (in our case $F = G$)

$$|D_\chi(1+x)F(1+x,\chi)| \ll |F(1+x,\chi)| = o(F(1+x))$$

as $x \to 0+$. Hence for each $\chi \neq \chi_0 \pmod q$ we have $D_\chi(1+x)F(1+x,\chi) = o(F(1+x))$ as $x \to 0+$. Thus

$$\frac{1}{\phi(q)} \sum_{\chi \in X} D_\chi(1+x)F(1+x,\chi) = o(F(1+x)) \quad \text{as } x \to 0+.$$

It follows that

$$F[\beta](1+x) \sim \frac{1}{\phi(q)} D(1+x)F(1+x) \quad \text{as } x \to 0+.$$

From the expression (9) for $D(s)$, we get

$$D(1) = \prod_{p^m \,||\, q} \alpha(p^{m-1})(\alpha(p) - 1).$$

Since $\alpha(p) \neq 1$ for any p dividing q, we have $D(1) \neq 0$ and

$$F[\beta](1+x) \sim \frac{1}{\phi(q)} D(1)F(1+x) \quad \text{as } x \to 0+. \tag{11}$$

Consider the second case. Since $D_\chi(s)$ are Dirichlet polynomials we have

$$D_\chi(s)F(1+x,\chi) \ll |F(1+x,\chi)| \quad \text{as } x \to \delta$$

for any character $\chi \in X$. By Lemma 14, we have

$$|F(1+x,\chi)| \le \prod_{p \,|\, q_0} \left|1 - \frac{\alpha(p)}{p^{1+x}}\right| |F(1+x)|,$$

where q_0 is the modulus of χ. Since $\alpha(p) = p^{1+\delta}$ for any prime p dividing q_0, the product tends to 0 as $x \to \delta$ and

$$F(1+x,\chi) = o(|F(1+x)|) \quad \text{as } x \to \delta.$$

Again we derive that

$$F[\beta](1+x) \sim \frac{1}{\phi(q)} D(1+x)F(1+x) \quad \text{as } x \to \delta.$$

From the expression (9) for $D(s)$, we get

$$D(1+\delta) = \prod_{p^m \,||\, q} \alpha(p^{m-1}) p^{-\delta(m-1)} (\alpha(p) p^{-\delta} - 1)$$

and since $\alpha(p) = p^{1+\delta}$ for every $p \,|\, q$, we have $D(1+\delta) = \phi(q)$. Therefore

$$F[\beta](1+x) \sim F(1+x) \quad \text{as } x \to \delta. \qquad \square$$

4. Proof of the theorems

Proof of Theorem 2. Assume first that assumptions (1) and (2) of Theorem 1 and assumption (3.1) of Theorem 2 are satisfied. Under these conditions, we have established that $F[\beta](s)$ is analytic on $\{\Re s > 1\}$ and $\sum_p \alpha(p)p^{1+x} \to +\infty$ as $x \to 0+$. We derive that

$$\ln F(1+x) = -\sum_p \ln\left(1 - \frac{\alpha(p)}{p^{1+x}}\right) \geq \sum_p \frac{\alpha(p)}{p^{1+x}} \to +\infty$$

as $x \to 0+$. By Proposition 17 we obtain $F[\beta](1+x) \to \infty$. By Proposition 16 with $\sigma_0 = 1$ we obtain the Ω-estimates.

Assume now that assumptions (1) and (2) of Theorem 1 and assumption (3.2) are satisfied. Under these conditions, we have established that $F[\beta](s)$ is analytic on $\{\Re s > 1\}$ and $\sum_p \alpha(p)/p^{1+x} \to +\infty$ as $x \to 0+$. We choose a set P_0 of $m \geq 0$ primes p satisfying $\alpha(p) = p$. The Euler product $F_0(s) = \prod_{p \notin P_0}(1 - \alpha(p)p^{-s})^{-1}$ is convergent for any s with $\Re s > 1$ and we have

$$\ln F_0(1+x) = -\sum_{p \in P_0} \ln\left(1 - \frac{\alpha(p)}{p^{1+x}}\right) \geq \sum_{p \in P_0} \frac{\alpha(p)}{p^{1+x}} \geq -\epsilon \ln x + c - m$$

as $x \to 0+$. On the other hand, for any $p \in P_0$ we have

$$\left(1 - \frac{\alpha(p)}{p^{1+x}}\right)^{-1} = (1 - e^{-x \ln p})^{-1} \geq \frac{1}{x \ln p}.$$

We derive that

$$F(1+x) \geq \frac{e^{c-m}}{\prod_{p \in P_0} \ln p} x^{-(\epsilon+m)}$$

when x is sufficiently close to 0. By Proposition 17 we obtain

$$F[\beta](1+x) \geq C \frac{e^{c-m}}{\prod_{p \in P_0} \ln p} x^{-(\epsilon+m)} \to \infty.$$

By Proposition 16 with $\sigma_0 = 1$ we obtain the Ω-estimates.

Assume now that assumptions (1) and (2) of Theorem 1 and assumption (3.3) of Theorem 2 are satisfied. Under these conditions, we have established that $F[\beta](s)$ is meromorphic over $\Re s > 1$, and that for any $\epsilon > 0$ there are finitely many primes p such that $\alpha(p) = p^{1+\epsilon}$ is finite and nonempty. We can set m as its cardinal. We deduce also that there exists $\delta_0 < \delta$ such that the Euler product $F_0(s) = \prod_{p \in P_0}(1 - \alpha(p)p^{-s})^{-1}$ converges for any s with $\Re s > \delta_0$. In particular, $F_0(1+\delta)$ is well-defined. On the other hand, for any $p \in P_0$ we have

$$\left(1 - \frac{\alpha(p)}{p^{1+x}}\right)^{-1} = (1 - e^{(x-\delta)\ln p})^{-1} \sim \frac{1}{(x-\delta)\ln p}$$

as $x \to \delta$. We derive

$$F(1+x) \sim \frac{F_0(1+\delta)}{\prod_{p \in P_0} \ln p} (x-\delta)^{-m}$$

as $x \to \delta$. By Proposition 17 we obtain

$$F[\beta](1+x) \sim C(x-\delta)^{-m} \to \infty$$

as $x \to \delta$. By Proposition 16 with $\sigma_0 = 1+\delta$ we obtain the Ω-estimates. □

Proof of Theorem 1. Using Theorem 2 it has been proven that the point $e(l/p^m)$ is a singular point of $\mathfrak{A}(z)$ for l with $(l, p) = 1$ and $m \geq 0$, if p is chosen such that $\alpha(p) \neq 1$ if $\delta = 0$ or $\alpha(p) = p^{1+\delta}$ if $\delta > 0$. This provides a dense set of singular points on the unit circle, unless $\alpha(p) = 1$ for any p. □

Now we can prove the generalization of Theorem 1.

Theorem 18. *Let $\alpha(n)$ be a positive completely multiplicative function and $\sigma_0 \in \mathbb{R}$ such that*:

(1) *For any nonprincipal Dirichlet character χ the following series diverges*:

$$\sum_p \alpha(p) \frac{1 - \Re\chi(p)}{p^{\sigma_0}}.$$

(2) *For any $\sigma > \sigma_0$, the following series converges*:

$$\sum_p \frac{\alpha(p)}{p^{\sigma}}.$$

If the series $\sum_{n=1}^{\infty} \alpha(n) z^n$ has a nonsingular point on the unit circle, then $\alpha(n) = n^{\sigma_0 - 1}$.

Note that the series $\sum_{n=1}^{\infty} n^s z^n$ is the polylogarithm of order s which admits an analytical extension beyond the unit circle for any $s \in \mathbb{C}$.

Proof. We can reduce to the case where $\sigma_0 > 0$: If the series $\sum_{n=1}^{\infty} \alpha(n) z^n$ has a nonsingular point on the unit circle, that is, an analytical extension beyond the unit circle, its derivative has the same radius of convergence and the same domain of analyticity. Therefore we can apply the theorem to $\sum_{n=1}^{+\infty} \alpha'(n) z^n$ with $\alpha'(n) = n\alpha(n)$. The function $\alpha'(n)$ satisfies the same assumptions as $\alpha(n)$ except with the abscissa $\sigma_0' = \sigma_0 + 1$ instead of σ_0. Therefore, using the homogeneity differential operator sufficiently many times, we can consider the function $n^k \alpha(n)$ with the abscissa $\sigma_0 + k$, where $k > -\sigma_0$. Applying the theorem for the abscissa $\sigma_0 + k > 0$, we deduce that $n^k \alpha(n) = n^{\sigma_0 + k - 1}$ for all n, that is, $\alpha(n) = n^{\sigma_0 - 1}$.

If $\sigma_0 > 0$, we can instead study $\tilde{\alpha}(n) = n^{\sigma_0 - 1} \alpha(n)$. The associated Dirichlet series $\widetilde{F}[\beta](s)$ satisfies $\widetilde{F}[\beta](s) = F[\beta](s + \sigma_0 - 1)$. If $\alpha(n)$ satisfies the assumptions of Theorem 18, then $\tilde{\alpha}(n)$ satisfies those of Theorem 1. We can deduce that there is a dense set of β such that $\widetilde{F}[\beta](s)$ is analytical over $\Re s > \sigma_0$ and with $F[\beta](1+x) \to \infty$ as $x \to 0+$. By translation, we derive that for the same β, $F[\beta](s)$ is analytical over $\{\Re s > \sigma_0\}$ and with $F[\beta](\sigma_0 + x) \to +\infty$ as $x \to 0+$. By Proposition 16, $e(\beta)$ is a singular point of $\sum_{n=1}^{\infty} \alpha(n) z^n$ for a dense set of β, unless $\tilde{\alpha}(n) = 1$ for all n, that is, unless $\alpha(n) = n^{\sigma_0 - 1}$. □

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