Model Theory

$M \models T$

Kim-independence in positive logic

Jan Dobrowolski and Mark Kamsma

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An important dividing line in the class of unstable theories is being $NSOP_1$, which is more general than being simple. In $NSOP_1$ theories forking independence may not be as well behaved as in stable or simple theories, so it is replaced by another independence notion, called Kim-independence. We generalise Kim-independence over models in $NSOP_1$ theories to positive logic — a proper generalisation of full first-order logic where negation is not built in, but can be added as desired. For example, an important application is that we can add hyperimaginary sorts to a positive theory to get another positive theory, preserving $NSOP_1$ and various other properties. We prove that, in a thick positive $NSOP_1$ theory, Kim-independence over existentially closed models has all the nice properties that it is known to have in an $NSOP_1$ theory in full first-order logic. We also provide a Kim–Pillay style theorem, characterising which thick positive theories are $NSOP_1$ by the existence of a certain independence relation. Furthermore, this independence relation must then be the same as Kim-independence. Thickness is the mild assumption that being an indiscernible sequence is type-definable.

In full first-order logic Kim-independence is defined in terms of Morley sequences in global invariant types. These may not exist in thick positive theories. We solve this by working with Morley sequences in global Lascar-invariant types, which do exist in thick positive theories. We also simplify certain tree constructions that were used in the study of Kim-independence in full first-order logic. In particular, we only work with trees of finite height.

1. Introduction

The study of (ternary) independence relations in model theory goes back to Shelah's notion of forking independence, which is an abstract generalisation of classical independence notions such as linear independence in vector spaces and algebraic independence in algebraically closed fields. Forking independence was initially used to study stable theories, in which it enjoys particularly nice properties. It was later discovered that forking independence can be useful in studying the broader class of simple theories, as it retains most of its features in that class [Kim 1998; Kim and Pillay 1997]. Moreover, the fundamental properties of forking independence in simple theories, such as transitivity, symmetry, and local character, fail in *all* nonsimple theories, which suggested that forking independence might not be so useful in studying any broader class of theories. On the other hand, some natural examples of nonsimple theories admitting useful notions of independence have been known, including the theories of infinite-dimensional vector spaces with a generic bilinear form [Granger 1999], ω-free PAC fields [Chatzidakis 2002; 2008], and random parametrised equivalence relations. Inspired by some ideas of Kim [2009], and building on [Chernikov and Ramsey 2016], Kaplan and Ramsey [2020] defined the notion of Kim-independence (denoted by $\underset{\smile}{\overset{K}{}}$), and they have proved that in NSOP_1 theories — a class containing all simple theories and, among many others, the three nonsimple theories mentioned above [Chernikov and Ramsey 2016, Section 6] — it satisfies over models all the main properties of forking independence in simple theories except base-monotonicity.

The goal of this paper is to generalise the theory of Kim-independence in NSOP_1 theories to the class of thick positive theories. Positive model theory, introduced in [Ben-Yaacov 2003a; Ben Yaacov and Poizat 2007] (with some ideas in a similar direction present also in [Hrushovski 1998] and [Pillay 2000]), provides a framework generalising that of full first-order logic and allows the study of a wider range of objects using model-theoretic techniques. An important class of such objects, which motivated the work undertaken in [Ben-Yaacov 2003a], is that of the hyperimaginary extensions T^{heq} of theories T in full first-order logic. In the context of NSOP_1 theories, elimination of hyperimaginaries has been assumed in [Kim 2021] in order to carry out a construction of weak canonical bases. It was asked there (in the discussion following Definition 4.1) whether T^{heq} satisfies the existence axiom for forking independence provided that T does. We observe that this is indeed true (Theorem 10.20), which might be helpful in eliminating the assumption of elimination of hyperimaginaries in [Kim 2021] by working with Kim-independence in T^{heq}.

Haykazyan and Kirby [2021] studied the theory ECEF of existentially closed exponential fields, and working with an arbitrary JEP-refinement (which, intuitively,

corresponds to a completion of an incomplete theory in full first-order logic), they have found an invariant ternary relation satisfying over models the following properties: strong finite character, existence, monotonicity, symmetry, and independence theorem. They have also proved that, for any positive theory, the existence of such a relation implies $NSOP_1$, so in particular the JEP-refinements of ECEF are $NSOP_1$. As in the full first-order setting, a natural question whether every positive $NSOP_1$ theory admits a ternary relation satisfying these properties arises.

Another class of examples of nonsimple $NSOP_1$ theories in positive logic comes from a recent work [d'Elbée et al. 2021], where d'Elbée, Kaplan and Neuhauser show that for any integral domain R all JEP-refinements of the theory $F_{R\text{-module}}$ of fields with a generic R-submodule are $NSOP_1$ but not simple. In particular, this applies to the theory of algebraically closed fields of characteristic zero with a generic additive subgroup.

We work under the mild assumption that the theory is thick. This means that being an indiscernible sequence is type-definable. Theories in full first-order logic, and their hyperimaginary extensions, are always thick. The theories ECEF and $F_{R\text{-module}}$ mentioned above are also thick.

Main results. The main results of our paper state that in every thick $NSOP_1$ theory, Kim-independence satisfies: symmetry (Theorem 6.5), the (strong) independence theorem (Theorems 7.7 and 7.15), transitivity (Theorem 8.4) and local character (Corollary 9.7), as well as invariance under automorphisms, existence, extension, monotonicity and (strong) finite character. Moreover, we prove a Kim–Pillay style theorem: in any thick positive theory T, if there exists a ternary relation $\underset{\smile}{\vert}$ satisfying all the above properties, then T is $NSOP_1$ and $\underset{\smile}{\vert} = \underset{\smile}{\vert}^K$ (Theorem 9.1).

Challenges. In contrast to the full first-order setting, in a positive theory, a type over an existentially closed model may fail to have an invariant global extension. This is a fundamental obstacle to generalising Kim-independence to the positive setting, as the original definition of it relies on existence of invariant extensions in the full first-order setting. We show, however, that in a thick theory any type over an existentially closed model M extends to a global M-Lascar-invariant type. We define Kim-independence in an arbitrary thick positive theory replacing the use of invariant types by Lascar-invariant types.

One of the difficulties in adapting the results of [Kaplan and Ramsey 2020; 2021] to the positive setting is that the tree-modelling property [Kim et al. 2014, Theorem 4.3], on which most of the constructions there rely, is not available in the positive setting. This forced us in particular to work only with trees of finite height, which turns out to be enough due to compactness and a careful choice of the global types with which we work. Consequently, we substitute the notion of a tree Morley sequence used in [Kaplan and Ramsey 2020] with a weaker notion

of a parallel-Morley sequence. In particular, we do not have a counterpart of the chain condition [Kaplan and Ramsey 2020, Corollary 5.15] for parallel-Morley sequences, which causes some additional technical difficulties in our proof of the strong independence theorem.

Our proofs yield in particular alternative proofs of the results in full first-order logic on Kim-independence not using any combinatorial tools other than the Ramsey theorem: while we do use the Erdős–Rado theorem to extract indiscernible sequences, in the full first-order setting this can be always replaced by the standard use of the Ramsey theorem; the technique of extracting strongly indiscernible trees from s-indiscernible trees from [Kaplan and Ramsey 2020] relying on the Erdős–Rado theorem is not used by us.

Overview. In Section 2 we review some basic terminology and facts about positive logic and NSOP$_1$ theories, and we make some observations which are used throughout the paper. In Section 3 we define a notion of a Morley sequence in a global Lascar-invariant type, and we prove some basic properties of these. In Section 4 we define Kim-dividing in an arbitrary thick NSOP$_1$ theory, we give several characterisations of Kim-dividing and we establish some basic properties of Kim-independence. In Section 5 we develop some tools which we later use in certain tree constructions: the EM-modelling property, which is a weak version of the modelling property used in [Kaplan and Ramsey 2020], parallel-Morley sequences, which serve as our substitute for the notion of a tree Morley sequence from [Kaplan and Ramsey 2020], and q-spread-outness, which is a variant of the spread-outness used in [Kaplan and Ramsey 2020]. Sections 6, 7 and 8 contain the proofs of the main properties of Kim-independence in thick positive theories: symmetry, independence theorem and transitivity, and Section 9 is dedicated to proving a Kim–Pillay-style characterisation of the NSOP$_1$ property among thick positive theories by existence of an abstract independence relation satisfying certain properties, and the characterisation of Kim-independence in NSOP$_1$ theories as the only relation satisfying them. In Section 10 we describe in detail some examples of thick NSOP$_1$ theories: Poizat's example of a thick non-semi-Hausdorff theory, (JEP refinements of) the positive theory of existentially closed exponential fields studied in [Haykazyan and Kirby 2021], and the hyperimaginary extensions of NSOP$_1$ theories.

2. Preliminaries

In this section we recall the basics of positive logic that we need in this paper. For a more extensive treatment we refer to [Ben-Yaacov 2003a; Poizat and Yeshkeyev 2018].

Throughout the paper variables will be of arbitrary (possibly infinite) length, unless stated otherwise.

Definition 2.1. Fix a signature $\mathcal{L}$. A *positive existential formula* in $\mathcal{L}$ is one that is obtained from combining atomic formulas using $\wedge$, $\vee$, $\top$, $\bot$ and $\exists$. An *h-inductive sentence* is a sentence of the form $\forall x(\varphi(x) \to \psi(x))$, where $\varphi(x)$ and $\psi(x)$ are positive existential formulas. A *positive theory* is a set of h-inductive sentences.

Note that every positive existential formula $\varphi(x)$ is equivalent to one of the form $\exists y \psi(x, y)$, where $\psi(x, y)$ is positive quantifier-free. Positive existential sentences and their negations can be used as axioms in a positive theory, since $\forall x \varphi(x)$ and $\forall x \neg \varphi(x)$ are equivalent to $\forall x(\top \to \varphi(x))$ and $\forall x(\varphi(x) \to \bot)$ respectively.

As in full first-order logic, we will assume that $\mathcal{L}$ contains a symbol $=$ interpreted in every $\mathcal{L}$-structure as equality.

Remark 2.2. We can study full first-order logic as a special case of positive logic. This is done through a process called *Morleyisation*. For this we add a relation symbol $R_\varphi(x)$ to our language for every formula $\varphi(x)$ in full first-order logic. Then we have our theory (inductively) express that $R_\varphi(x)$ and $\varphi(x)$ are equivalent. This way every formula in full first-order logic is (equivalent to) a relation symbol, and thus in particular to a positive existential formula.

Many definitions later in this section simplify in this case to familiar concepts. Every homomorphism will be an elementary embedding, and thus in particular an immersion. So every model will be an e.c. model. A theory has JEP if and only if it is complete, and the JEP-refinements correspond to completions.

Since we will only be considering full first-order logic as a special case of positive logic, we will make the following convention.

Convention 2.3. Whenever we say "formula" or "theory" we will mean "positive existential formula" and "positive theory" respectively, unless explicitly stated otherwise. This also means that every formula and theory we consider will be implicitly assumed to be positive (existential).

In full first-order logic we consider elementary embeddings because they preserve and reflect truth of full first-order formulas. Since we do not have negation in positive logic, there is a difference between preserving and reflecting truth of positive existential formulas.

Definition 2.4. A function $f : M \to N$ between $\mathcal{L}$-structures is called a *homomorphism* if for every $\varphi(x)$ and every $a \in M$ we have

$$M \models \varphi(a) \implies N \models \varphi(f(a)).$$

We call f an *immersion* if additionally the converse implication holds for all $\varphi(x)$ and all $a \in M$.

In positive model theory we study the existentially closed models.

Definition 2.5. We call a model M of T an *existentially closed model* or an *e.c. model* if the following equivalent conditions hold:

(i) Every homomorphism $f : M \to N$ with $N \models T$ is an immersion.

(ii) For every $a \in M$ and $\varphi(x)$ such that there is a homomorphism $f : M \to N$ with $N \models T$ and $N \models \varphi(f(a))$, we have that $M \models \varphi(a)$.

(iii) For every $a \in M$ and $\varphi(x)$ such that $M \not\models \varphi(a)$ there is $\psi(x)$ with $T \models \neg\exists x (\varphi(x) \wedge \psi(x))$ and $M \models \psi(a)$.

Fact 2.6. *Let T be some theory.*

(i) *Unions*: *The union of a chain of (e.c.) models is an (e.c.) model.*

(ii) *Amalgamation*: *If one of $M_1 \leftarrow M \to M_2$ is an immersion then there are $M_1 \to N \leftarrow M_2$ making the relevant square commute. In particular, every e.c. model is an amalgamation base.*

(iii) *Existential completion*: *For every $M \models T$ there is a homomorphism $f : M \to N$, where N is an e.c. model of T.*

(iv) *Compactness*: *Let $\Sigma(x)$ be a set of positive existential formulas and suppose that for every finite $\Sigma_0(x) \subseteq \Sigma(x)$ there is $M \models T$ with $a \in M$ such that $M \models \Sigma_0(a)$. Then there is an e.c. model N of T with $a \in N$ such that $N \models \Sigma(a)$.*

In the statement of compactness, Fact 2.6(iv), we have explicitly mentioned positive existential formulas because it is crucial that we cannot use all formulas from full first-order logic in $\Sigma(x)$. This is actually one of the big obstacles in this paper. We provide two examples to indicate how full compactness can fail.

Example 2.7. Consider the theory T with a symbol for inequality and ω many disjoint unary predicates $P_n(x)$. Then e.c. models of T are precisely those which consist of ω-many disjoint infinite sets, one for each predicate. If we had full compactness then the set

$$\Sigma(x) = \{\neg P_n(x) : n < \omega\}$$

would have a realisation in some e.c. model, which is impossible.

Example 2.8. It could happen that there is a definable set that is infinite and bounded. This does not contradict compactness: it just means that inequality is not positively definable on that set. Such situations might arise when adding hyperimaginaries as real elements, which can be done in positive logic (see Section 10C).

Definition 2.9. We say that a theory T has the *joint embedding property* or *JEP* if the following equivalent conditions hold:

(i) For any two models M_1 and M_2 there are homomorphisms $M_1 \to N \leftarrow M_2$.

(ii) If $T \models \neg\varphi \vee \neg\psi$ then $T \models \neg\varphi$ or $T \models \neg\psi$.

For a theory T we call an extension T' of T a *JEP-refinement* of T if it has JEP and every e.c. model of T' is also an e.c. model of T.

As suggested in Remark 2.2, having JEP is like requiring the theory to be complete. We can always find a JEP-refinement (a "completion") by taking the set of h-inductive sentences that are true in some e.c. model.

Fix a sufficiently large cardinal $\bar{\kappa}$. We will say a set is *small* if it is of cardinality smaller than $\bar{\kappa}$.

Convention 2.10. We will assume our theory T has JEP so we can work in a *monster model* $\mathfrak{M}$ (sometimes also called a universal domain), that is, a model which is:

- Existentially closed: $\mathfrak{M}$ is an e.c. model.

- Very homogeneous: Any partial immersion $f : \mathfrak{M} \to \mathfrak{M}$ with small domain and codomain extends to an automorphism on all of $\mathfrak{M}$.

- Very saturated: Any finitely satisfiable small set of formulas over $\mathfrak{M}$ is satisfiable in $\mathfrak{M}$.

We will assume all parameter sets considered to be small, except when we consider the monster model as a parameter set. We will use lowercase Latin letters $a, b, \ldots$ for (possibly small infinite) tuples inside the monster model and uppercase Latin letters $A, B, \ldots$ for (small) parameter sets inside the monster model. We will use letters M and N when these sets are e.c. models.

As is common, we use the notation $\models \varphi(a)$ to abbreviate $\mathfrak{M} \models \varphi(a)$.

The above also means that the right notion of a type in positive model theory is that of a positive existential type. That is, we write $\mathrm{tp}(a/B)$ for the set of all positive existential formulas over B satisfied by a. So we have $\mathrm{tp}(a/B) = \mathrm{tp}(a'/B)$ if and only if there is an automorphism $f : \mathfrak{M} \to \mathfrak{M}$ fixing B such that $f(a) = a'$. We also write $a \equiv_B a'$ in this case. By a type (over A) in T we will always mean a maximal consistent with T set of positive existential formulas (over A). By a partial type (over A) in T we will mean any consistent set of positive existential formulas (over A).

There are some subtle differences in possible definitions of saturatedness; see for example [Poizat and Yeshkeyev 2018, Section 2.4]. We are only interested in e.c. models, so for us it will mean the following. Constructing models of a certain level of saturation is then standard.

Definition 2.11. Let M be an e.c. model of some theory T. We say that M is κ-*saturated* if for every $A \subseteq M$ with $|A| < \kappa$ we have that a set $\Sigma(x)$ of formulas over A is satisfiable in M if and only if it is finitely satisfiable in M.

Fact 2.12. *For any* $\kappa \geq |A| + |T|$ *there is a* κ^+-*saturated* $N \supseteq A$ *with* $|N| \leq 2^\kappa$.

The following definitions are taken from [Ben-Yaacov 2003a; 2003c]. We added the notion of being Boolean.

Definition 2.13. Let T be a theory and work in a monster model. We call T

- *Boolean* if every formula in full first-order logic is equivalent to a positive existential formula, modulo T;

- *Hausdorff* if for any two distinct types $p(x)$ and $q(x)$ there are $\varphi(x) \notin p(x)$ and $\psi(x) \notin q(x)$ such that $\models \forall x (\varphi(x) \vee \psi(x))$;

- *semi-Hausdorff* if equality of types is type-definable, so there is a partial type $\Omega(x, y)$ such that $\operatorname{tp}(a) = \operatorname{tp}(b)$ if and only if $\models \Omega(a, b)$;

- *thick* if being an indiscernible sequence is type-definable, so there is a partial type $\Theta((x_i)_{i<\omega})$ such that $(a_i)_{i<\omega}$ is indiscernible if and only if $\models \Theta((a_i)_{i<\omega})$.

Remark 2.14. The reason for the name Hausdorff is that this corresponds to the type spaces being Hausdorff, where formulas correspond to closed sets. The name thick is based on the notion of thick formulas, which were originally defined in the setting of full first-order logic (see also [Ben-Yaacov 2003c]).

The name Boolean comes from the fact that the Lindenbaum–Tarski algebra of positive existential formulas forms a Boolean algebra, and this is in fact an equivalent assertion. In [Haykazyan 2019] these theories are called "positively model complete", but we think this name is more descriptive.

Through Morleyisation, Boolean theories are essentially the same as theories in full first-order logic, and so we will treat them as the same. The list of properties in Definition 2.13 is really a hierarchy, so Boolean implies Hausdorff implies semi-Hausdorff implies thick.

Definition 2.15. Let a and a' be two tuples, and let B be any parameter set. We write $d_B(a, a') \leq n$ if there are $a = a_0, a_1, \ldots, a_n = a'$ such that a_i and a_{i+1} are on a B-indiscernible sequence for all $0 \leq i < n$.

Fact 2.16 [Ben-Yaacov 2003c, Proposition 1.5]. *A theory is thick if and only if the property* "$d_B(x, x') \leq n$" *is type-definable over* B *for all* B *and* $n < \omega$.

The following appears as [Pillay 2000, Lemma 3.1] and [Ben-Yaacov 2003b, Lemma 1.2].

Lemma 2.17. *Let A be any parameter set, κ any cardinal, and let $\lambda = \beth_{(2^{|T|+|A|+\kappa})^+}$. Then for any sequence $(a_i)_{i<\lambda}$ of κ-tuples there is an A-indiscernible sequence $(b_i)_{i<\omega}$ such that for all $n < \omega$ there are $i_1 < \cdots < i_n < \lambda$ with $b_1 \cdots b_n \equiv_A a_{i_1} \cdots a_{i_n}$.*

Definition 2.18. In the notation of Lemma 2.17 we say that $(b_i)_{i<\omega}$ is *based on* the sequence $(a_i)_{i<\lambda}$ (over A).

Often the parameter set A will be clear from the context (it will be the set that the new sequence is indiscernible over), so we may leave out the "over A".

Definition 2.19. We write $\lambda_\kappa := \beth_{(2^\kappa)^+}$ for any cardinal κ and $\lambda_T := \lambda_{|T|}$.

Lemma 2.20. *Let M be a λ_T-saturated e.c. model of a thick theory. Then $a \equiv_M b$ implies $d_M(a, b) \leq 2$.*

Proof. By thickness, $d_M(x, y) \leq 1$ is M-type-definable. Let $\varphi(x, y)$ be a finite conjunction of formulas in $d_M(x, y) \leq 1$. It is enough to show that $\varphi(x, a) \wedge \varphi(x, b)$ is satisfiable, because then the partial type "$d_M(x, a) \leq 1$ and $d_M(x, b) \leq 1$" is finitely satisfiable.

Since φ is just a formula, we may as well assume a and b to be finite. Let m denote the (finite) part of M that appears in φ. By λ_T-saturatedness of M there is a sequence $(a_i)_{i<\lambda_T}$ in M such that $a_i(a_j)_{j<i} \equiv_m a(a_j)_{j<i}$ for all $i < \lambda_T$. Using Lemma 2.17 we then find m-indiscernible $(a_i')_{i<\omega}$ based on $(a_i)_{i<\lambda_T}$. So $\models \varphi(a_0', a_1')$, and thus there are $i_0 < i_1 < \lambda_T$ such that $M \models \varphi(a_{i_0}, a_{i_1})$. By construction we have $a_{i_1}a_{i_0} \equiv_m aa_{i_0}$, so $\models \varphi(a_{i_0}, a)$. Since $a \equiv_M b$ and $a_{i_0} \in M$ we also have $\models \varphi(a_{i_0}, b)$. $\square$

Lemma 2.21. *Let T be a thick theory. Let $B \supseteq A$ and κ any cardinal, and set $\lambda = \lambda_{|T|+|B|+\kappa}$. Then for any A-indiscernible sequence $(a_i)_{i<\lambda}$ of κ-tuples, there is B-indiscernible $(a_i')_{i<\lambda}$ based on $(a_i)_{i<\lambda}$ such that $d_A((a_i)_{i<\lambda}, (a_i')_{i<\lambda}) \leq 1$.*

Proof. By Lemma 2.17 there is B-indiscernible $(b_i)_{i<\omega}$ based on $(a_i)_{i<\lambda}$. Extend this to B-indiscernible $(b_i)_{i<\lambda}$. Define

$$\Sigma((x_i)_{i<\lambda}) = \text{tp}((b_i)_{i<\lambda}/B) \cup \text{"}d_A((x_i)_{i<\lambda}, (a_i)_{i<\lambda}) \leq 1\text{"},$$

and let $\Sigma_0(x_{i_1}, \ldots, x_{i_n}) \subseteq \Sigma((x_i)_{i<\lambda})$ be finite, only mentioning parameters in B and $a_{i_1}, \ldots, a_{i_n}$. Let $j_1 < \cdots < j_n < \lambda$ be such that $a_{j_1} \cdots a_{j_n} \equiv_B b_1 \cdots b_n \equiv_B b_{i_0} \cdots b_{i_n}$. It follows from the proof of Lemma 2.17 that we may choose j_1 to be arbitrarily large below λ, so we may assume $j_1 > i_n$. Then $a_{j_1} \cdots a_{j_n}$ realises Σ_0. By compactness we find the required $(a_i')_{i<\lambda}$ as a realisation of Σ. $\square$

The definition of dividing in positive theories is the same as in full first-order logic [Pillay 2000; Ben-Yaacov 2003b]. Following [Pillay 2000] we have to adjust forking to allow infinite disjunctions because compactness can no longer guarantee disjunctions to be finite.

Definition 2.22. We say that a partial type $\Sigma(x, b)$ *divides over* C if there is a C-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 \equiv_C b$ such that $\bigcup_{i<\omega} \Sigma(x, b_i)$ is inconsistent.

We say $\Sigma(x, b)$ *forks over* C if there is a (possibly infinite) set of formulas $\Phi(x)$ with parameters, each of which divides over C, such that $\Sigma(x, b)$ implies $\bigvee \Phi(x)$.

We write $a \underset{C}{\overset{d}{\downarrow}} b$ (or $a \underset{C}{\overset{f}{\downarrow}} b$) if $\text{tp}(a/Cb)$ does not divide (fork) over C.

Remark 2.23. We have that $\mathrm{tp}(a/Cb)$ divides over C if and only if there is a formula $\varphi(x, b) \in \mathrm{tp}(a/Cb)$ that divides over C. This follows directly from compactness. Note that for forking this is no longer necessarily true, because the disjunction may be infinite so we cannot apply compactness.

For a type p over a set B and a subset $A \subseteq B$, the restriction of p to A is a type over A which we denote by $p|_A$. We recall the notions of an heir and a coheir, which also make sense in positive logic.

Definition 2.24. Let $M \subseteq B$, and let $p = \mathrm{tp}(a/B)$ be a type over B. We say that p is a *coheir* of $p|_M$, and write $a \underset{M}{\overset{u}{\smile}} B$, if p is finitely satisfiable in M. We say that p is an *heir* of $p|_M$ if for every formula $\varphi(x, y)$, with parameters in M, and every $b \in B$ such that $\varphi(x, b) \in p$ there is some $b' \in M$ such that $\varphi(x, b') \in p$. In this case we write $a \underset{M}{\overset{h}{\smile}} B$.

Remark 2.25. As in full first-order logic, we have $A \underset{M}{\overset{u}{\smile}} B$ if and only if $B \underset{M}{\overset{h}{\smile}} A$.

In Proposition 3.13 we compare the above notions of independence further.

We recall that $2^{<\omega}$ is the set of all finite sequences of zeroes and ones. For $\eta, \nu \in 2^{<\omega}$ we write $\eta \preceq \nu$ if ν continues the sequence η. We write $\eta^\frown \nu$ for concatenation, so for example $\eta^\frown 0$ is the sequence η with a 0 concatenated to it.

Definition 2.26. Let T be a theory, and let $\varphi(x, y)$ be a formula. We say that $\varphi(x, y)$ has SOP$_1$ if there are $\psi(y_1, y_2)$ and $(a_\eta)_{\eta \in 2^{<\omega}}$ such that:

 (i) For every $\sigma \in 2^\omega$ the set $\{\varphi(x, a_{\sigma|n}) : n < \omega\}$ is consistent.

 (ii) $\psi(y_1, y_2)$ implies that $\varphi(x, y_1) \wedge \varphi(x, y_2)$ is inconsistent, that is,

$$T \models \forall y_1 y_2 \neg [\psi(y_1, y_2) \wedge \exists x (\varphi(x, y_1) \wedge \varphi(x, y_2))].$$

 (iii) For every $\eta, \nu \in 2^{<\omega}$ such that $\eta^\frown 0 \preceq \nu$ we have $\models \psi(a_{\eta^\frown 1}, a_\nu)$.

We say that T is NSOP$_1$ if no formula has SOP$_1$.

Remark 2.27. The idea of introducing the inconsistency witness $\psi(y_1, y_2)$ is due to Haykazyan and Kirby [2021]. In full first-order logic we can just take $\psi(y_1, y_2)$ to be $\neg \exists x (\varphi(x, y_1) \wedge \varphi(x, y_2))$, so we see that the definitions coincide there. The point of having ψ is that the inconsistency in (iii) is again definable by a single formula for all relevant η and ν. This enables us to apply compactness to make the tree $(a_\eta)_{\eta \in 2^{<\omega}}$ as big as we wish.

The following lemma, or rather its contrapositive, is what will actually be useful to us. If, in an NSOP$_1$ theory, we have two sequences that are "parallel to each other" in a certain way then we can transfer consistency for a formula along one sequence to the other. We will therefore give it the name "parallel sequences lemma".

Lemma 2.28 (parallel sequences lemma). *Suppose that $\varphi(x, y)$ is a formula, and $(\bar{c}_i) = (c_{i,0}, c_{i,1})_{i \in I}$ is an infinite indiscernible sequence satisfying*

(i) $c_{i,0} \equiv_{\bar{c}_{<i}} c_{i,1}$ *for all $i \in I$;*

(ii) $\{\varphi(x; c_{i,0}) : i \in I\}$ *is consistent;*

(iii) $\{\varphi(x; c_{i,1}) : i \in I\}$ *is inconsistent.*

Then T has SOP_1.

Proof. This is the same as [Kaplan and Ramsey 2020, Lemma 2.3] and that proof mostly goes through. We sketch a few small changes that are needed. Obviously we already start with an indiscernible sequence and by compactness we can freely change the order type of I preserving properties (i)–(iii). Then in the claim in that proof we need to make the array $(a_{i,0}, a_{i,1})$ sufficiently long. This can easily be done by elongating the original indiscernible sequence $(\bar{c}_i)$. Then we can find an indiscernible sequence based on $(\bar{a}_i) = (a_{i,0}, a_{i,1})$. Note that properties (1)–(3) in that claim are preserved by this operation. The reason for all this is because we need to start with an indiscernible sequence in [Kaplan and Ramsey 2020, Lemma 2.2] as well. Then the rest of that proof goes through. Finally, inconsistency of $\{\varphi(x, c_{l,1}), \chi(x, d_{l',0})\}$ should be witnessed by some formula (similarly for [Kaplan and Ramsey 2020, Lemma 2.2]), but the existence of such a witness easily follows from the construction of χ. $\square$

3. Global Lascar-invariant types

The definition of Lascar strong types from the first-order setting easily generalises to (thick) positive logic; see [Pillay 2000, Definition 3.13, Lemma 3.15] and [Ben-Yaacov 2003b, Lemma 1.38].[1]

Definition 3.1. We say a and b have the same *Lascar strong type over A*, and write $a \equiv_A^{\mathrm{Ls}} b$, if the following equivalent conditions hold:

(i) $d_A(a, b) \leq n$ for some $n < \omega$.

(ii) For each bounded A-invariant equivalence relation $E(x, y)$ we have $E(a, b)$.

(iii) There are λ_T-saturated e.c. models $M_1, \ldots, M_n$, each containing A, and $a = a_0, \ldots, a_n = b$ such that $a_i \equiv_{M_{i+1}} a_{i+1}$ for all $0 \leq i < n$.

We write $\mathrm{Lstp}(a/A)$ for the $\equiv_A^{\mathrm{Ls}}$-equivalence class of a.

Lemma 3.2. *The conditions in Definition 3.1 are equivalent in a thick theory.*

---

[1]Simplicity is assumed in [Ben-Yaacov 2003b, Lemma 1.38] but not used in the equivalence of the properties we mention. It is used for what is (iii) there.

Proof. The equivalence of (i) and (ii) is proved in both [Pillay 2000, Lemma 3.15] and [Ben-Yaacov 2003b, Lemma 1.38]. So we prove (i) $\Longleftrightarrow$ (iii).

(i) $\Longrightarrow$ (iii): Let $a = a_0, \ldots, a_n = b$ such that a_i and a_{i+1} are on an A-indiscernible sequence. Let $0 \leq i < n$, let $(a'_j)_{j<\omega}$ be an A-indiscernible sequence with $a'_0 a'_1 = a_i a_{i+1}$, and let $M \supseteq A$ be some λ_T-saturated model. By Lemma 2.17 and an automorphism there is $M_{i+1} \equiv_A M$ such that $(a'_j)_{j<\omega}$ is M_{i+1}-indiscernible. So in particular $a_i \equiv_{M_{i+1}} a_{i+1}$, as required.

(iii) $\Longrightarrow$ (i): By Lemma 2.20 $a_i \equiv_{M_i} a_{i+1}$ implies that $\mathrm{d}_{M_i}(a_i, a_{i+1}) \leq 2$ and as $A \subseteq M_i$ we are done. $\square$

Definition 3.1(iii) allows for the following definition.

Definition 3.3. Let $\mathrm{Aut}_f(\mathfrak{M}/A)$ be the group generated by

$$\bigcup \{\mathrm{Aut}(\mathfrak{M}/M) : M \text{ is a } \lambda_T\text{-saturated model and } A \subseteq M\}.$$

We call its elements *Lascar strong automorphisms*. It is clear that in a thick theory $a \equiv_A^{\mathrm{Ls}} b$ precisely when there is $f \in \mathrm{Aut}_f(\mathfrak{M}/A)$ such that $f(a) = b$.

Remark 3.4. If T is semi-Hausdorff we may replace "λ_T-saturated model" by "e.c. model" in Definition 3.1 and Lemma 3.2; see [Ben-Yaacov 2003c, Proposition 3.13].

Convention 3.5. Recall that a *global type* is a type over the monster model $\mathfrak{M}$. Building on Convention 2.10 about the monster model, we will use lowercase Greek letters $\alpha, \beta, \ldots$ for realisations of global types (in a bigger monster).

Definition 3.6. A global type q is called *A-Ls-invariant*, short for *A-Lascar-invariant*, if for a realisation $\alpha \models q$ we have that $b \equiv_A^{\mathrm{Ls}} b'$ implies $\alpha b \equiv_A^{\mathrm{Ls}} \alpha b'$.

Note that this definition does not depend on the choice of α. If α' is any other realisation of q, then $\alpha \equiv_{\mathfrak{M}} \alpha'$. So there is an automorphism f of the bigger monster over $\mathfrak{M}$ with $f(\alpha) = \alpha'$. So if $b \equiv_A^{\mathrm{Ls}} b'$ then $\alpha b \equiv_A^{\mathrm{Ls}} \alpha b'$ and therefore $f(\alpha) f(b) \equiv_{f(A)}^{\mathrm{Ls}} f(\alpha) f(b')$, which is just $\alpha' b \equiv_A^{\mathrm{Ls}} \alpha' b'$, since f fixes $\mathfrak{M}$.

Remark 3.7. Let q be any global type in a thick theory, let $\alpha \models q$, and let A be any (small) parameter set. Then there is $a \in \mathfrak{M}$ with $a \equiv_A^{\mathrm{Ls}} \alpha$. To see this, let $M \supseteq A$ be a λ_T-saturated model, and take any $a \models q|_M$.

Lemma 3.8. *Suppose that q is a global A-Ls-invariant type in a thick theory. Then:*

(i) *For any $f \in \mathrm{Aut}(\mathfrak{M}/A)$ the type $f(q)$ is A-Ls-invariant.*

(ii) *For any $B \supseteq A$, q is also B-Ls-invariant.*

Proof. Point (i) is straightforward. We prove (ii). Let $\alpha \models q$ and $b \equiv_B^{\mathrm{Ls}} b'$. Then there are λ_T-saturated models $M_1, \ldots, M_n$, all containing B, and $b = b_0, \ldots, b_n = b'$ such that $b_i \equiv_{M_{i+1}} b_{i+1}$ for all $0 \leq i < n$. Letting $0 \leq i < n$, it is enough to

show $\alpha b_i \equiv_{M_{i+1}} \alpha b_{i+1}$. We have $b_i M_{i+1} \equiv_A^{\mathrm{Ls}} b_{i+1} M_{i+1}$, so by A-Ls-invariance, $\alpha b_i M_{i+1} \equiv_A^{\mathrm{Ls}} \alpha b_{i+1} M_{i+1}$, which implies the desired result. $\qquad \square$

Lemma 3.9. *Let T be thick, and let $p = \mathrm{tp}(a/B)$ be a coheir over $M \subseteq B$. Then there is a global M-Ls-invariant type extending p.*

Proof. Define
$$\Gamma(x) = p(x) \cup \bigcup \{\mathrm{d}_M(xc, xc') \le 1 : c, c' \in \mathfrak{M} \text{ with } \mathrm{d}_M(c, c') \le 1\}.$$

We claim that $\Gamma(x)$ is consistent. For finite $p_0(x) \subseteq p(x)$ there is $d \in M$ such that $d \models p_0$. Then for any c and c' with $\mathrm{d}_M(c, c') \le 1$ we have that $\mathrm{d}_M(dc, dc') \le 1$ because d is in M. Any maximal extension of $\Gamma(x)$ will be a desired global M-Ls-invariant type. $\qquad \square$

Definition 3.10. For $A \subseteq B$ we say that $\mathrm{Lstp}(c/B)$ *extends* $\mathrm{Lstp}(c'/A)$ if $c \equiv_A^{\mathrm{Ls}} c'$.

Corollary 3.11. *In a thick theory we have that $\mathrm{Lstp}(a/M)$ extends to a global M-Ls-invariant type for any a and M.*

Proof. By Lemma 3.9 we have that $p = \mathrm{tp}(a/M)$ extends to some global M-Ls-invariant type q. For $\alpha \models q$ let $a' \equiv_M^{\mathrm{Ls}} \alpha$. Then there is $f \in \mathrm{Aut}(\mathfrak{M}/M)$ such that $f(a') = a$. So by Lemma 3.8(i), $f(q)$ is global M-Ls-invariant and is exactly what we need. $\qquad \square$

Definition 3.12. For a type $p = \mathrm{tp}(a/Cb)$ write $a \underset{C}{\overset{i\mathrm{Ls}}{\downarrow}} b$ if there is a global C-Ls-invariant extension of p.

Proposition 3.13. *In any thick theory T we have*
$$a \underset{C}{\overset{u}{\downarrow}} b \implies a \underset{C}{\overset{i\mathrm{Ls}}{\downarrow}} b \implies a \underset{C}{\overset{f}{\downarrow}} b \implies a \underset{C}{\overset{d}{\downarrow}} b.$$

Proof. This is standard, but we write out the arguments to check they hold with the slightly changed definitions for positive logic. The first implication is precisely Lemma 3.9, while the last implication is direct from the definition of dividing and forking.

We prove the middle implication. Assume $a \underset{C}{\overset{i\mathrm{Ls}}{\downarrow}} b$ and suppose for a contradiction that $p(x) = \mathrm{tp}(a/Cb)$ forks over C. Let $\Phi(x)$ be a set of formulas that all divide over C such that $p(x)$ implies $\bigvee \Phi(x)$. Let q be a global C-Ls-invariant extension of p, and let $\alpha \models q$. Then there must be $\varphi(x, d) \in \Phi(x)$ such that $\models \varphi(\alpha, d)$. Let $(d_i)_{i < \omega}$ be C-indiscernible with $d_0 = d$. For all $i < \omega$ we have $d \equiv_C^{\mathrm{Ls}} d_i$ and thus $\alpha d \equiv_C^{\mathrm{Ls}} \alpha d_i$. So in particular $\alpha \models \{\varphi(x, d_i) : i < \omega\}$, which contradicts that $\varphi(x, d)$ divides over C. $\qquad \square$

In the remainder of this section we will develop tensoring of global Ls-invariant types. This comes down to verifying that the usual constructions for global invariant types (see, e.g., [Simon 2015, Section 2.2.1]) work when we carefully replace types by Lascar strong types everywhere.

Lemma 3.14. *Suppose that T is thick, q is a global A-Ls-invariant type and $p = \mathrm{Lstp}(a^*/A)$. Then, for $\beta \models q$, the set*

$$R_{p,q}(A) = \{(a, b) \in \mathfrak{M} : a \equiv^{\mathrm{Ls}}_A a^* \text{ and } b \equiv^{\mathrm{Ls}}_{Aa} \beta\}$$

is (the set of realisations of) a Lascar strong type over A.

Proof. Clearly this does not depend on the choice of a^* or β. The set is nonempty, as for any $b \equiv^{\mathrm{Ls}}_{Aa^*} \beta$ we have $(a^*, b) \in R_{p,q}(A)$.

Let $(a, b), (a', b') \in R_{p,q}(A)$. Then $a \equiv^{\mathrm{Ls}}_A a^* \equiv^{\mathrm{Ls}}_A a'$, so by A-Ls-invariance $ab \equiv^{\mathrm{Ls}}_A a\beta \equiv^{\mathrm{Ls}}_A a'\beta \equiv^{\mathrm{Ls}}_A a'b'$. Conversely, suppose $(a, b) \in R_{p,q}(A)$ and $ab \equiv^{\mathrm{Ls}}_A a'b'$. Then $a' \equiv^{\mathrm{Ls}}_A a \equiv^{\mathrm{Ls}}_A a^*$. Furthermore, by A-Ls-invariance $\beta ab \equiv^{\mathrm{Ls}}_A \beta a'b'$, so applying an automorphism to $b \equiv^{\mathrm{Ls}}_{Aa} \beta$ we get $b' \equiv^{\mathrm{Ls}}_{Aa'} \beta$ and thus $(a', b') \in R_{p,q}(A)$. □

Theorem 3.15. *Suppose T is thick with global A-Ls-invariant types q and r. Then there is a unique global A-Ls-invariant type $q \otimes r$ such that for any $\alpha \models q$, $\beta \models r$ and $(\alpha', \beta') \models q \otimes r$, the following are equivalent for all $B \supseteq A$ and all a and b:*

(i) *$ab \equiv^{\mathrm{Ls}}_B \alpha'\beta'$.*

(ii) *$a \equiv^{\mathrm{Ls}}_B \alpha$ and $b \equiv^{\mathrm{Ls}}_{Ba} \beta$.*

In particular, this implies that also $\alpha' \models q$ and $\beta' \models r$.

Proof. Throughout, let $\alpha \models q$ and $\beta \models r$. For $B \supseteq A$, denote by q_B the Lascar strong type $\mathrm{Lstp}(\alpha/B)$. By Lemma 3.8(ii) and Lemma 3.14, we have a well-defined Lascar strong type $R_{q_B,r}(B)$.

Claim. *For $A \subseteq B \subseteq C$ we have $R_{q_C,r}(C) \subseteq R_{q_B,r}(B)$.*

Proof of claim. Let $(a, b) \in R_{q_C,r}(C)$. Then $a \equiv^{\mathrm{Ls}}_C \alpha$ and $b \equiv^{\mathrm{Ls}}_{Ca} \beta$. Hence $a \equiv^{\mathrm{Ls}}_B \alpha$ and $b \equiv^{\mathrm{Ls}}_{Ba} \beta$, so $(a, b) \in R_{q_B,r}(B)$. □

For $M \supseteq A$ a λ_T-saturated model $R_{q_M,r}(M)$ corresponds to the usual syntactic type over M. So viewing $R_{q_M,r}(M)$ as a set of formulas over M, we get, by the claim, that the following is a well-defined global type:

$$q \otimes r := \bigcup \{R_{q_M,r}(M) : M \text{ is a } \lambda_T\text{-saturated model and } A \subseteq M\}.$$

First we verify that $q \otimes r$ satisfies the universal property we claimed. So let $(\alpha', \beta') \models q \otimes r$ and $B \supseteq A$. Let $M \supseteq B$ be a λ_T-saturated model and pick $a'b' \equiv^{\mathrm{Ls}}_M \alpha'\beta'$. Then by construction $(a', b') \in R_{q_M,r}(M)$ and so by the claim $(a', b') \in R_{q_B,r}(B)$. So for any a and b we have $ab \equiv^{\mathrm{Ls}}_B \alpha'\beta'$ if and only if $ab \equiv^{\mathrm{Ls}}_B a'b'$ if and only if $(a, b) \in R_{q_B,r}(B)$ if and only if $a \equiv^{\mathrm{Ls}}_B \alpha$ and $b \equiv^{\mathrm{Ls}}_{Ba} \beta$.

Uniqueness follows because any global type satisfying this universal property must restrict to $R_{q_M,r}(M) = (q \otimes r)|_M$ for all λ_T-saturated $M \supseteq A$.

Finally we prove A-Ls-invariance. Let $d \equiv_A^{\mathrm{Ls}} d'$, and pick a and b in $\mathfrak{M}$ such that $ab \equiv_{Add'}^{\mathrm{Ls}} \alpha'\beta'$. So $a \equiv_{Add'}^{\mathrm{Ls}} \alpha'$ and thus, by A-Ls-invariance of q,

$$ad \equiv_A^{\mathrm{Ls}} \alpha'd \equiv_A^{\mathrm{Ls}} \alpha'd' \equiv_A^{\mathrm{Ls}} ad'.$$

Then A-Ls-invariance of r gives us $\beta'ad \equiv_A^{\mathrm{Ls}} \beta'ad'$. From the universal property we get $b \equiv_{Add'a}^{\mathrm{Ls}} \beta'$, so $abd \equiv_A^{\mathrm{Ls}} abd'$. Because, by assumption, $ab \equiv_{Add'}^{\mathrm{Ls}} \alpha'\beta'$, we conclude that $\alpha'\beta'd \equiv_A^{\mathrm{Ls}} \alpha'\beta'd'$ and we are done. □

Lemma 3.16. *For any global A-Ls-invariant types p, q, r in a thick theory we have*:

(i) *Associativity*: $(p \otimes q) \otimes r = p \otimes (q \otimes r)$.

(ii) *Monotonicity*: *For any $q'(x_0) = q(x_0, x_1)|_{x_0} \subseteq q(x_0, x_1)$ and any $r'(y_0) = r(y_0, y_1)|_{y_0} \subseteq r(y_0, y_1)$, we have $q' \otimes r' \subseteq q \otimes r$.*

Proof. (i) Let $(\alpha, \beta, \gamma) \models (p \otimes q) \otimes r$ and $(\alpha', \beta', \gamma') \models p \otimes (q \otimes r)$. We will prove that $\alpha\beta\gamma \equiv_B^{\mathrm{Ls}} \alpha'\beta'\gamma'$ for all $B \supseteq A$. Let $abc \equiv_B^{\mathrm{Ls}} \alpha\beta\gamma$. Then $b \equiv_{Ba}^{\mathrm{Ls}} \beta$ and $c \equiv_{Bab}^{\mathrm{Ls}} \gamma$. So we have $bc \equiv_{Ba}^{\mathrm{Ls}} \beta'\gamma'$. Since also $a \equiv_B^{\mathrm{Ls}} \alpha$ we thus conclude that $abc \equiv_B^{\mathrm{Ls}} \alpha'\beta'\gamma'$.

(ii) Let $(\alpha, \beta) = ((\alpha_0, \alpha_1), (\beta_0, \beta_1)) \models q \otimes r$, and let $ab \equiv_B^{\mathrm{Ls}} \alpha\beta$, where $B \supseteq A$ is arbitrary. Then in particular $a_0 \equiv_B^{\mathrm{Ls}} \alpha_0$ and $b_0 \equiv_{Ba_0}^{\mathrm{Ls}} \beta_0$. So if we let $(\alpha', \beta') \models q' \otimes r'$ then $\alpha_0\beta_0 \equiv_B^{\mathrm{Ls}} a_0b_0 \equiv_B^{\mathrm{Ls}} \alpha'\beta'$. So $(\alpha_0, \beta_0) \models q' \otimes r'$ and we are done. □

Definition 3.17. For a global A-Ls-invariant type, we define $q^{\otimes\delta}$ for an ordinal $\delta \geq 1$ by induction as follows:

- $q^{\otimes 1} = q$,
- $q^{\otimes\delta+1} = q^{\otimes\delta} \otimes q$,
- $q^{\otimes\delta} = \bigcup_{\gamma < \delta} q^{\otimes\gamma}$ when δ is a limit.

A *Morley sequence in q (over A)* is a sequence $(a_i)_{i<\delta}$ such that $(a_i)_{i<\delta} \equiv_A^{\mathrm{Ls}} (\alpha_i)_{i<\delta}$, where $(\alpha_i)_{i<\delta} \models q^{\otimes\delta}$.

Note that we define Morley sequences in terms of Lascar strong types here. So saying that $(a_i)_{i<\omega}$ is a Morley sequence in q over A is generally a stronger statement than just saying $(a_i)_{i<\omega} \models q^{\otimes\omega}|_A$. Of course, if A is a λ_T-saturated model in a thick theory then the two coincide.

Lemma 3.18. *Suppose that q is a global A-Ls-invariant type, and let $(\alpha_i)_{i<\delta} \models q^{\otimes\delta}$. Then for any strictly increasing sequence $(i_\eta)_{\eta<\gamma}$ in δ we have that $(\alpha_{i_\eta})_{\eta<\gamma} \models q^{\otimes\gamma}$.*

Proof. From the construction of $q^{\otimes\delta}$ it is clear that for $\gamma < \delta$ and $(\alpha_i)_{i<\delta} \models q^{\otimes\delta}$ we have $(\alpha_i)_{i<\gamma} \models q^{\otimes\gamma}$.

We prove the lemma by induction to γ. The base case and the limit step are easy, so we prove the successor step. So suppose that $(\alpha_{i_\eta})_{\eta<\gamma} \models q^{\otimes\gamma}$. We will prove that $(\alpha_{i_\eta})_{\eta<\gamma}\alpha_{i_\gamma} \equiv_B^{\mathrm{Ls}} \alpha_{<\gamma}\alpha_\gamma$ for all $B \supseteq A$. Let $a_{\leq i_\gamma} \equiv_B^{\mathrm{Ls}} \alpha_{\leq i_\gamma}$. Then in particular $(a_{i_\eta})_{\eta<\gamma} \equiv_B^{\mathrm{Ls}} (\alpha_{i_\eta})_{\eta<\gamma}$ and $a_{i_\gamma} \equiv_{B(a_{i_\eta})_{\eta<\gamma}}^{\mathrm{Ls}} \alpha_{i_\gamma}$. By the induction hypothesis

and the universal property this means $(a_{i_\eta})_{\eta<\gamma} a_{i_\gamma} \equiv^{\mathrm{Ls}}_B \alpha_{<\gamma} \alpha_\gamma$, which concludes the successor step. $\qquad\square$

By Lemma 3.18, $(a_i)_{i<\delta} \models q^{\otimes\delta}|_A$ if and only if $(a_{i_1}, \ldots, a_{i_n}) \models q^{\otimes n}|_A$ for all $i_1 < \cdots < i_n < \delta$. From this perspective it makes sense to make the following convention, even though we technically have not defined $q^{\otimes I}$ for arbitrary linear orders I.

Convention 3.19. Let I be any linear order, and let q be a global A-Ls-invariant type. Then by $(a_i)_{i\in I} \models q^{\otimes I}|_A$ we mean that for any $i_1 < \cdots < i_n$ in I we have $(a_{i_1}, \ldots, a_{i_n}) \models q^{\otimes n}|_A$.

Proposition 3.20. *For any Morley sequence $(a_i)_{i<\delta}$ in a global A-Ls-invariant type q the following hold*:

(i) *For all $i < \delta$, $a_i \equiv^{\mathrm{Ls}}_{Aa_{<i}} \alpha$, where $\alpha \models q$.*

(ii) *$(a_i)_{i<\delta}$ is A-indiscernible.*

Proof. We first prove (i). Let $(\alpha_i)_{i<\delta} \models q^{\otimes\delta}$ and $i < \delta$. Then $a_{<i} a_i \equiv^{\mathrm{Ls}}_A \alpha_{<i} \alpha_i$. As $\alpha_{<i}\alpha_i \models q^{\otimes i} \otimes q$, the universal property yields $a_i \equiv^{\mathrm{Ls}}_{Aa_{<i}} \alpha_i$, as required.

For (ii), consider any $i_1 < \cdots < i_n < \delta$. By Lemma 3.18, $\alpha_{i_1} \cdots \alpha_{i_n} \equiv_{\mathfrak{M}} \alpha_1 \cdots \alpha_n$, so in particular $\alpha_{i_1} \cdots \alpha_{i_n} \equiv^{\mathrm{Ls}}_A \alpha_1 \cdots \alpha_n$. As $(a_i)_{i<\delta} \equiv_A (\alpha_i)_{i<\delta}$, we conclude that $a_{i_1} \cdots a_{i_n} \equiv_A a_1 \cdots a_n$. $\qquad\square$

4. Kim-dividing

The idea of Kim-dividing is to restrict dividing witnesses to nonforking Morley sequences. Proving the existence of such sequences over arbitrary sets turns out to be difficult, and is in fact an open problem for NSOP_1 theories in full first-order logic; see [Dobrowolski et al. 2022, Remark 2.6, Question 6.6]. In [Kaplan and Ramsey 2020] this is solved by using Morley sequences in some global invariant type. In full first-order logic any type over a model extends to a global invariant type. In positive logic we need to assume the theory to be semi-Hausdorff to find global invariant extensions [Ben-Yaacov 2003c, Lemma 3.11], because they may not exist otherwise (see Section 10A). In the more general setting of thick positive theories we can always find global Ls-invariant extensions and the notion of a Morley sequence makes sense in such a global Ls-invariant type; see Section 3. Since we can generally only extend types over e.c. models to global Ls-invariant types, we will consider Kim-dividing only over e.c. models (compare Question 10.21).

Definition 4.1. Let $\Sigma(x, b)$ be a partial type in a thick theory, possibly with parameters in M, and let q be a global M-Ls-invariant extension of $\mathrm{tp}(b/M)$. We say that $\Sigma(x, b)$ *q-divides over* M if for any (equivalently, some) Morley sequence $(b_i)_{i<\omega}$ in q (over M) the set $\bigcup_{i<\omega} \Sigma(x, b_i)$ is inconsistent.

By compactness q-dividing does not depend on the length of the Morley sequence, as long as it is infinite.

Proposition 4.2. *Let T be thick, let q be a global M-Ls-invariant extension of $\mathrm{tp}(b/M)$ and write $p(x, y) = \mathrm{tp}(ab/M)$. Then the following are equivalent*:

  (i) *The type $p(x, b)$ does not q-divide.*

 (ii) *For any $f \in \mathrm{Aut}(\mathfrak{M}/M)$ the type $p(x, b)$ does not $f(q)$-divide.*

(iii) *For any (equivalently, some) $(b_i)_{i<\omega} \models q^{\otimes\omega}|_M$ the set $\bigcup_{i<\omega} p(x, b_i)$ is consistent.*

(iv) *There is an Ma-indiscernible sequence $(b_i)_{i<\omega} \models q^{\otimes\omega}|_M$ with $b_0 = b$.*

Proof. (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii): This follows because consistency of $\bigcup_{i<\omega} p(x, b_i)$ only depends on $\mathrm{tp}((b_i)_{i<\omega}/M)$, together with the fact that given a Morley sequence $(b_i)_{i<\omega}$ in q we have that $(f(b_i))_{i<\omega}$ is a Morley sequence in $f(q)$.

(i) $\Longrightarrow$ (iv): Let $(b_i)_{i<\lambda}$ be a Morley sequence in q for big enough λ. Let a^* realise $\bigcup_{i<\lambda} p(x, b_i)$, and let $(b_i')_{i<\omega}$ be Ma^*-indiscernible, based on $(b_i)_{i<\lambda}$. So there is $i < \lambda$ such that $a^* b_0' \equiv_M a^* b_i \equiv_M ab$. Let $(b_i'')_{i<\omega}$ with $b_0'' = b$ be such that $a(b_i'')_{i<\omega} \equiv_M a^*(b_i')_{i<\omega}$. Then $(b_i'')_{i<\omega}$ is Ma-indiscernible. Furthermore, since $(b_i)_{i<\lambda}$ was already M-indiscernible, we have $(b_i'')_{i<\omega} \equiv_M (b_i')_{i<\omega} \equiv_M (b_i)_{i<\omega}$, so $(b_i'')_{i<\omega} \models q^{\otimes\omega}|_M$.

(iv) $\Longrightarrow$ (iii): For such an Ma-indiscernible sequence $(b_i)_{i<\omega}$ we have, for all $i < \omega$, $ab = ab_0 \equiv_M ab_i$. So a realises $\bigcup_{i<\omega} p(x, b_i)$. $\square$

Proposition 4.3. *Let T be thick, let $\Sigma(x, b)$ be a partial type with parameters in M, and let q be a global M-Ls-invariant extension of $\mathrm{tp}(b/M)$. If $\Sigma(x, b)$ does not q-divide over M then there is a complete $p(x, b) \supseteq \Sigma(x, b)$ that does not q-divide over M.*

Proof. Let $(b_i)_{i<\lambda} \models q^{\otimes\lambda}|_M$ with $b_0 = b$. Then there is some $a \models \bigcup_{i<\lambda} \Sigma(x, b_i)$. Then, assuming we chose λ large enough, there is some $i_0 < \lambda$ such that for infinitely many $i < \lambda$ we have $ab_i \equiv_M ab_{i_0}$. Set $p(x, y) = \mathrm{tp}(ab_{i_0}/M)$. Then $p(x, b_{i_0})$ does not q-divide, while also $\Sigma(x, b_{i_0}) \subseteq p(x, b_{i_0})$. By invariance $p(x, b)$ does not q-divide. $\square$

The following lemma is the core of the connection between Kim-dividing and NSOP$_1$ theories. It tells us that q-dividing does not depend on the global Lascar-invariant type q. More discussion on the origins of this lemma can be found in [Kaplan and Ramsey 2020]. Briefly put, Kim [1998, Proposition 2.1] proved that in simple theories a formula divides with respect to every Morley sequence if and only if it divides with respect to some Morley sequence. The lemma below is an analogue of that for NSOP$_1$ theories.

Proposition 4.4 (Kim's lemma). *If T is thick NSOP_1, then q-dividing does not depend on q. That is, if q and r are global M-invariant types extending $\mathrm{tp}(b/M)$ then a partial type $\Sigma(x, b)$ q-divides if and only if it r-divides.*

Proof. This is essentially the proof of [Kaplan and Ramsey 2020, Proposition 3.15], adapted to the thick positive logic setting. By Proposition 4.2(ii) we may assume that q and r extend $\mathrm{Lstp}(b/M)$. Suppose that $\Sigma(x, b)$ does not q-divide, while it r-divides. We will prove that T has SOP_1. Let $(\bar{b}_i)_{i<\omega} = (b_{i,0}, b_{i,1})_{i<\omega}$ be a Morley sequence in $q \otimes r$. By Lemma 3.16(ii) and induction, $(b_{i,0})_{i<\omega}$ and $(b_{i,1})_{i<\omega}$ are Morley sequences in q and r respectively.

Since $\Sigma(x, b)$ r-divides, the set $\bigcup_{i<\omega} \Sigma(x, b_{i,1})$ is inconsistent. So by compactness there is an M-formula $\varphi(x, y) \in \Sigma(x, y)$ such that $\{\varphi(x, b_{i,1}) : i < \omega\}$ is inconsistent. Because $\Sigma(x, b)$ does not q-divide we have that $\{\varphi(x, b_{i,0}) : i < \omega\}$ is consistent.

We wish to apply the parallel sequences lemma (Lemma 2.28) to $\varphi(x, y)$ and $(\bar{b}_i)_{i<\omega^{\mathrm{op}}}$, where ω^{op} carries the opposite order of ω. So we are left to prove that $b_{i,0} \equiv_{M\bar{b}_{>i}} b_{i,1}$ for all $i < \omega$. We do so by proving that $b_{i,0}(\bar{b}_i)_{i<j<n} \equiv_M b_{i,1}(\bar{b}_i)_{i<j<n}$ for all $i < n < \omega$. Let $(\bar{\beta}_i)_{i<\omega} \models (q \otimes r)^{\otimes\omega}$. By Lemma 3.16(i) we have $(q \otimes r)^{\otimes n} = (q \otimes r)^{\otimes i+1} \otimes (q \otimes r)^{\otimes n-i-1}$. So we have $\bar{\beta}_{<n} \models (q \otimes r)^{\otimes i+1} \otimes (q \otimes r)^{\otimes n-i-1}$ and because $\bar{b}_{<n} \equiv_M^{\mathrm{Ls}} \bar{\beta}_{<n}$ we have $(\bar{b}_j)_{i<j<n} \equiv_{M\bar{b}_{\leq i}}^{\mathrm{Ls}} (\bar{\beta}_j)_{i<j<n}$. As $b_{i,0} \equiv_M^{\mathrm{Ls}} b \equiv_M^{\mathrm{Ls}} b_{i,1}$, we get, by M-Ls-invariance, that $b_{i,0}(\bar{\beta}_j)_{i<j<n} \equiv_M^{\mathrm{Ls}} b_{i,1}(\bar{\beta}_j)_{i<j<n}$. Putting the two together yields the required result. □

Definition 4.5. We say $\Sigma(x, b)$ *Kim-divides* (*over M*) if it q-divides for some global M-Ls-invariant q that extends $\mathrm{tp}(b/M)$. We write $a \underset{M}{\overset{K}{\downarrow}} b$ when $\mathrm{tp}(a/Mb)$ does not Kim-divide over M and call this *Kim-independence*.

Remark 4.6. By Lemma 3.9 we can extend any type over an e.c. model M in a thick theory to a global M-Ls-invariant type. So assuming NSOP_1, we have by Proposition 4.4 that $\mathrm{tp}(a/Mb)$ Kim-divides if and only if it q-divides for *any* global M-invariant extension q of $\mathrm{tp}(b/M)$.

In some constructions it will be necessary to stay within the same Lascar strong type. For this we introduce the technical tool of q-Ls-dividing.

Definition 4.7. Let T be thick, and let q be a global M-Ls-invariant extension of $\mathrm{Lstp}(b/M)$. We say that $\mathrm{Lstp}(a/Mb)$ *does not q-Ls-divide* (*over M*) if there is a Morley sequence $(b_i)_{i<\omega}$ in q with $b_0 = b$ that is Ma-indiscernible.

Remark 4.8. The length of the Morley sequence does not matter in Definition 4.7, as long as it is infinite. However, the argument here takes a little more care than for q-dividing.

One direction is clear: if there is an Ma-indiscernible Morley sequence $(b_i)_{i<\delta}$ in q for some $\delta \geq \omega$, then we can just take an initial segment. For the other direction

we let $N \supseteq M$ be λ_T-saturated and $(b_i)_{i<\omega} \models q^{\otimes\omega}|_N$. Then $(b_i)_{i<\omega}$ is a Morley sequence in q. Applying a Lascar strong automorphism we find $a'b_0 \equiv^{\mathrm{Ls}}_M ab$ such that $(b_i)_{i<\omega}$ is Ma'-indiscernible. Let n be such that $\mathrm{d}_M(a'b_0, ab) \le n$. Consider the set of formulas

$$q^{\otimes\delta}|_N((y_i)_{i<\delta}) \cup \text{``}(xy_i)_{i<\delta} \text{ is } M\text{-indiscernible''} \cup \mathrm{d}_M(xy_0, ab) \le n.$$

This set is finitely satisfiable, and hence it has a realisation. So we find an Ma''-indiscernible Morley sequence $(b'_i)_{i<\delta}$ in q with $a''b'_0 \equiv^{\mathrm{Ls}}_M ab$. The result follows by applying a Lascar strong automorphism.

Lemma 4.9. *Let T be thick, and let q be a global M-Ls-invariant extension of $\mathrm{Lstp}(b/M)$. A type $p = \mathrm{tp}(a/Mb)$ does not q-divide if and only if there is a realisation $a' \models p$ such that $\mathrm{Lstp}(a'/Mb)$ does not q-Ls-divide.*

Proof. The right-to-left direction is clear by Proposition 4.2(iv). For the other direction we let $(b'_i)_{i<\omega}$ be a Morley sequence in q with $b'_0 = b$. By Proposition 4.2(iv) there is $(b_i)_{i<\omega} \models q^{\otimes\omega}|_M$ that is Ma-indiscernible with $b_0 = b$. Pick a' such that $a'(b'_i)_{i<\omega} \equiv_M a(b_i)_{i<\omega}$ and we are done. $\square$

Corollary 4.10. *Let T be thick, and let q be a global M-Ls-invariant extension of $\mathrm{Lstp}(b/M)$. Suppose that there is $M \subseteq N \subseteq b$ such that N is λ_T-saturated. Then $\mathrm{tp}(a/Mb)$ does not q-divide if and only if $\mathrm{Lstp}(a/Mb)$ does not q-Ls-divide.*

Proof. By Lemma 4.9 we only need to prove the left-to-right direction. So suppose that $\mathrm{tp}(a/Mb)$ does not q-divide. Then there is a' with $a' \equiv_{Mb} a$ such that $\mathrm{Lstp}(a'/Mb)$ does not q-Ls-divide. In particular, we have that $a'b \equiv_N ab$, so $a'b \equiv^{\mathrm{Ls}}_M ab$. It follows that $\mathrm{Lstp}(a/Mb)$ does not q-Ls-divide. $\square$

Proposition 4.11. *In a thick NSOP_1 theory Kim-independence always satisfies the following properties*:

(i) *Strong finite character*: *If $a \not\underset{M}{\overset{K}{\smile}} b$, then there is a formula $\varphi(x, b, m)$ in $\mathrm{tp}(a/Mb)$ such that for any $a' \models \varphi(x, b, m)$ we have $a' \not\underset{M}{\overset{K}{\smile}} b$.*

(ii) *Existence over models*: $a \underset{M}{\overset{K}{\smile}} M$.

(iii) *Monotonicity*: $aa' \underset{M}{\overset{K}{\smile}} bb' \Longrightarrow a \underset{M}{\overset{K}{\smile}} b$.

Proof. All follow directly from the definitions, using compactness for (i). $\square$

Remark 4.12. Let T be a thick theory. Then Kim-dividing implies dividing because any Morley sequence in some q is in particular an indiscernible sequence. So by Proposition 3.13,

$$a \underset{M}{\overset{u}{\smile}} b \implies a \underset{M}{\overset{i\mathrm{Ls}}{\smile}} b \implies a \underset{M}{\overset{f}{\smile}} b \implies a \underset{M}{\overset{d}{\smile}} b \implies a \underset{M}{\overset{K}{\smile}} b.$$

Proposition 4.13. *Let T be a thick theory and M an e.c. model of T, and let a, b, c be tuples. Let also $q(x, y)$ be a global M-Ls-invariant extension of $\mathrm{Lstp}(bc/M)$*

and write $r(x) = q|_x$. If $\mathrm{Lstp}(a/Mb)$ does not r-Ls-divide then there is $c^*b \equiv^{\mathrm{Ls}}_M cb$ such that $\mathrm{Lstp}(a/Mbc^*)$ does not q-Ls-divide.

Proof. Let $(b_i, c_i)_{i<\lambda}$ be a Morley sequence over M in q for some big enough λ. Since $(b_i)_{i<\lambda}$ is a Morley sequence over M in r and $\mathrm{Lstp}(a/Mb)$ does not r-divide there is a' with $a'b_0 \equiv^{\mathrm{Ls}}_M ab$ such that $(b_i)_{i<\lambda}$ is Ma'-indiscernible.

Let $f \in \mathrm{Aut}_f(\mathfrak{M}/M)$ be such that $f(a'b_0) = ab$ and put $(b'_i, c'_i) = (f(b_i), f(c_i))$. Then $b'_0 = b$, $(b'_i)_{i<\lambda}$ is Ma-indiscernible and $(b'_i, c'_i)_{i<\lambda}$ is a Morley sequence over M in q.

Let $M' \supseteq Ma$ be λ_T-saturated and use Lemma 2.21 to find M'-indiscernible $(b''_i, c''_i)_{i<\lambda}$ based on $(b'_i, c'_i)_{i<\lambda}$ and such that $\mathrm{d}_M((b''_i, c''_i)_{i<\lambda}, (b'_i, c'_i)_{i<\lambda}) \leq 1$. In particular, $(b''_i, c''_i)_{i<\lambda}$ is a Morley sequence over M in q. Let $i < \lambda$ be such that $b''_0 \equiv_{M'} b'_i$. Then $b''_0 \equiv^{\mathrm{Ls}}_{Ma} b'_i \equiv^{\mathrm{Ls}}_{Ma} b'_0 = b$. So there is $g \in \mathrm{Aut}_f(\mathfrak{M}/Ma)$ such that $g(b''_0) = b$. Set $c^* = g(c''_0)$. So $bc^* \equiv^{\mathrm{Ls}}_M b''_0 c''_0 \equiv^{\mathrm{Ls}}_M b'_0 c'_0 \equiv^{\mathrm{Ls}}_M b_0 c_0 \equiv^{\mathrm{Ls}}_M bc$. Finally, since $(g(b''_i), g(c''_i))_{i<\lambda}$ is a Morley sequence over M in q starting with bc^* that is Ma-indiscernible, we conclude that $\mathrm{Lstp}(a/Mbc^*)$ does not q-Ls-divide. $\square$

Corollary 4.14 (extension). *In a thick NSOP_1 theory we have that if $a \mathop{\smile\hskip-0.9em\lower0.35ex\hbox{$\vert$}}\nolimits^K_M b$ then for any c there is $c' \equiv^{\mathrm{Ls}}_{Mb} c$ such that $a \mathop{\smile\hskip-0.9em\lower0.35ex\hbox{$\vert$}}\nolimits^K_M bc'$.*

Proof. We first prove a weaker version where we conclude $c' \equiv_{Mb} c$ instead of $c' \equiv^{\mathrm{Ls}}_{Mb} c$.

Let $q(x, y)$ be an M-Ls-invariant extension of $\mathrm{Lstp}(bc/M)$ and write $r(x) = q|_x$, where x matches b. Since $a \mathop{\smile\hskip-0.9em\lower0.35ex\hbox{$\vert$}}\nolimits^K_M b$ there is $a'b \equiv_M ab$ such that $\mathrm{Lstp}(a'/Mb)$ does not r-Ls-divide. By Proposition 4.13 we thus find $bc^* \equiv^{\mathrm{Ls}}_M bc$ such that $\mathrm{Lstp}(a'/Mbc^*)$ does not q-Ls-divide. Letting c' be such that $abc' \equiv_M a'bc^*$, then c' satisfies $a \mathop{\smile\hskip-0.9em\lower0.35ex\hbox{$\vert$}}\nolimits^K_M bc'$ and furthermore we have $bc' \equiv_M bc^* \equiv_M bc$.

Now we use the weaker version to prove the full version. Let $N \supseteq Mb$ be some λ_T-saturated model. By the above we can find $N' \equiv_{Mb} N$ such that $a \mathop{\smile\hskip-0.9em\lower0.35ex\hbox{$\vert$}}\nolimits^K_M N'$. Then using the above again we find $c' \equiv_{N'} c$ such that $a \mathop{\smile\hskip-0.9em\lower0.35ex\hbox{$\vert$}}\nolimits^K_M N'c'$. Since $Mb \subseteq N'$ we thus get $c' \equiv^{\mathrm{Ls}}_{Mb} c$ and $a \mathop{\smile\hskip-0.9em\lower0.35ex\hbox{$\vert$}}\nolimits^K_M bc'$, as required. $\square$

5. EM-modelling and parallel-Morley sequences

In this section we will introduce some tools which will be useful later in certain tree constructions.

Definition 5.1 [Kim et al. 2014, Defintion 2.1]. The *Shelah language*

$$L_s = \{\trianglelefteq, \wedge, <_{\mathrm{lex}}, (P_\alpha)_{\alpha<\omega}\}$$

consists of binary relation symbols $\trianglelefteq$ and $<_{\mathrm{lex}}$, a binary function symbol $\wedge$, and unary relation symbols P_α. We will consider a tree $\omega^{\leq k}$ (with $k < \omega$) as

an L_s-structure, where $\trianglelefteq$ is interpreted as the containment relation, $<_{\text{lex}}$ as the lexicographic order, $\wedge$ as the meet function and P_α as the α-th level of the tree.

Definition 5.2 [Kim et al. 2014, Definition 3.7]. Let I be an arbitrary index structure and C an arbitrary set of parameters. The EM-*type* of a tuple $A = (a_i)_{i \in I}$ over C is the partial type in variables $(x_i)_{i \in I}$, consisting of all the formulas of the form $\varphi(x_{\bar{i}})$ over C (where $\bar{i}$ is a tuple in I) satisfying the following property: $\models \varphi(a_{\bar{j}})$ holds whenever $\bar{j}$ is a tuple in I with $\text{qftp}_I(\bar{j}) = \text{qftp}_I(\bar{i})$. We let $\text{EM}_I(A/C)$ denote this partial type.

In particular, we write $\text{EM}_s(A/C)$ (respectively, $\text{EM}_<(A/C)$) for $\text{EM}_I(A/C)$, where I is considered as an L_s-structure (respectively, a $\{<\}$-structure).

Definition 5.3. Let I be an index structure, and let $A = (a_i)_{i \in I}$ and $B = (b_i)_{i \in I}$ be I-indexed tuples of compatible parameters. We will say that A is EM_I-*based on B over C* if $\text{EM}_I(A/C) \supseteq \text{EM}_I(B/C)$.

Corollary 5.4. *If A is any set of parameters, then for any compatible sequence $(a_i)_{i<\omega}$ there is an A-indiscernible sequence $(b_i)_{i<\omega}$ which is $\text{EM}_<$-based on $(a_i)_{i<\omega}$ over A.*

Proof. By compactness there is a sequence $(a_i')_{i<\lambda_{|T|+|A|+|a_0|}}$ which is $\text{EM}_<$-based on $(a_i)_{i<\omega}$ over A. Then by Lemma 2.17 there is an A-indiscernible sequence $(b_i)_{i<\omega}$ which is $\text{EM}_<$-based on $(a_i')_{i<\lambda_{|T|+|A|+|a_0|}}$ over A, hence $\text{EM}_<$-based on $(a_i)_{i<\omega}$ over A. $\qquad\square$

In what follows we consider $\omega^{\leq k}$ as an L_s-structure (see Definition 5.1). We will only work with trees of width ω, as we will only need those, but everything naturally works for arbitrary (infinite) widths.

Definition 5.5. We call a tree $(a_\eta)_{\eta \in \omega^{\leq k}}$ *s-indiscernible over C* if for any $\bar{\eta}, \bar{\nu} \subseteq \omega^{\leq k}$ such that $\bar{\eta} \equiv_{\text{qf}} \bar{\nu}$ we have that $a_{\bar{\eta}} \equiv_C a_{\bar{\nu}}$.

Lemma 5.6. *Suppose $\bar{\eta} = (\eta_0, \ldots, \eta_{n-1}) \equiv_{\text{qf}} \bar{\nu} = (\nu_0, \ldots, \nu_{n-1})$ are tuples of elements of $\omega^{\leq k}$ for some $k < \omega$. Then there exists a sequence I of n-tuples of elements of $\omega^{\leq k}$ such that $\bar{\eta} \frown I$ and $\bar{\nu} \frown I$ are qf-indiscernible sequences in $\omega^{\leq k}$.*

Proof. Let $l < \omega$ be such that $\bar{\eta}, \bar{\nu} \subseteq \{\varnothing\} \cup \{\xi \in \omega^{\leq k} \setminus \{\varnothing\} : \xi(0) < l\}$. For every $0 < m < \omega$ choose a tuple $\bar{\chi}^m \subseteq \{\varnothing\} \cup \{\xi \in \omega^{\leq k} \setminus \{\varnothing\} : ml \leq \xi(0) < (m+1)l\}$ such that $\bar{\chi}^m \equiv_{\text{qf}} \bar{\eta} \equiv_{\text{qf}} \bar{\nu}$ (for example, for every $n' < n$ put $\chi_{n'}^m(0) = \eta_{n'}(0) + ml$ and $\chi_{n'}^m(i) = \eta_{n'}(i)$ for every $0 < i \leq k$). Finally, put $I = (\bar{\chi}^m)_{0<m<\omega}$. $\qquad\square$

Corollary 5.7. *If T is thick then s-indiscernibility is type-definable, i.e., for every $k < \omega$ and a tuple of variables y there is a partial type $\pi((x_\eta)_{\eta \in \omega^{\leq k}}, y)$ over $\varnothing$ such that for all D with $|D| = |y|$, $((a_\eta)_{\eta \in \omega^{\leq k}}, D) \models \pi$ if and only if $(a_\eta)_{\eta \in \omega^{\leq k}}$ is s-indiscernible over D.*

More specifically, we can take $\pi((x_\eta)_{\eta \in \omega^{\leq k}}, y)$ to be the partial type that expresses that for any $(\eta_0, \ldots, \eta_{n-1}) \equiv_{\mathrm{qf}} (\nu_0, \ldots, \nu_{n-1})$ the Lascar distance of $(x_{\eta_0}, \ldots, x_{\eta_{n-1}})$ and $(x_{\nu_0}, \ldots, x_{\nu_{n-1}})$ over y is at most 2.

Proof. Let π be as above. Consider arbitrary $(a_\eta)_{\eta \in \omega^{\leq k}}$ and D. If $((a_\eta)_{\eta \in \omega^{\leq k}}, D) \models \pi$ then $(a_\eta)_{\eta \in \omega^{\leq k}}$ is indiscernible over D, as being at Lascar distance at most 2 over D implies equality of types over D.

Conversely, if $((a_\eta)_{\eta \in \omega^{\leq k}}, D)$ is s-indiscernible over D and

$$\bar{\eta} = (\eta_0, \ldots, \eta_{n-1}) \equiv_{\mathrm{qf}} \bar{\nu} = (\nu_0, \ldots, \nu_{n-1}),$$

then with $I = (\bar{\chi}^m)_{0 < m < \omega}$ given by Lemma 5.6 we have that $a_{\bar{\eta}} \frown (a_{\bar{\chi}^m})_{0 < m < \omega}$ and $a_{\bar{\nu}} \frown (a_{\bar{\chi}^m})_{0 < m < \omega}$ are both indiscernible sequences over D, so $a_{\bar{\eta}}$ and $a_{\bar{\nu}}$ are at Lascar distance at most 2 over D. $\qquad\square$

We now adapt the proof of [Kim et al. 2014, Theorem 4.3] to obtain the EM_s-modelling property for positive logic.

Proposition 5.8. *Suppose T is thick and consider an arbitrary set of parameters D and $k < \omega$. Then for any tree $A = (a_\eta)_{\eta \in \omega^{\leq k}}$ of compatible tuples there is an s-indiscernible over D tree $C = (c_\eta)_{\eta \in \omega^{\leq k}}$ which is EM_s-based on A over D.*

Proof. We proceed by induction on k. The case $k = 0$ is trivial. Suppose the assertion holds for some k and consider any $A = (a_\eta)_{\eta \in \omega^{\leq k+1}}$. For any $i < \omega$ consider an $\omega^{\leq k}$-indexed tree $A_i := (a_{i \frown \eta})_{\eta \in \omega^{\leq k}}$. Using the inductive hypothesis we choose inductively for each $i < \omega$ a tree $B_i = (b_\eta^i)_{\eta \in \omega^{\leq k}}$ which is s-indiscernible over $Da_\varnothing B_{<i} A_{>i}$ and EM_s-based on A_i over $Da_\varnothing B_{<i} A_{>i}$. Let $B = (b_\eta)_{\eta \in \omega^{\leq k+1}}$, where $b_\varnothing = a_\varnothing$ and $b_{i \frown \xi} = b_\xi^i$ for every $i < \omega$ and $\xi \in \omega^{\leq k}$.

Claim. *B_i is s-indiscernible over $Db_\varnothing B_{\neq i}$ for every $i < \omega$.*

Proof of claim. Fix $i < \omega$. We will show by induction on j that B_i is s-indiscernible over $Db_\varnothing B_{<i} B_{i+1} \cdots B_{j-1} A_{\geq j}$ for every $j > i$, which is enough by Corollary 5.7. For $j = i+1$ this follows directly from the choice of B_i. Now suppose the assertion holds for some $j > i$. By Corollary 5.7 there is a type $\pi((x_\eta)_{\eta \in \omega^{\leq k}}, \bar{y})$ over $D' := Db_\varnothing B_{<i} B_{i+1} \cdots B_{j-1} A_{>j}$, where $\bar{y} = (y_\eta)_{\eta \in \omega^{\leq k}}$, expressing that $(x_\eta)_{\eta \in \omega^{\leq k}}$ is s-indiscernible over $D'\bar{y}$. Then $B_i A_j \models \pi$. Note that the type $\pi(B_i, \bar{y})$ is invariant under all permutations of $\bar{y}$, and therefore if $\varphi(y_{\eta_0}, \ldots, y_{\eta_{n-1}}) \in \pi(B_i, \bar{y})$ then $\varphi(y_{\nu_0}, \ldots, y_{\nu_{n-1}}) \in \mathrm{tp}(A_j/D'B_i)$ for all $\nu_0, \ldots, \nu_{n-1} \in \omega^{\leq k}$. In particular, $\pi(B_i, \bar{y}) \subseteq \mathrm{EM}_s(A_j/D'B_i)$. Thus, by the choice of B_j, we have that $\pi(B_i, \bar{y}) \subseteq \mathrm{EM}_s(B_j/D'B_i)$, so in particular $B_i B_j \models \pi$. Hence B_i is indiscernible over $D'B_j = Db_\varnothing B_{<i} B_{i+1} \cdots B_j A_{\geq j+1}$, as required. $\qquad\square$

Claim. *B is EM_s-based on A over D.*

Proof of claim. Consider any $i < \omega$ and the trees $E = (e_\eta)_{\eta \in \omega^{\leq k+1}}$ and $F = (f_\eta)_{\eta \in \omega^{\leq k+1}}$ given by

$$e_\varnothing = f_\varnothing = a_\varnothing, \quad e_{j \frown \eta} = \begin{cases} b_{j \frown \eta} & \text{for } j < i, \\ a_{j \frown \eta} & \text{for } j \geq i, \end{cases} \quad \text{and} \quad f_{j \frown \eta} = \begin{cases} b_{j \frown \eta} & \text{for } j \leq i, \\ a_{j \frown \eta} & \text{for } j > i. \end{cases}$$

We will prove that $\pi_0 := \mathrm{EM}_s(E/D) \subseteq \mathrm{EM}_s(F/D) =: \pi_1$, which clearly is sufficient to prove the claim. Let $\bar{x} = (x_\eta)_{\eta \in \omega^{\leq k+1}}$ be a tuple of variables compatible with the a_η's. We naturally view π_0 and π_1 as partial types in the variable $\bar{x}$. Consider any formula

$$\varphi(x_{\eta_0}, \ldots, x_{\eta_l}, x_{\eta_{l+1}}, \ldots, x_{\eta_{l'}}) \in \pi_0$$

over D with

$$\eta_0, \ldots, \eta_l \in K_i := \{i \frown \xi : \xi \in \omega^{\leq k}\} \quad \text{and} \quad \eta_{l+1}, \ldots, \eta_{l'} \in \omega^{\leq k+1} \setminus K_i.$$

We will be done if we show

$$\models \varphi(f_{\eta_0}, \ldots, f_{\eta_{l'}}).$$

Write $\eta_t = i \frown \xi_t$ for $t = 0, 1, \ldots, l$. For any $\xi'_0, \ldots, \xi'_l \in \omega^{\leq k}$ with

$$\mathrm{qftp}_{L_s}(\xi'_0, \ldots, \xi'_l) = \mathrm{qftp}_{L_s}(\xi_0, \ldots, \xi_l),$$

we have

$$\mathrm{qftp}_{L_s}(\eta_0, \ldots, \eta_{l'}) = \mathrm{qftp}_{L_s}(i \frown \xi_0, \ldots, i \frown \xi_l, \eta_{l+1}, \ldots, \eta_{l'})$$
$$= \mathrm{qftp}_{L_s}(i \frown \xi'_0, \ldots, i \frown \xi'_l, \eta_{l+1}, \ldots, \eta_{l'}),$$

so, as $\varphi \in \pi_0$, we get that $\models \varphi(e_{i \frown \xi'_0}, \ldots, e_{i \frown \xi'_l}, e_{\eta_{l+1}}, \ldots, e_{\eta_{l'}})$. This shows that

$$\varphi(y_{\xi_0}, \ldots, y_{\xi_l}, e_{\eta_{l+1}}, \ldots, e_{\eta_{l'}}) \in \mathrm{EM}_s(A_i/a_\varnothing A_{<i} B_{>i}),$$

where A_i is naturally indexed by $\omega^{\leq k}$, so, by the choice of B_i, we get that

$$\models \varphi(b^i_{\xi_0}, \ldots, b^i_{\xi_l}, e_{\eta_{l+1}}, \ldots, e_{\eta_{l'}}).$$

Because

$$(b^i_{\xi_0}, \ldots, b^i_{\xi_l}, e_{\eta_{l+1}}, \ldots, e_{\eta_{l'}}) = (f_{i \frown \xi_0}, \ldots, f_{i \frown \xi_l}, f_{\eta_{l+1}}, \ldots, f_{\eta_{l'}}) = (f_{\eta_0}, \ldots, f_{\eta_{l'}}),$$

this means that $\models \varphi(f_{\eta_0}, \ldots, f_{\eta_{l'}})$, as required. □

By Corollary 5.4 we find a sequence $(C_i)_{i < \omega} = ((c^i_\eta)_{\eta \in \omega^{\leq k}})_{i < \omega}$ which is $\mathrm{EM}_<$-based on $(B_i)_{i < \omega}$ over $Db_\varnothing$ and indiscernible over $Db_\varnothing$. Let $C = (c_\eta)_{\eta \in \omega^{\leq k+1}}$ be given by $c_\varnothing = b_\varnothing$ and $c_{i \frown \xi} = c^i_\xi$ for any $\xi \in \omega^{\leq k}$ and $i < \omega$. By the first claim on page 76 and Corollary 5.7 we get that C_i is s-indiscernible over $C_{\neq i} D c_\varnothing$ for every $i < \omega$, which, together with $Dc_\varnothing$-indiscernibility of $(C_i)_{i < \omega}$, easily gives that C is s-indiscernible over D (as in [Kim et al. 2014]). It is left to prove:

Claim. *C is EM_s-based on B (and hence on A) over D.*

Proof of claim. Consider any formula $\varphi(x_{i_1 \frown \xi_1}, \ldots, x_{i_l \frown \xi_l}, x_\varnothing) \in \mathrm{EM}_s(B/D)$ with $i_1, \ldots, i_l \in \omega$ and $\xi_1, \ldots, \xi_l \in \omega^{\leq k}$. Then for every $j_1, \ldots, j_l \in \omega$ with $\mathrm{qftp}_{\{<\}}(j_1, \ldots, j_l) = \mathrm{qftp}_{\{<\}}(i_1, \ldots, i_l)$, we have that

$$\mathrm{qftp}_{L_s}(j_1 \frown \xi_1, \ldots, j_l \frown \xi_l, \varnothing) = \mathrm{qftp}_{L_s}(i_1 \frown \xi_1, \ldots, i_l \frown \xi_l, \varnothing),$$

so $\models \varphi(b_{j_1 \frown \xi_1}, \ldots, b_{j_l \frown \xi_l}, b_\varnothing)$. This means that

$$\varphi(x_{i_1 \frown \xi_1}, \ldots, x_{i_l \frown \xi_l}, b_\varnothing) \in \mathrm{EM}_<((B_i)_{i<\omega}/b_\varnothing D),$$

and therefore, by the choice of C, we have that $\models \varphi(c_{i_1 \frown \xi_1}, \ldots, c_{i_l \frown \xi_l}, c_\varnothing)$, and thus $\varphi(x_{i_1 \frown \xi_1}, \ldots, x_{i_l \frown \xi_l}, x_\varnothing) \in \mathrm{EM}_s(C/D)$, as required. $\qquad\square$

Definition 5.9. Let I be a linearly ordered set. For a global M-Ls-invariant type q, we will call a sequence $(a_i)_{i \in I}$ a *parallel-Morley sequence in q over M* if there is some $(b_i)_{i \in I} \models q^{\otimes I}|_M$ such that the pair (a_i, b_i) starts an $M a_{>i} b_{>i}$-indiscernible sequence for every $i \in I$. We will say that $(a_i)_{i \in I}$ is a parallel-Morley sequence in $\mathrm{tp}(a/M)$ if it is a parallel-Morley sequence in some global M-Ls-invariant type $q \supseteq \mathrm{tp}(a/M)$.

In the semi-Hausdorff case we can replace the condition "(a_i, b_i) starts an $M a_{>i} b_{>i}$-indiscernible sequence" by "$a_i \equiv_{M a_{>i} b_{>i}} b_i$". The reason for which we need the stronger condition in thick theories is that equality of types is not necessarily type-definable there, so some of the compactness arguments below would not work with the weaker condition.

Note that a parallel-Morley sequence is *not* required to be indiscernible. The reason for the name "parallel-Morley sequence" is because such a sequence is parallel to a Morley sequence, in the sense of the parallel sequences lemma (Lemma 2.28). We make this precise in Corollary 5.11, for which we first slightly reformulate the parallel sequences lemma.

Lemma 5.10. *Let T be thick and suppose $\varphi(x, y)$ is a formula and $(c_{i,0}, c_{i,1})_{i \in I}$ is an infinite sequence of pairs with $(c_{i,1})_{i \in I}$ indiscernible, such that*

(i) *for every $i \in I$, the pair $(c_{i,0}, c_{i,1})$ starts a $c_{>i,0} c_{>i,1}$-indiscernible sequence;*

(ii) *$\{\varphi(x; c_{i,0}) : i \in I\}$ is consistent;*

(iii) *$\{\varphi(x; c_{i,1}) : i \in I\}$ is inconsistent.*

Then T has SOP_1.

Proof. We may assume the tuples $c_{i,0}$ and $c_{i,1}$ to be finite. As $(c_{i,1})_{i \in I}$ is indiscernible and $\{\varphi(x, c_{i,1}) : i \in I\}$ is inconsistent, there is some $\psi(y_1, \ldots, y_k)$ that implies $\neg\exists x(\varphi(x, y_1) \wedge \cdots \wedge \varphi(x, y_k))$ such that for any $i_1 < \cdots < i_k \in I$ we have $\models \psi(c_{i_1,1}, \ldots, c_{i_k,1})$. Call this ψ-inconsistent. By compactness there is a sequence of pairs $(\bar{c}'_i)_{i<\lambda_T} = (c'_{i,0}, c'_{i,1})_{i<\lambda_T}$ such that $(c'_{i,0}, c'_{i,1})$ starts a $\bar{c}'_{>i}$-indiscernible
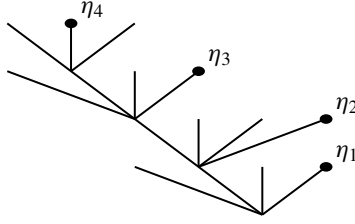
Figure 1. An example of η_i's from Definition 5.12.

sequence for every $i < \lambda_T$, $\{\varphi(x, c'_{i,0}) : i < \lambda_T\}$ is consistent and $\{\varphi(x, c'_{i,1}) : i < \lambda_T\}$ is ψ-inconsistent. Then an indiscernible sequence based on $(\bar{c}'_i)_{i<\lambda_T}$ will satisfy the assumptions of Lemma 2.28, so T has SOP$_1$. $\square$

By Kim's lemma (Proposition 4.4) and Lemma 5.10 we easily get the following.

Corollary 5.11. *Suppose T is thick* NSOP$_1$ *with an e.c. model M, $\Sigma(x, b)$ is a partial type, I is an infinite linearly ordered set, and $(b_i)_{i \in I}$ a parallel-Morley sequence in* tp(b/M). *If $\bigcup \{\Sigma(x, b_i) : i \in I\}$ is consistent then $\Sigma(x, b)$ does not Kim-divide over M. If $(b_i)_{i \in I}$ is indiscernible over M, then the converse also holds.*

Definition 5.12. Let M be an e.c. model and q a global M-Ls-invariant type.

(i) We say that a tree $(c_\eta)_{\eta \in \omega^{\leq k}}$ is *q-spread-out over M* if for any $\eta_1 \in \omega^1$, $\eta_2 \in \omega^2, \ldots, \eta_k \in \omega^k$ such that

$$\eta_1 >_{\text{lex}} \eta_2 >_{\text{lex}} \cdots >_{\text{lex}} \eta_k \quad \text{and} \quad (\forall l < l' \leq k)(\eta_{l'} \wedge \eta_l \in \omega^{l-1}),$$

we have that $(c_{\eta_k}, \ldots, c_{\eta_1})$ is a Morley sequence in q over M.

(ii) We say that $(c_\eta)_{\eta \in \omega^{\leq k}}$ is *weakly q-spread-out over M* if $(c_{\eta_k}, \ldots, c_{\eta_1}) \models q^{\otimes k}|_M$ for η_i's as in (i).

Clearly q-spread-outness implies weak q-spread-outness. We will freely use the above definition for trees of parameters indexed by trees naturally isomorphic to trees of the form $\omega'^{\leq k'}$, e.g., subtrees of $\omega^{\leq k}$ consisting of all nodes extending a fixed node.

The point of the conditions on the η_i's in Definition 5.12 is that this is quantifier-free definable by an L_s-formula. This is useful for preservation when EM$_s$-basing trees on one another, as we do in the following lemma.

Lemma 5.13. *Let k be a natural number, M an e.c. model and q a global M-Ls-invariant type.*

(i) *If $((c_{i \frown \eta})_{\eta \in \omega^{\leq k-1}})_{i < \omega}$ is a Morley sequence in a global M-Ls-invariant type $r(x, z) \supseteq q(x)$ over M, where x corresponds to the elements c_i and where $(c_{0 \frown \eta})_{\eta \in \omega^{\leq k-1}}$ is q-spread-out over M then also $(c_\eta)_{\eta \in \omega^{\leq k}}$ is q-spread-out over M for any choice of root $c_\varnothing$.*
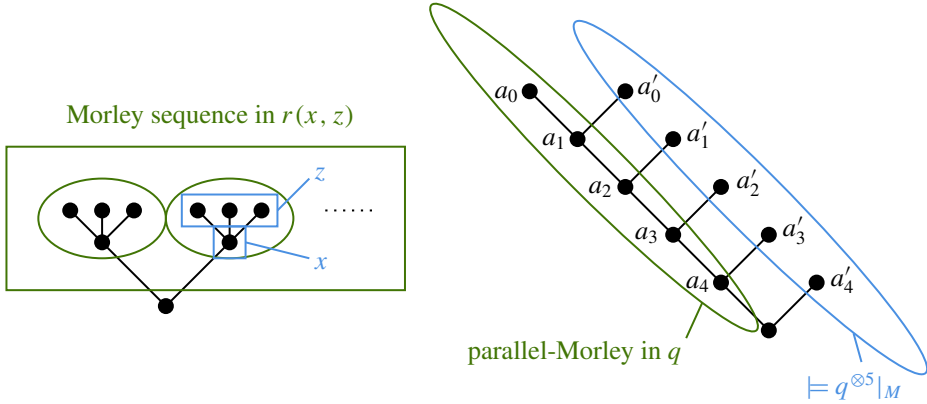
Figure 2. Lemma 5.13(i), left, and Lemma 5.13(iii), right.

(ii) *If $(c_\eta)_{\eta \in \omega^{\leq k}}$ is weakly q-spread-out over M and*

$$(c'_\eta)_{\eta \in \omega^{\leq k}} \models \mathrm{EM}_s((c_\eta)_{\eta \in \omega^{\leq k}}/M),$$

then also $(c'_\eta)_{\eta \in \omega^{\leq k}}$ is weakly q-spread-out over M.

(iii) *If $(c_\eta)_{\eta \in \omega^{\leq k}}$ is weakly q-spread-out over M and s-indiscernible over M, then for $a_i = c_{0^{k-i}}$ we have that $(a_i)_{i<k}$ is a parallel-Morley sequence in q over M.*

Proof. (i) Let $\eta_k \in \omega^k, \ldots, \eta_1 \in \omega^1$ be such that

$$\eta_1 >_{\text{lex}} \cdots >_{\text{lex}} \eta_k \quad \text{and} \quad (\forall l < l' \leq k)(\eta_{l'} \wedge \eta_l \in \omega^{l-1}).$$

We will prove that $(c_{\eta_k}, \ldots, c_{\eta_1})$ is a Morley sequence in q. For each $\ell \geq 2$, let $\beta_\ell \in \omega^1$ be such that $\eta_\ell \trianglerighteq \beta_\ell$. For every $\ell > 2$, we have by assumption that $\eta_2 \wedge \eta_\ell = \eta_2|_1 = \beta_2$, and hence $\beta_\ell = \beta_2 =: \beta$ (and $\eta_1 >_{\text{lex}} \beta$ as $\eta_1 >_{\text{lex}} \eta_2$). In particular, $(c_{\eta_k}, \ldots, c_{\eta_2})$ is contained in $(c_{\beta \frown \eta})_{\eta \in \omega^{\leq k-1}}$, which has the same Lascar strong type over M as $(c_{0 \frown \eta})_{\eta \in \omega^{\leq k-1}}$. So, as $(c_{0 \frown \eta})_{\eta \in \omega^{\leq k-1}}$ is q-spread-out by assumption, $(c_{\eta_k}, \ldots, c_{\eta_2})$ is a Morley sequence in q. As $((c_{i \frown \eta})_{\eta \in \omega^{\leq k-1}})_{i<\omega}$ is a Morley sequence in r, we have that $(c_{\eta_1 \frown \eta})_{\eta \in \omega^{\leq k-1}}$, which contains c_{η_1}, has the same Lascar strong type over $M(c_{\beta \frown \eta})_{\eta \in \omega^{\leq k-1}}$, which contains $Mc_{\eta_k} \cdots c_{\eta_2}$, as some realisation of r. Since $q(x) = r|_x$ we see that c_{η_1} has the same Lascar strong type over $Mc_{\eta_k}, \ldots, c_{\eta_2}$ as some realisation of q. So we conclude that $(c_{\eta_k}, \ldots, c_{\eta_1})$ is indeed a Morley sequence in q.

(ii) This holds because the condition on $(\eta_1, \ldots, \eta_k)$ in the definition of weak q-spread-outness is expressible by a quantifier-free L_s-formula.

(iii) Put $a'_i := c_{0^{k-i-1} \frown 1}$ for $i < k$. Then $(a'_i)_{i<k} \models q^{\otimes k}|_M$ by weak q-spread-outness, and (a_i, a'_i) starts an $M_{a_{>i} a'_{>i}}$-indiscernible sequence for each $i < k$ by s-indiscernibility. $\qquad \square$

6. Symmetry

Lemma 6.1 (chain condition). *Let T be a thick NSOP_1 theory, and let M be an e.c. model. Let $(b_i)_{i<\kappa}$ be a Morley sequence in some global M-Ls-invariant $q(x)$. If $(b_i)_{i<\kappa}$ is Ma-indiscernible then $a \underset{M}{\overset{K}{\smile}} (b_i)_{i<\kappa}$.*

Proof. We will prove that $a \underset{M}{\overset{K}{\smile}} b_{i_1} \cdots b_{i_k}$ for all $i_1 < \cdots < i_k < \kappa$. This is indeed enough by finite character. By Ma-indiscernibility of $(b_i)_{i<\kappa}$ we may assume $\{i_1, \ldots, i_k\} = \{0, \ldots, k-1\}$.

We have $(b_i)_{i<\omega} \equiv_M^{\mathrm{Ls}} (\beta_i)_{i<\omega}$ for some $(\beta_i)_{i<\omega} \models q^{\otimes\omega}$. Define the tuple $\gamma_i = (\beta_{ik}, \beta_{ik+1}, \ldots, \beta_{ik+k-1})$ for all $i < \omega$. Then $(\gamma_i)_{i<\omega} \models (q^{\otimes k})^{\otimes\omega}$ by associativity of tensoring (Lemma 3.16). We let $c_i = (b_{ik}, b_{ik+1}, \ldots, b_{ik+k-1})$ for all $i < \omega$. Then $(c_i)_{i<\omega} \equiv_M^{\mathrm{Ls}} (\gamma_i)_{i<\omega}$. So $(c_i)_{i<\omega}$ is a Morley sequence in $q^{\otimes k}$ over M and $(c_i)_{i<\omega}$ is Ma-indiscernible. So $\mathrm{tp}(a/Mc_0) = \mathrm{tp}(a/Mb_0 \cdots b_{k-1})$ does not $q^{\otimes k}$-divide, and thus $a \underset{M}{\overset{K}{\smile}} b_0 \cdots b_{k-1}$, as required. $\qquad\square$

Definition 6.2. Suppose M is an e.c. model, q a global type extending $\mathrm{Lstp}(a/M)$ and λ a cardinal. We will say that the extension $q \supseteq \mathrm{Lstp}(a/M)$ satisfies $(*)_\lambda$ if for every c with $|c| \leq \lambda$ there is a global M-Ls-invariant type $r(x, y) \supseteq \mathrm{Lstp}(ac/M)$ extending $q(x)$ (in particular, q is M-Ls-invariant).

Lemma 6.3. *For any e.c. model M, tuple a and cardinal λ there is $q \supseteq \mathrm{Lstp}(a/M)$ satisfying $(*)_\lambda$.*

Proof. Let M, a and λ be as in the statement. Choose a small tuple d such that for any c with $|c| \leq \lambda$ there is some $d' \subseteq d$ with $\mathrm{Lstp}(ad'/M) = \mathrm{Lstp}(ac/M)$ (this is possible as the number of Lascar types of tuples of fixed length over M is bounded by Lemma 2.20). Now take a global M-Ls-invariant extension $r(x, y)$ of $\mathrm{Lstp}(ad/M)$, where x corresponds to a. Then $q := r|_x$ is an extension of $\mathrm{Lstp}(a/M)$ satisfying $(*)_\lambda$. $\qquad\square$

Remark 6.4. If $q \supseteq \mathrm{Lstp}(a/M)$ is finitely satisfiable in M then it satisfies $(*)_\lambda$ for any cardinal λ [Mennuni 2020, Lemma 3.4]. However, finitely satisfiable extensions may not exist in thick theories.

Theorem 6.5 (symmetry). *In a thick NSOP_1 theory, $a \underset{M}{\overset{K}{\smile}} b$ implies $b \underset{M}{\overset{K}{\smile}} a$.*

Proof. We may assume that b enumerates a λ_T-saturated model containing M. If this is not the case, let $N \supseteq Mb$ be a λ_T-saturated model. By extension, Corollary 4.14, we find $N' \equiv_{Mb} N$ such that $a \underset{M}{\overset{K}{\smile}} N'$. Now we replace b by N' and we continue the proof.

Set $\lambda = |ab|$. By Lemma 6.3 we can choose a global extension $q \supseteq \mathrm{Lstp}(a/M)$ satisfying $(*)_\lambda$. Let $p(y, a) = \mathrm{tp}(b/Ma)$. We will show that there is a parallel-Morley sequence $(a_i)_{i<\omega}$ in q over M such that $\bigcup_{i<\omega} p(y, a_i)$ is consistent, which

is enough by Corollary 5.11. All the properties we wish $(a_i)_{i<\omega}$ to have are type-definable. It is thus enough to find such a sequence of length k for every $k < \omega$.

So fix any $k < \omega$. By backward induction on $k' = k+1, k, \ldots, 1$ we will define trees $(c_\eta)_{\eta \in S_{k'}}$, where $S_{k'} = \{\xi \in \omega^{\leq k+1} : 0^{k'-1} \trianglelefteq \xi\}$. We will write $S_{k'}^*$ for $S_{k'}$ without the root, so $S_{k'}^* = S_{k'} - \{0^{k'-1}\}$. For each k' the tree $(c_\eta)_{\eta \in S_{k'}}$ will satisfy the following conditions:

(A1)$_{k'}$ $c_\eta c_\nu \equiv_M^{\mathrm{Ls}} ab$ for all $\nu \rhd \eta \in S_{k'}$ with $\nu \in \omega^{k+1}$ and $\eta \in \omega^{\leq k}$.

(A2)$_{k'}$ $(c_\eta)_{\eta \in S_{k'} \cap \omega^{\leq k}}$ is q-spread-out over M.

(A3)$_{k'}$ We have $c_{0^{k'-1}} \underset{M}{\overset{K}{\displaystyle\bigcup}} (c_\eta)_{\eta \in S_{k'}^*}$ (the root is independent from the rest).

For $k' = k+1$ we let t be a global M-Ls-invariant extension of $\mathrm{Lstp}(b/M)$. Since $a \underset{M}{\overset{K}{\displaystyle\bigcup}} b$ we have that $\mathrm{tp}(a/Mb)$ does not t-divide. By Corollary 4.10 and our assumption on b, this means that $\mathrm{Lstp}(a/Mb)$ does not t-Ls-divide. So we find an Ma-indiscernible Morley sequence $(c_{0^k \frown \alpha})_{\alpha < \omega}$ in t with $c_{0^{k+1}} = b$. By Lemma 6.1, we have that $a \underset{M}{\overset{K}{\displaystyle\bigcup}} (c_{0^k \frown \alpha})_{\alpha < \omega}$. So we pick $c_{0^k} = a$ and directly satisfy (A3)$_{k'}$. Condition (A2)$_{k'}$ is vacuous and (A1)$_{k'}$ follows directly from Ma-indiscernibility of $(c_{0^k \frown \alpha})_{\alpha < \omega}$ and the fact that $c_{0^{k+1}} = b$.

For the inductive step, suppose we have constructed $(c_\eta)_{\eta \in S_{k'}}$. By (A1)$_{k'}$ there is a tuple d such that $c_{0^{k'-1}}(c_\eta)_{\eta \in S_{k'}^*} \equiv_M^{\mathrm{Ls}} ad$. So, by $(*)_\lambda$, there is a global M-Ls-invariant type $r(x,z) \supseteq q(x)$ extending $\mathrm{Lstp}(c_{0^{k'-1}}(c_\eta)_{\eta \in S_{k'}^*}/M)$. By (A3)$_{k'}$ we have that $c_{0^{k'-1}} \underset{M}{\overset{K}{\displaystyle\bigcup}} (c_\eta)_{\eta \in S_{k'}^*}$. So since $b \subseteq (c_\eta)_{\eta \in S_{k'}^*}$ and using our assumption on b we have by Corollary 4.10 that $\mathrm{Lstp}(c_{0^{k'-1}}/M(c_\eta)_{\eta \in S_{k'}^*})$ does not $r|_z$-Ls-divide. By extension for Ls-dividing, Proposition 4.13, we find c such that $c(c_\eta)_{\eta \in S_{k'}^*} \equiv_M^{\mathrm{Ls}} c_{0^{k'-1}}(c_\eta)_{\eta \in S_{k'}^*}$ and $\mathrm{Lstp}(c/M(c_\eta)_{\eta \in S_{k'}})$ does not r-Ls-divide. So there is an Mc-indiscernible Morley sequence $((d_{\eta,i})_{\eta \in S_{k'}})_{i < \omega}$ in r such that $(d_{\eta,0})_{\eta \in S_{k'}} = (c_\eta)_{\eta \in S_{k'}}$. We set $c_{0^{k'-2}} = c$ and $c_{0^{k'-2} \frown i \frown \zeta} = d_{0^{k'-1} \frown \zeta, i}$. Again, using Lemma 6.1 we directly get (A3)$_{k'-1}$.

Now (A2)$_{k'-1}$ follows from Lemma 5.13(i). We verify (A1)$_{k'-1}$. Everything above the root consists of copies (via a Lascar strong automorphism over M) of $(c_\eta)_{\eta \in S_{k'}}$, so we only need to check that $c_{0^{k'-2}}c_\nu \equiv_M^{\mathrm{Ls}} ab$ for all $\nu \in S_{k'-1} \cap \omega^{k+1}$. By indiscernibility we may assume $\nu \in S_{k'} \cap \omega^{k+1}$. Then (A1)$_{k'-1}$ follows from (A1)$_{k'}$ and the fact that $c_{0^{k'-2}}(c_\eta)_{\eta \in S_{k'}^*} \equiv_M^{\mathrm{Ls}} c_{0^{k'-1}}(c_\eta)_{\eta \in S_{k'}^*}$.

Thus the inductive step, and hence the construction of the tree $(c_\eta)_{\eta \in \omega^{k+1}} = (c_\eta)_{\eta \in S_1}$, is completed.

Consider the following condition:

(A1$'$)$_1$ $c_\eta c_\nu \equiv_M ab$ for all $\nu \rhd \eta$ with $\nu \in \omega^{k+1}$ and $\eta \in \omega^{\leq k}$.

This condition is clearly implied by (A1)$_1$ as it is seen by the EM$_s$-type of $(c_\eta)_{\eta \in \omega^{\leq k+1}}$ over M. Let $(c'_\eta)_{\eta \in \omega^{\leq k+1}}$ be an s-indiscernible tree that is EM$_s$-based on $(c_\eta)_{\eta \in \omega^{\leq k+1}}$ over M, and we get that $(c'_\eta)_{\eta \in \omega^{\leq k+1}}$ satisfies (A1$'$)$_1$, and $(c'_\eta)_{\eta \in \omega^{\leq k}}$ is weakly q-spread-out over M by Lemma 5.13(ii).

Put $a_i = c'_{0^{k+1-i}}$. Then $(a_1, \ldots, a_k)$ is a parallel-Morley sequence in q over M by Lemma 5.13(iii), and by $(A1')_1$ we have that $\bigcup_{1 \leq i \leq k} p(y, a_i)$ is consistent because it is realised by $c'_{0^{k+1}}$. This completes the proof. $\qquad\square$

Lemma 6.6. *Let T be a thick theory. Suppose that $\varphi(x, y)$ has* SOP_1*, witnessed by $\psi(y_1, y_2)$. Then there is an e.c. model M and b_1, b_2, c_1, c_2 such that $c_1 \underset{M}{\overset{u}{\bigcup}} c_2$, $c_1 \underset{M}{\overset{u}{\bigcup}} b_1$, $c_2 \underset{M}{\overset{u}{\bigcup}} b_2$ and $b_1 c_1 \overset{\mathrm{Ls}}{\underset{M}{\equiv}} b_2 c_2$ and $\models \varphi(b_1, c_1) \wedge \varphi(b_2, c_2) \wedge \psi(c_1, c_2)$.*

Proof. The proof is mostly the same as that of [Haykazyan and Kirby 2021, Proposition A.7] but we have to adjust a few things throughout to get equality of Lascar strong types rather than just equality of types. As in that proof, we will use a Skolemisation technique for positive logic [Haykazyan and Kirby 2021, Lemma A.6]. In such a Skolemised theory the positively definable closure of any set is an e.c. model and the reduct of an e.c. model (to the original language) is an e.c. model (of the original theory). It is not directly clear whether this Skolemisation construction preserves thickness, but that is not a problem. Ultimately we are interested in Lascar strong types in our original theory. So even though we technically work in a Skolemised theory the (type-definable) predicate $\mathrm{d}(x, y) \leq 1$ should be taken as in our original theory.

Let κ be any cardinal. By compactness we find parameters $(a_\eta)_{\eta \in 2^{<\kappa}}$ such that

 (i) for every $\sigma \in 2^\kappa$ the set $\{\varphi(x, a_{\sigma|i}) : i < \kappa\}$ is consistent,

 (ii) for every $\eta, \nu \in 2^{<\kappa}$ such that $\eta^\frown 0 \preceq \nu$, we have $\models \psi(a_{\eta^\frown 1}, a_\nu)$.

For a big enough cardinal λ, we construct by induction a sequence $(\eta_i, \nu_i)_{i < \lambda}$ with $\eta_i, \nu_i \in 2^{<\kappa}$ such that

 (1) $\eta_i \trianglelefteq \eta_j$ and $\eta_i \trianglelefteq \nu_j$ for all $i < j < \lambda$,

 (2) $\eta_i \trianglerighteq (\eta_i \wedge \nu_i) \frown 0$, $\nu_i = (\eta_i \wedge \nu_i) \frown 1$, and (a_{η_i}, a_{ν_i}) starts an $a_{\eta_{<i}} a_{\nu_{<i}}$-indiscernible sequence for every $i < \lambda$.

Assume $(\eta_j, \nu_j)_{j<i}$ has been constructed and set $\eta = \bigcup_{j<i} \eta_j$. Assuming we chose κ large enough, then, by applying Lemma 2.17 to $(a_{\eta^\frown 0^\alpha \frown 1})_{\alpha>0}$, it follows that there are $0 < \alpha < \beta < \kappa$ such that $(a_{\eta^\frown 0^\alpha \frown 1}, a_{\eta^\frown 0^\beta \frown 1})$ starts an $\{\eta_j, \nu_j : j < i\}$-indiscernible sequence. We set $\nu_i = \eta^\frown 0^\alpha \frown 1$ and $\eta_i = \eta^\frown 0^\beta \frown 1$.

By (i) and (1), there is b_2 realising $\{\varphi(x, a_{\eta_i}) : i < \lambda\}$. Now let $(e_i, d_i)_{i < \omega+2}$ be indiscernible over b_2 based on $(a_{\eta_i}, a_{\nu_i})_{i<\lambda}$.

Let M be the positively definable closure of $\{e_i, d_i : i < \omega\}$. As discussed, we may assume M to be an e.c. model. Set $c_1 = d_\omega$ and $c_2 = e_{\omega+1}$. Then $c_1 \underset{\{e_i, d_i : i < \omega\}}{\overset{u}{\bigcup}} c_2$ and $c_2 \underset{\{e_i, d_i : i < \omega\}}{\overset{u}{\bigcup}} b_2$ by indiscernibility. So $c_1 \underset{M}{\overset{u}{\bigcup}} c_2$, $c_2 \underset{M}{\overset{u}{\bigcup}} b_2$ and $\models \varphi(b_2, c_2)$. By construction $c_1 c_2 = d_\omega e_{\omega+1} \equiv a_{\nu_{i_0}} a_{\eta_{i_1}}$ for some $i_0 < i_1 < \lambda$ and thus $\models \psi(c_1, c_2)$ by (ii), (1), and (2).

To find b_1 we first claim that $\mathrm{d}_M(e_\omega, d_\omega) \leq 1$. By compactness it suffices to prove that $\mathrm{d}_A(e_\omega, d_\omega) \leq 1$ for all finite $A \subseteq M$. By how we constructed M it then

suffices to prove that (e_ω, d_ω) starts an indiscernible sequence over $\{e_i, d_i : i < n\}$ for all $n < \omega$. To prove this last statement we let $i_0 < \cdots < i_{n+1} < \lambda$ be such that

$$e_0 d_0 \cdots e_n d_n e_\omega d_\omega \equiv a_{\eta_{i_0}} a_{v_{i_0}} \cdots a_{\eta_{i_n}} a_{v_{i_n}} a_{\eta_{i_{n+1}}} a_{v_{i_{n+1}}}.$$

By how we constructed $(\eta_i, v_i)_{i < \lambda}$ we have $(a_{\eta_{i_{n+1}}}, a_{v_{i_{n+1}}})$ starts an indiscernible sequence over $\{a_{\eta_{i_0}} a_{v_{i_0}} \cdots a_{\eta_{i_n}} a_{v_{i_n}}\}$. So the claim follows after applying the automorphism.

Now we leave the Skolemised theory and work in the original theory, in which $d(x, y) \leq 1$ corresponds to actually having Lascar distance one. We have that $c_2 = e_{\omega+1} \equiv^{\mathrm{Ls}}_M e_\omega \equiv^{\mathrm{Ls}}_M d_\omega = c_1$, so there is $f \in \mathrm{Aut}_f(\mathfrak{M}/M)$ such that $f(c_2) = c_1$. Let $b_1 = f(b_2)$. Then $c_2 b_2 \equiv^{\mathrm{Ls}}_M c_1 b_1$, and thus also $\models \varphi(b_1, c_1)$ and $c_1 \mathop{\smile}\limits_M^u b_1$, as required. $\qquad\square$

Theorem 6.7. *Let T be a thick theory. The following are equivalent:*

(i) *T is NSOP_1.*

(ii) *Symmetry: $a \mathop{\smile}\limits_M^K b$ implies $b \mathop{\smile}\limits_M^K a$.*

(iii) *Weak symmetry: $a \mathop{\smile}\limits_M^{i\mathrm{Ls}} b$ implies $b \mathop{\smile}\limits_M^K a$.*

Proof. Theorem 6.5 is precisely (i) $\Longrightarrow$ (ii). For (ii) $\Longrightarrow$ (iii) we just note that $a \mathop{\smile}\limits_M^{i\mathrm{Ls}} b$ implies $a \mathop{\smile}\limits_M^K b$. Finally, for (iii) $\Longrightarrow$ (i) we proceed as in [Kaplan and Ramsey 2020, Proposition 3.22] replacing their reference to [Chernikov and Ramsey 2016] by Lemma 6.6 and being careful about using global Ls-invariant types instead of just global invariant types.

We prove the contrapositive, so assume that T has SOP_1. Then, by Lemma 6.6, there is an e.c. model M and b_1, b_2, c_1, c_2 such that $c_1 \mathop{\smile}\limits_M^u c_2$, $c_1 \mathop{\smile}\limits_M^u b_1$, $c_2 \mathop{\smile}\limits_M^u b_2$ and $b_1 c_1 \equiv^{\mathrm{Ls}}_M b_2 c_2$. Furthermore, for $p(x, c_1) = \mathrm{tp}(b_1 c_1/M)$, we have that $p(x, c_1) \cup p(x, c_2)$ is inconsistent. In particular, we have that $\mathrm{Lstp}(c_1/Mc_2)$ extends to a global M-Ls-invariant q. Then as $c_1 \equiv^{\mathrm{Ls}}_M c_2$ there is a Morley sequence $(d_i)_{i < \omega}$ in q with $d_0 d_1 = c_2 c_1$. We thus have that $\bigcup\{p(x, d_i) : i < \omega\}$ is inconsistent. So $b_2 \mathop{\not\smile}\limits_M^K c_2$. Since also $c_2 \mathop{\smile}\limits_M^u b_2$ and therefore $c_2 \mathop{\smile}\limits_M^{i\mathrm{Ls}} b_2$, we see that weak symmetry fails and this concludes our proof. $\qquad\square$

7. Independence theorem

We recall the following facts. The first is the same as [Kaplan and Ramsey 2020, Lemma 7.4] and the second is the same as the claim in [Dobrowolski et al. 2022, Lemma 5.3]. Their proofs work in our setting as well.

Fact 7.1. *The following hold in any thick NSOP_1 theory.*

(i) *If $a \mathop{\smile}\limits_M^d bc$ and $b \mathop{\smile}\limits_M^K c$ then $ab \mathop{\smile}\limits_M^K c$.*

(ii) *If $a \mathop{\smile}\limits_M^K b$ and $a \mathop{\smile}\limits_M^K c$ then there is c' with $ac' \equiv_M ac$ such that $a \mathop{\smile}\limits_M^K bc'$.*

For the following lemma we borrow a trick from [Dobrowolski et al. 2022, Lemma 5.4].

Lemma 7.2. *Let T be thick* NSOP$_1$, *and let* $a \equiv^{\mathrm{Ls}}_M a'$, $a \mathrel{\underset{M}{\overset{K}{\smile}}} b$ *and* $a' \mathrel{\underset{M}{\overset{K}{\smile}}} c$. *Then there is c' such that* $ac' \equiv^{\mathrm{Ls}}_M a'c$ *and* $a \mathrel{\underset{M}{\overset{K}{\smile}}} bc'$.

Proof. Let c^* be such that $ac^* \equiv^{\mathrm{Ls}}_M a'c$, so $a \mathrel{\underset{M}{\overset{K}{\smile}}} c^*$. Let $N' \supseteq M$ be λ_T-saturated, and let q be a global M-Ls-invariant extension of $\mathrm{Lstp}(N'/M)$. Let N realise $q|_{Mabc^*}$, so we have $N \mathrel{\underset{M}{\overset{i\mathrm{Ls}}{\smile}}} abc^*$. By Fact 7.1(i), we then have $Na \mathrel{\underset{M}{\overset{K}{\smile}}} b$ and $Na \mathrel{\underset{M}{\overset{K}{\smile}}} c^*$. So by fact Fact 7.1(ii), we find c' with $Nac' \equiv_M Nac^*$ and $Na \mathrel{\underset{M}{\overset{K}{\smile}}} bc'$. We thus have $ac' \equiv^{\mathrm{Ls}}_M ac^* \equiv^{\mathrm{Ls}}_M a'c$, as required. $\qquad\square$

Definition 7.3. We write $b \mathrel{\underset{M}{\overset{*}{\smile}}} c$ to mean that $\mathrm{Lstp}(b/Mc)$ extends to a global M-Ls-invariant type $\mathrm{tp}(N/\mathfrak{M})$ for some $\beth_\omega(\lambda_T + |Mbc|)$-saturated model $N \supseteq M$. Extending $\mathrm{Lstp}(b/Mc)$ here means that there is some $\beta \in N$ with $\beta \equiv^{\mathrm{Ls}}_{Mc} b$.

The point of the enormous cardinal $\beth_\omega(\lambda_T + |Mbc|)$ is that we will want to find a λ_T-saturated model M' containing M and a copy of b in N, and then again some λ_T-saturated $M'' \supseteq M'$ inside N. By Fact 2.12 we can choose these λ_T-saturated models small enough so that this process can be repeated any finite number of times.

We easily see that $\mathrel{\overset{*}{\smile}}$ is invariant under automorphisms and, assuming thickness, that $b \mathrel{\underset{M}{\overset{*}{\smile}}} M$ for all M.

Lemma 7.4. *We have that $\mathrel{\overset{*}{\smile}}$ satisfies the following extension properties.*

(i) *Left extension*: *If $b \mathrel{\underset{M}{\overset{*}{\smile}}} c$ and $|d| < \beth_\omega(\lambda_T + |Mbc|)$, then there is $d' \equiv^{\mathrm{Ls}}_{Mb} d$ such that $bd' \mathrel{\underset{M}{\overset{*}{\smile}}} c$.*

(ii) *Right extension*: *If $b \mathrel{\underset{M}{\overset{*}{\smile}}} c$ and $|d| < \beth_\omega(\lambda_T + |Mbc|)$, then there is $d' \equiv^{\mathrm{Ls}}_{Mc} d$ such that $b \mathrel{\underset{M}{\overset{*}{\smile}}} cd'$.*

Proof. In both cases we assume $b \mathrel{\underset{M}{\overset{*}{\smile}}} c$. So let $q = \mathrm{tp}(N/\mathfrak{M})$ be a global M-Ls-invariant extension of $\mathrm{Lstp}(b/Mc)$ for some $\beth_\omega(\lambda_T + |Mbc|)$-saturated $N \supseteq M$.

We first prove left extension. Let $N' \equiv^{\mathrm{Ls}}_{Mc} N$ be in $\mathfrak{M}$. By moving things by a Lascar strong automorphism over Mc we may assume $b \in N'$. By Fact 2.12 there is $Mb \subseteq M' \subseteq N'$ where M' is λ_T-saturated and of cardinality $\leq 2^{\lambda_T + |Mb|}$. Let d' realise $\mathrm{tp}(d/M')$ in N'. Therefore, $d' \equiv^{\mathrm{Ls}}_{Mb} d$ while q also extends $\mathrm{Lstp}(bd'/Mc)$, so indeed $bd' \mathrel{\underset{M}{\overset{*}{\smile}}} c$.

Now we prove right extension. Let $\beta \in N$ be such that $\beta \equiv^{\mathrm{Ls}}_{Mc} b$. Pick $b' \in \mathfrak{M}$ such that $b' \equiv^{\mathrm{Ls}}_{Mcd} \beta$. Then clearly $b' \mathrel{\underset{M}{\overset{*}{\smile}}} cd$. We finish the proof by picking d' such that $bd' \equiv^{\mathrm{Ls}}_{Mc} b'd$. $\qquad\square$

Proposition 7.5 (weak independence theorem). *Let T be thick* NSOP$_1$. *Suppose that $a \equiv^{\mathrm{Ls}}_M a'$, $a \mathrel{\underset{M}{\overset{K}{\smile}}} b$, $a' \mathrel{\underset{M}{\overset{K}{\smile}}} c$ and $b \mathrel{\underset{M}{\overset{*}{\smile}}} c$. Then there is a'' with $a'' \equiv^{\mathrm{Ls}}_{Mb} a$ and $a'' \equiv^{\mathrm{Ls}}_{Mc} a'$ such that $a'' \mathrel{\underset{M}{\overset{K}{\smile}}} bc$.*

Proof. We may assume that b and c both enumerate a λ_T-saturated model containing M. If this is not the case, let $N \supseteq Mb$ be λ_T-saturated and such that $|N| < \beth_\omega(\lambda_T + |Mbc|)$. By left extension from Lemma 7.4 we then find $N' \equiv_{Mb}^{\mathrm{Ls}} N$ with $N' \underset{M}{\overset{*}{\bigcup}} c$. By Corollary 4.14 we find a_0 with $a_0 \equiv_{Mb}^{\mathrm{Ls}} a$ and $a_0 \underset{M}{\overset{K}{\bigcup}} N'$. Now we can replace a by a_0 and b by N' and continue the proof. The case for c is analogous.

By Lemma 7.2 there is c' such that $ac' \equiv_M^{\mathrm{Ls}} a'c$ and $a \underset{M}{\overset{K}{\bigcup}} bc'$. Apply left extension from Lemma 7.4 to $b \underset{M}{\overset{*}{\bigcup}} c$ and c' to find $c'' \equiv_{Mb}^{\mathrm{Ls}} c$ with $bc' \underset{M}{\overset{*}{\bigcup}} c''$. Let b^* be such that $b^* c'' \equiv_M^{\mathrm{Ls}} bc'$ and apply right extension from Lemma 7.4 to $bc' \underset{M}{\overset{*}{\bigcup}} c''$ and b^* to find $b'' \equiv_{Mc''}^{\mathrm{Ls}} b^*$ with $bc' \underset{M}{\overset{*}{\bigcup}} b''c''$. In particular, $b''c'' \equiv_M^{\mathrm{Ls}} bc'$, and $\mathrm{Lstp}(bc'/Mb''c'')$ extends to a global M-Ls-invariant type q. So there is a Morley sequence $(b_i c_i)_{i<\omega}$ in q with $(b_0, c_0) = (b'', c'')$ and $(b_1, c_1) = (b, c')$. As $a \underset{M}{\overset{K}{\bigcup}} bc'$, we can find a^* with $a^* b'' c'' \equiv_M abc'$ such that $(b_i c_i)_{i<\omega}$ is Ma^*-indiscernible. By construction we had $c'' \equiv_{Mb}^{\mathrm{Ls}} c$, so there is a Lascar strong automorphism σ over Mb such that $\sigma(c'') = c$. Setting $a'' = \sigma(a^*)$, we check that this is indeed the a'' we are looking for.

By the chain condition (Lemma 6.1), $a^* \underset{M}{\overset{K}{\bigcup}} (b_i c_i)_{i<\omega}$, so we have $a^* \underset{M}{\overset{K}{\bigcup}} bc''$, and $a'' \underset{M}{\overset{K}{\bigcup}} bc$ then follows by invariance. By Ma^*-indiscernibility we have $a''b \equiv_M a^*b \equiv_M a^*b'' \equiv_M ab$. We assumed b to enumerate a λ_T-saturated model, so indeed $a'' \equiv_{Mb}^{\mathrm{Ls}} a$. By construction of c' we have $a''c \equiv_M a^*c'' \equiv_M ac' \equiv_M a'c$. We assumed c to enumerate a λ_T-saturated model, so indeed $a'' \equiv_{Mc}^{\mathrm{Ls}} a'$, which concludes the proof. $\square$

Fact 7.6. *In a thick theory, if $N \supseteq M$ is $(2^{|M|+\lambda_T})^+$-saturated and q and r are global M-Ls-invariant types with $q|_N = r|_N$, then $q = r$.*

Proof. By Fact 2.12 there is $M \subseteq M' \subseteq N$ where M' is a λ_T-saturated model and $|M'| < (2^{|M|+\lambda_T})^+$. Let $\varphi(x, b)$ be any formula with parameters b. Let $b' \in N$ realise $\mathrm{tp}(b/M')$. Then $b \equiv_M^{\mathrm{Ls}} b'$. By M-Ls-invariance and $q|_N = r|_N$, we have

$$\varphi(x, b) \in q \quad \Longleftrightarrow \quad \varphi(x, b') \in q \quad \Longleftrightarrow \quad \varphi(x, b') \in r \quad \Longleftrightarrow \quad \varphi(x, b) \in r,$$

which concludes the proof. $\square$

Theorem 7.7 (independence theorem)**.** *Let T be a thick NSOP_1 theory. Suppose that $a \equiv_M^{\mathrm{Ls}} a'$, $a \underset{M}{\overset{K}{\bigcup}} b$, $a' \underset{M}{\overset{K}{\bigcup}} c$ and $b \underset{M}{\overset{K}{\bigcup}} c$. Then there is a'' with $a'' \equiv_{Mb}^{\mathrm{Ls}} a$, $a'' \equiv_{Mc}^{\mathrm{Ls}} a'$ and $a'' \underset{M}{\overset{K}{\bigcup}} bc$.*

Proof. We may assume that b and c both enumerate a λ_T-saturated model containing M. If this is not the case, let $N \supseteq Mb$ be λ_T-saturated. By extension (Corollary 4.14) and symmetry, then find $N' \equiv_{Mb}^{\mathrm{Ls}} N$ with $N' \underset{M}{\overset{K}{\bigcup}} c$. Applying extension again we find a_0 with $a_0 \equiv_{Mb}^{\mathrm{Ls}} a$ and $a_0 \underset{M}{\overset{K}{\bigcup}} N'$. Now we can replace a by a_0 and b by N' and continue the proof. The case for c is analogous.

Let $N_0 \supseteq M$ be $(2^{|M|+\lambda_T})^+$-saturated, and let κ be a big enough cardinal (depending only on $|N_0 bc|$). Pick some global M-Ls-invariant type $q(y, z)$ extending $\mathrm{Lstp}(bc/M)$ such that q also extends to a global M-Ls-invariant type $\mathrm{tp}(N/\mathfrak{M})$ for some saturated enough $N \supseteq M$ (depending only on κ). So there is β realising $q|_y$ with $\beta \equiv_M^{\mathrm{Ls}} b$. Let $(b_i c_i)_{i<\kappa}$ be a Morley sequence in q with $b_0 = b$, and let $b_\kappa \equiv_{M(b_i c_i)_{i<\kappa}}^{\mathrm{Ls}} \beta$. Then we have $b_i c_i \underset{M}{\overset{*}{\downarrow}} b_{<i} c_{<i}$ for all $i < \kappa$ and $b_\kappa \underset{M}{\overset{*}{\downarrow}} (b_i c_i)_{i<\kappa}$.

We will inductively construct a sequence $(b_i')_{i \leq \kappa}$ with $b_0' = b$ such that at step i,

(i) $c \underset{M}{\overset{K}{\downarrow}} b_{\leq i}'$,

(ii) $cb_i' \equiv_M^{\mathrm{Ls}} cb$,

(iii) $b_{\leq i}' \equiv_M^{\mathrm{Ls}} b_{\leq i}$.

The base case is already fixed: $b_0' = b$. So suppose we have constructed $b_{\leq i}'$. By induction hypothesis (iii) we can find $b^* b_{\leq i}' \equiv_M^{\mathrm{Ls}} b_{i+1} b_{\leq i}$. So $b^* \underset{M}{\overset{*}{\downarrow}} b_{\leq i}'$. Let c^* be such that $c^* b^* \equiv_M^{\mathrm{Ls}} cb$, so $c^* \underset{M}{\overset{K}{\downarrow}} b^*$. Therefore, also using (i) from the induction hypothesis, we can apply the weak independence theorem (Proposition 7.5) to find c' such that $c' \underset{M}{\overset{K}{\downarrow}} b_{\leq i}' b^*$, $c' \equiv_{Mb^*}^{\mathrm{Ls}} c^*$ and $c' \equiv_{Mb_{\leq i}'}^{\mathrm{Ls}} c$. We now pick b_{i+1}' to be such that $cb_{i+1}' \equiv_{Mb_{\leq i}'}^{\mathrm{Ls}} c'b^*$. Then indeed $c \underset{M}{\overset{K}{\downarrow}} b_{\leq i+1}'$. We also have

$$b_{\leq i}' b_{i+1}' \equiv_M^{\mathrm{Ls}} b_{\leq i}' b^* \equiv_M^{\mathrm{Ls}} b_{\leq i} b_{i+1}.$$

Finally,

$$cb_{i+1}' \equiv_M^{\mathrm{Ls}} c'b^* \equiv_M^{\mathrm{Ls}} c^* b^* \equiv_M^{\mathrm{Ls}} cb.$$

This concludes the successor step. For the limit stage we assume we have constructed $b_{<i}'$. We then have $c \underset{M}{\overset{K}{\downarrow}} b_{<i}'$ by finite character. We also have $b_{\leq j}' \equiv_M^{\mathrm{Ls}} b_{\leq j}$ for all $j < i$. So we have $b_{<i}' \equiv_M b_{<i}$. We assumed b to enumerate a λ_T-saturated model containing M, so because $b_0' = b = b_0$ we do in fact have $b_{<i}' \equiv_M^{\mathrm{Ls}} b_{<i}$. We then construct b_i' in an analogous way to the successor step.

We let $(c_i')_{i<\kappa}$ be such that $b_\kappa' (b_i' c_i')_{i<\kappa} \equiv_M^{\mathrm{Ls}} b_\kappa (b_i c_i)_{i<\kappa}$. So by M-Ls-invariance of $q|_y$ we have $\beta b_\kappa' (b_i' c_i')_{i<\kappa} \equiv_M^{\mathrm{Ls}} \beta b_\kappa (b_i c_i)_{i<\kappa}$ and thus by how we chose b_κ we have $b_\kappa' \equiv_{M(b_i' c_i')_{i<\kappa}}^{\mathrm{Ls}} \beta$.

Because $q \subseteq \mathrm{tp}(N/\mathfrak{M})$ for some saturated enough N, we can find

$$\beta\gamma (\beta_i, \gamma_i)_{i<\kappa} \equiv_M^{\mathrm{Ls}} b_\kappa' c (b_i', c_i')_{i<\kappa}$$

in N, where $\beta\gamma \models q$. Here we used that $b_\kappa' c \equiv_M^{\mathrm{Ls}} bc$. Set

$$q'((y_i, z_i)_{i<\kappa}, y, z) = \mathrm{tp}((\beta_i, \gamma_i)_{i<\kappa} \beta\gamma / \mathfrak{M}).$$

Then q' is global M-Ls-invariant because $\mathrm{tp}(N/\mathfrak{M})$ is global M-Ls-invariant. By Fact 7.6 and our choice of κ, we get that some global M-Ls-invariant type $q'|_{y_i z_i yz}$ occurs for κ many i (modulo identifying the variables for different i's). We now focus on a subsequence of length ω such that (after relabelling) $q'|_{y_i z_i yz}$ does not depend on i, and we forget about κ. We also relabel b_κ' to b'.

Claim 1. *In summary*:

(i) *We constructed a Morley sequence $(b_i' c_i')_{i<\omega}$ in q, where q is a global M-Ls-invariant extension of $\mathrm{Lstp}(bc/M)$.*

(ii) *For every $i < \omega$, we have $b_i' c \equiv_M^{\mathrm{Ls}} b'c \equiv_M^{\mathrm{Ls}} bc$.*

(iii) *Let $\beta \models q|_y$. Then $b' \equiv_{M(b_i' c_i')_{i<\omega}}^{\mathrm{Ls}} \beta$.*

(iv) *$q(y,z) \subseteq q'((y_i, z_i)_{i<\omega}, y, z)$ and q' is global M-Ls-invariant and extends $\mathrm{Lstp}((b_i', c_i')_{i<\omega} b'c/M)$.*

(v) *There is some sufficiently saturated N such that $q' \subseteq \mathrm{tp}(N/\mathfrak{M})$ and $\mathrm{tp}(N/\mathfrak{M})$ is M-Ls-invariant.*

(vi) *The type $q'|_{y_i z_i yz}$ does not depend on i, modulo identifying variables for different i's.*

Claim 2. *For every $k < \omega$, there are*

$$g_0 h_0 g_1 h_1 \cdots g_{k-1} h_{k-1} g_k, \quad g_0' h_0' g_1' h_1' \cdots g_{k-1}' h_{k-1}' \quad \text{and} \quad h_0'' g_1'' h_1'' \cdots g_{k-1}'' h_{k-1}'' g_k''$$

such that

(i) *$(g_i' h_i')_{i<k} \models (q'|_{y_0,z})^{\otimes k}|_M$,*

(ii) *$(h_i'' g_{i+1}'')_{i<k} \models (q'|_{z_0,y})^{\otimes k}|_M$,*

(iii) *$(g_i h_i, g_i' h_i')$ starts an $M g_{>i} h_{>i} g_{>i}' h_{>i}'$-indiscernible sequence for every $i < k$,*

(iv) *$(h_i g_{i+1}, h_i'' g_{i+1}'')$ starts an $M h_{>i} g_{>i+1} h_{>i}'' g_{>i+1}''$-indiscernible sequence for every $i < k$.*

We first prove that the theorem follows from Claim 2. We set $p_0(x, y) = \mathrm{tp}(ab/M)$ and $p_1(x, z) = \mathrm{tp}(a'c/M)$. We will prove that $p_0(x, b) \cup p_1(x, c)$ does not Kim-divide over M. This is enough, because by Proposition 4.3 we can then extend it to a complete type that does not Kim-divide over M. Since we assumed b and c to enumerate λ_T-saturated models containing M, any realisation a'' of that complete type is then what we needed to construct.

By compactness, we can find M-indiscernible $(g_i h_i g_i' h_i' g_i'' h_i'')_{i \in \mathbb{Z}}$ such that $(g_i' h_i')_{i \in \mathbb{Z}} \models (q'|_{y_0,z})^{\otimes \mathbb{Z}}|_M$ and $(h_i'' g_{i+1}'')_{i \in \mathbb{Z}} \models (q'|_{z_0,y})^{\otimes \mathbb{Z}}|_M$. Furthermore, we can make it so that for every $i \in \mathbb{Z}$ we have that

$$g_i h_i \equiv_{M g_{>i} h_{>i} g_{>i}' h_{>i}'} g_i' h_i' \quad \text{and} \quad h_i g_{i+1} \equiv_{M h_{>i} g_{>i+1} h_{>i}'' g_{>i+1}''} h_i'' g_{i+1}''.$$

We have $q'|_{y,z_0} \supseteq \mathrm{tp}(b'c_0'/M)$, by Claim 1(iv). So, by parts (iii) and (v) of Claim 1, we have that $b' \underset{M}{\overset{*}{\smile}} c_0'$. Then by Proposition 7.5 we have that $p_0(x, g_1'') \cup p_1(x, h_0'')$ does not Kim-divide. Then because $(h_i'' g_{i+1}'')_{i \geq n} \models (q'|_{z_0,y})^{\otimes \omega}|_M$ for all $n \in \mathbb{Z}$, we get that $\bigcup_{i \in \mathbb{Z}} p_0(x, g_{i+1}'') \cup p_1(x, h_i'')$ is consistent. By the parallel sequences lemma (Lemma 2.28) we thus have that $\bigcup_{i \in \mathbb{Z}} p_0(x, g_{i+1}) \cup p_1(x, h_i)$ is consistent. This is the same set as $\bigcup_{i \in \mathbb{Z}} p_0(x, g_i) \cup p_1(x, h_i)$. So again by the parallel sequences lemma

we get that $\bigcup_{i \in \mathbb{Z}} p_0(x, g_i') \cup p_1(x, h_i')$ is consistent. By parts (ii) and (iii) of Claim 1, we have that $q'|_{y_0, z}$ extends $\mathrm{Lstp}(bc/M)$. So we conclude that $p_0(x, b) \cup p_1(x, c)$ does not Kim-divide over M, as required.

We are left to verify Claim 2. We fix k and by backwards induction on $k' = 2k, 2k-1, \ldots, 1$ we will define trees $(d_\eta e_\eta)_{\eta \in S_{k'}}$ where $S_{k'} = \{\xi \in \omega^{\leq 2k+1} : 0^{k'-1} \trianglelefteq \xi\}$ such that for each k' the tree $(d_\eta e_\eta)_{\eta \in S_{k'}}$ satisfies the following condition:

$(\mathbf{P})_{k'}$ For every $\eta \in \omega^{\leq 2k-1}$ and $i < \omega$ such that $\eta \frown i \in S_{k'}$ we have that

$$(d_{\eta \frown i \frown j} e_{\eta \frown i \frown j})_{j < \omega} d_{\eta \frown i} e_{\eta \frown i} \equiv_{M(d_{\trianglerighteq \eta \frown i'} e_{\trianglerighteq \eta \frown i'})_{i' < i}}^{\mathrm{Ls}} (\beta_j \gamma_j)_{j < \omega} \beta \gamma.$$

Recall that $q' = \mathrm{tp}((\beta_j \gamma_j)_{j < \omega} \beta \gamma / \mathfrak{M})$. So in particular

$$(d_{\eta \frown j} e_{\eta \frown j})_{j < \omega} d_\eta e_\eta \equiv_M^{\mathrm{Ls}} (\beta_j \gamma_j)_{j < \omega} \beta \gamma \quad \text{for all } \eta \in \omega^{\leq 2k} \cap S_{k'}.$$

For $k' = 2k$ we let $(d_\eta e_\eta)_{\eta \in S_{2k}}$ just be $(b_i' c_i')_{i < \omega} b' c$. Suppose now that we have constructed $(d_\eta e_\eta)_{\eta \in S_{k'}}$. We have $(d_{0^{k'-1} \frown i} e_{0^{k'-1} \frown i})_{i < \omega} d_{0^{k'-1}} e_{0^{k'-1}} \equiv_M^{\mathrm{Ls}} (\beta_i \gamma_i)_{i < \omega} \beta \gamma$, by $(\mathbf{P})_{k'}$. So by Claim 1(v) there is a global M-Ls-invariant $r \supseteq q'$ such that r also extends $\mathrm{Lstp}((d_\eta e_\eta)_{\eta \in S_{k'}} / M)$. Here we match $(d_{0^{k'-1} \frown i} e_{0^{k'-1} \frown i})_{i < \omega} d_{0^{k'-1}} e_{0^{k'-1}}$ with the variables in q'. Let $((d_{\eta, i} e_{\eta, i})_{\eta \in S_{k'}})_{i < \omega}$ be a Morley sequence in r with $(d_{\eta, 0} e_{\eta, 0})_{\eta \in S_{k'}} = (d_\eta e_\eta)_{\eta \in S_{k'}}$. We set

$$d_{0^{k'-2} \frown i \frown \xi} e_{0^{k'-2} \frown i \frown \xi} = d_{0^{k'-1} \frown \xi, i} e_{0^{k'-1} \frown \xi, i} \quad \text{for all } i < \omega \text{ and } \xi \in \omega^{\leq 2k+2-k'}.$$

We directly get $(\mathbf{P})_{k'-1}$ for $\eta \in S_{k'} - \{0^{k'-2}\}$ by virtue of $((d_{\eta, i} e_{\eta, i})_{\eta \in S_{k'}})_{i < \omega}$ being a Morley sequence. By Claim 1(iv), we have that $(d_{0^{k'-2} \frown i} e_{0^{k'-2} \frown i})_{i < \omega}$ is a Morley sequence in q. So we can find $d_{0^{k'-2}} e_{0^{k'-2}}$ such that

$$(d_{0^{k'-2} \frown i} e_{0^{k'-2} \frown i})_{i < \omega} d_{0^{k'-2}} e_{0^{k'-2}} \equiv_M^{\mathrm{Ls}} (\beta_i \gamma_i)_{i < \omega} \beta \gamma,$$

and that concludes the construction of $(d_\eta e_\eta)_{\eta \in S_{k'-1}}$.

Similarly as in the proof of Lemma 5.13, we will now show by induction on $n \leq k$ that the following holds.

$(\mathbf{Q})_n$ Let $\eta_{2k-2m} \in \omega^{2k-2m}$ and $\nu_{2k-2m+1} \in \omega^{2k-2m+1}$ for $0 \leq m \leq n$. Suppose that $\eta_{2k-2m} \triangleleft \nu_{2k-2m+1}$ for all $0 \leq m \leq n$, $\eta_{2k} >_{\mathrm{lex}} \eta_{2k-2} >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} \eta_{2k-2n}$ and for all $0 \leq m' < m \leq n$ we have that $\eta_{2k-2m} \wedge \eta_{2k-2m'} \in \omega^{2k-2m-1}$. Then $(d_{\nu_{2k-2m+1}} e_{\nu_{2k-2m+1}} d_{\eta_{2k-2m}} e_{\eta_{2k-2m}})_{m \leq n}$ is a Morley sequence in $q'|_{y_0 z_0 yz}$.

For $n = 0$ this follows immediately from $(\mathbf{P})_1$ and Claim 1(vi). So suppose $(\mathbf{Q})_n$ holds for some $n < k$, and let $\eta_{2k-2m} \in \omega^{2k-2m}$ and $\nu_{2k-2m+1} \in \omega^{2k-2m+1}$ for $0 \leq m \leq n+1$ be as in the statement of $(\mathbf{Q})_{n+1}$. For any $m < n$ we have that $\eta_{2k-2m} \wedge \eta_{2k-2n-2} = \eta_{2k-2n-2}|_{2k-2n-3}$. Therefore we can write $\eta_{2k-2n-2} = \xi \frown i$ for some $\xi \in \omega^{2k-2n-3}$ and $i < \omega$. We then have $\eta_{2k-2m} \trianglerighteq \xi \frown i'$ for some $i' < i$ for all $m \leq n$. So it follows from $(\mathbf{P})_1$, Claim 1(vi) and the induction hypothesis that $(d_{\nu_{2k-2m+1}} e_{\nu_{2k-2m+1}} d_{\eta_{2k-2m}} e_{\eta_{2k-2m}})_{m \leq n+1}$ is a Morley sequence in $q'|_{y_0 z_0 yz}$.
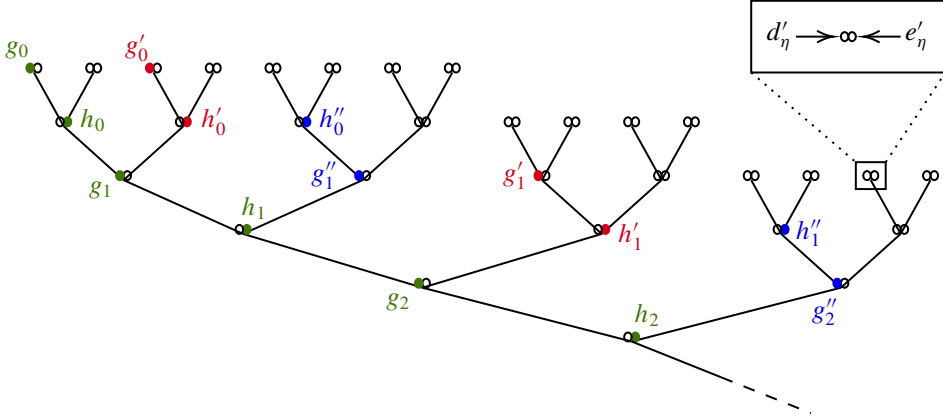
Figure 3. Choice of the $g_i h_i g_i' h_i' g_i'' h_i''$.

By exactly the same argument we also have the following condition. It differs from $(\mathbf{Q})_n$ in that the levels have been shifted by one (therefore we only consider it for $n < k$).

$(\mathbf{Q}')_n$ Let $\eta_{2k-2m-1} \in \omega^{2k-2m-1}$ and $\nu_{2k-2m} \in \omega^{2k-2m}$ for $0 \le m \le n$. Suppose that $\eta_{2k-2m-1} \vartriangleleft \nu_{2k-2m}$ for all $0 \le m \le n$, $\eta_{2k-1} >_{\text{lex}} \eta_{2k-3} >_{\text{lex}} \cdots >_{\text{lex}} \eta_{2k-2n-1}$ and for all $0 \le m' < m \le n$ we have that $\eta_{2k-2m-1} \wedge \eta_{2k-2m'-1} \in \omega^{2k-2m-2}$. Then $(d_{\nu_{2k-2m}} e_{\nu_{2k-2m}} d_{\eta_{2k-2m-1}} e_{\eta_{2k-2m-1}})_{m \le n}$ is a Morley sequence in $q'|_{y_0 z_0 yz}$.

Now let $(d_\eta' e_\eta')_{\eta \in \omega^{2k+1}}$ be an s-indiscernible over M tree which is EM_s-based on $(d_\eta e_\eta)_{\eta \in \omega^{2k+1}}$ over M. We put $g_i = d'_{0^{2(k-i)+1}}$ for $i \le k$, and for $i < k$ we put $h_i = e'_{0^{2(k-i)}}$, $g_i' = d'_{0^{2(k-i)-1} \frown 1 \frown 0}$, $h_i' = e'_{0^{2(k-i)-1} \frown 1}$, $g_{i+1}'' = d'_{0^{2(k-i-1)} \frown 1}$ and $h_i'' = e'_{0^{2(k-i-1)} \frown 1 \frown 0}$; see Figure 3. Then conditions (i) and (ii) from Claim 2 follow from $(\mathbf{Q})_k$ and $(\mathbf{Q}')_{k-1}$, while conditions (iii) and (iv) follow from s-indiscernibility. □

Now that we have proved the independence theorem, we first note some useful immediate consequences in Corollary 7.10. After that, the rest of this section will be devoted to proving a stronger version of the independence theorem, Theorem 7.15.

Definition 7.8. Let I be a linear order. We say that $(a_i)_{i \in I}$ is a $\underset{M}{\overset{K}{\bigcup}}$-*independent sequence* if $a_i \underset{M}{\overset{K}{\bigcup}} a_{<i}$ for every $i \in I$. We say that $(a_i)_{i \in I}$ is $\underset{M}{\overset{K}{\bigcup}}$-*Morley* if it is $\underset{M}{\overset{K}{\bigcup}}$-*independent* and M-indiscernible.

Lemma 7.9. *Let T be thick* NSOP_1 *with an e.c. model M, and let a, b, c be any tuples of parameters and x a tuple of variables. Then there exists a (partial) type $\Sigma(x, y)$ over Mab such that for any x and y we have that*

$$\models \Sigma(x, y) \quad \Longleftrightarrow \quad (y \equiv_{Mb} c) \wedge (xa \underset{M}{\overset{K}{\bigcup}} yb).$$

In particular, taking $y = \varnothing$, we get that the condition $xa \underset{M}{\overset{K}{\bigcup}} b$ is type-definable over Mab in the variable x.

Proof. Let $q(y, z)$ be a global M-Ls-invariant type extending $\mathrm{tp}(cb/M)$. Then, by Kim's lemma, for any $y \equiv_{Mb} c$ and any x, the condition $xa \underset{M}{\overset{K}{\downarrow}} yb$ is equivalent to

$$\exists(y_i z_i)_{i<\omega}\big(q^{\otimes\omega}|_M((y_i z_i)_{i<\omega}) \text{ and } y_0 z_0 = yb \text{ and } (y_i z_i)_{i<\omega} \text{ is } Max\text{-indiscernible}\big),$$

which is clearly a type-definable over Mab condition by thickness. $\qquad\square$

In particular, we get that being an $\underset{M}{\overset{K}{\downarrow}}$-independent sequence in a fixed type over M is type-definable over M in thick NSOP_1 theories. That is, for a linear order I, we can use the type

$$\bigcup_{i \in I} \Sigma(x_{<i}, x_i),$$

where Σ is as in Lemma 7.9. Then, by symmetry, Theorem 6.5, this (partial) type expresses exactly what we wanted.

Corollary 7.10. *Suppose T is thick* NSOP_1 *with an e.c. model M.*

(i) *If $a \underset{M}{\overset{K}{\downarrow}} b$ and $a \equiv_M^{\mathrm{Ls}} b$ then there exists an infinite M-indiscernible sequence starting with (a, b).*

(ii) *If $a \equiv_M^{\mathrm{Ls}} b$ then a and b are at Lascar distance at most 2 over M. In particular, Lascar equivalence over e.c. models is type-definable.*

(iii) *Generalised independence theorem: Let $(a_i)_{i<\kappa}$ be an $\underset{M}{\overset{K}{\downarrow}}$-independent sequence. Suppose $b_i \equiv_M^{\mathrm{Ls}} b$ and $b_i \underset{M}{\overset{K}{\downarrow}} a_i$ for every $i < \kappa$. Then there exists b' such that $b' a_i \equiv_M^{\mathrm{Ls}} b_i a_i$ for every $i < \kappa$ and $b' \underset{M}{\overset{K}{\downarrow}} (a_i)_{i<\kappa}$.*

Proof. (i) We can inductively find a sequence $(c_i)_{i<\omega}$ such that $c_0 c_1 = ab$, $c_i \equiv_M^{\mathrm{Ls}} b$, $c_i \underset{M}{\overset{K}{\downarrow}} c_{<i}$ and $c_i c_j \equiv_M ab$ for all $i < j < \omega$: indeed, if we have constructed $c_{\leq i}$ then by the independence theorem we can choose c_{i+1} such that $c_{i+1} \equiv_{Mc_{<i}}^{\mathrm{Ls}} c_i$, $c_i c_{i+1} \equiv_M^{\mathrm{Ls}} ab$ and $c_{i+1} \underset{M}{\overset{K}{\downarrow}} c_{\leq i}$.

By compactness we can find a sequence $(c_i')_{i<\lambda_{|T|+|Ma|}}$ with $c_i' c_j' \equiv_M ab$ for all $i < j < \lambda_{|T|+|Ma|}$. Choose an M-indiscernible sequence $(d_i)_{i<\omega}$ based on $(c_i')_{i<\lambda_{|T|+|Ma|}}$ over M. Then $d_0 d_1 \equiv_M ab$, so we conclude that the pair (a, b) starts an M-indiscernible sequence.

(ii) By extension (Corollary 4.14) we can choose $c \equiv_M^{\mathrm{Ls}} a$ with $c \underset{M}{\overset{K}{\downarrow}} ab$. By (i) we get that (a, c) and (b, c) both start M-indiscernible sequences.

(iii) We choose inductively a sequence $(b_j')_{j \leq \kappa}$ such that $b_j' a_i \equiv_M^{\mathrm{Ls}} b_i a_i$ for every $i < j$ and $b_j' \underset{M}{\overset{K}{\downarrow}} (a_i)_{i<j}$, so that we can put $b' := b_\kappa$. The successor step follows directly by the independence theorem, and the limit step follows by type-definability of Lascar equivalence over M, Lemma 7.9 and compactness. $\qquad\square$

Definition 7.11. We will say that a tree $(c_\eta)_{\eta \in \omega^{\leq k}}$ is *spread-out over M* if $(c_{\unrhd \eta \frown i})_{i<\omega}$ is a Morley sequence in some global M-Ls-invariant type for every $\eta \in \omega^{\leq k-1}$.

There are two differences between being spread-out over M and being q-spread-out over M (see Definition 5.12 for the latter). In the latter the global M-Ls-invariant type involved has to be q, while the former just requires some global M-Ls-invariant type. The second difference is in the sequence in the tree that is required to be a Morley sequence. In the former we consider a sequence of subtrees above some fixed node, all at the same level. In the latter we consider a sequence of nodes in the tree, one in every level (except for the root), as pictured in Figure 1.

The following lemma follows from the independence theorem exactly as in [Kaplan and Ramsey 2020, Lemma 6.2/Remark 6.3], so we omit the proof.

Fact 7.12. *Suppose that T is thick* NSOP$_1$*, M is an e.c. model, $a \underset{M}{\overset{K}{\smile}} b$, $(b_\eta)_{\eta \in \omega^{\leq k}}$ (with $k < \omega$) is a spread-out over M tree such that $b_\eta \underset{M}{\overset{K}{\smile}} b_{\triangleright \eta}$ and $b_\eta \equiv_M^{\mathrm{Ls}} b$ for every $\eta \in \omega^{\leq k}$. Then, writing $p(x, b) = tp(a/Mb)$, there exists $a' \models \bigcup_{\eta \in \omega^{\leq k}} p(x, b_\eta)$ with $a' \underset{M}{\overset{K}{\smile}} (b_\eta)_{\eta \in \omega^{\leq k}}$ and $a' \equiv_M^{\mathrm{Ls}} a$.*

Lemma 7.13. *Suppose that T is thick* NSOP$_1$*, M is an e.c. model, $b \equiv_M^{\mathrm{Ls}} b'$, $b \underset{M}{\overset{K}{\smile}} b'$ and I is a linear order with two distinct elements 0 and 1. Then there is a $\underset{M}{\overset{K}{\smile}}$-Morley parallel-Morley in* $\mathrm{tp}(b/M)$ *sequence $(b_i)_{i \in I}$ with $b_0 = b$ and $b_1 = b'$.*

Proof. By extension (Corollary 4.14) there is a λ_T-saturated model $N \supseteq Mb$ with $N \underset{M}{\overset{K}{\smile}} b'$. Then there is a λ_T-saturated model $N' \supseteq Mb'$ with $N' \equiv_M^{\mathrm{Ls}} N$. Hence, again by extension, we can find $N'' \equiv_{Mb'}^{\mathrm{Ls}} N'$ with $N \underset{M}{\overset{K}{\smile}} N''$. So replacing b and b' by N and N'' we may assume without loss of generality that b and b' are λ_T-saturated models containing M. Put $\lambda = |b|$ and (using Lemma 6.3) choose a global M-Ls-invariant extension q of $\mathrm{Lstp}(b'/M)$ satisfying $(*)_\lambda$.

We claim that it is enough to show that for any $1 < k < \omega$ there is a $\underset{M}{\overset{K}{\smile}}$-independent parallel-Morley sequence $(a_i)_{i<k}$ in q over M with $a_i \equiv_M^{\mathrm{Ls}} b'$ and $a_i a_j \equiv_M bb'$ for all $i < j < k$: indeed, if we show this, then, as all these conditions are type-definable by Lemma 7.9 and Corollary 7.10(ii), we can find by compactness a $\underset{M}{\overset{K}{\smile}}$-independent over M parallel-Morley sequence $(a_i)_{i<\lambda_{|T|+|b|}}$ in q over M with $a_i a_j \equiv_M bb'$ for each $i < j$, and then taking an M-indiscernible sequence indexed by I which is based on $(a_i)_{i<\lambda_{|T|+|Mb|}}$ over M and moving it by an automorphism to guarantee that $b_0 b_1 = bb'$ (note this may change q) will do the job.

So fix any $1 < k < \omega$ and put $p = tp(b'/Mb)$. By backward induction on $k' = k+1, k, \dots, 1$ we will define trees $(c_\eta)_{\eta \in S_{k'}}$ where $S_{k'} := \{\xi \in \omega^{\leq k} : 0^{k'-1} \trianglelefteq \xi\}$ such that for each k' the tree $(c_\eta)_{\eta \in S_{k'}}$ is spread-out over M and satisfies the following conditions:

(A1)$_{k'}$ $c_\eta c_\nu \equiv_M bb'$ for any $\nu, \eta \in S_{k'}$ with $\nu \triangleleft \eta$ and $c_\eta \equiv_M^{\mathrm{Ls}} b'$ for any $\eta \in S_{k'}$.

(A2)$_{k'}$ $(c_\eta)_{\eta \in S_{k'}}$ is q-spread-out over M.

(A3)$_{k'}$ $c_\eta \underset{M}{\overset{K}{\smile}} c_{\triangleright \eta}$ for every $\eta \in S_{k'}$.

For $k' = k + 1$ putting $c_{0^k} = b'$ works. Now suppose we are done for some $k' \leq k + 1$. By Fact 7.12 we can find $c' \models \bigcup_{\eta \in S_{k'}} p(x, c_\eta)$ with $c' \equiv_M^{\mathrm{Ls}} b'$ and $c' \underset{M}{\overset{K}{\smile}} (c_\eta)_{\eta \in S_{k'}}$. By (A1)$_{k'}$ there is a tuple d such that $c_{0^{k'-1}}(c_\eta)_{\eta \in S_{k'}^*} \equiv_M^{\mathrm{Ls}} b'd$. Now, by $(*)_\lambda$ there is some global M-Ls-invariant type $r(x, z) \supseteq q(x)$ which extends $\mathrm{Lstp}(b'd/M) = \mathrm{Lstp}(c_{0^{k'-1}}(c_\eta)_{\eta \in S_{k'}^*}/M)$. Also, as $c' \underset{M}{\overset{K}{\smile}} (c_\eta)_{\eta \in S_{k'}}$ and c_η's are λ_T-saturated models (as b' is), we get by Corollary 4.10 that $\mathrm{Lstp}(c'/M(c_\eta)_{\eta \in S_{k'}})$ does not $r(x, z)$-Ls-divide over M. Hence, there is an Mc'-indiscernible Morley sequence $I := ((c_{\eta,i})_{\eta \in S_{k'}})_{i < \omega}$ in $r(x, z)$ over M with $c_{\eta,0} = c_\eta$ for each $\eta \in S_{k'}$. By the chain condition (Lemma 6.1) we have that $c' \underset{M}{\overset{K}{\smile}} I$. Thus, putting $c_{0^{k'-2} \frown i \frown \zeta} := c_{0^{k'-1} \frown \zeta, i}$ for all $i < \omega$, $\zeta \in \omega^{\leq k+1-k'}$, and $c_{0^{k'-2}} := c'$, we immediately get that the tree $(c_\eta)_{\eta \in S_{k'-1}}$ satisfies (A3)$_{k'-1}$. (A1)$_{k'-1}$ follows from (A1)$_{k'}$, the choice of c' and Mc' indiscernibility of I. (A2)$_{k'-1}$ follows from (A2)$_{k'}$ and Lemma 5.13(i). This completes the inductive construction.

Letting $(c'_\eta)_{\eta \in \omega^{\leq k}}$ be an s-indiscernible over M tree which is EM$_s$-based on $(c_\eta)_{\eta \in \omega^{\leq k}}$ over Mb', we get that $(c'_\eta)_{\eta \in \omega^{\leq k}}$ satisfies (A1)$_1$ and (A3)$_1$ (by Lemma 7.9 and Corollary 7.10(ii)) and is weakly q-spread-out over M by Lemma 5.13(ii).

Put $a_i := c'_{0^{k-i}}$ for $i < k$. Then by Lemma 5.13(iii) we have that $(a_i)_{i < k}$ is parallel-Morley in q over M. Also, $a_i a_j \equiv_M bb'$ for all $i < j < k$ by (A1)$_1$, and $(a_i)_{i < k}$ is $\underset{M}{\overset{K}{\smile}}$-independent over M by (A3)$_1$. This completes the proof. $\qquad \square$

Lemma 7.14 (chain condition for $\overset{K}{\smile}$-Morley sequences). *Suppose T is thick* NSOP$_1$ *with an e.c. model M, $(d_i)_{i \in I}$ is an infinite $\underset{M}{\overset{K}{\smile}}$-Morley sequence and $a \underset{M}{\overset{K}{\smile}} d_{i_0}$ for some $i_0 \in I$. Then there exists $a^* d_{i_0} \equiv_M^{\mathrm{Ls}} a d_{i_0}$ such that $(d_i)_{i \in I}$ is indiscernible over Ma^* and $a^* \underset{M}{\overset{K}{\smile}} (d_i)_{i \in I}$.*

Proof. By compactness, there exists a $\underset{M}{\overset{K}{\smile}}$-Morley sequence $(d''_i)_{i < \lambda}$ such that $(d_i)_{i \in I} \frown (d''_i)_{i < \lambda}$ is M-indiscernible, where $\lambda = \lambda_{|T|+|Mad_0|+|I|}$. As $d_{i_0} \equiv_M^{\mathrm{Ls}} d''_0$, $a \underset{M}{\overset{K}{\smile}} d_{i_0}$ and $(d''_i)_{i < \lambda}$ is $\overset{K}{\smile}$-independent over M, we get by Corollary 7.10(iii) that there exists a' with $a' d''_i \equiv_M^{\mathrm{Ls}} a d_{i_0}$ for every $i < \lambda$ and $a' \underset{M}{\overset{K}{\smile}} (d''_i)_{i < \lambda}$. Let $(d'_i)_{i \in I}$ be an Ma'-indiscernible sequence based on $(d''_i)_{i < \lambda}$ over $Maa'(d_i)_{i \in I}$. Therefore (by finite character and invariance of $\overset{K}{\smile}$), $a' \underset{M}{\overset{K}{\smile}} (d'_i)_{i \in I}$, $(d'_i)_{i \in I} \equiv_M^{\mathrm{Ls}} (d_i)_{i \in I}$ (as $(d_i)_{i \in I} \frown (d'_i)_{i \in I}$ is indiscernible over M), and $a' d'_{i_0} \equiv_M^{\mathrm{Ls}} a d_{i_0}$. Hence, letting f be a Lascar strong automorphism over M sending $(d'_i)_{i \in I}$ to $(d_i)_{i \in I}$ and putting $a^* = f(a')$ we get that $a^* \underset{M}{\overset{K}{\smile}} (d_i)_{i \in I}$ and $(d_i)_{i \in I}$ is Ma^*-indiscernible. Also $a^* d_{i_0} \equiv_M^{\mathrm{Ls}} a' d'_{i_0} \equiv_M^{\mathrm{Ls}} a d_{i_0}$, as required. $\qquad \square$

Theorem 7.15 (strong independence theorem). *Suppose T is thick* NSOP$_1$ *with an e.c. model M, $a_0 \underset{M}{\overset{K}{\smile}} b$, $a_1 \underset{M}{\overset{K}{\smile}} c$, $b \underset{M}{\overset{K}{\smile}} c$, and $a_0 \equiv_M^{\mathrm{Ls}} a_1$. Then there exists an a such that $a \equiv_{Mb}^{\mathrm{Ls}} a_0$, $a \equiv_{Mc}^{\mathrm{Ls}} a_1$, $a \underset{M}{\overset{K}{\smile}} bc$, $b \underset{M}{\overset{K}{\smile}} ac$ and $c \underset{M}{\overset{K}{\smile}} ab$.*

Proof. By a similar trick as at the start of the proof of Theorem 7.7 we may assume that b and c enumerate λ_T-saturated models containing M.

By the independence theorem there is a_2 with $a_2 \equiv^{\mathrm{Ls}}_{Mb} a_0$, $a_2 \equiv^{\mathrm{Ls}}_{Mc} a_1$ and $a_2 \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} bc$. By extension (Corollary 4.14) there is $b' \equiv^{\mathrm{Ls}}_{Mc} b$ such that $b \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} b'c$, and thus $b'c \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} b$ by symmetry. By extension again, there is $c' \equiv^{\mathrm{Ls}}_{Mb} c$ with $b'c \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} bc'$. As $b'c \equiv^{\mathrm{Ls}}_{M} bc \equiv^{\mathrm{Ls}}_{M} bc'$, we get by Lemma 7.13 that there is a $\mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M}$-Morley parallel-Morley in $\mathrm{tp}(bc/M)$ sequence $I = (b_i, c_i)_{i \in \mathbb{Z}}$ with $b_0 c_0 = bc'$ and $b_1 c_1 = b'c$. As $a_2 \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} bc$, we get by Lemma 7.14 that there is some a such that $abc' \equiv^{\mathrm{Ls}}_{M} a_2 bc$, I is Ma-indiscernible and $a \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} I$.

Then, by monotonicity, $a \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} bc$. We also have $ab \equiv^{\mathrm{Ls}}_{M} a_2 b \equiv^{\mathrm{Ls}}_{M} a_0 b$, and by indiscernibility, $ac \equiv^{\mathrm{Ls}}_{M} ac' \equiv^{\mathrm{Ls}}_{M} a_2 c \equiv^{\mathrm{Ls}}_{M} a_1 c$. Since b and c were assumed to enumerate λ_T-saturated models we get $a \equiv^{\mathrm{Ls}}_{Mb} a_0$ and $a \equiv^{\mathrm{Ls}}_{Mc} a_1$. Also, $(b_i)_{i \le 0}$ is an Mac-indiscernible parallel-Morley sequence in $\mathrm{tp}(b/M)$ with $b_0 = b$, which gives $b \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} ac$ by Corollary 5.11. Similarly, as $(c_i)_{i \ge 1}$ is an Mab-indiscernible parallel-Morley sequence in $\mathrm{tp}(c/M)$ with $c_1 = c$, we get that $c \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} ab$. $\qquad\square$

8. Transitivity

Lemma 8.1. *If $M \subseteq N$ are e.c. models of a thick NSOP_1 theory, $a \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} N$, and μ is a small cardinal, then there is a parallel-Morley in $\mathrm{tp}(a/N)$ sequence $(a_i)_{i \in \mu}$ with $a_0 = a$ such that $a_i \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} Na_{<i}$ for every $i < \mu$.*

Proof. Put $\lambda = |Na| + \aleph_0$ and (using Lemma 6.3) choose a global N-Ls-invariant extension q of $\mathrm{Lstp}(a/N)$ satisfying $(*)_\lambda$.

By Lemma 7.9, compactness, finite character of Kim-independence, and an automorphism, it is enough to find for any given $k < \omega$ a parallel-Morley sequence $(a_i)_{i < k}$ in q over N such that $a_i \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} Na_{<i}$ for every $i < k$.

So fix any $k < \omega$. By backward induction on $k' = k+1, k, \ldots, 1$ we will construct trees $(c_\eta)_{\eta \in S_{k'}}$, where $S_{k'} := \{\xi \in \omega^{\le k} : 0^{k'-1} \trianglelefteq \xi\}$, such that for each k' the tree $(c_\eta)_{\eta \in S_{k'}}$ satisfies the following conditions:

(A1)$_{k'}$ For any $\eta \in S_{k'}$ we have $c_\eta \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} Nc_{\rhd \eta}$ and $c_\eta \equiv^{\mathrm{Ls}}_{N} a$.

(A2)$_{k'}$ $(c_\eta)_{\eta \in S_{k'}}$ is q-spread-out over N.

For $k' = k+1$ we let $c_{0^k} = a$. For the inductive step, suppose we are done for some k'. By (A1)$_{k'}$ we have $c_{0^{k'-1}} \equiv^{\mathrm{Ls}}_{N} a$, so by $(*)_\lambda$ there is a global N-Ls-invariant type $r(x, y) \supseteq q(x)$ extending $\mathrm{Lstp}(c_{0^{k'-1}}, (c_\eta)_{\eta \in S^*_{k'}}/N)$ where x corresponds to $c_{0^{k'-1}}$. Choose a Morley sequence $I := ((c_{\eta,i})_{\eta \in S_{k'}})_{i < \omega}$ in $r(x, y)$ over N with $c_{\eta,0} = c_\eta$ for each $\eta \in S_{k'}$. By extension (Corollary 4.14) there is $c' \equiv^{\mathrm{Ls}}_{N} a$ with $c' \mathop{\smile\hskip-0.8em\vert\,}\nolimits^{K}_{M} NI$. Put $c_{0^{k'-2} \frown i \frown \zeta} := c_{0^{k'-1} \frown \zeta, i}$ for all $i < \omega$, $\zeta \in \omega^{\le k+1-k'}$, and $c_{0^{k'-2}} := c'$. Then (A2)$_{k'-1}$ follows by Lemma 5.13(i), whereas (A1)$_{k'-1}$ with $\eta \in S^*_{k'-1}$ follows by invariance of Kim-independence, and (A1)$_{k'-1}$ with $\eta = 0^{k'-2}$ follows by the choice of $c_{0^{k'-2}} = c'$. Thus the inductive step, and hence the construction of the tree $(c_\eta)_{\eta \in \omega^{\le k}} = (c_\eta)_{\eta \in S_1}$, is completed.

Letting $(c'_\eta)_{\eta \in \omega^{\leq k}}$ be an s-indiscernible over N tree that is EM_s-based on $(c_\eta)_{\eta \in \omega^{\leq k}}$ over Na, we get that $(c'_\eta)_{\eta \in \omega^{\leq k}}$ satisfies $(A1)_1$ by Lemma 7.9 and Corollary 7.10(ii), and is weakly q-spread-out over N by Lemma 5.13(ii). Thus, by Lemma 5.13(iii), putting $a_i = c'_{0^{k-i}}$ for $i < k$ we get a parallel-Morley sequence $(a_i)_{i<k}$ in q over N satisfying the requirements. $\qquad\square$

Lemma 8.2. *Suppose T is thick NSOP_1 and $M \subseteq N$ are e.c. models of T. If $a \underset{M}{\overset{K}{\downarrow}} N$ and $c \underset{M}{\overset{K}{\downarrow}} N$ then there is $c' \equiv^{\mathrm{Ls}}_N c$ such that $ac' \underset{M}{\overset{K}{\downarrow}} N$ and $a \underset{N}{\overset{K}{\downarrow}} c'$.*

Proof. By Lemma 7.9 there is a type $\Gamma(x; N, a)$ that is equivalent to the condition $ax \underset{M}{\overset{K}{\downarrow}} N$. By Lemma 8.1 there is a parallel-Morley in $\mathrm{tp}(a/N)$ sequence $(a_i)_{i<\lambda_T}$ with $a_0 = a$ such that $a_i \underset{M}{\overset{K}{\downarrow}} Na_{<i}$ for every $i < \lambda_T$. Replacing $(a_i)_{i<\lambda_T}$ with an N-indiscernible sequence based on it over N and moving by an automorphism (to keep $a_0 = a$), we may assume $(a_i)_{i<\lambda_T}$ is N-indiscernible.

Claim. $\bigcup_{i<\lambda_T} \Gamma(x; N, a_i)$ *has a realisation c'' such that $c'' \equiv^{\mathrm{Ls}}_N c$.*

Proof of claim. By induction on $n < \omega$ we will find $c_n \equiv^{\mathrm{Ls}}_N c$ such that $c_n \underset{M}{\overset{K}{\downarrow}} Na_{<n}$ and $c_n \models \bigcup_{i<n} \Gamma(x; N, a_i)$, which is enough by compactness, N-indiscernibility of $(a_i)_{i<\lambda_T}$ and Corollary 7.10(ii). For $n = 0$ put $c_0 = c$. Assume we have found c_n and find by extension (Corollary 4.14) some $c' \equiv^{\mathrm{Ls}}_M c$ such that $c' \underset{M}{\overset{K}{\downarrow}} a_n$. By Theorem 7.15, there exists c_{n+1} with $c_{n+1}a_{<n} \equiv^{\mathrm{Ls}}_N c_n a_{<n}$, $c_{n+1}a_n \equiv^{\mathrm{Ls}}_M c'a_n$, $c_{n+1} \underset{M}{\overset{K}{\downarrow}} Na_{<n+1}$ and $a_n c_{n+1} \underset{M}{\overset{K}{\downarrow}} Na_{<n}$. In particular, $c_{n+1} \equiv^{\mathrm{Ls}}_N c_n \equiv^{\mathrm{Ls}}_N c$ and $c_{n+1} \models \bigcup_{i<n+1} \Gamma(x; N, a_i)$. $\qquad\square$

Let c'' be given by the claim, and let $(a'_i)_{i<\omega}$ be an Nc''-indiscernible sequence based on $(a_i)_{i<\lambda_T}$ over $Nc''a$. Then $a'_0 \equiv^{\mathrm{Ls}}_N a$ (as $a_i \equiv^{\mathrm{Ls}}_N a$ for every $i < \lambda_T$), so there is a Lascar strong automorphism f over N sending a'_0 to $a = a_0$. Put $c' := f(c'')$. Then $(f(a'_i))_{i<\omega}$ is an Nc'-indiscernible parallel-Morley sequence in $\mathrm{tp}(a/N)$ starting with a, so $c' \underset{N}{\overset{K}{\downarrow}} a$ by Corollary 5.11. Also, $c' \models \Gamma(x; N, a)$, so $ac' \underset{M}{\overset{K}{\downarrow}} N$ by the choice of Γ, and we are done. $\qquad\square$

Lemma 8.3. *Suppose T is thick NSOP_1 with e.c. models $M \subseteq N$ and $a \underset{M}{\overset{K}{\downarrow}} N$. Then there is a $\underset{N}{\overset{K}{\downarrow}}$-Morley parallel-Morley in $\mathrm{tp}(a/M)$ sequence $(a_i)_{i<\omega}$ with $a = a_0$.*

Proof. By extension (Corollary 4.14) we may assume that a is a λ_T-saturated model extending M. By Lemma 6.3 there is a global M-Ls-invariant extension $q(x) \supseteq \mathrm{tp}(a/M)$ satisfying the property $(*)_\lambda$ with $\lambda = |a| + \aleph_0$. We claim that it is enough to find for any given $k < \omega$ a parallel-Morley sequence $(a_i)_{i<k}$ in q over M such that $a_i \underset{N}{\overset{K}{\downarrow}} a_{<i}$ and $a_i \equiv_N a$ for every $i < k$: indeed, if we prove this, then, since the condition $(a_i \equiv_N a) \wedge (a_i \underset{N}{\overset{K}{\downarrow}} a_{<i})$ is type-definable by Lemma 7.9, we can find by compactness such a sequence of length $\lambda_{|T|+|Na|}$. Then taking an N-indiscernible sequence based on $(a_i)_{i<\lambda_{|T|+|Na|}}$ over N and moving it by an automorphism we obtain a desired sequence.

So fix any $k < \omega$. By backward induction on $k' = k+1, k, \ldots, 1$ we will define trees $(c_\eta)_{\eta \in S_{k'}}$, where $S_{k'} := \{\xi \in \omega^{\leq k} : 0^{k'-1} \trianglelefteq \xi\}$, such that for each k' the tree $(c_\eta)_{\eta \in S_{k'}}$ satisfies the following conditions:

(A1)$_{k'}$ For any $\eta \in S_{k'}$ we have $c_\eta \underset{N}{\overset{K}{\downarrow}} c_{\rhd \eta}$ and $c_\eta \equiv_N^{\mathrm{Ls}} a$.

(A2)$_{k'}$ $(c_\eta)_{\eta \in S_{k'}}$ is q-spread-out over M.

(A3)$_{k'}$ $(c_\eta)_{\eta \in S_{k'}} \underset{M}{\overset{K}{\downarrow}} N$.

For $k' = k+1$, we let $c_{0^k} = a$. For the inductive step, suppose we are done for some k'. By $(*)_\lambda$ and (A1)$_{k'}$ there is a global M-invariant type $r(x, y)$ extending $\mathrm{Lstp}(c_{0^{k'-1}}, (c_\eta)_{\eta \in S_{k'}^*}/M)$ and $q(x)$. As c_η's are λ_T-saturated models, we get, by (A3)$_{k'}$ and Corollary 4.10, that $\mathrm{Lstp}(N/(c_\eta)_{\eta \in S_{k'}})$ does not r-Ls-divide over M. Thus there is an N-indiscernible Morley sequence $I = ((c_{\eta,i})_{\eta \in S_{k'}})_{i < \omega}$ in $r(x, y)$ over M with $c_{\eta,0} = c_\eta$ for each $\eta \in S_{k'}$ and $I \underset{M}{\overset{K}{\downarrow}} N$. By Lemma 8.2 there is $a' \equiv_N^{\mathrm{Ls}} a$ such that $a' \underset{N}{\overset{K}{\downarrow}} I$ and $a'I \underset{M}{\overset{K}{\downarrow}} N$. Put $c_{0^{k'-2} \frown i \frown \zeta} := c_{0^{k'-1} \frown \zeta, i}$ for all $i < \omega$, $\zeta \in \omega^{\leq k+1-k'}$, and $c_{0^{k'-2}} := a'$. Then we get (A2)$_{k'-1}$ by Lemma 5.13(i), we get (A1)$_{k'-1}$ using that $a' \underset{N}{\overset{K}{\downarrow}} I$, and (A3)$_{k'-1}$ holds as $a'I \underset{M}{\overset{K}{\downarrow}} N$. Thus the inductive step, and hence the construction of the tree $(c_\eta)_{\eta \in \omega^{\leq k}} = (c_\eta)_{\eta \in S_1}$, is completed.

Letting $(c'_\eta)_{\eta \in \omega^{\leq k}}$ be an s-indiscernible over N tree which is EM_s-based on $(c_\eta)_{\eta \in \omega^{\leq k}}$ over Na, we get that $(c'_\eta)_{\eta \in \omega^{\leq k}}$ is weakly q-spread-out over M by Lemma 5.13(ii) and satisfies (A1)$_1$ by Lemma 7.9 and Corollary 7.10(ii). Thus putting $a_i = c'_{0^{k-i}}$ for $i < k$ we get by Lemma 5.13(iii) a parallel-Morley sequence $(a_i)_{i < k}$ in q over M satisfying the requirements. $\qquad \square$

Theorem 8.4 (transitivity). *Suppose T is thick NSOP_1 with models $M \subseteq N$. If $a \underset{M}{\overset{K}{\downarrow}} N$ and $a \underset{N}{\overset{K}{\downarrow}} c$, then $a \underset{M}{\overset{K}{\downarrow}} Nc$.*

Proof. By Lemma 8.3 there is a $\underset{N}{\overset{K}{\downarrow}}$-Morley parallel-Morley in $\mathrm{tp}(a/M)$ sequence $I = (a_i)_{i < \omega}$ with $a_0 = a$. Because $a \underset{N}{\overset{K}{\downarrow}} c$, we get by Lemma 7.14 an Nc-indiscernible sequence $I' = (a'_i)_{i < \omega} \equiv_{Na} I$. As I' is also parallel-Morley in $\mathrm{tp}(a/M)$ and $a'_0 = a$, we get by Corollary 5.11 that $Nc \underset{M}{\overset{K}{\downarrow}} a$, so, by symmetry, we are done. $\qquad \square$

9. Kim–Pillay style theorem

Theorem 9.1. *Let T be a thick positive theory. Then T is NSOP_1 if and only if there is an automorphism invariant ternary relation $\downarrow$ on small subsets of the monster model, only allowing e.c. models in the base, satisfying the following properties:*

FINITE CHARACTER *If $a \underset{M}{\downarrow} b_0$ for all finite $b_0 \subseteq b$, then $a \underset{M}{\downarrow} b$.*

EXISTENCE *$a \underset{M}{\downarrow} M$ for any model M.*

MONOTONICITY *$aa' \underset{M}{\downarrow} bb'$ implies $a \underset{M}{\downarrow} b$.*

SYMMETRY *$a \underset{M}{\downarrow} b$ implies $b \underset{M}{\downarrow} a$.*

LOCAL CHARACTER *Let a be a finite tuple and $\kappa > |T|$ be regular. Then for every continuous chain $(M_i)_{i<\kappa}$ with $|M_i| < \kappa$ for all i, there is $i < \kappa$ such that $a \underset{M_i}{\cancel{\smile}} M$, where $M = \bigcup_{i<\kappa} M_i$.*

INDEPENDENCE THEOREM *If $a \underset{M}{\smile} b$, $a' \underset{M}{\smile} c$ and $b \underset{M}{\smile} c$ with $a \equiv^{\mathrm{Ls}}_M a'$, then there is a'' such that $a''b \equiv^{\mathrm{Ls}}_M ab$, $a''c \equiv^{\mathrm{Ls}}_M a'c$ and $a'' \underset{M}{\smile} bc$.*

EXTENSION *If $a \underset{M}{\smile} b$, then for any c there is $a' \equiv_{Mb} a$ such that $a' \underset{M}{\smile} bc$.*

TRANSITIVITY *If $a \underset{M}{\smile} N$ and $a \underset{N}{\smile} b$ with $M \subseteq N$, then $a \underset{M}{\smile} Nb$.*

Furthermore, in this case, $\smile = \smile^K$.

The properties in Theorem 9.1 are not as strong as they could be. For example, we actually proved the strong independence theorem for $\smile^K$; see Theorem 7.15. The slightly simpler formulation of the properties in Theorem 9.1 is easier to verify for an arbitrary independence relation $\smile$. Then it follows immediately from $\smile = \smile^K$ that such an independence relation $\smile$ also satisfies the stronger formulations.

Remark 9.2. In the existing Kim–Pillay style theorems for full first-order logic, [Kaplan and Ramsey 2020, Theorem 9.1; 2021, Theorem 6.11] and [Chernikov et al. 2020, Theorem 5.1], there are still various properties that mention syntax. Our Theorem 9.1 is completely syntax-free. One syntax-dependent property is mentioned in all of the above theorems, and is called STRONG FINITE CHARACTER: if $a \underset{M}{\cancel{\smile}} b$ then there is $\varphi(x, b, m) \in \mathrm{tp}(a/Mb)$ such that for any $a' \models \varphi(x, b, m)$ we have $a' \underset{M}{\cancel{\smile}} b$.

We could replace FINITE CHARACTER and LOCAL CHARACTER in Theorem 9.1 by STRONG FINITE CHARACTER. Obviously STRONG FINITE CHARACTER implies FINITE CHARACTER and modulo the other properties it also implies LOCAL CHARACTER by Lemmas 9.5 and 9.6.

Remark 9.3. To conclude that a theory is NSOP$_1$ it is enough to find an independence relation with the properties STRONG FINITE CHARACTER, EXISTENCE, MONOTONICITY, SYMMETRY and INDEPENDENCE THEOREM; see [Haykazyan and Kirby 2021, Theorem 6.4]. However, that does not guarantee that the independence relation is also Kim-independence; see [Kaplan and Ramsey 2020, Remark 9.39] for an example (already in full first-order logic). We also point out that [Haykazyan and Kirby 2021, Theorem 6.4] says nothing about the properties that Kim-independence generally has in NSOP$_1$ theories. Finally, our proof is also different because we do not rely on the syntactic property STRONG FINITE CHARACTER.

Remark 9.4. We point out a minor difference between Theorem 7.7 and INDEPENDENCE THEOREM in Theorem 9.1. In the former we get $a'' \equiv^{\mathrm{Ls}}_{Mb} a$, which is

generally stronger than the $a''b \equiv_M^{\mathrm{Ls}} ab$ in the latter (and similar for c). Again, the reason is that the latter is easier to verify. Definitely in semi-Hausdorff theories, because then $a''b \equiv_M^{\mathrm{Ls}} ab$ is equivalent to $a''b \equiv_M ab$, so we do not have to worry about Lascar strong types. For a concrete example of this, see Fact 10.3(i). The only place where INDEPENDENCE THEOREM is used, namely, to get consistency along a certain sequence, we only need this weaker version.

Lemma 9.5. *Let $\underset{}{\downarrow}$ satisfy* STRONG FINITE CHARACTER, EXISTENCE, MONO-TONICITY *and* SYMMETRY. *Then a $\underset{M}{\overset{u}{\downarrow}} b$ implies $a \underset{M}{\downarrow} b$.*

Proof. Exactly as in [Chernikov and Ramsey 2016, Proposition 5.8]. $\qquad\square$

Lemma 9.6. *Let $\underset{}{\downarrow}$ be as in Lemma 9.5. Then it satisfies* LOCAL CHARACTER.

Proof. By Lemma 9.5, the proof from [Kaplan et al. 2019, Theorem 3.2] applies. Our formulation of local character then follows. $\qquad\square$

Corollary 9.7 (local character). *In a thick* NSOP$_1$ *theory Kim-independence satisfies* LOCAL CHARACTER.

Remark 9.8. In [Kaplan et al. 2019] there are also different formulations of LOCAL CHARACTER, for example in terms of club sets of $[M]^{|T|}$. Since their arguments apply directly, these formulations also hold for Kim-independence in any thick NSOP$_1$ theory.

The next definition is based on the notion of isi-dividing from [Kamsma 2020].

Definition 9.9. We say that a type $p(x, b) = \mathrm{tp}(a/Cb)$ *long divides* over C if there is μ such that for every $\lambda \geq \mu$ there is a sequence $(b_i)_{i < \lambda}$ with $b_i \equiv_C b$ for all $i < \lambda$ such that for some $\kappa < \lambda$ and every $I \subseteq \lambda$ with $|I| \geq \kappa$ we have that $\bigcup_{i \in I} p(x, b_i)$ is inconsistent. We write $a \underset{C}{\overset{\mathrm{ld}}{\downarrow}} b$ if $\mathrm{tp}(a/Cb)$ does not long divide over C.

There is a close connection between long dividing and dividing. Even though we do not need this connection in our proofs, it is still interesting to explore it. Dividing implies long dividing. Given an indiscernible sequence that witnesses dividing of a type p, we can use compactness to make it as long as we wish. So we find arbitrarily long sequences where p is inconsistent along any infinite subsequence, so p long divides. The converse is not so clear to us.

Question 9.10. Does long dividing imply dividing?

At least if we assume the existence of a proper class of Ramsey cardinals then the answer is positive. To see this, suppose that p long divides, and let λ be a big enough Ramsey cardinal. Then there is some sequence $(b_i)_{i < \lambda}$ witnessing that p long divides. Since we assumed λ to be Ramsey there is a cofinal $I \subseteq \lambda$ such that $(b_i)_{i \in I}$ is indiscernible. By the definition of long dividing, $\bigcup_{i \in I} p(x, b_i)$ is then inconsistent and so we conclude that p divides.

Lemma 9.11. *We have that* $a \underset{C}{\overset{i\text{Ls}}{\downharpoonright}} b$ *implies* $a \underset{C}{\overset{\text{ld}}{\downharpoonright}} b$.

Proof. Let $p(x, y) = \text{tp}(ab/C)$, and let λ be any regular cardinal bigger than the number of Lascar strong types over C (compatible with b). Let $(b_i)_{i<\lambda}$ be any sequence in $\text{tp}(b/C)$. By choice of λ there must be $I \subseteq \lambda$ such that $b_i \equiv_C^{\text{Ls}} b_j$ for all $i, j \in I$ and $|I| = \lambda$. Pick some $i_0 \in I$, and let a' be such that $a'b_{i_0} \equiv_C ab$. By assumption $a \underset{C}{\overset{i\text{Ls}}{\downharpoonright}} b$, so $a' \underset{C}{\overset{i\text{Ls}}{\downharpoonright}} b_{i_0}$. Let $q \supseteq \text{tp}(a'/Cb_{i_0})$ be a global C-Ls-invariant extension, and let $\alpha \models q$. Then $\alpha b_i \equiv_C^{\text{Ls}} \alpha b_{i_0}$ for all $i \in I$, so $\bigcup\{p(x, b_i) : i \in I\}$ is consistent. $\square$

Definition 9.12. Let $\downharpoonright$ be some independence relation, and let $(a_i)_{i<\kappa}$ be some sequence. Suppose furthermore that there is a continuous chain $(M_i)_{i<\kappa}$ of e.c. models, with $M \subseteq M_0$, such that $a_{<i} \subseteq M_i$ and $a_i \underset{M}{\downharpoonright} M_i$ for all $i < \kappa$. Then we call $(M_i)_{i<\kappa}$ an $\underset{M}{\downharpoonright}$-*independence chain (for* $(a_i)_{i<\kappa}$*).*

Remark 9.13. Let $\downharpoonright$ be an independence relation satisfying EXISTENCE and EXTENSION, let a be any tuple, and let M be any model. Then as usual we can inductively build arbitrarily long sequences $(a_i)_{i<\kappa}$ together with an $\underset{M}{\downharpoonright}$-independence chain $(M_i)_{i<\kappa}$, such that $a \equiv_M a_i$ for all $i < \kappa$.

The following is adapted from one half of the original Kim–Pillay theorem, and occurs in [Kamsma 2020, Theorem 1.1]. We just have to check that the use of base-monotonicity can be replaced with our more carefully formulated form of local character.

Proposition 9.14. *Let* $\downharpoonright$ *be as in Theorem 9.1. Then* $a \underset{M}{\overset{\text{ld}}{\downharpoonright}} b$ *implies* $a \underset{M}{\downharpoonright} b$.

Note that we will actually not need INDEPENDENCE THEOREM here.

Proof. It follows directly from the definition of long dividing that $\downharpoonright^{\text{ld}}$ has monotonicity on the left side. So by FINITE CHARACTER and SYMMETRY we may assume a to be finite.

By Remark 9.13 we find a sequence $(b_i)_{i<\kappa}$ with an $\underset{M}{\downharpoonright}$-independence chain $(M_i)_{i<\kappa}$ such that $b \equiv_M b_i$ for all $i < \kappa$. Picking the right $\kappa > (|T| + |M|)^+$, there must be $I \subseteq \kappa$ with order type $(|T| + |M|)^+$ such that $\bigcup_{i \in I} p(x, b_i)$ is consistent, where $p(x, y) = \text{tp}(ab/M)$. Let a' be a realisation of this set. By MONOTONICITY and downward Löwenheim–Skolem, we may assume that $(M_i)_{i \in I}$ is a continuous chain with $|M_i| \leq |T| + |M|$ for all $i \in I$. Then by LOCAL CHARACTER there is $i_0 \in I$ such that $a' \underset{M_{i_0}}{\downharpoonright} M_I$, where $M_I = \bigcup_{i \in I} M_i$. By MONOTONICITY we have $a' \underset{M_{i_0}}{\downharpoonright} b_{i_0}$ and by construction we also have $b_{i_0} \underset{M}{\downharpoonright} M_{i_0}$. So by SYMMETRY and TRANSITIVITY we obtain $a' \underset{M}{\downharpoonright} b_{i_0}$. The result now follows since $a'b_{i_0} \equiv_M ab$. $\square$

We note that in the above proof it is relevant that we work with long dividing instead of dividing. This is because the application of LOCAL CHARACTER only really makes sense if the chain consists of e.c. models, as we only allow e.c. models in the base. At the same time we need those e.c. models to form an independence

chain for the rest of the proof to work. If we would try to follow the same proof just for dividing then we would have to work with indiscernible sequences. Finding an indiscernible $\mathop{\smile}\limits^{}$-independent sequence is not an issue. This can be done as usual: we first build a very long $\mathop{\smile}\limits^{}$-independent sequence and then base an indiscernible sequence on it. This preserves being $\mathop{\smile}\limits^{}$-independent due to FINITE CHARACTER, but it does not carry over the independence chain. In long dividing this is not an issue, because we work directly with the very long sequence we constructed. So any "decorations", such as the independence chain, are then at our disposal.

The following lemma and its proof are a weaker version of the chain condition for $\mathop{\smile}\limits^{K}$-Morley sequences (Lemma 7.14) that works for long enough $\mathop{\smile}\limits^{K}$-independent sequences.

Lemma 9.15. *Let T be a thick* NSOP$_1$ *theory. Suppose that $a \mathop{\smile}\limits_{M}^{K} b$. Let $(b_i)_{i<\kappa}$ be an $\mathop{\smile}\limits_{M}^{K}$-independent sequence, where κ is a regular cardinal larger than the number of Lascar strong types over M (compatible with b) and where $b \equiv_M b_i$ for all $i < \kappa$. Then there is $I \subseteq \kappa$ with $|I| = \kappa$ such that $\bigcup_{i \in I} p(x, b_i)$ does not Kim-divide (and is thus consistent), where $p(x, y) = \mathrm{tp}(ab/M)$.*

Proof. By the choice of κ there is $I \subseteq \kappa$ with $|I| = \kappa$ such that $b_i \equiv_M^{\mathrm{Ls}} b_j$ for all $i, j \in I$. We conclude by the generalised independence theorem (Corollary 7.10(iii)). $\qquad\square$

Proof of Theorem 9.1. We already proved that $\mathop{\smile}\limits^{K}$ has all the listed properties if T is NSOP$_1$. So now we assume that we have an abstract independence relation $\mathop{\smile}\limits^{}$ satisfying the listed properties and we prove that $\mathop{\smile}\limits^{} = \mathop{\smile}\limits^{K}$ and that T is NSOP$_1$.

The direction $a \mathop{\smile}\limits_{M} b \Longrightarrow a \mathop{\smile}\limits_{M}^{K} b$ holds. This proof is based on the proof of the same direction in [Kaplan and Ramsey 2020, Theorem 9.1]. Let $p(x, b) = \mathrm{tp}(a/Mb)$, and let q be any global M-Ls-invariant extension of $\mathrm{tp}(b/M)$. Then a Morley sequence $(b_i)_{i<\omega}$ in q is a $\mathop{\smile}\limits_{M}$-Morley sequence by Lemma 9.11 and Proposition 9.14. By the standard INDEPENDENCE THEOREM argument we thus find that $\bigcup_{i<\omega} p(x, b_i)$ is consistent, and thus $a \mathop{\smile}\limits_{M}^{K} b$.

The theory T is NSOP$_1$. We prove weak symmetry as in Theorem 6.7. So suppose $a \mathop{\smile}\limits_{M}^{i\mathrm{Ls}} b$. Then combining Lemma 9.11 and Proposition 9.14 again we get $a \mathop{\smile}\limits_{M} b$. So by SYMMETRY we have $b \mathop{\smile}\limits_{M} a$ and then $b \mathop{\smile}\limits_{M}^{K} a$ follows from the above.

The direction $a \mathop{\smile}\limits_{M}^{K} b \Longrightarrow a \mathop{\smile}\limits_{M} b$ holds. This proof is based on the proof of the same direction in [Chernikov et al. 2020, Theorem 5.1]. By Remark 9.13 we obtain a long enough sequence $(b_i)_{i<\kappa}$ with an $\mathop{\smile}\limits_{M}$-independence chain $(M_i)_{i<\kappa}$ and $b_i \equiv_M b$ for all $i < \kappa$. By the above $(M_i)_{i<\kappa}$ is also an $\mathop{\smile}\limits_{M}^{K}$-independence chain. So by Lemma 9.15 there is $I \subseteq \kappa$ with order type κ such that $\bigcup_{i \in I} p(x, b_i)$ is consistent, where $p(x, b) = \mathrm{tp}(a/Mb)$. Let a' be a realisation of this set. By deleting an end segment, MONOTONICITY and downward Löwenheim–Skolem we may assume that $(M_i)_{i \in I}$ is a continuous chain with $|M_i| \leq |T| + |M|$ for all $i \in I$ and I has order

type $(|T|+|M|)^+$. By LOCAL CHARACTER there is $i_0 \in I$ such that $a' \mathrel{\smash{\underset{M_{i_0}}{\textstyle\bigcup}}} M_I$, where $M_I = \bigcup_{i \in I} M_i$, and therefore $a' \mathrel{\smash{\underset{M_{i_0}}{\textstyle\bigcup}}} b_{i_0}$. We also have $b_{i_0} \mathrel{\smash{\underset{M}{\textstyle\bigcup}}} M_{i_0}$, and thus by SYMMETRY and TRANSITIVITY we get $a' \mathrel{\smash{\underset{M}{\textstyle\bigcup}}} b_{i_0}$, and hence $a \mathrel{\smash{\underset{M}{\textstyle\bigcup}}} b$. $\square$

10. Examples

In this section we present some examples of thick NSOP$_1$ theories. First, we recall Poizat's example of a thick non-semi-Hausdorff theory (which is bounded hence NSOP$_1$). Next, we look at (the JEP refinements of) the positive theory of existentially closed exponential fields, which was shown to be NSOP$_1$ in [Haykazyan and Kirby 2021] by constructing a suitable independence relation. We deduce from the known results that this theory is Hausdorff (hence thick), and then we show that Kim-independence coincides in it with the independence relation studied in [Haykazyan and Kirby 2021]. Finally, we show that NSOP$_1$ is preserved under taking hyperimaginary extensions; in particular, the hyperimaginary extension of an arbitrary NSOP$_1$ theory in full first-order logic is a Hausdorff NSOP$_1$ theory.

Let us also briefly mention the class of nonsimple NSOP$_1$ thick theories found recently in [d'Elbée et al. 2021]. For any integral domain R, the authors consider in the language of rings enriched by a predicate P and constants for elements of R the theory $F_{R\text{-module}}$: the theory of fields together with the quantifier-free diagram of R and where P defines an R-submodule. By [d'Elbée et al. 2021, Theorem 4.2, Theorem 4.8], for any integral domain R, the theory $F_{R\text{-module}}$ is nonsimple and NSOP$_1$ in the sense of positive logic. Also, by [d'Elbée et al. 2021, Remark 4.9] it is thick and Kim-independence in the sense of our paper coincides there with weak independence, as defined in [d'Elbée et al. 2021, Definition 4.4]. In the particular case $R = \mathbb{Z}$ this shows that the theory of algebraically closed fields of characteristic zero with a generic additive subgroup, which is known to be noncompanionable by [d'Elbée 2021a, Remark 1.20], is nonsimple and NSOP$_1$ in positive logic (see also [d'Elbée 2021b, Remark 5.35]).

10A. *A thick, non-semi-Hausdorff theory.* The following is an example of a thick non-semi-Hausdorff theory from [Poizat 2010, Section 4]. Consider a language $L = \{P_n, R_n : n < \omega\} \cup \{r\}$ where P_n's and R_n's are unary relation symbols and r a binary relation symbol. Let $M = \{a_n, b_n : n < \omega\}$ be an L-structure with $a_0, b_0, a_1, a_2, \ldots$ pairwise distinct, in which P_n is interpreted as $\{a_n, b_n\}$, R_n as the complement of P_n, and r as the symmetric antireflexive relation $\{(a_n, b_n), (b_n, a_n) : n < \omega\}$. Let T be the h-inductive theory of the structure M. Then the models of T are bounded (in fact any e.c. extension of M adds at most two new points), so T is thick (and also NSOP$_1$). However, T is not semi-Hausdorff. In fact, it was observed by Rosario Mennuni that the unique nonalgebraic maximal type over M does not have any global M-invariant extensions. This shows that, in the definition

of Kim-independence in thick theories, it is necessary to work with Ls-invariant types rather than just invariant types. This is also an example where having the same type over an e.c. model does not guarantee having the same Lascar strong type (over that model).

10B. *Existentially closed exponential fields.* In [Haykazyan and Kirby 2021] the class of existentially closed exponential fields is studied using positive logic. They prove that this is NSOP_1 by providing a nice enough independence relation. We verify that this independence relation is indeed Kim-independence.

Definition 10.1. An *exponential field* or *E-field* is a field of characteristic zero with a group homomorphism E from the additive group to the multiplicative group. We call such a field an *EA-field* if it is also an algebraically closed field. We can axiomatise EA-fields by a positive theory and call this theory $T_{\text{EA-field}}$. The existentially closed exponential fields are then the e.c. models of $T_{\text{EA-field}}$.

Our definition is slightly different from [Haykazyan and Kirby 2021] where they consider the class of e.c. models of just the theory of E-fields. However, these classes of e.c. models coincide; see [Haykazyan and Kirby 2021, Proposition 3.3] and the discussion after it.

There are also many different JEP-refinements; see [Haykazyan and Kirby 2021, Corollary 4.6]. To work in a monster model we need to fix one such JEP-refinement. This is not an issue, since everything we discuss here works in any JEP-refinement.

Definition 10.2 [Haykazyan and Kirby 2021, Definition 5.1]. For any set A write $\langle A \rangle^{\text{EA}}$ for the smallest EA-subfield containing A. We define an independence notion $\smile$ by

$$A \underset{C}{\smile} B \iff \langle AC \rangle^{\text{EA}} \underset{\langle C \rangle^{\text{EA}}}{\overset{\text{ACF}}{\smile}} \langle BC \rangle^{\text{EA}},$$

where $\smile^{\text{ACF}}$ is the usual independence relation in algebraically closed fields.

Note that the independence relation $\smile$ actually makes sense over arbitrary sets. It would be interesting to compare this once Kim-independence over arbitrary sets has been developed in positive logic (see Question 10.21 below). For now we will restrict ourselves to working over e.c. models.

Fact 10.3. *We recall the following facts about $T_{\text{EA-field}}$.*

  (i) *The independence relation $\smile$ satisfies* STRONG FINITE CHARACTER, EXISTENCE, MONOTONICITY, SYMMETRY, INDEPENDENCE THEOREM.

 (ii) *Any span $F_1 \leftarrow F \rightarrow F_2$ of embeddings of EA-fields can be amalgamated in such a way that, after embedding the result into the monster model, $F_1 \underset{F}{\smile} F_2$.*

(iii) *For EA-fields F_1 and F_2, if $\mathrm{qftp}(F_1) = \mathrm{qftp}(F_2)$ then $\mathrm{tp}(F_1) = \mathrm{tp}(F_2)$.*

Proof. (i) This is [Haykazyan and Kirby 2021, Theorem 6.5]. They do not mention Lascar strong types in their formulation of INDEPENDENCE THEOREM. However, as we will see in Proposition 10.4, the theory is Hausdorff, so the types over e.c. models are Lascar strong types.

(ii) This is [Haykazyan and Kirby 2021, Theorem 4.3]. The fact that $F_1 \underset{F}{\bigcup} F_2$ is not mentioned there, but it is direct from their proof.

(iii) This follows directly from (ii).   □

To apply our theorem, Theorem 9.1, we need to verify a few more things.

Proposition 10.4. *The theory $T_{\text{EA-field}}$ is Hausdorff.*

Proof. Let T_k be the set of all h-inductive sentences that are true in all e.c. models of $T_{\text{EA-field}}$. By [Poizat and Yeshkeyev 2018, Theorem 8], being Hausdorff is equivalent to the models of T_k being amalgamation bases. By Fact 10.3(ii), the models of $T_{\text{EA-field}}$ are already amalgamation bases, so the models of T_k are in particular also amalgamation bases. So we conclude that $T_{\text{EA-field}}$ is indeed Hausdorff.   □

Note that Hausdorff is the best we can get, because [Haykazyan and Kirby 2021, Corollary 3.8] tells us that $T_{\text{EA-field}}$ cannot be Boolean. They prove this by showing that in every e.c. model F of $T_{\text{EA-field}}$ we have for all $a \in F$ that

$$a \in \mathbb{Z} \quad \Longleftrightarrow \quad F \models \forall x (E(x) = 1 \to E(ax) = 1),$$

so if the theory were Boolean this would contradict compactness.

Proposition 10.5. *The independence relation $\bigcup$ in $T_{\text{EA-field}}$ satisfies EXTENSION and TRANSITIVITY.*

Proof. We first prove TRANSITIVITY. Let $A \underset{B}{\bigcup} C$ and $A \underset{C}{\bigcup} D$ with $B \subseteq C$. So we have $\langle AB \rangle^{\text{EA}} \underset{\langle B \rangle^{\text{EA}}}{\overset{\text{ACF}}{\bigcup}} \langle BC \rangle^{\text{EA}}$, which is just

$$\langle AB \rangle^{\text{EA}} \underset{\langle B \rangle^{\text{EA}}}{\overset{\text{ACF}}{\bigcup}} \langle C \rangle^{\text{EA}}.$$

We also have $\langle AC \rangle^{\text{EA}} \underset{\langle C \rangle^{\text{EA}}}{\overset{\text{ACF}}{\bigcup}} \langle CD \rangle^{\text{EA}}$, and therefore, by monotonicity of ACF-independence,

$$\langle AB \rangle^{\text{EA}} \underset{\langle C \rangle^{\text{EA}}}{\overset{\text{ACF}}{\bigcup}} \langle CD \rangle^{\text{EA}}.$$

Then by transitivity of ACF-independence the result follows.

Now we prove EXTENSION. Let $a \underset{C}{\bigcup} b$ and let d be arbitrary. From the definition, we get $a \underset{C}{\bigcup} Cb$. We apply Fact 10.3(ii) to $\langle Cab \rangle^{\text{EA}} \supseteq \langle Cb \rangle^{\text{EA}} \subseteq \langle Cbd \rangle^{\text{EA}}$, and we can embed the amalgamation in the monster in such a way that $\langle Cbd \rangle^{\text{EA}}$ remains the same. So we get some EA-field F with $\text{qftp}(F/\langle Cb \rangle^{\text{EA}}) = \text{qftp}(\langle Cab \rangle^{\text{EA}}/\langle Cb \rangle^{\text{EA}})$ and $F \underset{\langle Cb \rangle^{\text{EA}}}{\bigcup} \langle Cbd \rangle^{\text{EA}}$, which simplifies to $F \underset{Cb}{\bigcup} Cbd$. By Fact 10.3(iii) and restricting ourselves to the copy $a' \in F$ of a we thus have $\text{tp}(a'/Cb) = \text{tp}(a/Cb)$. So we get $a' \underset{C}{\bigcup} Cb$ and $a' \underset{Cb}{\bigcup} Cbd$, and $a' \underset{C}{\bigcup} bd$ follows from TRANSITIVITY and MONOTONICITY.   □

Corollary 10.6. *The independence relation* $\underset{\smile}{\downarrow}$ *in* $T_{\text{EA-field}}$ *is the same as Kim-independence over e.c. models.*

Proof. This is a direct application of Theorem 9.1, using Remark 9.2 to replace LOCAL CHARACTER by STRONG FINITE CHARACTER. □

10C. *Hyperimaginaries.* One of the main motivations for studying positive logic in [Ben-Yaacov 2003a] was to be able to add hyperimaginaries in the same way we usually add imaginaries. It is well known that by doing so we leave the framework of full first-order logic, for example because we might get a bounded infinite definable set. However, we do stay within the framework of positive logic. We show that adding hyperimaginaries as real elements does not essentially change anything. So working with hyperimaginaries in positive logic requires no special treatment.

The construction in this section is based on [Ben-Yaacov 2003a, Example 2.16], but we work things out in far greater detail. This then allows us to prove that certain properties are invariant under adding hyperimaginaries.

We fix the following things throughout the rest of this section. First, we fix a positive theory T in a signature $\mathcal{L}$ with monster model $\mathfrak{M}$. For simplicity we assume $\mathcal{L}$ is single sorted (extending this to the multisorted setting is straightforward). Let $\mathcal{E}$ be a set of partial types (over $\varnothing$) $E(x, y)$, where x and y are (possibly infinite but small) tuples of variables, such that each E defines an equivalence relation in $\mathfrak{M}$.

Definition 10.7. We define the *hyperimaginary language* $\mathcal{L}_{\mathcal{E}}$ as a multisorted extension of $\mathcal{L}$. The sort of $\mathcal{L}$ will be called the *real sort* and is denoted by S_{real}. Then for each $E \in \mathcal{E}$ we add a sort S_E, called a *hyperimaginary sort*. For a variable y of sort S_E we denote by y_r a tuple of variables of the real sort, matching the length of the representatives of the E-equivalence classes.

For all $E_1, \dots, E_n \in \mathcal{E}$ we add a relation symbol $R_\varphi(x, y_1, \dots, y_n)$ of sort $S_{\text{real}}^{|x|} \times S_{E_1} \times \cdots \times S_{E_n}$ for each $\mathcal{L}$-formula $\varphi(x, y_{1,r}, \dots, y_{n,r})$.

In the above definition, not all variables in $\varphi(x, y_{1,r}, \dots, y_{n,r})$ need to actually appear in the formula. In particular, it is not problem for the $y_{i,r}$ to be infinite tuples. Similarly, when we write something like $\exists y_r \varphi(y_r)$, then we really only quantify over the variables that actually appear in φ.

Definition 10.8. We extend $\mathfrak{M}$ to an $\mathcal{L}_{\mathcal{E}}$-structure $\mathfrak{M}^{\mathcal{E}}$ as follows. The real sort S_{real} is just $\mathfrak{M}$, and for each $E \in \mathcal{E}$ the sort S_E is $\mathfrak{M}^\alpha / E$, where α is the length of the tuples of free variables in E. From now on we will use the shorthand notation $\mathfrak{M}/E$ and not mention α. For $E_1, \dots, E_n \in \mathcal{E}$ and $\varphi(x, y_{1,r}, \dots, y_{n,r})$ we interpret the relation symbol R_φ as follows. We let $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(a, c_1, \dots, c_n)$ if and only if there are representatives $b_1, \dots, b_n$ of $c_1, \dots, c_n$ such that $\mathfrak{M} \models \varphi(a, b_1, \dots, b_n)$.

For a real tuple b and some $E \in \mathcal{E}$ we will write $[b]$ for the corresponding hyperimaginary in $\mathfrak{M}/E$. To prevent cluttering of notation, we will actually also

use the notation $[b]$ for a tuple of hyperimaginaries. This notation leaves implicit which sort(s) $[b]$ belongs to, but that should not be a problem in what follows.

Definition 10.9. We define the $\mathcal{L}_{\mathcal{E}}$-theory $T^{\mathcal{E}}$ as the set of all h-inductive $\mathcal{L}_{\mathcal{E}}$-sentences true in $\mathfrak{M}^{\mathcal{E}}$.

In this construction, $\mathfrak{M}^{\mathcal{E}}$ will be a monster model of $T^{\mathcal{E}}$ (Theorem 10.15). Being Hausdorff/semi-Hausdorff/thick is preserved under adding hyperimaginaries (Theorem 10.17). We have that T is NSOP$_1$ if and only if $T^{\mathcal{E}}$ is NSOP$_1$ (Theorem 10.18). So in particular this means that if we start with an NSOP$_1$ theory T in full first-order logic, viewed as a positive theory, then $T^{\mathcal{E}}$ is a Hausdorff (and thus thick) NSOP$_1$ theory, and all our results apply. Finally, we also have that T satisfies the existence axiom for forking if and only if $T^{\mathcal{E}}$ satisfies the existence axiom for forking (Theorem 10.20).

We set up our construction in such a way that we can add any set $\mathcal{E}$ of hyper-imaginaries. If we wish to study $\mathfrak{M}^{\mathrm{heq}}$, where we have added all hyperimaginaries, we would have to add a proper class of hyperimaginaries. We can formalise this by taking $\mathcal{E}$ to be the set of all equivalence relations $E(x, y)$ where $|x| \leq |T|$. Then, by [Ben-Yaacov 2003c, Corollary 3.3], every possible hyperimaginary is interdefinable with a set of hyperimaginaries in $\mathcal{E}$. So we can take $\mathfrak{M}^{\mathrm{heq}}$ and T^{heq} to be $\mathfrak{M}^{\mathcal{E}}$ and $T^{\mathcal{E}}$.

Lemma 10.10. *Let $\varphi(x, y)$ be an $\mathcal{L}_{\mathcal{E}}$-formula, where x is a tuple of real variables and y is a tuple of hyperimaginary variables. Then there is a set of $\mathcal{L}$-formulas $\Sigma_{\varphi}(x, y_r)$ such that $\mathfrak{M} \models \Sigma_{\varphi}(a, b)$ if and only if $\mathfrak{M}^{\mathcal{E}} \models \varphi(a, [b])$.*

Proof. We first assume that $\varphi(x, y)$ is of the form

$$\exists wz\Big(\psi(x, w) \wedge \varepsilon(y, z) \wedge \bigwedge_{i \in I} R_{\chi_i}(x, w, y, z)\Big).$$

Here w is a tuple of real variables and z a tuple of hyperimaginary variables. The formula $\psi(x, w)$ is an $\mathcal{L}$-formula and $\varepsilon(y, z)$ is a conjunction of equalities of hyperimaginaries.

We define the partial type Γ_{φ} as follows. For each $i \in I$ we introduce tuples of real variables y_i and z_i matching y_r and z_r respectively. We let $E_{\varepsilon}(y_r, z_r)$ be the union of partial types in $\mathcal{E}$ expressing $\varepsilon([y_r], [z_r])$, and we close E_{ε} under conjunctions. Then we set

$\Gamma_{\varphi}(x, y_r, w, z_r, (y_i)_{i \in I}, (z_i)_{i \in I})$

$$= \Big\{\psi(x, w) \wedge \epsilon(y_r, z_r) \wedge \bigwedge_{i \in I} \chi_i(x, w, y_i, z_i) : \epsilon \in E_{\varepsilon}\Big\} \tag{1}$$

$$\cup \bigcup \{E_y(y_r, y_i) : i \in I\} \tag{2}$$

$$\cup \bigcup \{E_z(z_r, z_i) : i \in I\}. \tag{3}$$

Here E_y and E_z are the equivalence relations corresponding to the hyperimaginary variables y and z respectively.

Let $\Sigma_\varphi(x, y_r)$ express the following:

$$\exists w z_r (y_i)_{i \in I} (z_i)_{i \in I} \Gamma_\varphi(x, y_r, w, z_r, (y_i)_{i \in I}, (z_i)_{i \in I}).$$

Now suppose that a and b are such that $\mathfrak{M} \models \Sigma_\varphi(a, b)$. Then we find realisations such that

$$\mathfrak{M} \models \Gamma_\varphi(a, b, c, d, (b_i)_{i \in I}, (d_i)_{i \in I}).$$

Then (2) and (3) tell us that $[b] = [b_i]$ and $[d] = [d_i]$ for all $i \in I$, while (1) guarantees that $\mathfrak{M}^{\mathcal{E}} \models \varphi(a, [b])$. This proves the forward direction and the converse is straightforward by just taking representatives of the hyperimaginaries that are involved.

We assumed φ to be of a particular form. Since every formula can be written as a disjunction of regular formulas (i.e., formulas built using conjunction and existential quantification), we are only left an induction step for disjunction. So let $\varphi_1(x, y)$ and $\varphi_2(x, y)$ with $\Sigma_{\varphi_1}(x, y_r)$ and $\Sigma_{\varphi_2}(x, y_r)$ be given. We define $\Sigma_{\varphi_1 \vee \varphi_2}(x, y_r)$ as

$$\{\psi_1 \vee \psi_2 : \psi_1 \in \Sigma_{\varphi_1}, \psi_2 \in \Sigma_{\varphi_2}\}.$$

One easily checks that $\mathfrak{M} \models \Sigma_{\varphi_1 \vee \varphi_2}(a, b)$ precisely when $\mathfrak{M} \models \Sigma_{\varphi_1}(a, b)$ or $\mathfrak{M} \models \Sigma_{\varphi_2}(a, b)$ or both, and the result follows. $\square$

Lemma 10.11. *Let $\Gamma(x, y)$ be a set of $\mathcal{L}_{\mathcal{E}}$-formulas, where x is a tuple of real variables and y is a tuple of hyperimaginary variables. Then there is a set of $\mathcal{L}$-formulas $\Sigma_\Gamma(x, y_r)$ such that $\mathfrak{M} \models \Sigma_\Gamma(a, b)$ if and only if $\mathfrak{M}^{\mathcal{E}} \models \Gamma(a, [b])$.*

Proof. Define

$$\Sigma_\Gamma(x, y_r) = \bigcup_{\varphi \in \Gamma} \Sigma_\varphi(x, y_r),$$

where Σ_φ is as in Lemma 10.10. $\square$

Lemma 10.12. *If $\mathrm{tp}(a[b]) = \mathrm{tp}(a'[b'])$ then there is b'' such that $\mathrm{tp}(ab) = \mathrm{tp}(a'b'')$ and $[b'] = [b'']$.*

Proof. Define

$$\Sigma(x, y) = \mathrm{tp}_{\mathcal{L}}(ab) \cup E(b', y).$$

It is enough to prove that $\Sigma(a', y)$ is finitely satisfiable. Let $\varphi(x, y) \in \mathrm{tp}_{\mathcal{L}}(ab)$. Then $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(a, [b])$, so $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(a', [b'])$. So there is $b'' \in \mathfrak{M}$ with $\mathfrak{M} \models E(b', b'')$ and $\mathfrak{M} \models \varphi(a', b'')$, as required. $\square$

Lemma 10.13. *For every tuple of hyperimaginary variables y there is a partial $\mathcal{L}_{\mathcal{E}}$-type $\Xi(y_r, y)$ such that $\mathfrak{M}^{\mathcal{E}} \models \Xi(a, [a'])$ if and only if $[a] = [a']$.*

Proof. We define

$$\Xi(y_r, y) = \{R_\varepsilon(y_r, y) : \varepsilon \in E\},$$

where E is the equivalence relation corresponding to y. The right-to-left direction

is clear. For the forward direction we suppose that $\mathfrak{M}^{\mathcal{E}} \models \Xi(a, [a'])$. Consider the partial type

$$\Gamma(y_r) = E(a, y_r) \cup E(y_r, a').$$

For any $\varepsilon(a, y_r) \in E(a, y_r)$ we have $\mathfrak{M}^{\mathcal{E}} \models R_\varepsilon(a, [a'])$. So there must be $a^* \in \mathfrak{M}$ such that $[a^*] = [a']$ and $\mathfrak{M} \models \varepsilon(a, a^*)$. Therefore, $\mathfrak{M} \models \varepsilon(a, a^*) \wedge E(a^*, a')$. We thus see that Γ is finitely satisfiable, so there is a realisation a''. We conclude that $[a] = [a''] = [a']$. $\qquad \square$

Lemma 10.14. *Any automorphism $f : \mathfrak{M} \to \mathfrak{M}$ uniquely extends to an automorphism $f^{\mathcal{E}} : \mathfrak{M}^{\mathcal{E}} \to \mathfrak{M}^{\mathcal{E}}$ by setting $f^{\mathcal{E}}([b]) = [f(b)]$.*

Proof. It is straightforward to check that $f^{\mathcal{E}}$ is well defined and bijective. We need to show that $f^{\mathcal{E}}$ preserves and reflects truth of the new relation symbols in $\mathcal{L}_{\mathcal{E}}$ (preservation of equality is just saying that $f^{\mathcal{E}}$ is well defined). Suppose that $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(a, [b])$. By definition there is b' such that $[b'] = [b]$ and $\mathfrak{M} \models \varphi(a, b')$. Then $\mathfrak{M} \models \varphi(f(a), f(b'))$ and hence $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(f(a), [f(b')])$, which is just $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(f^{\mathcal{E}}(a), f^{\mathcal{E}}([b]))$. The converse follows in a similar way.

Finally we check uniqueness of $f^{\mathcal{E}}$. Suppose that $g : \mathfrak{M}^{\mathcal{E}} \to \mathfrak{M}^{\mathcal{E}}$ also extends f. For $[b] \in \mathfrak{M}^{\mathcal{E}}$ we have $\mathfrak{M}^{\mathcal{E}} \models \Xi(b, [b])$ by Lemma 10.13. So if g is an automorphism we must have $\mathfrak{M}^{\mathcal{E}} \models \Xi(g(b), g([b]))$, which means that $g([b]) = [g(b)] = [f(b)]$, as required. $\qquad \square$

Theorem 10.15. *The structure $\mathfrak{M}^{\mathcal{E}}$ is a monster model of $T^{\mathcal{E}}$.*

Proof. We prove that $\mathfrak{M}^{\mathcal{E}}$ is e.c. and is just as saturated and homogeneous as $\mathfrak{M}$. So let κ be such that $\mathfrak{M}$ is κ-saturated and κ-homogeneous. Note that this means that κ is definitely bigger than the length of any tuple representing a hyperimaginary.

Existentially closed: We will use Definition 2.5(iii). Suppose that $\mathfrak{M}^{\mathcal{E}} \not\models \varphi(a, [b])$. Then $\mathfrak{M} \not\models \Sigma_\varphi(a, b)$, where Σ_φ is from Lemma 10.11. Therefore, there exists $\psi(x, y_r) \in \Sigma_\varphi(x, y_r)$ such that $\mathfrak{M} \not\models \psi(a, b)$. Because $\mathfrak{M}$ is e.c. we find $\chi(x, y_r)$ with $T \models \neg \exists x y_r (\psi(x, y_r) \wedge \chi(x, y_r))$ and $\mathfrak{M} \models \chi(a, b)$. Thus $\mathfrak{M}^{\mathcal{E}} \models R_\chi(a, [b])$. We will conclude by proving that $\mathfrak{M}^{\mathcal{E}} \models \neg \exists x y (\varphi(x, y) \wedge R_\chi(x, y))$. Suppose for a contradiction that there are a' and b' such that $\mathfrak{M}^{\mathcal{E}} \models \varphi(a', [b']) \wedge R_\chi(a', [b'])$. Then there is b'' with $[b'] = [b'']$ and $\mathfrak{M} \models \chi(a', b'')$. So $\mathfrak{M}^{\mathcal{E}} \models \varphi(a', [b''])$ and thus $\mathfrak{M} \models \Sigma_\varphi(a', b'')$. We then get that $\mathfrak{M} \models \psi(a', b'') \wedge \chi(a', b'')$, which cannot happen.

Saturation: Let $\Gamma(x, y, c, [d])$ be a finitely satisfiable partial $\mathcal{L}_{\mathcal{E}}$-type with $|c[d]| < \kappa$. Let $\Sigma_\Gamma(x, y, c, d)$ be the set of $\mathcal{L}$-formulas from Lemma 10.11. By the construction there we have

$$\Sigma_\Gamma(x, y, c, d) = \bigcup_{\varphi \in \Gamma} \Sigma_\varphi(x, y, c, d),$$

where Σ_φ is as in Lemma 10.10. So finite satisfiability of $\Gamma(x, y, c, [d])$ implies

finite satisfiability of $\Sigma_\Gamma(x, y, c, d)$. We thus find $a, b \in \mathfrak{M}$ with $\mathfrak{M} \models \Sigma_\Gamma(a, b, c, d)$ and hence $\mathfrak{M}^{\mathcal{E}} \models \Gamma(a, [b], c, [d])$.

Homogeneity: If $\mathrm{tp}(a[b]) = \mathrm{tp}(a'[b'])$, then by Lemma 10.12 there is b'' such that $[b''] = [b']$ and $\mathrm{tp}(ab) = \mathrm{tp}(a'b'')$. Let $f : \mathfrak{M} \to \mathfrak{M}$ be an automorphism with $f(ab) = a'b''$. Then, by Lemma 10.14, we find $f^{\mathcal{E}} : \mathfrak{M}^{\mathcal{E}} \to \mathfrak{M}^{\mathcal{E}}$ with $f^{\mathcal{E}}(a[b]) = f(a)[f(b)] = a'[b''] = a'[b']$, as required. $\qquad\square$

Lemma 10.16. *A sequence $(a_i[b_i])_{i \in I}$ is indiscernible if and only if there are representatives b_i' of $[b_i]$ such that $(a_i b_i')_{i \in I}$ is indiscernible.*

Proof. We first prove the left-to-right direction. By compactness we may assume I to be long enough. We can find indiscernible $(a_i^* b_i^*)_{i \in I}$ based on $(a_i b_i)_{i \in I}$. Let $p((x_i y_{i,r})_{i \in I}) = \mathrm{tp}((a_i^* b_i^*)_{i \in I})$ and define the type

$$\Gamma = p((a_i y_{i,r})_{i \in I}) \cup \{\Xi(y_{i,r}, [b_i]) : i \in I\}.$$

Then a realisation of Γ is precisely what we need, so we prove that Γ is finitely satisfiable. That is, for $i_1 < \cdots < i_n \in I$, we will produce a realisation of Γ restricted to the variables $y_{i_1, r}, \ldots, y_{i_n, r}$ and parameters $a_{i_1}, \ldots, a_{i_n}, [b_{i_1}], \ldots, [b_{i_n}]$. By construction there are $j_1 < \cdots < j_n \in I$ such that $\mathrm{tp}(a_{i_1}^* b_{i_1}^* \cdots a_{i_n}^* b_{i_n}^*) = \mathrm{tp}(a_{j_1} b_{j_1} \cdots a_{j_n} b_{j_n})$. As $\mathrm{tp}(a_{i_1}[b_{i_1}] \cdots a_{i_n}[b_{i_n}]) = \mathrm{tp}(a_{j_1}[b_{j_1}] \cdots a_{j_n}[b_{j_n}])$, by Lemma 10.12 we can find $b_{i_1}' \cdots b_{i_n}'$ with $\mathrm{tp}(a_{i_1} b_{i_1}' \cdots a_{i_n} b_{i_n}') = \mathrm{tp}(a_{j_1} b_{j_1} \cdots a_{j_n} b_{j_n})$ while also $[b_{i_k}'] = [b_{i_k}]$ for all $1 \leq k \leq n$. So $b_{i_1}' \cdots b_{i_n}'$ is the desired realisation of Γ restricted to $y_{i_1, r}, \ldots, y_{i_n, r}$ and $a_{i_1}, \ldots, a_{i_n}, [b_{i_1}], \ldots, [b_{i_n}]$.

For the right-to-left direction, we note that, for any $i_1 < \cdots < i_n \in I$ and $j_1 < \cdots < j_n \in I$, we have

$$\Sigma_{\mathrm{tp}(a_{i_1}[b_{i_1}] \cdots a_{i_n}[b_{i_n}])} \subseteq \mathrm{tp}(a_{i_1} b_{i_1}' \cdots a_{i_n} b_{i_n}') = \mathrm{tp}(a_{j_1} b_{j_1}' \cdots a_{j_n} b_{j_n}').$$

So $\mathrm{tp}(a_{i_1}[b_{i_1}] \cdots a_{i_n}[b_{i_n}]) \subseteq \mathrm{tp}(a_{j_1}[b_{j_1}] \cdots a_{j_n}[b_{j_n}])$, and the claim follows by maximality of types. $\qquad\square$

Theorem 10.17. *The following properties of T are preserved when adding hyperimaginaries*:

- *Hausdorff*,
- *semi-Hausdorff*,
- *thick*.

That is, if T has the property then $T^{\mathcal{E}}$ has it as well.

Proof. <u>Hausdorff</u>: Let $a[b] \not\equiv a'[b']$. Then there exists $\varphi \in \mathrm{tp}(a[b])$ such that $\varphi \notin \mathrm{tp}(a'[b'])$. So there is a negation $\psi \in \mathrm{tp}(a'[b'])$ of φ. By Lemma 10.11 we have that Σ_φ and Σ_ψ are consistent while $\Sigma_\varphi \cup \Sigma_\psi$ is inconsistent.

Fix some type q of T such that $\Sigma_\psi \subseteq q$. We will produce formulas α_q and β_q such that $\Sigma_\varphi \cup \{\alpha_q\}$ is inconsistent, $\beta_q \notin q$ and $T \models \forall x y_r (\alpha_q(x, y_r) \vee \beta_q(x, y_r))$. Let $p \supseteq \Sigma_\varphi$ be a type of T. Then because T is Hausdorff there are formulas χ_p and θ_p such that $\chi_p \notin p$ and $\theta_p \notin q$, while $T \models \forall x y_r(\chi_p(x, y_r) \vee \theta_p(x, y_r))$. Then $\Sigma_\varphi \cup \{\chi_p : p \supseteq \Sigma_\varphi\}$ is inconsistent, and therefore there are $p_1, \ldots, p_n$ such that $\Sigma_\varphi \cup \{\chi_{p_1} \wedge \cdots \wedge \chi_{p_n}\}$ is inconsistent. We can now take α_q to be $\chi_{p_1} \wedge \cdots \wedge \chi_{p_n}$ and β_q to be $\theta_{p_1} \vee \cdots \vee \theta_{p_n}$.

Now $\Sigma_\psi \cup \{\beta_q : q \supseteq \Sigma_\psi\}$ is inconsistent. So there are $q_1, \ldots, q_k$ such that $\Sigma_\psi \cup \{\beta_{q_1} \wedge \cdots \wedge \beta_{q_k}\}$ is inconsistent. We set $\beta = \beta_{q_1} \wedge \cdots \wedge \beta_{q_k}$ and $\alpha = \alpha_{q_1} \vee \cdots \vee \alpha_{q_n}$. We then also have that $\Sigma_\varphi \cup \{\alpha\}$ is inconsistent and $T \models \forall x y_r(\alpha(x, y_r) \vee \beta(x, y_r))$.

Now consider the formulas $R_\alpha(x, y)$ and $R_\beta(x, y)$. By construction we have $T^\mathcal{E} \models \forall x y(R_\alpha(x, y) \vee R_\beta(x, y))$. We claim that $R_\alpha \notin \operatorname{tp}(a[b])$. Suppose for a contradiction that $\mathfrak{M}^\mathcal{E} \models R_\alpha(a, [b])$. Then there is b^* with $[b^*] = [b]$ such that $\mathfrak{M} \models \alpha(a, b^*)$. Since $\varphi \in \operatorname{tp}(a[b]) = \operatorname{tp}(a[b^*])$, we also have $\mathfrak{M} \models \Sigma_\varphi(a, b^*)$, contradicting that $\Sigma_\varphi \cup \{\alpha\}$ is inconsistent. So indeed $R_\alpha \notin \operatorname{tp}(a[b])$. Analogously we get that $R_\beta \notin \operatorname{tp}(a'[b'])$, which concludes the proof that $T^\mathcal{E}$ is Hausdorff.

Semi-Hausdorff: Suppose that equality of $\mathcal{L}$-types is type-definable by a partial $\mathcal{L}$-type Ω. Then for a tuple x of real variables and a tuple y of hyperimaginary variables, we consider the partial $\mathcal{L}_\mathcal{E}$-type $\Omega^\mathcal{E}(xy, x'y')$ that expresses the following:

$$\exists y_r y_r'(\Xi(y_r, y) \wedge \Xi(y_r', y') \wedge \Omega(x y_r, x' y_r')).$$

We claim that $\Omega^\mathcal{E}$ expresses equality of $\mathcal{L}_\mathcal{E}$-types.

If $\mathfrak{M}^\mathcal{E} \models \Omega^\mathcal{E}(a[b], a'[b'])$ then we find c and c' such that

$$\mathfrak{M}^\mathcal{E} \models \Xi(c, [b]) \wedge \Xi(c', [b']) \wedge \Omega(ac, a'c').$$

By Lemma 10.13, $[c] = [b]$ and $[c'] = [b']$. Therefore, $\varphi \in \operatorname{tp}(a[b]) = \operatorname{tp}(a[c])$ if and only if $\Sigma_\varphi \subseteq \operatorname{tp}(ac) = \operatorname{tp}(a'c')$ if and only if $\varphi \in \operatorname{tp}(a'[c']) = \operatorname{tp}(a'[b'])$. So $\operatorname{tp}(a[b]) = \operatorname{tp}(a'[b'])$, as required.

Conversely, if $\operatorname{tp}(a[b]) = \operatorname{tp}(a'[b'])$ then by Lemma 10.12 we find b'' such that $[b''] = [b']$ and $\operatorname{tp}(ab) = \operatorname{tp}(a'b'')$. Hence, $\models \Xi(b, [b]) \wedge \Xi(b'', [b']) \wedge \Omega(ab, a'b'')$.

Thick: Let Θ express indiscernibility of a sequence of real tuples. Then

$$\exists (y_{i,r})_{i<\omega} \Big(\Theta((x_i y_{i,r})_{i<\omega}) \wedge \bigwedge_{i<\omega} \Xi(y_{i,r}, y_i) \Big)$$

expresses indiscernibility of $(x_i y_i)_{i<\omega}$ in $T^\mathcal{E}$. Here we use that a sequence in $\mathfrak{M}^\mathcal{E}$ is indiscernible if and only if there is an indiscernible sequence of real representatives; see Lemma 10.16. □

Theorem 10.18. *The theory T is* NSOP_1 *if and only if $T^\mathcal{E}$ is* NSOP_1.

The technique in the proof of Theorem 10.18 can also be applied to other combinatorial properties, such as the order property, TP, TP_2, IP, etc. Of course, to do this, one first needs to write down a proper definition of these properties for positive logic, such as Definition 2.26 for SOP_1 or [Haykazyan and Kirby 2021, Definition 6.1] for TP_2.

Proof. One direction is trivial: if T has a formula with SOP_1, then so has $T^{\mathcal{E}}$.

We prove the other direction: supposing that $T^{\mathcal{E}}$ has a formula with SOP_1, we will show that T already has a formula with SOP_1. So let $\varphi(x, y; w, z)$ be an $\mathcal{L}_{\mathcal{E}}$-formula with SOP_1. Here x and w are tuples of real variables, and y and z are tuples of hyperimaginary variables. Let $(a_\eta[b_\eta] : \eta \in 2^{<\omega})$ and $\psi(w_1, z_1; w_2, z_2)$ be witnesses of SOP_1. Let $\Sigma_\varphi(x, y_r; w, z_r)$ and $\Sigma_\psi(w_1, z_{1,r}; w_2, z_{2,r})$ be as in Lemma 10.10. Then

$$\Sigma_\psi(w_1, z_{1,r}; w_2, z_{2,r}) \cup \Sigma_\varphi(x, y_r, w_1; z_{1,r}) \cup \Sigma_\varphi(x, y_r, w_2; z_{2,r})$$

is inconsistent. Hence there are finite $\varphi' \in \Sigma_\varphi$ and $\psi' \in \Sigma_\psi$ that are inconsistent with each other. That is,

$$T \models \neg\exists x y_r w_1 z_{1,r} w_2 z_{2,r} \big(\psi'(w_1, z_{1,r}, w_2, z_{2,r}) \wedge \varphi'(x, y_r, w_1, z_{1,r}) \\ \wedge \varphi'(x, y_r, w_2, z_{2,r}) \big). \quad (4)$$

As usual, any variables not actually appearing in the formulas should be ignored in the existential quantifier. We claim that φ' has SOP_1, which is witnessed by $(a_\eta b_\eta : \eta \in 2^{<\omega})$ and ψ'. We check the items in Definition 2.26.

(i) Let $\sigma \in 2^\omega$. Then $\{\varphi(x, y, a_{\sigma|_n}, [b_{\sigma|_n}]) : n < \omega\}$ is consistent. So there are c and $[d]$ such that $\models \varphi(c, [d], a_{\sigma|_n}, [b_{\sigma|_n}])$ for all $n < \omega$. That is, we have $\Sigma_\varphi(c, d, a_{\sigma|_n}, b_{\sigma|_n})$ for all $n < \omega$. In particular, $\{\varphi'(x, y_r, a_{\sigma|_n}, b_{\sigma|_n}) : n < \omega\}$ is consistent.

(ii) By construction; see (4).

(iii) Let $\eta, \nu \in \omega^{<\omega}$ such that $\eta^\frown 0 \preceq \nu$. Then $\models \psi(a_{\eta^\frown 1}, [b_{\eta^\frown 1}], a_\nu, [b_\nu])$, so $\models \Sigma_\psi(a_{\eta^\frown 1}, b_{\eta^\frown 1}, a_\nu, b_\nu)$ and in particular $\models \psi'(a_{\eta^\frown 1}, b_{\eta^\frown 1}, a_\nu, b_\nu)$. $\square$

Definition 10.19. We say that a theory satisfies the *existence axiom for forking* if $\mathrm{tp}(a/B)$ does not fork over B for any a and B.

Theorem 10.20. *The theory T satisfies the existence axiom for forking if and only if $T^{\mathcal{E}}$ satisfies the existence axiom for forking.*

Proof. One direction is immediate: anything witnessing forking in T will also be in $T^{\mathcal{E}}$. We prove the other direction. So assume there is $\mathrm{tp}(a[b]/C[D])$ that forks over $C[D]$. That is, it implies a (possibly infinite) disjunction $\bigvee_{i \in I} \varphi_i(xy, e^i[f^i])$ with $\varphi_i(xy, e^i[f^i])$ dividing over $C[D]$ for each $i \in I$. For each $i \in I$ we let $(e^i_j[f^i_j])_{j \in J}$ be a long enough $C[D]$-indiscernible sequence with $e^i_0[f^i_0] = e^i[f^i]$

such that $\{\varphi_i(xy, e^i_j[f^i_j]) : j \in J\}$ is inconsistent. By Lemma 10.12 we may assume that $e^i_j f^i_j \equiv e^i f^i$ for every $j \in J$. We claim that $\Sigma_{\varphi_i}(x, y_r, e^i, f^i)$ (see Lemma 10.10) divides over CD for all $i \in I$. Note that Σ_{φ_i} may contain parameters from CD.

To prove the claim let k be such that $\{\varphi_i(xy, e^i_j[f^i_j]) : j \in J_0\}$ is inconsistent for all $J_0 \subseteq J$ with $|J_0| = k$. So $\bigcup_{j \in J_0} \Sigma_{\varphi_i}(x, y_r, e^i_j, f^i_j)$ is inconsistent for all such J_0. Let $(e_n f_n)_{n<\omega}$ be a CD-indiscernible sequence based on $(e^i_j f^i_j)_{j \in J}$ over CD. Then there are $j_1 < \cdots < j_k \in J$ such that $e_1 f_1 \cdots e_k f_k \equiv_{CD} e^i_{j_1} f^i_{j_1} \cdots e^i_{j_k} f^i_{j_k}$, and therefore $\bigcup_{n<\omega} \Sigma_{\varphi_i}(x, y_r, e_n, f_n)$ is inconsistent. We conclude that $\Sigma_{\varphi_i}(x, y_r, e^i, f^i)$ divides over CD, as claimed.

By the claim there is $\psi_i(x, y_r, e^i, f^i)$ that is implied by $\Sigma_{\varphi_i}(x, y_r, e^i, f^i)$ such that $\psi_i(x, y_r, e^i, f^i)$ divides over CD, for all $i \in I$. Let $p = \text{tp}(a[b]C[D])$. Then $\Sigma_p(x, y_r, C, D)$ implies $\bigvee_{i \in I} \Sigma_{\varphi_i}(x, y_r, e^i, f^i)$. We thus have that $\Sigma_p(x, y_r, C, D)$ implies $\bigvee_{i \in I} \psi_i(x, y_r, e^i, f^i)$. So $\Sigma_p(x, y_r, C, D)$ forks over CD. $\qquad\square$

In the discussion following Definition 4.1 in [Kim 2021] it is stated that one may produce results for Kim-independence for the hyperimaginary extension M^{heq} of a first-order structure M parallel with those for first-order structures, provided that M^{heq} satisfies the existence axiom for forking (which, by the above theorem, is equivalent to the assumption that T satisfies this axiom). More generally, one can ask if our results on Kim-independence over models in thick NSOP_1 theories can be extended to arbitrary base sets assuming the existence axiom for forking:

Question 10.21. Suppose T is a thick positive NSOP_1 theory satisfying the existence axiom for forking. Can $\overset{K}{\underset{\ }{\smile}}$ be extended to an automorphism-invariant ternary relation between arbitrary small sets which satisfies the properties listed in Theorem 9.1?

Acknowledgements

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JAN DOBROWOLSKI:

dobrowol@math.uni.wroc.pl
Institute for Mathematical Logic and Basic Research, University of Münster, Germany

and

Instytut Matematyczny, Uniwersytetu Wrocławskiego, Poland
Current address: Department of Mathematics, University of Manchester, United Kingdom

MARK KAMSMA:

m.kamsma@uea.ac.uk
School of Mathematics, University of East Anglia, Norwich, United Kingdom

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