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# Square-full primitive roots in arithmetic progressions

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An asymptotic estimate for the number of positive primitive roots which are square-full integers in arithmetic progressions is derived. The employed method combines two techniques and is based on the character-sum method involving two characters; one character is to take care of being a primitive root, based on a result of Shapiro, and the other character is to take care of being square-full, based on a result of Munsch.

## 1. Introduction

An integer $n > 1$ is called square-full if in its canonical prime representation each prime appears with exponent $\geq 2$. The integer 1 is square-full by convention. For a positive real number $x$, let $Q_2(x)$ denote the number of square-full integers that are $\leq x$. The oldest known work related to $Q_2(x)$ is due to Erdős and Szekeres [1934], who proved that

$$Q_2(x) = \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} x^{\frac{1}{2}} + O(x^{\frac{1}{3}}).$$

This was later refined by Bateman and Grosswald [1958], who replaced the error term by

$$\frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)} x^{\frac{1}{3}} + O(x^{\frac{1}{6}} \exp(-C(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}))$$

for some absolute constant $C > 0$. There have been many works on the improvement of the error term; see, e.g., [Cai 1997; Cao 1994; 1997; Liu 1994; Suryanarayana and Sitaramachandra Rao 1973; Wu 1998; 2001]. Regarding square-full integers in an arithmetic progression, Liu and Zhang [2013] used Perron's formula and properties of the Dirichlet $L$-functions to study the character sums over square-full numbers and gave an asymptotic formula for $Q_2(x : \ell, q)$, the number of square-full integers which are congruent to $\ell$ modulo an integer $q$ and not exceeding $x$. One year later, Munsch [2014] applied the Pólya–Vinogradov inequality to bound the character sums over square-full integers and improved the results obtained by Liu and Zhang by showing that for all $\varepsilon > 0$ we have

$$Q_2(x; \ell, q) = \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} \frac{A_{\ell,q}}{q} x^{\frac{1}{2}} + \frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)} \frac{B_{\ell,q}}{q} x^{\frac{1}{3}} + O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}} q^{\frac{11}{32}+\varepsilon}), \tag{1-1}$$

where $A_{\ell,q}$, $B_{\ell,q}$ are constants depending on certain $L$-functions. Chan and Tsang [2013] used the Dirichlet hyperbola method and Burgess bound on character sums to study this problem. Later, Chan [2015] improved their results. Character sums over square-full integers are also prominent in [Liu and

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Zhang 2013; Munsch 2014]. The second author [Srichan 2013] used the exponent-pair method to study the appearance of square-full integers in an arithmetic progression and showed that the right-hand side of (1-1) can be strengthened to

$$\frac{L(\frac{3}{2},\chi_0)+\sum_{\chi_1}\bar{\chi}_1(\ell)L(\frac{3}{2},\chi_1)}{qL(3,\chi_0)}x^{\frac{1}{2}}+\frac{L(\frac{2}{3},\chi_0)+\sum_{\chi_2}\bar{\chi}_2(\ell)L(\frac{2}{3},\chi_2)}{qL(2,\chi_0)}x^{\frac{1}{3}}+O(q^{\frac{1}{2}+\varepsilon}x^{\frac{1}{6}}), \tag{1-2}$$

where $\chi_0$, $\chi_1$ and $\chi_2$ denote the principal, quadratic and cubic characters modulo $q$, respectively. Character sums over square-full integers play a significant role in the proof of (1-2).

For relatively prime integers $a, m$ with $m \geq 1$, if the smallest positive integer $f$ such that $a^f \equiv 1 \bmod m$ satisfies $f = \phi(m)$ (the Euler totient), then $a$ is called a primitive root mod $m$. Shapiro [1983, Sections 8.5–8.6] also used the character-sum method to obtain the following estimates related to the number of primitive roots modulo an odd prime $p$:

- The number of positive primitive roots mod $p$ that are $\leq x$ is

$$\frac{\phi(p-1)}{p-1}(x+O(2^{\omega(p-1)}p^{\frac{1}{2}}\log p)),$$

where $\omega(n)$ denotes the number of distinct prime factors of $n$.

- For integers $k > 0$, $\ell$ with $\gcd(p,k) = 1$, the number of positive primitive roots mod $p$ that are $\leq x$ and $\equiv \ell \bmod k$ is

$$\frac{\phi(p-1)}{p-1}\left(\frac{x}{k}+O(2^{\omega(p-1)}p^{\frac{1}{2}}\log p)\right).$$

- The number of positive primitive roots mod $p$ that are $\leq x$ which are square-full is

$$\frac{\phi(p-1)}{p-1}(cx^{\frac{1}{2}}+O(x^{\frac{1}{3}}2^{\omega(p-1)})p^{\frac{1}{6}}(\log p)^{\frac{1}{3}}), \tag{1-3}$$

where $c$ is a constant.

Liu and Zhang [2005], using Perron's formula, improved the error term in (1-3) to $O(x^{1/4+\varepsilon}p^{9/44+\varepsilon})$. Munsch and Trudgian [2018] improved this result by showing that (1-3) can be replaced by

$$\frac{\phi(p-1)}{p-1}\left(\left(1+\frac{1}{p}+\frac{1}{p^2}\right)^{-1}\frac{C_p x^{\frac{1}{2}}}{\zeta(3)}+O(x^{\frac{1}{3}}(\log x)p^{\frac{1}{9}}(\log p)^{\frac{1}{6}}2^{\omega(p-1)})\right), \tag{1-4}$$

where $C_p \gg p^{-1/(8\sqrt{e})}$. Recently, Srichan [2020] used the exponent-pair method and the lemmas used in the proof of Theorem 2.1 in [Srichan 2013] to further refine the estimate (1-4) with the following result: for a given odd prime $p \leq x^{1/5}$, the number of square-full integers which are primitive roots mod $p$ and $\leq x$ is equal to

$$\frac{\phi(p-1)}{p}\left\{\left(\frac{L(\frac{3}{2},\chi_0)-L(\frac{3}{2},\chi_1)}{L(3,\chi_0)}\right)x^{\frac{1}{2}}+\left(\frac{L(\frac{2}{3},\chi_0)-L(\frac{2}{3},\chi_2^2)}{L(2,\chi_0)}\right)x^{\frac{1}{3}}\right\} \\ +O(x^{\frac{1}{6}}\phi(p-1)3^{\omega_{1,3}(p-1)}p^{\frac{1}{2}+\varepsilon}). \tag{1-5}$$

Here, $\chi_0$, $\chi_1 \neq \chi_0$, and $\chi_2 \neq \chi_0$ denote, respectively, the principal, quadratic and cubic characters mod $p$, and $\omega_{1,3}(n)$ denotes the number of distinct primes $\equiv 1 \pmod 3$ which are divisors of $n$.

In the present work, we derive an asymptotic estimate for the number of primitive roots mod $p$ which are square-full in an arithmetic progression. Shapiro [1983] was first to give an asymptotic formula for the number of primitive roots mod $p$ in an arithmetic progression by showing for given integers $k > 0$, $\ell$, and prime $p$ with $\gcd(p,k) = 1$, the number of positive primitive roots modulo $p$ that are congruent to $\ell$ mod $k$ and not exceeding $x$ is equal to

$$\frac{\phi(p-1)}{p-1}\left(\frac{x}{k} + O(2^{\omega(p-1)} p^{\frac{1}{2}} \log p)\right).$$

Throughout, let $\varepsilon$ be a fixed sufficiently small positive constant, let $\phi(n)$ be Euler's totient, let $\mu(n)$ be the Möbius function, and for a given odd prime $p$ let

$$T_2(n) = \begin{cases} 1 & \text{if } n \text{ is a square-full primitive root mod } p, \\ 0 & \text{otherwise} \end{cases} \tag{1-6}$$

be the characteristic function of the primitive roots modulo $p$ which are square-full integers. Our main result is:

**Theorem 1.1.** *Given an integer $q \geq 2$, an integer $0 < \ell < q$ with* $\gcd(\ell, q) = 1$, *and a given odd prime $p$ such that $p \nmid q$, we have*

$$\sum_{\substack{n \leq x \\ n \equiv \ell \bmod q}} T_2(n) = \frac{\phi(p-1)}{\phi(p)\phi(q)}\left(A_{p,q} x^{\frac{1}{2}} + B_{p,q} x^{\frac{1}{3}} + O((pq)^{\frac{11}{32}+\varepsilon} x^{\frac{1}{4}} (\log x)^{\frac{1}{2}} 2^{\omega(p-1)} \phi(q))\right),$$

*where*

$$A_{p,q} = \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} \prod_{p_1 \mid pq} \left(\frac{1 - p_1^{-1}}{1 + p_1^{-\frac{3}{2}}}\right) - \sum_{\lambda \in Y_1} \frac{L\left(\frac{3}{2}, \lambda\right)}{\zeta(3)} \left(1 + \frac{1}{p} + \frac{1}{p^2}\right)^{-1} H_q\left(\tfrac{1}{2}, \lambda\right) \tag{1-7}$$

$$+ \sum_{\chi \in X_1} \bar{\chi}(\ell) \frac{L\left(\frac{3}{2}, \chi\right)}{\zeta(3)} \prod_{p_1 \mid q} \left(1 + \frac{1}{p_1} + \frac{1}{p_1^2}\right)^{-1} F_p\left(\tfrac{1}{2}, \chi\right) \tag{1-8}$$

$$- \sum_{\substack{\chi \in X_1 \\ \lambda \in Y_1}} \bar{\chi}(\ell) \frac{L\left(\frac{3}{2}, \chi\lambda\right)}{\zeta(3)} \prod_{p_1 \mid pq} (1 - p_1^{-3})^{-1} \frac{\phi(pq)}{pq}, \tag{1-9}$$

$$B_{p,q} = \frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)} \prod_{p_1 \mid pq} \left(\frac{1 - p_1^{-\frac{2}{3}}}{1 + p_1^{-1}}\right) - \frac{1}{2} \sum_{\lambda \in Y_2} \frac{L\left(\frac{2}{3}, \lambda^2\right)}{\zeta(2)} \left(1 + \frac{1}{p}\right)^{-1} H_q\left(\tfrac{1}{3}, \lambda\right) \tag{1-10}$$

$$+ \sum_{\chi \in X_2} \bar{\chi}(\ell) \frac{L\left(\frac{2}{3}, \chi^2\right)}{\zeta(2)} \prod_{p_1 \mid q} \left(1 + \frac{1}{p_1}\right)^{-1} F_p\left(\tfrac{1}{3}, \chi\right) \tag{1-11}$$

$$- \frac{1}{2} \sum_{\substack{\chi \in X_2 \\ \lambda \in Y_2}} \bar{\chi}(\ell) \frac{L\left(\frac{2}{3}, \chi^2\lambda^2\right)}{\zeta(2)} \prod_{p_1 \mid pq} (1 + p_1^{-1}), \tag{1-12}$$

*with*

$$H_q(s, \lambda) = \prod_{p_1 \mid q} \frac{1 - \lambda^2(p_1) p_1^{-2s}}{1 + \lambda^3(p_1) p_1^{-3s}}, \qquad F_p(s, \chi) = \frac{1 - \chi^2(p) p^{-2s}}{1 + \chi^3(p) p^{-3s}},$$

*the products being over primes $p_1$. Here, $X_1$, $X_2$ denote the set of quadratic, respectively, cubic characters mod $q$, while $Y_1$, $Y_2$ denote the set of quadratic, respectively, cubic characters mod $p$.*

Our approach is based mostly on that of [Munsch 2014]. However, in contrast to the analysis in that paper, which deals with character sums involving one character, here we work with character sums involving two characters; one to take care of square-full integers (the approach in [Munsch 2014]) and the other to take care of primitive roots (the approach in [Shapiro 1983]). This leads inevitably to handling many more subcases. The subcases with contributions towards the main terms arise from one of the characters being principal and are presented with detailed proofs, while proofs for those subcases with contributions only towards the error terms are given more tersely.

## 2. Lemmas

We collect in this section, several auxiliary results used in the proof of the theorem.

**Lemma 2.1** [Shapiro 1983, Lemma 8.5.1]. *For a given odd prime $p$, the characteristic function indicating if $n$ is a primitive root mod $p$ satisfies*

$$\frac{\phi(p-1)}{p-1}\sum_{d\mid p-1}\frac{\mu(d)}{\phi(d)}\sum_{\lambda\in\Gamma_d}\lambda(n)=\begin{cases}1 & \text{if } n \text{ is a primitive root mod } p,\\ 0 & \text{otherwise,}\end{cases}$$

*where $\Gamma_d$ denotes the set of characters of the character group mod $p$ that are of order $d$.*

The next lemma gives special cases of the well-known as Pólya–Vinogradov inequality, taken from [Iwaniec and Kowalski 2004, Theorem 12.5, p. 324].

**Lemma 2.2.** *For a real $x \ge 1$, an integer $q \ge 2$, and a Dirichlet character $\chi$, let*

$$S_\chi(x)=\sum_{1\le n\le x}\chi(n).$$

*For any nonprincipal character $\chi$ mod $q$, we have:*

(I) [Iwaniec and Kowalski 2004, Theorem 12.5, p. 324]

$$|S_\chi(x)|\le 6\sqrt{q}\log q.$$

(II) [Burgess 1962; Iwaniec and Kowalski 2004, Theorem 12.6, p. 326]

$$|S_\chi(x)|\ll x^{\frac{1}{2}}q^{\frac{3}{16}+\varepsilon}. \tag{2-1}$$

(III) [Iwaniec and Kowalski 2004, equation (12.58)] *If $q$ is prime, then*

$$|S_\chi(x)|\ll x^{\frac{1}{2}}q^{\frac{3}{16}}(\log q)^{\frac{1}{2}}. \tag{2-2}$$

For two characters $\chi$ mod $q$ and $\lambda$ mod $p$, we define our main sum of the product of two characters over square-free integers:

$$V(x;\chi,\lambda):=\sum_{n\le x}\mu(n)^2\chi(n)\lambda(n). \tag{2-3}$$

**Lemma 2.3.** *Let $\chi$ and $\lambda$ be nontrivial characters mod $q$ and mod $p$, respectively. We have*

$$|V(x;\chi,\lambda)| \ll \begin{cases} x^{\frac{1}{2}}(pq)^{\frac{1}{4}}(\log pq)^{\frac{1}{2}}, \\ x^{\frac{1}{2}}(\log x)(pq)^{\frac{3}{16}+\varepsilon}. \end{cases}$$

*Proof.* Since $\mu^2(n) = \sum_{d^2 \mid n} \mu(d)$, we get

$$\begin{aligned} V(x;\chi,\lambda) &= \sum_{md^2 \le x} \mu(d)\chi(m)\chi(d^2)\lambda(m)\lambda(d^2) \\ &= \sum_{d \le x^{1/2}} \mu(d)\chi^2(d)\lambda^2(d) \sum_{m \le x/d^2} \chi(m)\lambda(m). \end{aligned} \tag{2-4}$$

To obtain the first bound, we introduce $H \ge 1$ and split the sum into two parts

$$V(x;\chi,\lambda) = \sum_{d \le H} \mu(d)\chi^2(d)\lambda^2(d) \sum_{m \le x/d^2} \chi(m)\lambda(m) + \sum_{H < d \le x^{1/2}} \mu(d)\chi^2(d)\lambda^2(d) \sum_{m \le x/d^2} \chi(m)\lambda(m).$$

Using Lemma 2.2, since $\chi \cdot \lambda$ is a character mod $pq$, we have

$$|V(x;\chi,\lambda)| \ll \sum_{d \le H} \sqrt{pq}\,\log pq + \sum_{H < d \le x^{1/2}} x/d^2 \ll H\sqrt{pq}\log pq + \frac{x}{H}.$$

Choosing

$$H = \lfloor x^{\frac{1}{2}}(pq)^{-\frac{1}{4}}(\log pq)^{-\frac{1}{2}} \rfloor,$$

we obtain

$$|V(x;\chi,\lambda)| \ll x^{\frac{1}{2}}(pq)^{\frac{1}{4}}(\log pq)^{\frac{1}{2}},$$

which is the first bound. To obtain the second bound, we apply (2-1) in Lemma 2.2 to (2-4) to get

$$|V(x;\chi,\lambda)| \ll \sum_{d \le x^{1/2}} \left(\frac{x}{d^2}\right)^{\frac{1}{2}} (pq)^{\frac{3}{16}+\varepsilon} \ll x^{\frac{1}{2}}(\log x)(pq)^{\frac{3}{16}+\varepsilon}. \qquad \square$$

One can also obtain Lemma 2.3 by using Lemma 2.3 in [Munsch 2014] and the fact that if $\chi_1$ is a character mod $q_1$ and $\chi_2$ is a character mod $q_2$, their product $\chi_1\chi_2$ is a character mod $\mathrm{lcm}(q_1, q_2)$.

By proofs similar to those of Lemmas 2.5 and 2.7 in [Munsch 2014], we obtain:

**Lemma 2.4.** *Let $q$ be an integer $\ge 2$, let $p$ be an odd prime with $p \nmid q$, let $\chi, \lambda$ be nonprincipal characters mod $q$ and $p$, respectively, and let $H$ be a positive integer. Then*

$$\sum_{\substack{n \le x \\ \gcd(n,pq)=1}} 1 = \frac{\phi(pq)}{pq}x + O(\tau(pq)), \tag{2-5}$$

$$\sum_{\substack{n \le x \\ \gcd(n,pq)=1}} \mu^2(n) = \frac{x}{\zeta(2)} \prod_{p_1 \mid pq} \left(1 + \frac{1}{p_1}\right)^{-1} + O(x^{\frac{1}{2}}\tau(pq)), \tag{2-6}$$

$$\sum_{n \le H} \frac{\mu^2(n)\chi(n)\lambda(n)}{n^{\frac{3}{2}}} = \frac{L(\frac{3}{2}, \chi\lambda)}{L(3, \chi^2\lambda^2)} + O(H^{-1}(\log H)(pq)^{\frac{3}{16}+\varepsilon}), \tag{2-7}$$

$$\sum_{n\le H}\frac{\chi(n)\lambda(n)}{n^{\frac{2}{3}}}=L\big(\tfrac{2}{3},\chi\lambda\big)+O(H^{-\frac{2}{3}}(\log pq)(pq)^{\frac{1}{2}}), \tag{2-8}$$

*where $\tau(n)$ denotes the number of positive divisors of $n$, and the product runs over primes $p_1 \mid pq$.*

For any character $\chi \bmod q$, we define

$$Q_2(x;\chi):=\sum_{n\le x}\alpha(n)\chi(n),\quad \alpha(n):=\begin{cases}1 & \text{if } n \text{ is a square-full integer,}\\ 0 & \text{otherwise,}\end{cases} \tag{2-9}$$

which is the character sum of square-full integers not exceeding $x$. The next lemma is [Munsch 2014, Lemma 1.3].

**Lemma 2.5.** *Let $p > 3$ be an integer, let $\lambda$ be a Dirichlet character mod $p$, and let $\lambda_0$ be the trivial character mod $p$. Then*

$$Q_2(x;\lambda)$$
$$=\begin{cases}\dfrac{\zeta(\frac{3}{2})}{\zeta(3)}\displaystyle\prod_{p_1\mid p}\Big(\frac{1-p_1^{-1}}{1+p_1^{-\frac{3}{2}}}\Big)x^{\frac{1}{2}}+\frac{\zeta(\frac{2}{3})}{\zeta(2)}\prod_{p_1\mid p}\Big(\frac{1-p_1^{-\frac{2}{3}}}{1+p_1^{-1}}\Big)x^{\frac{1}{3}}+O(x^{\frac{1}{6}+\varepsilon}p^{\varepsilon}) & \text{if } \lambda=\lambda_0,\\ \dfrac{L(\frac{3}{2},\chi)}{\zeta(3)}\displaystyle\prod_{p_1\mid p}\Big(1+\frac{1}{p_1}+\frac{1}{p_1^2}\Big)^{-1}x^{\frac{1}{2}}+O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}p^{\frac{3}{32}+\varepsilon}) & \text{if } \lambda \text{ is a quadratic character,}\\ \dfrac{L(\frac{2}{3},\chi^2)}{\zeta(2)}\displaystyle\prod_{p_1\mid p}\Big(1+\frac{1}{p_1}\Big)^{-1}x^{\frac{1}{3}}+O(x^{\frac{1}{4}}p^{\frac{1}{4}+\varepsilon}) & \text{if } \lambda \text{ is a cubic character,}\\ O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}p^{\frac{11}{32}+\varepsilon}) & \text{if } \lambda^2\ne\lambda_0 \text{ and } \lambda^3\ne\lambda_0.\end{cases}$$

By exactly the same steps of proof as that of [Munsch 2014, p. 562], we extract:

**Lemma 2.6.** *Let $p, q$ be positive integers with $pq > 3$, let $\xi^{(j)}$ $(j = 1, 2)$ be Dirichlet characters mod $pq$, and let $\xi_0$ be the trivial character mod $pq$. If $\xi^{(1)} \ne \xi_0$ and $\xi^{(2)} \ne \xi_0$, then*

$$\sum_{a\le x^{1/2}}\xi^{(1)}(a)\sum_{b\le (x/a^2)^{1/3}}\mu^2(b)\xi^{(2)}(b)\ll x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}(pq)^{\frac{11}{32}+\varepsilon}.$$

## 3. Proof of Theorem 1.1

Keeping the notation as in the statement of Theorem 1.1, let

$$Q_p(x;\ell,q):=\sum_{\substack{n\le x\\ n\equiv \ell \bmod q}}T_2(n) \tag{3-1}$$

denote the number of square-full integers $n \equiv \ell \bmod q$ not exceeding $x$ which are primitive roots mod $p$. By the orthogonality relation for Dirichlet characters mod $q$, we have

$$Q_p(x;\ell,q)=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\bar{\chi}(\ell)\sum_{n\le x}T_2(n)\chi(n). \tag{3-2}$$

Using (1-6), the definition of $T_2(n)$, together with Lemma 2.1 and (2-9), we have

$$Q_p(x;\ell,q)=\frac{\phi(p-1)}{(p-1)\phi(q)}\sum_{d\mid p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi \bmod q}\bar{\chi}(\ell)\sum_{\lambda\in\Gamma_d}\sum_{n\le x}\alpha(n)\chi(n)\lambda(n),$$

where the sum for $\lambda$ runs over the Dirichlet characters mod $p$ of order $d$. For brevity, let

$$T(x;\chi,\lambda):=\sum_{n\le x}\alpha(n)\chi(n)\lambda(n), \tag{3-3}$$

so that

$$Q_p(x;\ell,q)=\frac{\phi(p-1)}{(p-1)\phi(q)}\sum_{d\mid p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi \bmod q}\bar{\chi}(\ell)\sum_{\lambda\in\Gamma_d}T(x;\chi,\lambda). \tag{3-4}$$

The Euler product formula for $L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$ leads to the Dirichlet series of the function $\alpha(n)\chi(n)\lambda(n)$ as

$$\sum_{n=1}^{\infty}\alpha(n)\chi(n)\lambda(n)n^{-s}=\frac{L(2s,\chi^2\lambda^2)L(3s,\chi^3\lambda^3)}{L(6s,\chi^6\lambda^6)}. \tag{3-5}$$

Our task now is to derive asymptotic estimates for $T(x;\chi,\lambda)$, the sum of the product of two characters over square-full integers. There are three main cases. Case 1 deals with $\chi=\chi_0$, the principal character mod $q$. Case 2 deals with $\lambda=\lambda_0$, the principle character mod $p$ and Case 3 is when both $\chi$ and $\lambda$ are nonprincipal characters with respect to their moduli. The subcases run through all possible shapes of the two characters.

Case 1: $\chi=\chi_0$, the principal character mod $q$. If $\lambda$ is any Dirichlet character modulo $p$, then

$$\begin{aligned}
&\sum_{n=1}^{\infty}\alpha(n)\chi_0(n)\lambda(n)n^{-s}\\
&\quad=\frac{L(2s,\chi_0\lambda^2)L(3s,\chi_0\lambda^3)}{L(6s,\chi_0\lambda^6)}\\
&\quad=\prod_{p_1}\biggl(\frac{(1-\chi_0(p_1)\lambda^2(p_1)p_1^{-2s})(1-\chi_0(p_1)\lambda^3(p_1)p_1^{-3s})}{1-\chi_0(p_1)\lambda^6(p_1)p_1^{-6s}}\biggr)^{-1}\\
&\quad=\prod_{\substack{p_1\\ p_1\nmid q}}\biggl(\frac{(1-\lambda^2(p_1)p_1^{-2s})(1-\lambda^3(p_1)p_1^{-3s})}{1-\lambda^6(p_1)p_1^{-6s}}\biggr)^{-1}\\
&\quad=\prod_{p_1}\biggl(\frac{(1-\lambda^2(p_1)p_1^{-2s})(1-\lambda^3(p_1)p_1^{-3s})}{1-\lambda^6(p_1)p_1^{-6s}}\biggr)^{-1}\prod_{p_1\mid q}\frac{(1-\lambda^2(p_1)p_1^{-2s})(1-\lambda^3(p_1)p_1^{-3s})}{1-\lambda^6(p_1)p_1^{-6s}}\\
&\quad=\frac{L(2s,\lambda^2)L(3s,\lambda^3)}{L(6s,\lambda^6)}H_q(s,\lambda)=\biggl(\sum_{n=1}^{\infty}\alpha(n)\lambda(n)n^{-s}\biggr)H_q(s,\lambda),
\end{aligned}$$

where the last product runs over primes $p_1\mid q$, and

$$H_q(s,\lambda)=\prod_{p_1\mid q}\frac{1-\lambda^2(p_1)p_1^{-2s}}{1+\lambda^3(p_1)p_1^{-3s}}:=\sum_{n=1}^{\infty}h_q(n,\lambda)n^{-s}. \tag{3-6}$$

Equating coefficients of the Dirichlet series, summing over $n$, and making use of (3-3) and (2-9) leads to

$$\begin{aligned} T(x:\chi_0,\lambda) &= \sum_{n\le x}\alpha(n)\chi_0(n)\lambda(n) = \sum_{n\le x}\sum_{de=n} h_q(d,\lambda)\alpha(e)\lambda(e) \\ &= \sum_{d\le x} h_q(d,\lambda)\sum_{e\le x/d}\alpha(e)\lambda(e) = \sum_{d\le x} h_q(d,\lambda)Q_2\Big(\frac{x}{d},\lambda\Big). \end{aligned} \tag{3-7}$$

We next obtain asymptotic formulas for (3-7) by applying Lemma 2.5 to $Q_2(x/d;\lambda)$.

<u>Subcase 1.1</u>: $\lambda=\lambda_0$, the principal character mod $p$. We get, using also (3-6),

$$\begin{aligned} T(x;\chi_0,\lambda_0) &= \sum_{d\le x} h_q(d,\lambda_0)Q_2\Big(\frac{x}{d},\lambda_0\Big) \\ &= \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\prod_{p_1\mid p}\bigg(\frac{1-p_1^{-1}}{1+p_1^{-\frac{3}{2}}}\bigg)x^{\frac{1}{2}}\sum_{d\le x}\Big(\frac{h_q(d,\lambda_0)}{d}\Big)^{\frac{1}{2}} \\ &\quad + \frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)}\prod_{p_1\mid p}\bigg(\frac{1-p_1^{-\frac{2}{3}}}{1+p_1^{-1}}\bigg)x^{\frac{1}{3}}\sum_{d\le x}\Big(\frac{h_q(d,\lambda_0)}{d}\Big)^{\frac{1}{3}} + O\bigg(x^{\frac{1}{6}+\varepsilon}p^{\varepsilon}\sum_{d\le x}\Big(\frac{h_q(d,\lambda_0)}{d}\Big)^{\frac{1}{6}+\varepsilon}\bigg) \\ &= \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\prod_{p_1\mid pq}\bigg(\frac{1-p_1^{-1}}{1+p_1^{-\frac{3}{2}}}\bigg)x^{\frac{1}{2}} + \frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)}\prod_{p_1\mid pq}\bigg(\frac{1-p_1^{-\frac{2}{3}}}{1+p_1^{-1}}\bigg)x^{\frac{1}{3}} + O(x^{\frac{1}{6}+\varepsilon}p^{\varepsilon}). \end{aligned}$$

The contribution from $T(x;\chi_0,\lambda_0)$ towards (3-4) is equal to

$$\begin{aligned} &\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(1)}{\phi(1)}\bar{\chi}_0(\ell)T(x;\chi_0,\lambda_0) \\ &\quad = \frac{\phi(p-1)}{\phi(p)\phi(q)}\bigg(\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\prod_{p_1\mid pq}\bigg(\frac{1-p_1^{-1}}{1+p_1^{-\frac{3}{2}}}\bigg)x^{\frac{1}{2}} + \frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)}\prod_{p_1\mid pq}\bigg(\frac{1-p_1^{-\frac{2}{3}}}{1+p_1^{-1}}\bigg)x^{\frac{1}{3}} + O(x^{\frac{1}{6}+\varepsilon}p^{\varepsilon})\bigg). \end{aligned} \tag{3-8}$$

<u>Subcase 1.2</u>: $\lambda^2=\lambda_0,\ \lambda\neq\lambda_0$, i.e, $\lambda$ a quadratic character mod $p$. Similar reasoning as in the last case leads to

$$T(x;\chi_0,\lambda) = \frac{L\left(\frac{3}{2},\lambda\right)}{\zeta(3)}\Big(1+\frac{1}{p}+\frac{1}{p^2}\Big)^{-1}H_q\left(\tfrac{1}{2},\lambda\right)x^{\frac{1}{2}} + O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}p^{\frac{3}{32}+\varepsilon}).$$

The contribution from $T(x;\chi_0,\lambda)$ towards (3-4) is equal to

$$\begin{aligned} &\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(2)}{\phi(2)}\bar{\chi}_0(\ell)\sum_{\lambda\in Y_1}T(x;\chi_0,\lambda) \\ &\quad = -\frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\lambda\in Y_1}\bigg(\frac{L\left(\frac{3}{2},\lambda\right)}{\zeta(3)}\Big(1+\frac{1}{p}+\frac{1}{p^2}\Big)^{-1}H_q\left(\tfrac{1}{2},\lambda\right)x^{\frac{1}{2}} + O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}p^{\frac{3}{32}+\varepsilon})\bigg). \end{aligned} \tag{3-9}$$

<u>Subcase 1.3</u>: $\lambda^3=\lambda_0,\ \lambda\neq\lambda_0$, i.e., $\lambda$ a cubic character mod $p$. As before, similar calculation yields

$$T(x;\chi_0,\lambda) = \frac{L\left(\frac{2}{3},\lambda^2\right)}{\zeta(2)}\Big(1+\frac{1}{p}\Big)^{-1}H_q\left(\tfrac{1}{3},\lambda\right)x^{\frac{1}{3}} + O(x^{\frac{1}{4}}p^{\frac{1}{4}+\varepsilon}).$$

The contribution from $T(x;\chi_0,\lambda)$ towards (3-4) is equal to

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(3)}{\phi(3)}\bar{\chi}_0(\ell)\sum_{\lambda\in Y_2}T(x;\chi_0,\lambda) = -\frac{\phi(p-1)}{2\phi(p)\phi(q)}\sum_{\lambda\in Y_2}\biggl(\frac{L\bigl(\frac{2}{3},\lambda^2\bigr)}{\zeta(2)}\Bigl(1+\frac{1}{p}\Bigr)^{-1}H_q\bigl(\tfrac{1}{3},\lambda\bigr)x^{\frac{1}{3}}+O(x^{\frac{1}{4}}p^{\frac{1}{4}+\varepsilon})\biggr). \tag{3-10}$$

Subcase 1.4: $\lambda^2\neq\lambda_0,\ \lambda^3\neq\lambda_0$. Similar calculation gives

$$T(x;\chi_0,\lambda)=O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}p^{\frac{11}{32}+\varepsilon}).$$

The contribution from $T(x;\chi_0,\lambda)$ towards (3-4) is equal to

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\substack{d\mid p-1\\ d>3}}\frac{\mu(d)}{\phi(d)}\bar{\chi}_0(\ell)\sum_{\lambda\in\Gamma_d\setminus\{Y_1\cup Y_2\}}T(x;\chi_0,\lambda) = \frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\substack{d\mid p-1\\ d>3}}\frac{\mu(d)}{\phi(d)}\sum_{\lambda\in\Gamma_d\setminus\{Y_1\cup Y_2\}}O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}p^{\frac{11}{32}+\varepsilon}). \tag{3-11}$$

Case 2: $\lambda=\lambda_0$, the principal character mod $p$, and $\chi\neq\chi_0$. From (3-5), steps similar to those at the beginning of Case 1 yield

$$\begin{aligned}\sum_{n=1}^{\infty}\alpha(n)\chi(n)\lambda_0(n)n^{-s} &= \frac{L(2s,\chi^2\lambda_0)L(3s,\chi^3\lambda_0)}{L(6s,\chi^6\lambda_0)}\\ &=\prod_{p_1}\biggl(\frac{(1-\chi^2(p_1)\lambda_0(p_1)p_1^{-2s})(1-\chi^3(p_1)\lambda_0(p_1)p_1^{-3s})}{1-\chi^6(p_1)\lambda_0(p_1)p_1^{-6s}}\biggr)^{-1}\\ &=\prod_{\substack{p_1\\ p_1\neq p}}\biggl(\frac{(1-\chi^2(p_1)p_1^{-2s})(1-\chi^3(p_1)p_1^{-3s})}{1-\chi^6(p_1)p_1^{-6s}}\biggr)^{-1}\\ &=\frac{(1-\chi^2(p)p^{-2s})(1-\chi^3(p)p^{-3s})}{1-\chi^6(p)p^{-6s}}\prod_{p_1}\biggl(\frac{(1-\chi^2(p_1)p_1^{-2s})(1-\chi^3(p_1)p_1^{-3s})}{1-\chi^6(p_1)p_1^{-6s}}\biggr)^{-1}\\ &=\frac{L(2s,\chi^2)L(3s,\chi^3)}{L(6s,\chi^6)}F_p(s,\chi)=\biggl(\sum_{n=1}^{\infty}\alpha(n)\chi(n)n^{-s}\biggr)F_q(s,\chi),\end{aligned}$$

where

$$F_p(s,\chi)=\frac{1-\chi^2(p)p^{-2s}}{1+\chi^3(p)p^{-3s}}:=\sum_{n=1}^{\infty}f_p(n,\chi)n^{-s},$$

and consequently,

$$T(x;\chi,\lambda_0)=\sum_{d\le x}f_p(d,\chi)Q_2\Bigl(\frac{x}{d},\chi\Bigr). \tag{3-12}$$

Subcase 2.1: $\chi^2=\chi_0,\ \chi\neq\chi_0$. We have

$$T(x;\chi,\lambda_0)=\frac{L\big(\frac{3}{2},\chi\big)}{\zeta(3)}\prod_{p_1\mid q}\Big(1+\frac{1}{p_1}+\frac{1}{p_1^2}\Big)^{-1}F_p\big(\tfrac{1}{2},\chi\big)x^{\frac{1}{2}}+O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}q^{\frac{3}{32}+\varepsilon}).$$

The contribution from $T(x;\chi,\lambda_0)$ towards (3-4) is equal to

$$\begin{aligned}\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(1)}{\phi(1)}&\sum_{\chi\in X_1}\bar{\chi}(\ell)T(x;\chi,\lambda_0)\\ &=\frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\chi\in X_1}\bar{\chi}(\ell)\bigg(\frac{L\big(\frac{3}{2},\chi\big)}{\zeta(3)}\prod_{p_1\mid q}\Big(1+\frac{1}{p_1}+\frac{1}{p_1^2}\Big)^{-1}F_p\big(\tfrac{1}{2},\chi\big)x^{\frac{1}{2}}+O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}q^{\frac{3}{32}+\varepsilon})\bigg).\end{aligned}\tag{3-13}$$

Subcase 2.2: $\chi^3=\chi_0,\ \chi\neq\chi_0$. We have

$$T(x;\chi,\lambda_0)=\frac{L\big(\frac{2}{3},\chi^2\big)}{\zeta(2)}\prod_{p_1\mid q}\Big(1+\frac{1}{p_1}\Big)^{-1}F_p\big(\tfrac{1}{3},\chi\big)x^{\frac{1}{3}}+O(x^{\frac{1}{4}}q^{\frac{1}{4}+\varepsilon}).$$

The contribution from $T(x;\chi,\lambda_0)$ towards (3-4) is equal to

$$\begin{aligned}\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(1)}{\phi(1)}&\sum_{\chi\in X_2}\bar{\chi}(\ell)T(x;\chi,\lambda_0)\\ &=\frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\chi\in X_2}\bar{\chi}(\ell)\bigg(\frac{L\big(\frac{2}{3},\chi^2\big)}{\zeta(2)}\prod_{p_1\mid q}\Big(1+\frac{1}{p_1}\Big)^{-1}F_p\big(\tfrac{1}{3},\chi\big)x^{\frac{1}{3}}+O(x^{\frac{1}{4}}q^{\frac{1}{4}+\varepsilon})\bigg).\end{aligned}\tag{3-14}$$

Subcase 2.3: $\chi^2\neq\chi_0,\ \chi^3\neq\chi_0$. We have

$$T(x;\chi,\lambda_0)=O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}q^{\frac{11}{32}+\varepsilon}).$$

The contribution from $T(x;\chi,\lambda_0)$ towards (3-4) is equal to

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(1)}{\phi(1)}\sum_{\chi\notin X_1\cup X_2}\bar{\chi}(\ell)T(x;\chi,\lambda_0)=\frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\chi\notin X_1\cup X_2}\bar{\chi}(\ell)O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}q^{\frac{11}{32}+\varepsilon}).\tag{3-15}$$

Case 3: $\chi\neq\chi_0,\ \lambda\neq\lambda_0$, i.e., both are nonprincipal characters.

Subcase 3.1: $\chi^2=\chi_0,\ \lambda^2=\lambda_0$, i.e., both are quadratic characters. We proceed as in [Munsch 2014, p. 560]. Since $n$ is square-full, we can uniquely write $n=a^2b^3$ with $b$ square-free, and so

$$\begin{aligned}T(x;\chi,\lambda)&=\sum_{n\le x}\alpha(n)\chi(n)\lambda(n)=\sum_{a^2b^3\le x}\mu^2(b)\chi(a^2b^3)\lambda(a^2b^3)\\ &=\sum_{b\le x^{1/3}}\mu^2(b)\chi(b^3)\lambda(b^3)\sum_{a\le(x/b^3)^{1/2}}\chi(a^2)\lambda(a^2)=\sum_{b\le x^{1/3}}\mu^2(b)\chi(b^3)\lambda(b^3)\sum_{\substack{a\le(x/b^3)^{1/2}\\ \gcd(a,pq)=1}}1.\end{aligned}$$

For any real $H \geq 1$, we have

$$\begin{aligned} T(x;\chi,\lambda) &= \sum_{b\leq H}\mu^2(b)\chi(b^3)\lambda(b^3)\sum_{\substack{a\leq(x/b^3)^{1/2}\\ \gcd(a,pq)=1}}1+\sum_{H<b\leq x^{1/3}}\mu^2(b)\chi(b^3)\lambda(b^3)\sum_{\substack{a\leq(x/b^3)^{1/2}\\ \gcd(a,pq)=1}}1\\ &= \sum_{b\leq H}\mu^2(b)\chi(b^3)\lambda(b^3)\sum_{\substack{a\leq(x/b^3)^{1/2}\\ \gcd(a,pq)=1}}1+\sum_{\substack{a\leq(x/H^3)^{1/2}\\ \gcd(a,pq)=1}}\sum_{H<b\leq(x/a^2)^{1/3}}\mu^2(b)\chi(b^3)\lambda(b^3). \end{aligned}$$

The first term is bounded using (2-5), the character shapes and (2-7):

$$\begin{aligned} &\sum_{b\leq H}\mu^2(b)\chi(b^3)\lambda(b^3)\sum_{\substack{a\leq(x/b^3)^{1/2}\\ \gcd(a,pq)=1}}1\\ &\quad=\sum_{b\leq H}\mu^2(b)\chi(b)\lambda(b)\left(\frac{\phi(pq)}{pq}\frac{x^{\frac12}}{b^{\frac32}}+O(\tau(pq))\right)\\ &\quad=\frac{L\left(\frac32,\chi\lambda\right)}{\zeta(3)}\prod_{p_1\mid pq}(1-p_1^{-3})^{-1}\frac{\phi(pq)}{pq}x^{\frac12}+O(x^{\frac12}(pq)^{\frac{3}{16}+\varepsilon}(\log H)H^{-1})+O(H(pq)^{\varepsilon}). \end{aligned}$$

The second term is bounded using the character shapes and the second bound in Lemma 2.3:

$$\sum_{\substack{a\leq(x/H^3)^{1/2}\\ \gcd(a,pq)=1}}\sum_{H<b\leq(x/a^2)^{1/3}}\mu^2(b)\chi(b^3)\lambda(b^3)\ll(pq)^{\frac{3}{16}+\varepsilon}x^{\frac16}(\log x)\sum_{a\leq(x/H^3)^{1/2}}a^{-\frac13}\ll(pq)^{\frac{3}{16}+\varepsilon}x^{\frac12}(\log x)H^{-1}.$$

Choosing $H=x^{1/4}(\log x)^{1/2}(pq)^{3/32}$, we have

$$T(x;\chi,\lambda)=\frac{L\left(\frac32,\chi\lambda\right)}{\zeta(3)}\prod_{p_1\mid pq}(1-p_1^{-3})^{-1}\frac{\phi(pq)}{pq}x^{\frac12}+O(x^{\frac14}(\log x)^{\frac12}(pq)^{\frac{3}{32}+\varepsilon}).$$

The contribution from $T(x;\chi,\lambda)$ towards (3-4) is equal to

$$\begin{aligned} &\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(2)}{\phi(2)}\sum_{\chi\in X_1}\bar\chi(\ell)\sum_{\lambda\in Y_1}T(x;\chi,\lambda)\\ &=-\frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\chi\in X_1}\bar\chi(\ell)\sum_{\lambda\in Y_1}\left(\frac{L\left(\frac32,\chi\lambda\right)}{\zeta(3)}\prod_{p_1\mid pq}(1-p_1^{-3})^{-1}\frac{\phi(pq)}{pq}x^{\frac12}+O(x^{\frac14}(\log x)^{\frac12}(pq)^{\frac{3}{32}+\varepsilon})\right). \end{aligned}\tag{3-16}$$

Subcase 3.2: $\chi^3=\chi_0$, $\lambda^3=\lambda_0$, i.e., both are cubic characters. We proceed as in [Munsch 2014, p. 561] using the character shapes, for any real $H\geq 1$, to get

$$\begin{aligned} T(x;\chi,\lambda)&=\sum_{n\leq x}\alpha(n)\chi(n)\lambda(n)=\sum_{a^2b^3\leq x}\mu^2(b)\chi(a^2b^3)\lambda(a^2b^3)\\ &=\sum_{a\leq x^{1/2}}\chi(a^2)\lambda(a^2)\sum_{b\leq(x/a^2)^{1/3}}\mu^2(b)\chi(b)^3\lambda(b)^3=\sum_{a\leq x^{1/2}}\chi(a^2)\lambda(a^2)\sum_{\substack{b\leq(x/a^2)^{1/3}\\ \gcd(b,pq)=1}}\mu^2(b)\\ &=\sum_{a\leq H}\chi(a^2)\lambda(a^2)\sum_{\substack{b\leq(x/a^2)^{1/3}\\ \gcd(b,pq)=1}}\mu^2(b)+\sum_{H<a\leq x^{1/2}}\chi(a^2)\lambda(a^2)\sum_{\substack{b\leq(x/a^2)^{1/3}\\ \gcd(b,pq)=1}}\mu^2(b). \end{aligned}\tag{3-17}$$

The first term in (3-17) is bounded by using (2-6) and (2-8):

$$\begin{aligned}\sum_{a\le H}\chi(a^2)\lambda(a^2)&\sum_{\substack{b\le(x/a^2)^{1/3}\\ \gcd(b,pq)=1}}\mu^2(b)\\
&=\sum_{a\le H}\chi(a^2)\lambda(a^2)\biggl(\frac{x^{\frac13}}{a^{\frac23}\zeta(2)}\prod_{p_1\mid pq}\Bigl(1+\frac{1}{p_1}\Bigr)^{-1}+O(x^{\frac16}a^{-\frac13}\tau(pq))\biggr)\\
&=\frac{x^{\frac12}}{\zeta(2)}\prod_{p_1\mid pq}\Bigl(1+\frac{1}{p_1}\Bigr)^{-1}\sum_{a\le H}\frac{\chi^2(a)\lambda^2(a)}{a^{\frac23}}+O(x^{\frac16}(pq)^{\varepsilon}H^{\frac23})\\
&=\frac{L\bigl(\frac23,\chi^2\lambda^2\bigr)}{\zeta(2)}\prod_{p_1\mid pq}(1+p_1^{-1})^{-1}x^{\frac13}+O(x^{\frac16}(pq)^{\varepsilon}H^{\frac23})+O(x^{\frac13}(pq)^{\frac12}(\log pq)H^{-\frac23}).\end{aligned}$$

The second term in (3-17) is bounded by inverting the summation and using Lemma 2.2(I):

$$\begin{aligned}\sum_{H<a\le x^{\frac12}}\chi(a^2)\lambda(a^2)\sum_{\substack{b\le(x/a^2)^{1/3}\\ \gcd(b,pq)=1}}\mu^2(b)&=\sum_{\substack{b\le(x/H^2)^{1/3}\\ \gcd(b,pq)=1}}\mu^2(b)\sum_{H<a\le(x/b^3)^{1/2}}(\chi\lambda)^2(a)\\
&\ll(pq)^{\frac12}(\log pq)x^{\frac13}H^{-\frac23}.\end{aligned}$$

Thus,

$$T(x;\chi,\lambda)=\frac{L\bigl(\frac23,\chi^2\lambda^2\bigr)}{\zeta(2)}\prod_{p_1\mid pq}(1+p_1^{-1})^{-1}x^{\frac13}+O(x^{\frac16}(pq)^{\varepsilon}H^{\frac23})+O(x^{\frac13}(pq)^{\frac12}(\log pq)H^{-\frac23}).$$

Choosing $H=x^{1/8}(pq)^{3/8}$, we obtain

$$T(x;\chi,\lambda)=\frac{L\bigl(\frac23,\chi^2\lambda^2\bigr)}{\zeta(2)}\prod_{p_1\mid pq}(1+p_1^{-1})^{-1}x^{\frac13}+O(x^{\frac14}(pq)^{\frac14+\varepsilon}).$$

The contribution from $T(x;\chi,\lambda)$ towards (3-4) is equal to

$$\begin{aligned}\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(3)}{\phi(3)}&\sum_{\chi\in X_2}\bar\chi(\ell)\sum_{\lambda\in Y_2}T(x;\chi,\lambda)\\
&=-\frac{\phi(p-1)}{2\phi(p)\phi(q)}\sum_{\chi\in X_2}\bar\chi(\ell)\sum_{\lambda\in Y_2}\biggl(\frac{L\bigl(\frac23,\chi^2\lambda^2\bigr)}{\zeta(2)}\prod_{p_1\mid pq}(1+p_1^{-1})^{-1}x^{\frac13}+O(x^{\frac14}(pq)^{\frac14+\varepsilon})\biggr).\end{aligned}\tag{3-18}$$

Subcase 3.3: $\chi^2\ne\chi_0,\ \chi^3\ne\chi_0,\ \lambda^2\ne\lambda_0,\ \lambda^3\ne\lambda_0$, i.e., both are nonquadratic and noncubic characters. We proceed as in [Munsch 2014, p. 562]. Similar to the last subcase, using the character shapes, for any real $H\ge1$, we have

$$\begin{aligned}T(x;\chi,\lambda)&=\sum_{a\le x^{1/2}}\chi(a^2)\lambda(a^2)\sum_{b\le(x/a^2)^{1/3}}\mu^2(b)\chi(b^3)\lambda(b^3)\\
&=\sum_{a\le H}\chi(a^2)\lambda(a^2)\sum_{b\le(x/a^2)^{1/3}}\mu^2(b)\chi(b^3)\lambda(b^3)+\sum_{H<a\le x^{1/2}}\chi(a^2)\lambda(a^2)\sum_{b\le(x/a^2)^{1/3}}\mu^2(b)\chi(b^3)\lambda(b^3).\end{aligned}$$

The first term is bounded using the second bound in Lemma 2.3:

$$\sum_{a\le H}\chi(a^2)\lambda(a^2)\sum_{b\le(x/a^2)^{1/3}}\mu^2(b)(\chi\lambda)^3(b)\ll(pq)^{\frac{3}{16}+\varepsilon}x^{\frac16}(\log x)\sum_{a\le H}a^{-\frac13}$$
$$\ll(pq)^{\frac{3}{16}+\varepsilon}x^{\frac16}(\log x)H^{\frac23}.$$

The second term is bounded using Lemma 2.2:

$$\sum_{H<a\le x^{1/2}}\chi(a^2)\lambda(a^2)\sum_{b\le(x/a^2)^{1/3}}\mu^2(b)\chi(b^3)\lambda(b^3)=\sum_{b\le(x/H^2)^{1/3}}\mu^2(b)\chi(b^3)\lambda(b^3)\sum_{H<a\le(x/b^3)^{1/2}}(\chi\lambda)^2(a)$$
$$\ll\sqrt{pq}\,\log(pq)x^{\frac13}H^{-\frac23}.$$

Thus,

$$T(x;\chi,\lambda)=O((pq)^{\frac{3}{16}+\varepsilon}x^{\frac16}(\log x)H^{\frac23})+O((pq)^{\frac12}\log(pq)x^{\frac13}H^{-\frac23}).$$

Choosing $H=x^{1/8}(pq)^{15/64}(\log x)^{-3/4}$, we obtain

$$T(x;\chi,\lambda)=O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12}).$$

The contribution from $T(x;\chi,\lambda)$ towards (3-4) is equal to

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\substack{d\mid p-1\\ d>3}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\notin X_1\cup X_2}\bar\chi(\ell)\sum_{\lambda\in\Gamma_d\setminus Y_1\cup Y_2}T(x;\chi,\lambda)$$
$$=\frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\substack{d\mid p-1\\ d>3}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\notin X_1\cup X_2}\bar\chi(\ell)\sum_{\lambda\in\Gamma_d\setminus Y_1\cup Y_2}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12}).\tag{3-19}$$

Subcase 3.4: $\chi^2=\chi_0$, $\lambda^3=\lambda_0$. Since $\chi$ and $\lambda$ are nonprincipal characters mod $q$ and mod $p$, respectively, with prime $p\nmid q$, in this subcase the product $\chi\lambda$ can be considered as a nonprincipal character mod $pq$. Thus,

$$T(x;\chi,\lambda)=\sum_{a\le x^{1/2}}\chi(a^2)\lambda(a^2)\sum_{b\le(x/a^2)^{1/3}}\mu^2(b)\chi(b^3)\lambda(b^3)$$
$$=\sum_{a\le x^{1/2}}(\chi_0\lambda^2)(a)\sum_{b\le(x/a^2)^{1/3}}\mu^2(b)(\chi^3\lambda_0)(b)=\sum_{a\le x^{1/2}}\xi^{(1)}(a)\sum_{b\le(x/a^2)^{1/3}}\mu^2(b)\xi^{(2)}(b),\tag{3-20}$$

where $\xi^{(1)}:=\chi_0\lambda^2$ and $\xi^{(2)}:=\chi^3\lambda_0$ are Dirichlet characters modulo $pq$. Since $\xi^{(1)}$ and $\xi^{(2)}$ are nonprincipal characters mod $pq$, by Lemma 2.6, we have

$$T(x;\chi,\lambda)=O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12})\tag{3-21}$$

and the contribution from $T(x;\chi,\lambda)$ towards (3-4) is equal to

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(2)}{\phi(2)}\sum_{\chi\in X_1}\bar\chi(\ell)\sum_{\lambda\in Y_2}T(x;\chi,\lambda)=-\frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\chi\in X_1}\bar\chi(\ell)\sum_{\lambda\in Y_2}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12}).\tag{3-22}$$

The remaining subcases can all be treated in a manner similar to Subcase 3.4 to yield the same estimate (3-21) for $T(x;\chi,\lambda)$, and their contributions from $T(x;\chi,\lambda)$ towards (3-4) are listed below.

Subcase 3.5: $\chi^2=\chi_0,\ \lambda^2\neq\lambda_0,\ \lambda^3\neq\lambda_0$.

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\substack{d\,|\,p-1\\ d>3}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in X_1}\bar{\chi}(\ell)\sum_{\lambda\in\Gamma_d\setminus Y_1\cup Y_2}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}). \tag{3-23}$$

Subcase 3.6: $\chi^3=\chi_0,\ \lambda^2=\lambda_0,\ \lambda\neq\lambda_0$.

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(2)}{\phi(2)}\sum_{\chi\in X_2}\bar{\chi}(\ell)\sum_{\lambda\in Y_1}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}). \tag{3-24}$$

Subcase 3.7: $\chi^3=\chi_0,\ \lambda^2\neq\lambda_0, \lambda^3\neq\lambda_0$.

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}\sum_{\substack{d\,|\,p-1\\ d>3}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in X_2}\bar{\chi}(\ell)\sum_{\lambda\in\Gamma_d\setminus Y_1\cup Y_2}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}). \tag{3-25}$$

Subcase 3.8: $\chi^2\neq\chi_0, \chi^3\neq\chi_0,\ \lambda^2=\lambda_0, \lambda\neq\lambda_0$.

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(2)}{\phi(2)}\sum_{\chi\notin X_1\cup X_2}\bar{\chi}(\ell)\sum_{\lambda\in Y_1}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}). \tag{3-26}$$

Subcase 3.9: $\chi^2\neq\chi_0,\ \chi^3\neq\chi_0,\ \lambda^3=\lambda_0$.

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}\frac{\mu(3)}{\phi(3)}\sum_{\chi\notin X_1\cup X_2}\bar{\chi}(\ell)\sum_{\lambda\in Y_2}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}). \tag{3-27}$$

The largest contribution towards the sum in (3-4), i.e., the term containing

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}x^{\frac{1}{2}},$$

comes from (3-8) in Subcase 1.1, (3-9) in Subcase 1.2, (3-13) in Subcase 2.1, and (3-16) in Subcase 3.1, with coefficient equal to $A_{p,q}$ as displayed in (1-7).

The second largest contribution in (3-4), i.e., the term containing

$$\frac{\phi(p-1)}{\phi(p)\phi(q)}x^{\frac{1}{3}},$$

comes from (3-8) in Subcase 1.1, (3-10) in Subcase 1.3, (3-14) in Subcase 2.2, and (3-18) in Subcase 3.2 with coefficient equal to $B_{p,q}$ as displayed in (1-10).

The contribution from the error terms in (3-4) coming from Subcases 1.1–1.4 is, apart from the factor

$$\frac{\phi(p-1)}{\phi(p)\phi(q)},$$

equal to

$$O(x^{\frac{1}{6}+\varepsilon}p^{\varepsilon})-\sum_{\lambda\in Y_1}O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}p^{\frac{3}{32}+\varepsilon})-\frac{1}{2}\sum_{\lambda\in Y_2}O(x^{\frac{1}{4}}p^{\frac{1}{4}+\varepsilon})+\sum_{\substack{d\,|\,p-1\\ d>3}}\frac{\mu(d)}{\phi(d)}\sum_{\lambda\in\Gamma_d\setminus Y_1\cup Y_2}O(x^{\frac{1}{4}}(\log x)^{\frac{1}{2}}p^{\frac{11}{32}+\varepsilon}).$$

The contribution from the error terms in (3-4) coming from Subcases 2.1–2.3 is, apart from the factor

$$\frac{\phi(p-1)}{\phi(p)\phi(q)},$$

equal to

$$\sum_{\chi\in X_1}\bar{\chi}(\ell)O(x^{\frac14}(\log x)^{\frac12}q^{\frac{3}{32}+\varepsilon})+\sum_{\chi\in X_2}\bar{\chi}(\ell)O(x^{\frac14}q^{\frac14+\varepsilon})+\sum_{\chi\notin X_1\cup X_2}\bar{\chi}(\ell)O(x^{\frac14}(\log x)^{\frac12}q^{\frac{11}{32}+\varepsilon}).$$

The contribution from the error terms in (3-4) coming from Subcases 3.1–3.9 is, apart from the factor

$$\frac{\phi(p-1)}{\phi(p)\phi(q)},$$

equal to

$$\begin{aligned}
&-\sum_{\chi\in X_1}\bar{\chi}(\ell)\sum_{\lambda\in Y_1}O(x^{\frac14}(\log x)^{\frac12}(pq)^{\frac{3}{32}+\varepsilon})\\
&-\frac12\sum_{\chi\in X_2}\bar{\chi}(\ell)\sum_{\lambda\in Y_2}O(x^{\frac14}(pq)^{\frac14+\varepsilon})\\
&+\sum_{\substack{d\mid p-1\\ d>3}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\notin X_1\cup X_2}\bar{\chi}(\ell)\sum_{\lambda\in\Gamma_d\setminus Y_1\cup Y_2}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12})\\
&-\sum_{\chi\in X_1}\bar{\chi}(\ell)\sum_{\lambda\in Y_2}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12})\\
&+\sum_{\substack{d\mid p-1\\ d>3}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in X_1}\bar{\chi}(\ell)\sum_{\lambda\in\Gamma_d\setminus Y_1\cup Y_2}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12})\\
&-\sum_{\chi\in X_2}\bar{\chi}(\ell)\sum_{\lambda\in Y_1}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12})\\
&+\sum_{\substack{d\mid p-1\\ d>3}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in X_2}\bar{\chi}(\ell)\sum_{\lambda\in\Gamma_d\setminus Y_1\cup Y_2}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12})\\
&-\sum_{\chi\notin X_1\cup X_2}\bar{\chi}(\ell)\sum_{\lambda\in Y_1}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12})\\
&-\frac12\sum_{\chi\notin X_1\cup X_2}\bar{\chi}(\ell)\sum_{\lambda\in Y_2}O((pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12}).
\end{aligned}$$

Taking all the subcases into account, the error term is

$$\begin{aligned}
&\ll (pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12}\sum_{\substack{d\mid p-1\\ d>3}}\frac{|\mu(d)|}{\phi(d)}\sum_{\chi \bmod q}|\bar{\chi}(\ell)|\sum_{\lambda\in\Gamma_d}1\\
&\ll (pq)^{\frac{11}{32}+\varepsilon}x^{\frac14}(\log x)^{\frac12}2^{\omega(p-1)}\phi(q),
\end{aligned}$$

using the estimates

$$\sum_{\lambda\in\Gamma_d}1=O(\phi(d)),\qquad \sum_{\chi \bmod q}|\bar{\chi}(\ell)|=O(\phi(q)),\qquad \sum_{\substack{d\mid p-1\\ d>3}}|\mu(d)|=O(2^{\omega(p-1)}),$$

and the theorem is proved.

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