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multiply transitive action on an abelian group**

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Dedicated to Tuna Altınel in celebration of his freedom

We investigate the configuration where a group of finite Morley rank acts definably and generically m -transitively on an elementary abelian p -group of Morley rank n , where p is an odd prime, and $m \geq n$. We conclude that $m = n$, and the action is equivalent to the natural action of $\mathrm{GL}_n(F)$ on F^n for some algebraically closed field F . This strengthens one of our earlier results, and partially answers two problems posed by Borovik and Cherlin in 2008.

1. Introduction

This is the fourth and concluding work in a series of papers, which began with [Berkman and Borovik 2011; 2012; 2018]. All were aimed at proving the following theorem, but they handled different stages of the proof, each using a completely different approach and technique.

Theorem 1.1. *Let G be a group of finite Morley rank, V an elementary abelian p -group of Morley rank n , and p an odd prime. Assume that G acts on V faithfully, definably and generically m -transitively with $m \geq n$. Then $m = n$ and there is an algebraically closed field F such that $V \simeq F^n$, $G \simeq \mathrm{GL}_n(F)$, and the action is the natural action.*

In [Berkman and Borovik 2018], the same theorem was proven under the extra assumption that the action of G on V is generically *sharply* m -transitive. In this paper, we prove the generic sharpness of the action of G on V under the hypothesis of Theorem 1.1. Then Theorem 1.1 follows from the previous result [Berkman and Borovik 2018, Theorem 1]. We use the technique developed in [Borovik 2020] for analysis of actions of certain subgroups of G specifically for the needs of the present project; see Section 3A.

Theorem 1.1 gives partial confirmations to the following two conjectures; note that the latter is implicit in the former.

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Conjecture 1.2 [Altinel et al. 2008, Problem 37, p. 536; Borovik and Cherlin 2008, Problem 13]. *Let G be a connected group of finite Morley rank acting faithfully, definably, and generically n -transitively on a connected abelian group V of Morley rank n . Then V has a structure of an n -dimensional vector space over an algebraically closed field F of Morley rank 1, and G is $\mathrm{GL}_n(F)$ in its natural action on F^n .*

Conjecture 1.3 [Borovik and Cherlin 2008, Problem 12]. *Let G be a connected group of finite Morley rank acting faithfully, definably, and generically t -transitively on an abelian group V of Morley rank n . Then $t \leq n$.*

The cases when V is a torsion-free abelian group or an elementary abelian 2-group require completely different approaches and methods and are handled in our next paper. But even that result will not be the end of the story, since it appears to be an almost inevitable step in any proof of the following conjecture.

Conjecture 1.4 [Altinel et al. 2008, Problem 36, p. 536; Borovik and Cherlin 2008, Problem 9]. *Let G be a connected group of finite Morley rank acting faithfully, definably, transitively, and generically $(n+2)$ -transitively on a set Ω of Morley rank n . Then the pair (G, Ω) is equivalent to the projective linear group $\mathrm{PGL}_{n+1}(F)$ acting on the projective space $\mathbb{P}^n(F)$ for some algebraically closed field F .*

Indeed, the group $F^n \rtimes \mathrm{GL}_n(F)$ is the stabiliser of a point in the action of $\mathrm{PGL}_{n+1}(F)$ on $\mathbb{P}^n(F)$.

Altinel and Wiscons [2018; 2019] have already made important contributions towards a solution to the above conjecture. The importance of Conjecture 1.4 has been recently highlighted in [Freitag and Moosa 2021].

General discussion and a survey of results on actions of groups of finite Morley rank can be found in [Borovik and Deloro 2019]. Terminology and notation follow [Altinel et al. 2008; Borovik and Nesin 1994; Borovik and Cherlin 2008].

2. Useful facts

In what follows, (G, X) is an infinite permutation group of finite Morley rank.

Definition. Let Y be a definable subset of X . If $\mathrm{rk}(X \setminus Y) < \mathrm{rk}(X)$ then Y is called a *strongly generic* subset of X . We will simply call it a *generic subset*. If G acts transitively on a generic subset of X , then we say G acts *generically transitively* on X . If the induced action of G on X^n is generically transitive, then we say G acts *generically n -transitively* on X .

The following two facts show that connectedness assumptions are superfluous in our context.

Fact 2.1. *If G acts generically m -transitively on a group X , where $m \geq \mathrm{rk}(X)$, then X is a connected group.*

Proof. If $m \geq 2$, this is a special case of [Borovik and Cherlin 2008, Lemma 1.8]. When $m = 1$, note that the generic orbit, say $A \subseteq X$, is cofinite in X . Since G fixes X° and A setwise, G also fixes $X^\circ \cap A$ setwise. The transitivity of G on A implies $A \subseteq X^\circ$. Hence $X = X^\circ$, since A is cofinite. $\square$

Fact 2.2 [Altinel and Wiscons 2018, Lemma 4.10]. *If G acts n -transitively on X , and X is of degree 1, then G° also acts n -transitively on X .*

For any prime p , recall that a connected solvable p -group of bounded exponent is called a p -unipotent group, and a divisible abelian p -group is called a p -torus.

As the following two facts show, the structure of Sylow 2-subgroups in groups of finite Morley rank is well understood.

Fact 2.3 [Altinel et al. 2008, Propositions I.6.11, I.6.4, I.6.2]. *Sylow 2-subgroups of a group of finite Morley rank are conjugate. Moreover, if S is a Sylow 2-subgroup of a group of finite Morley rank, then $S^\circ = U * T$, where U is a definable 2-unipotent group, and T is a 2-torus. In particular, Sylow 2-subgroups in groups of finite Morley rank are locally finite.*

Fact 2.4 [Borovik et al. 2007a; Altinel et al. 2008, Theorem IV.4.1]. *Sylow 2-subgroups of a connected group of finite Morley rank are either trivial or infinite.*

The following is a structure theorem for nilpotent groups of finite Morley rank.

Fact 2.5 [Borovik and Nesin 1994, Theorem 6.8]. *Let G be a nilpotent group of finite Morley rank. Then G is the central product $D * B$, where D and B are definable characteristic subgroups of G , D is divisible, and B has bounded exponent.*

We gather below some facts about solvable groups of finite Morley rank which will be used in our proof of Theorem 1.1.

Fact 2.6. *Let M be a connected solvable group of finite Morley rank. Then the following hold:*

- (a) *The commutator subgroup $[M, M]$ is connected and nilpotent.*
- (b) *The group M can be written as a product $M = [M, M]C$, where C is a connected nilpotent subgroup.*
- (c) *If M is of bounded exponent, then M is nilpotent.*

Proof. These follow from [Borovik and Nesin 1994, Theorem 6.8], [Altinel et al. 2008, Corollary I.8.30], and [Altinel et al. 2008, Lemma I.5.5], respectively. $\square$

Next, we list some results about various configurations where groups act on groups.

Fact 2.7 [Berkman and Borovik 2018, Fact 2.12]. *Let V be a connected abelian group and E an elementary abelian 2-group of order 2^m acting definably and*

faithfully on V . Assume $m \geq n = \text{rk}(V)$ and V contains no involutions. Then $m = n$ and $V = V_1 \oplus \cdots \oplus V_n$, where

- (a) every subgroup V_i for $i = 1, \dots, n$ is connected, has Morley rank 1 and is E -invariant.

Moreover,

- (b) each V_i for $i = 1, \dots, n$ is a weight space of E ; that is, there exists a nontrivial homomorphism $\rho_i : E \rightarrow \{\pm 1\}$ such that

$$V_i = \{v \in V \mid v^e = \rho_i(e) \cdot v \text{ for all } e \in E\}.$$

Proof. Statements can be found in [Berkman and Borovik 2018], whose proofs refer to [Berkman and Borovik 2012, Lemma 7.1]. $\square$

Assume that G acts on a group V such that the only infinite definable invariant subgroup of V is itself under this action. Then we say G acts on V *minimally*, or V is *G -minimal*.

Fact 2.8 [Berkman and Borovik 2018, Proposition 2.18]. *Let V be a connected abelian group and $\Sigma = \mathbb{Z}_2^m \rtimes \text{Sym}_m$ act definably and faithfully on V . Assume $m \geq \text{rk}(V)$ and V contains no involutions. Then Σ acts on V minimally.*

Fact 2.9 (Zilber [Borovik and Nesin 1994, Theorem 9.1]). *Let A and V be connected abelian groups of finite Morley rank such that A acts on V definably, $C_A(V) = 1$ and V is A -minimal. Then there exists an algebraically closed field K and a definable subgroup $S \leq K^*$ such that the action $A \curvearrowright V$ is definably equivalent to the natural action of S on K^+ .*

Fact 2.10 [Altinel et al. 2008, Lemma I.8.2]. *Let G be a connected solvable group acting on an abelian group V . If V is G -minimal, then G' acts trivially on V .*

Recall that if a group has no nontrivial p -elements, we call it a $p^\perp$ -group. A connected divisible abelian group is called a *torus*, and a torus A is called *good* if every definable subgroup of A is the definable hull of its torsion elements.

Fact 2.11 [Altinel et al. 2008, Proposition I.11.7]. *If a connected solvable $p^\perp$ -group A acts faithfully on an abelian p -group V , then A is a good torus.*

Fact 2.12 [Altinel et al. 2008, Proposition I.8.5]. *Let p be a prime. Assume $V \trianglelefteq G$ is a definable solvable subgroup that contains no p -unipotent subgroup, and $U \leq G$ is a definable connected p -group of bounded exponent. Then $[U, V] = 1$.*

Fact 2.13 [Altinel et al. 2008, Lemma I.4.5]. *A definable group of automorphisms of an infinite field of finite Morley rank is trivial.*

3. Definable actions on elementary abelian p -groups

In this section, V is a connected elementary abelian p -group of finite Morley rank and X is a finite group acting on V definably. We use additive notation for the group operation on V and treat V as a vector space over $\mathbb{F}_p$.

It is convenient to work with the ring R generated by X in $\text{End } V$. It is finite and its elements are definable endomorphisms; R is traditionally called the enveloping algebra (over $\mathbb{F}_p$) of the action of X on V . We treat V as a right R -module.

If $v \in V$, the set

$$vR = \{vr : r \in R\}$$

is an R -submodule, which is called the *cyclic submodule generated by v* . Of course all cyclic submodules contain less than $|R|$ elements, and therefore, there are finitely many possibilities for the isomorphism type of each of them.

3A. Coprime actions. In this subsection, we assume that X is a finite p' -group acting on V . Recall that a torsion group is called a p' -group, if it has no nontrivial p -elements.

We recall some generalities from representation theory. Applying Maschke's theorem to the action of X on R by right multiplication, we see that R is a semisimple $\mathbb{F}_p$ -algebra and that every finite R -submodule in V is semisimple, that is, a direct sum of simple modules.

The following important (but easy) result (which generalises [Borovik 2020, Theorem 5]) now follows immediately.

Theorem 3.1. *Let V be a connected elementary abelian p -group of finite Morley rank, X a finite p' -group acting on V definably, and R the enveloping algebra over $\mathbb{F}_p$ for the action of X on V . Assume that $A_1, A_2, \dots, A_m$ is the complete list of nontrivial simple submodules for R in V , up to isomorphism. Then*

$$\text{rk } V \geq m.$$

Proof. For each $i = 1, \dots, m$, write

$$V_i = \{v \in R : \text{all simple submodules of } vR \text{ are isomorphic to } A_i\}.$$

It is easy to see that all the V_i are definable submodules of V and

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_m.$$

Since V is connected, all the V_i are connected. Hence, being a nontrivial, definable, connected submodule, each V_i has Morley rank at least 1. Therefore, $\text{rk}(V) \geq m$. $\square$

Problem 3.2. *It would be interesting to remove from Theorem 3.1 the assumption that X is a p' -group and prove the following:*

If $A_1, A_2, \dots, A_m$ are nontrivial simple pairwise nonisomorphic R -modules appearing as sections W/U for some definable R -modules $U < W \leq V$, then

$$\text{rk } V \geq m.$$

3B. p -Group actions. The following is folklore, and this elegant and short proof was suggested by the referee.

Fact 3.3. *Let V be a connected elementary abelian p -group of finite Morley rank.*

- (a) *If x is a p -element and $\langle x \rangle$ acts on V definably, then $[V, x]$ is a proper subgroup of V . In particular, if $\text{rk}(V) = 1$, then the action is trivial.*
- (b) *If P is a p -torus which acts on V definably, then the action is trivial.*

Proof. (a) We will work in $\text{End } V$. Let $x \in \text{End } V$ of order p^k . Since $(x - 1)^{p^k} = x^{p^k} - 1 = 0$, we get a descending chain of definable subgroups

$$V \geq V(x - 1) \geq V(x - 1)^2 \geq \dots$$

which reaches 0 in at most p^k steps. Thus, the chain does not become stationary before it reaches 0. Therefore, $V(x - 1) = [V, x]$ is a proper subgroup in V .

(b) Since V has finite Morley rank, for any p -element x acting definably on V the above chain reaches 0 in at most $\text{rk}(V)$ steps. Therefore, if $p^k \geq \text{rk}(V)$ then $V(x^{p^k} - 1) = V(x - 1)^{p^k} = 0$. Since P is a p -torus, for any $y \in P$, there exists $x \in P$ such that $y = x^{p^k}$. Hence, $V(y - 1) = V(x^{p^k} - 1) = 0$, and we are done. $\square$

4. Preliminary results

Throughout this section, we assume G and V are groups of finite Morley rank, V is a connected elementary abelian p -group of Morley rank n , where p is an odd prime, and G acts on V definably and faithfully.

Lemma 4.1. *Let H a definable connected subgroup of G , and $q \neq p$ a prime number. Then H does not contain any definable connected q -groups of bounded exponent. In particular, if H has an involution, then the connected component of any of its Sylow 2-subgroups is a 2-torus.*

Proof. Combine Facts 2.12, 2.3 and 2.4. $\square$

4A. Groups of p -unipotent type. Following [Borovik et al. 2007b], we shall call a group K a *group of p -unipotent type*, if every definable connected solvable subgroup in K is a nilpotent p -group of bounded exponent. We still work under the assumptions of this section.

Proposition 4.2. *Let K be a definable subgroup in G which contains no good tori. Then K is a torsion group of p -unipotent type. In addition, K does not contain nontrivial definable divisible abelian subgroups.*

Proof. First note that by Fact 3.3, K contains no nontrivial p -tori. Therefore, every connected definable solvable $p^\perp$ -subgroup in K is trivial by Fact 2.11.

Now it is easy to see that very definable divisible abelian subgroup in K is trivial. Indeed, if such a subgroup, say A , contains a nontrivial p -element, then it contains a nontrivial p -torus, which is impossible by the above paragraph. Hence A is a $p^\perp$ -group and is trivial again by above.

Next, notice that every element in K is of finite order. Indeed, if $x \in K$ is of infinite order, then the connected component $d(x)^\circ$ of the definable closure of $\langle x \rangle$ is a divisible abelian group, which contradicts the above paragraph.

By Fact 2.5, $M = BD$, where B and D are connected, B is of bounded exponent and D is divisible. However, $D = 1$ by above, and B is a p -group by Lemma 4.1. Therefore, every definable connected nilpotent subgroup M in K is a p -group of bounded exponent.

By Fact 2.6(b), if M is a connected solvable subgroup in K , then $M = [M, M]C$ where C is a connected nilpotent subgroup.

Finally, we will prove that K is of p -unipotent type. Let M be a definable connected solvable subgroup of K . Then by above, $M = [M, M]C$, where C is a connected nilpotent subgroup. By Fact 2.6(a), $[M, M]$ is also connected and nilpotent. Hence both subgroups are p -groups of bounded exponent by above; therefore, so is M . Now the nilpotency of M follows from Fact 2.6(c). $\square$

4B. Basis of induction. Connected groups acting faithfully and definably on abelian groups of Morley rank $n \leq 3$ are well understood. To prove these special cases of our theorem, the following results will be used.

Fact 4.3 [Deloro 2009]. *Let G be a connected nonsolvable group acting faithfully on a connected abelian group V . If $\text{rk}(V) = 2$, then there exists an algebraically closed field K such that the action $G \curvearrowright V$ is equivalent to $\text{GL}_2(K) \curvearrowright K^2$ or $\text{SL}_2(K) \curvearrowright K^2$.*

Fact 4.4 [Borovik and Deloro 2016; Frécon 2018]. *Let G be a connected nonsolvable group acting faithfully and minimally on an abelian group V . If $\text{rk}(V) = 3$ then there exists an algebraically closed field K such that $V = K^3$ and G is isomorphic to either $\text{PSL}_2(K) \times Z(G)$ or $\text{SL}_3(K) * Z(G)$. The action is the adjoint action in the former case, and the natural action in the latter case.*

4C. Throwback to pseudoreflexion actions. To exclude the case when G in our Theorem 1.1 is not connected, we will need a result which uses concepts from one of our earlier papers [Berkman and Borovik 2012]. A special case of this result, when G is connected, was stated as [Berkman and Borovik 2012, Corollary 1.3].

Proposition 4.5. *Let G be a group of finite Morley rank acting definably and faithfully on an elementary abelian p -group V of Morley rank n , where p is an*

odd prime. Assume that G contains a definable subgroup $G^\sharp \simeq \mathrm{GL}_n(F)$ for an algebraically closed field F of characteristic p . Assume also that V is definably isomorphic to the additive group of the F -vector space F^n and $G^\sharp$ acts on V as on its canonical module. Then $G^\sharp = G$.

Proof. Observe first that $\mathrm{rk} F^n = n$ implies $\mathrm{rk} F = 1$. Pseudoreflexion subgroups in the sense of [Berkman and Borovik 2012] are connected definable abelian subgroups $R < G^\sharp$ such that $V = [V, R] \oplus C_V(R)$ and R acts transitively on the nonzero elements of $[V, R]$. By Fact 2.9, one can immediately conclude that $R \simeq F^*$ and $[V, R] \simeq F^+$. Therefore $\mathrm{rk} R = 1 = \mathrm{rk}[V, R]$ in our case.

It is easy to see that pseudoreflexion subgroups in $G^\sharp = \mathrm{GL}_n(F)$ are one-dimensional (in the sense of the theory of algebraic groups) tori of the form, in a suitable coordinate system in F^n ,

$$R = \{\mathrm{diag}(x, 1, \dots, 1) \mid x \in F, x \neq 0\},$$

and all pseudoreflexion subgroups in $G^\sharp$ are conjugate in $G^\sharp$.

If R is a pseudoreflexion subgroup in $G^\sharp$, consider the subgroup $\langle R^G \rangle$ generated in G by all G -conjugates of R , which is a normal definable subgroup generated by pseudoreflexion subgroups. In view of [Berkman and Borovik 2012, Theorem 1.2], $G^\sharp = \langle R^G \rangle$ is normal in G .

We will use induction on $n \geq 1$. When $n = 1$, $G^\sharp = R \simeq F^*$ and $V = [V, R] \simeq F^+$. The subring generated by R in the ring of definable endomorphisms of V is a field by Schur's lemma, which we will denote by E . Since G normalises R , G acts as a group of field automorphisms on E . Hence, by Fact 2.13, $G = C_G(R)$. Thus, G acts linearly on $V \simeq F^+$, and therefore, $G = F^* = R = G^\sharp$.

Now assume $n \geq 2$. By the Frattini argument, $G = G^\sharp N_G(R)$. Write $H = N_G(R)$ and $H^\sharp = N_{G^\sharp}(R)$. It is well known that $H^\sharp = R \times L$, where $L \simeq \mathrm{GL}_{n-1}(F)$ centralises $[V, R]$ and acts on $C_V(R) \simeq F^{n-1}$ as on a canonical module. Obviously, H leaves $[V, R]$ and $C_V(R)$ invariant. Note that the action of H/R on $C_V(R)$ is faithful. Indeed, if K/R is the kernel, then K acts faithfully on $[V, R]$ with a normal subgroup $R \cong F^*$. This brings us to the base of induction, which was discussed above. Hence $K = R$. Thus, H/R contains a definable subgroup

$$H^\sharp/R = (R \times L)/R \simeq \mathrm{GL}_{n-1}(F);$$

by the inductive assumption, $H/R = H^\sharp/R$, and hence $H = H^\sharp$. Therefore, $G = G^\sharp H = G^\sharp H^\sharp = G^\sharp$. $\square$

5. Proof of Theorem 1.1

We present a complete proof of Theorem 1.1 in this section. Therefore, we work under the following assumptions.

We have a group of finite Morley rank G acting definably, faithfully, and generically m -transitively on a connected elementary abelian p -group V of Morley rank n , with p an odd prime and $m \geq n$.

First note that V is connected by Fact 2.1. Another crucial observation is that G° also acts definably, faithfully, and generically m -transitively on V by Fact 2.2. Therefore, in view of Proposition 4.5, it will suffice to prove Theorem 1.1 in the special case when $G = G^\circ$ is connected.

Therefore, from now on we assume that G is connected.

5A. The core configuration. We will focus now on a group-theoretic configuration at the heart of Theorem 1.1.

The generic m -transitivity of G on V means that there is a generic subset A of V^m on which G acts transitively. We fix an element $\bar{a} = (a_1, \dots, a_m) \in A$, and write

$$V_0 = \langle a_1, \dots, a_m \rangle.$$

From now on, we denote by K the pointwise stabilizer, and by H the setwise stabilizer of $\{\pm a_1, \dots, \pm a_m\}$ in G .

In $\bar{H} = H/K$ we have m involutions $\bar{e}_i$ for $i = 1, \dots, m$ defined by their action on $a_1, \dots, a_m$:

$$a_j^{\bar{e}_i} = \begin{cases} -a_j & \text{if } i = j, \\ a_j & \text{otherwise.} \end{cases}$$

Lemma 5.1. *The group $\bar{E}_m = \langle \bar{e}_1, \dots, \bar{e}_m \rangle$ is an elementary abelian group of order 2^m and $H/K \simeq \Sigma_m = \bar{E}_m \rtimes \text{Sym}_m$, where Sym_m permutes the generators $\bar{e}_1, \dots, \bar{e}_m$ of $\bar{E}_m$.*

Notice that the group Σ_m is the *hyperoctahedral* group of degree m , which prominently features in the theory of algebraic groups as the reflection group of type BC_m . This fact is not used in this paper, but is likely to pop up in some of our future work.

Proof. Since G acts generically m -transitively on V , the proof of [Berkman and Borovik 2018, Lemma 3.1] can be repeated in this context as well, and we obtain the desired result. $\square$

5B. Essential subgroups and ample subgroups. Let D be the full preimage in H of the subgroup $\bar{E}_m < H/K$. At this point, we temporarily forget about the ambient group G and generic transitivity and focus on the group H and its subgroups D and K .

For a subgroup $X \leq H$, we write $X_D = X \cap D$ and $X_K = X \cap K$.

We shall call a definable subgroup $X \leq H$ *ample* if $KX = H$.

A definable subgroup $X \leq D$ is *essential* if $KX = D$. Equivalently, a definable subgroup $X < G$ is essential if

- X leaves invariant the set $\{\pm a_1, \dots, \pm a_m\}$ (which is equivalent to $X \leq H$) and, consequently, the subgroup V_0 ;
- X_K is the pointwise stabiliser of $\bar{a}$ in X (and consequently $X_K = C_X(V_0)$), and $X/X_K \simeq \bar{E}_m$ acts on V_0 as on the canonical module $\mathbb{Z}_p^m$ for $\bar{E}_m$ and leaves invariant the subgroups

$$A_1 = \langle a_1 \rangle, \dots, A_m = \langle a_m \rangle.$$

Notice that $X = H$ is an ample subgroup. Obviously, if X is ample, then X_D is essential.

The following lemma summarises the application of representation theory of finite groups in our context.

Lemma 5.2. *If X is a finite essential subgroup and X_K is a p' -group, then $X_K = 1$. Also, in that case, $m = n$.*

Proof. Since X_K is a p' -group, X is also a p' -group because $p \neq 2$ and X/X_K is a 2-group which covers $D/K = \bar{E}_m$.

Let R be the enveloping algebra of X .

Notice that X (hence R) acts on each subgroup $A_1, \dots, A_m$ irreducibly. Moreover, each representation is different, because the A_i are cyclic groups of order p and, among the elements $\bar{e}_1, \dots, \bar{e}_m$, only $\bar{e}_i$ inverts A_i . By Theorem 3.1, $m = n$ and $V = V_1 \oplus \dots \oplus V_n$, where in the modules V_i each simple R -submodule is isomorphic to A_i . But X_K acts trivially on each A_i , hence acts trivially on each V_i and therefore on V . This means $X_K = 1$. $\square$

Notice that in the next lemma we do not assume that X is finite, and therefore we continue to accept the possibility that $m > n$.

Lemma 5.3. *If X is an essential subgroup then X_K is a $2^\perp$ -group.*

Proof. Let S be a Sylow 2-subgroup in X . If X_K is not a $2^\perp$ -group, then $X_K \cap S \neq 1$. Take a nontrivial element $s \in X_K \cap S$ and elements $s_1, \dots, s_m$ in S whose images in X/X_K generate $X/X_K \simeq \bar{E}_m$. By Fact 2.3, S is locally finite, so $s, s_1, \dots, s_m$ generate a finite 2-subgroup, say Y , in S . Obviously, $X_K Y = X$, and hence Y is essential and $s = 1$ by Lemma 5.2, a contradiction. $\square$

We can now characterise essential subgroups.

Lemma 5.4. *Let E be a Sylow 2-subgroup in D . Then $KE = D$, $E_K = 1$, and $E \simeq \bar{E}_m$. In particular, E is an essential subgroup.*

Moreover, essential subgroups of D are exactly those definable subgroups which contain one of the Sylow 2-subgroups of D .

Proof. Since E is a Sylow 2-subgroup in D , so is KE/K in $D/K \simeq \bar{E}_m$, thus $D = KE$, that is, E is essential. By Lemma 5.3, E_K is a $2^\perp$ -group, so it is trivial. Since $K/E_K \simeq \bar{E}_m$, the first statement follows. The second statement is clear. $\square$

Now we obtain the equality $m = n$ in the general case.

Lemma 5.5. $m = n$.

Proof. Because H contains an elementary abelian 2-subgroup of order 2^m by Lemma 5.4, Fact 2.7 (or Theorem 3.1) gives us $m = n$. $\square$

Lemma 5.6. K contains no good tori.

Proof. Assume the contrary, and let T be a maximal good torus in K . By conjugacy of maximal good tori [Altinel et al. 2008, Proposition IV.1.15], and the Frattini argument, we have $D = KN_D(T)$ and thus $N_D(T)$ is an essential subgroup and contains a Sylow 2-subgroup E of D . Let q be a prime such that T has q -torsion and Q the maximal elementary abelian q -subgroup in T . Obviously, E normalises Q and QE is an essential subgroup. By Lemma 5.2, $Q = 1$, a contradiction. $\square$

Lemma 5.7. K is a torsion group of p -unipotent type.

Proof. This is an immediate consequence of Lemma 5.6 and Proposition 4.2. $\square$

Lemma 5.8. Let E be a Sylow 2-subgroup in D . Then $X = N_H(E)$ is an ample subgroup. Moreover, $X_K = 1$ and $X \simeq \Sigma_n$.

Proof. By the Frattini argument, $DX = H$, and thus X is an ample subgroup. Since $X_K \triangleleft X$, $E \triangleleft X$, and $X_K \cap E = 1$ by Lemma 5.3, we have $[X_K, E] = 1$. If x is a p' -element in X_K , then the subgroup $\langle x \rangle \times E$ is essential and therefore $x = 1$ by Lemma 5.2. Hence X_K is a p -group.

Take the weight decomposition of V with respect to E :

$$V = V_1 \oplus \cdots \oplus V_n, \text{ where } i = 1, 2, \dots, n.$$

Note that every element in X_K leaves every one-dimensional space V_i invariant. Since X_K is a p -group, Fact 3.3 is applicable, and thus we conclude that X_K fixes each V_i , and hence V , elementwise. Therefore $X_K = 1$, which, in its turn, implies $X \simeq \Sigma_n$. $\square$

5C. An almost final configuration: the ample subgroup $K^\circ \Sigma$.

Lemma 5.9. If $m \geq 2$ then G is not solvable.

Proof. If G is solvable, by Fact 2.8, we know that Fact 2.10 is applicable, so $G' = 1$ and G is abelian. However, for $m \geq 2$, Σ_m is not abelian. $\square$

At the heart of our proof of Theorem 1.1, there is a Core Configuration. We set it up by writing $Q = K^\circ$ and $X = QN_H(E)$ and listing the properties of X which we have established so far.

Core Configuration. • X is a group of finite Morley rank acting definably and faithfully on an elementary abelian p -group V with p odd of Morley rank $n \geq 3$.

- $Q \triangleleft X$ is a nontrivial connected definable subgroup of p -unipotent type; notice that Q is not necessarily nilpotent. We also denote it by Q_n .
- $\Sigma \simeq \Sigma_n$ is a subgroup of X which normalises Q . It will be convenient to denote it just by Σ_n .
- Finally, to emphasise the inductive nature of our setting, we may write, if necessary, $X = X_n$.

In the next Lemma 5.10 we analyse the Core Configuration on its own, without using any further information about G , and prove that $Q = 1$.

Lemma 5.10. *Under assumptions of the Core Configuration, $Q_n = 1$ for all $n \geq 3$.*

Proof. We proceed by induction on n . If $n = 3$, Q_3 is nilpotent in view of Facts 4.3 and 4.4, which give us a basis of induction on $n \geq 3$. For the inductive step for $n > 3$, take the involution e_n and consider

$$C_{Q\Sigma}(e_n) = C_Q^\circ(e_n)C_\Sigma(e_n).$$

Obviously, this group leaves invariant the eigenspaces V_n^+ and V_n^- . Write $Q_{n-1} = C_Q^\circ(e_n)$. Observe $C_\Sigma(e_n) = \Sigma_{n-1} \times \langle e_n \rangle$, and we write $X_{n-1} = Q_{n-1}\Sigma_{n-1}$.

For the inductive assumption, we have $Q_{n-1} = 1$. Then $C_Q^\circ(e_n) = Q_{n-1} = 1$ and Q_n is an abelian p -group. Assume $Q_n \neq 1$. Then $[V, Q_n]$ is a nontrivial proper connected subgroup of V and is Σ_n -invariant, which contradicts the minimality of the action of Σ_n on V , Fact 2.8. The contradiction shows $Q_n = 1$. $\square$

We can now return to the main proof.

Proposition 5.11. $K^\circ = 1$.

Proof. If $n \leq 3$, we know everything about G from Facts 4.3, 4.4, and Lemma 5.9, and in these cases, $K^\circ = 1$. If $n \geq 3$, the proposition follows from Lemma 5.10. $\square$

Corollary 5.12. *If X is an ample subgroup, then X_K is a finite group without involutions and $X = X_K \rtimes \Sigma$.*

5D. The final case: K is finite.

Lemma 5.13. $K = 1$ and $H = \Sigma$.

Proof. Let $Q \neq 1$ be a Sylow q -subgroup of K for a prime $q \neq p$. Then by the Frattini argument, $H = K N_H(Q)$. Write $X = N_H(Q)$. Then X is an ample subgroup. We can assume without loss of generality that $E < X$. Applying Lemma 5.2 to the essential group QE , we see that $Q = 1$. Hence if $K \neq 1$ then K is a p -group.

Consider the ample group $K\Sigma$; because of the minimality of the action of Σ on V (Fact 2.8), we have $K = 1$ by Fact 3.3(a). Hence $H = \Sigma$. $\square$

This completes the proof of Theorem 1.1.

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