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Groups of finite Morley rank with a generically multiply transitive action on an abelian group

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Dedicated to Tuna Altınel in celebration of his freedom

We investigate the configuration where a group of finite Morley rank acts definably and generically m-transitively on an elementary abelian p-group of Morley rank n, where p is an odd prime, and $m \geqslant n$. We conclude that $m = n$, and the action is equivalent to the natural action of $\mathrm{GL}_n(F)$ on F^n for some algebraically closed field F. This strengthens one of our earlier results, and partially answers two problems posed by Borovik and Cherlin in 2008.

1. Introduction

This is the fourth and concluding work in a series of papers, which began with [Berkman and Borovik 2011; 2012; 2018]. All were aimed at proving the following theorem, but they handled different stages of the proof, each using a completely different approach and technique.

Theorem 1.1. *Let G be a group of finite Morley rank, V an elementary abelian p-group of Morley rank n, and p an odd prime. Assume that G acts on V faithfully, definably and generically m-transitively with $m \geqslant n$. Then $m = n$ and there is an algebraically closed field F such that $V \simeq F^n$, $G \simeq \mathrm{GL}_n(F)$, and the action is the natural action.*

In [Berkman and Borovik 2018], the same theorem was proven under the extra assumption that the action of G on V is generically *sharply* m-transitive. In this paper, we prove the generic sharpness of the action of G on V under the hypothesis of Theorem 1.1. Then Theorem 1.1 follows from the previous result [Berkman and Borovik 2018, Theorem 1]. We use the technique developed in [Borovik 2020] for analysis of actions of certain subgroups of G specifically for the needs of the present project; see Section 3A.

Theorem 1.1 gives partial confirmations to the following two conjectures; note that the latter is implicit in the former.

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Conjecture 1.2 [Altınel et al. 2008, Problem 37, p. 536; Borovik and Cherlin 2008, Problem 13]. *Let G be a connected group of finite Morley rank acting faithfully, definably, and generically n-transitively on a connected abelian group V of Morley rank n. Then V has a structure of an n-dimensional vector space over an algebraically closed field F of Morley rank* 1, *and G is* $\mathrm{GL}_n(F)$ *in its natural action on F^n.*

Conjecture 1.3 [Borovik and Cherlin 2008, Problem 12]. *Let G be a connected group of finite Morley rank acting faithfully, definably, and generically t-transitively on an abelian group V of Morley rank n. Then $t \leqslant n$.*

The cases when V is a torsion-free abelian group or an elementary abelian 2-group require completely different approaches and methods and are handled in our next paper. But even that result will not be the end of the story, since it appears to be an almost inevitable step in any proof of the following conjecture.

Conjecture 1.4 [Altınel et al. 2008, Problem 36, p. 536; Borovik and Cherlin 2008, Problem 9]. *Let G be a connected group of finite Morley rank acting faithfully, definably, transitively, and generically* $(n+2)$*-transitively on a set Ω of Morley rank n. Then the pair* (G, Ω) *is equivalent to the projective linear group* $\mathrm{PGL}_{n+1}(F)$ *acting on the projective space* $\mathbb{P}^n(F)$ *for some algebraically closed field F.*

Indeed, the group $F^n \rtimes \mathrm{GL}_n(F)$ is the stabiliser of a point in the action of $\mathrm{PGL}_{n+1}(F)$ on $\mathbb{P}^n(F)$.

Altınel and Wiscons [2018; 2019] have already made important contributions towards a solution to the above conjecture. The importance of Conjecture 1.4 has been recently highlighted in [Freitag and Moosa 2021].

General discussion and a survey of results on actions of groups of finite Morley rank can be found in [Borovik and Deloro 2019]. Terminology and notation follow [Altınel et al. 2008; Borovik and Nesin 1994; Borovik and Cherlin 2008].

2. Useful facts

In what follows, (G, X) is an infinite permutation group of finite Morley rank.

Definition. Let Y be a definable subset of X. If $\mathrm{rk}(X \setminus Y) < \mathrm{rk}(X)$ then Y is called a *strongly generic* subset of X. We will simply call it a *generic subset*. If G acts transitively on a generic subset of X, then we say G acts *generically transitively* on X. If the induced action of G on X^n is generically transitive, then we say G acts *generically n-transitively* on X.

The following two facts show that connectedness assumptions are superfluous in our context.

Fact 2.1. *If G acts generically m-transitively on a group X, where $m \geqslant \mathrm{rk}(X)$, then X is a connected group.*

Proof. If $m \geqslant 2$, this is a special case of [Borovik and Cherlin 2008, Lemma 1.8]. When $m = 1$, note that the generic orbit, say $A \subseteq X$, is cofinite in X. Since G fixes X° and A setwise, G also fixes $X^\circ \cap A$ setwise. The transitivity of G on A implies $A \subseteq X^\circ$. Hence $X = X^\circ$, since A is cofinite. □

Fact 2.2 [Altınel and Wiscons 2018, Lemma 4.10]. *If G acts n-transitively on X, and X is of degree* 1, *then G° also acts n-transitively on X.*

For any prime p, recall that a connected solvable p-group of bounded exponent is called a *p-unipotent* group, and a divisible abelian p-group is called a *p-torus*.

As the following two facts show, the structure of Sylow 2-subgroups in groups of finite Morley rank is well understood.

Fact 2.3 [Altınel et al. 2008, Propositions I.6.11, I.6.4, I.6.2]. *Sylow* 2*-subgroups of a group of finite Morley rank are conjugate. Moreover, if S is a Sylow* 2*-subgroup of a group of finite Morley rank, then $S^\circ = U * T$, where U is a definable* 2*-unipotent group, and T is a* 2*-torus. In particular, Sylow* 2*-subgroups in groups of finite Morley rank are locally finite.*

Fact 2.4 [Borovik et al. 2007a; Altınel et al. 2008, Theorem IV.4.1]. *Sylow* 2*-subgroups of a connected group of finite Morley rank are either trivial or infinite.*

The following is a structure theorem for nilpotent groups of finite Morley rank.

Fact 2.5 [Borovik and Nesin 1994, Theorem 6.8]. *Let G be a nilpotent group of finite Morley rank. Then G is the central product $D*B$, where D and B are definable characteristic subgroups of G, D is divisible, and B has bounded exponent.*

We gather below some facts about solvable groups of finite Morley rank which will be used in our proof of Theorem 1.1.

Fact 2.6. *Let M be a connected solvable group of finite Morley rank. Then the following hold*:

(a) *The commutator subgroup $[M, M]$ is connected and nilpotent.*

(b) *The group M can be written as a product $M = [M, M]C$, where C is a connected nilpotent subgroup.*

(c) *If M is of bounded exponent, then M is nilpotent.*

Proof. These follow from [Borovik and Nesin 1994, Theorem 6. 8], [Altınel et al. 2008, Corollary I.8.30], and [Altınel et al. 2008, Lemma I.5.5], respectively. □

Next, we list some results about various configurations where groups act on groups.

Fact 2.7 [Berkman and Borovik 2018, Fact 2.12]. *Let V be a connected abelian group and E an elementary abelian* 2*-group of order 2^m acting definably and* *faithfully on* V. *Assume* $m \geqslant n = \mathrm{rk}(V)$ *and* V *contains no involutions. Then* $m = n$ *and* $V = V_1 \oplus \cdots \oplus V_n$, *where*

(a) *every subgroup* V_i *for* $i = 1, \ldots, n$ *is connected*, *has Morley rank* 1 *and is* E*-invariant.*

Moreover,

(b) *each* V_i *for* $i = 1, \ldots, n$ *is a weight space of* E; *that is*, *there exists a nontrivial homomorphism* $\rho_i : E \to \{\pm 1\}$ *such that*

$$V_i = \{v \in V \mid v^e = \rho_i(e) \cdot v \text{ for all } e \in E\}.$$

Proof. Statements can be found in [Berkman and Borovik 2018], whose proofs refer to [Berkman and Borovik 2012, Lemma 7.1]. □

Assume that G acts on a group V such that the only infinite definable invariant subgroup of V is itself under this action. Then we say G acts on V *minimally*, or V is G*-minimal*.

Fact 2.8 [Berkman and Borovik 2018, Proposition 2.18]. *Let* V *be a connected abelian group and* $\Sigma = \mathbb{Z}_2^m \rtimes \mathrm{Sym}_m$ *act definably and faithfully on* V. *Assume* $m \geqslant \mathrm{rk}(V)$ *and* V *contains no involutions. Then* Σ *acts on* V *minimally.*

Fact 2.9 (Zilber [Borovik and Nesin 1994, Theorem 9.1]). *Let* A *and* V *be connected abelian groups of finite Morley rank such that* A *acts on* V *definably*, $C_A(V) = 1$ *and* V *is* A*-minimal. Then there exists an algebraically closed field* K *and a definable subgroup* $S \leqslant K^*$ *such that the action* $A \curvearrowright V$ *is definably equivalent to the natural action of* S *on* K^+.

Fact 2.10 [Altınel et al. 2008, Lemma I.8.2]. *Let* G *be a connected solvable group acting on an abelian group* V. *If* V *is* G*-minimal*, *then* G' *acts trivially on* V.

Recall that if a group has no nontrivial p-elements, we call it a $p^\perp$*-group*. A connected divisible abelian group is called a *torus*, and a torus A is called *good* if every definable subgroup of A is the definable hull of its torsion elements.

Fact 2.11 [Altınel et al. 2008, Proposition I.11.7]. *If a connected solvable* $p^\perp$*-group* A *acts faithfully on an abelian* p*-group* V, *then* A *is a good torus.*

Fact 2.12 [Altınel et al. 2008, Proposition I.8.5]. *Let* p *be a prime. Assume* $V \trianglelefteq G$ *is a definable solvable subgroup that contains no* p*-unipotent subgroup*, *and* $U \leqslant G$ *is a definable connected* p*-group of bounded exponent. Then* $[U, V] = 1$.

Fact 2.13 [Altınel et al. 2008, Lemma I.4.5]. *A definable group of automorphisms of an infinite field of finite Morley rank is trivial.*

3. Definable actions on elementary abelian p-groups

In this section, V is a connected elementary abelian p-group of finite Morley rank and X is a finite group acting on V definably. We use additive notation for the group operation on V and treat V as a vector space over $\mathbb{F}_p$.

It is convenient to work with the ring R generated by X in $\operatorname{End} V$. It is finite and its elements are definable endomorphisms; R is traditionally called the enveloping algebra (over $\mathbb{F}_p$) of the action of X on V. We treat V as a right R-module.

If $v \in V$, the set

$$vR = \{vr : r \in R\}$$

is an R-submodule, which is called the *cyclic submodule generated by* v. Of course all cyclic submodules contain less than $|R|$ elements, and therefore, there are finitely many possibilities for the isomorphism type of each of them.

3A. *Coprime actions.* In this subsection, we assume that X is a finite p'-group acting on V. Recall that a torsion group is called a *p'-group*, if it has no nontrivial p-elements.

We recall some generalities from representation theory. Applying Maschke's theorem to the action of X on R by right multiplication, we see that R is a semisimple $\mathbb{F}_p$-algebra and that every finite R-submodule in V is semisimple, that is, a direct sum of simple modules.

The following important (but easy) result (which generalises [Borovik 2020, Theorem 5]) now follows immediately.

Theorem 3.1. *Let V be a connected elementary abelian p-group of finite Morley rank, X a finite p'-group acting on V definably, and R the enveloping algebra over $\mathbb{F}_p$ for the action of X on V. Assume that $A_1, A_2, \dots, A_m$ is the complete list of nontrivial simple submodules for R in V, up to isomorphism. Then*

$$\operatorname{rk} V \geqslant m.$$

Proof. For each $i = 1, \dots, m$, write

$$V_i = \{v \in R : \text{all simple submodules of } vR \text{ are isomorphic to } A_i\}.$$

It is easy to see that all the V_i are definable submodules of V and

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m.$$

Since V is connected, all the V_i are connected. Hence, being a nontrivial, definable, connected submodule, each V_i has Morley rank at least 1. Therefore, $\operatorname{rk}(V) \geqslant m$. □

Problem 3.2. *It would be interesting to remove from Theorem 3.1 the assumption that X is a p'-group and prove the following*:

If $A_1, A_2, \dots, A_m$ are nontrivial simple pairwise nonisomorphic R-modules appearing as sections W/U for some definable R-modules $U < W \leqslant V$, then

$$\operatorname{rk} V \geqslant m.$$

3B. ***p*-Group actions.** The following is folklore, and this elegant and short proof was suggested by the referee.

Fact 3.3. *Let V be a connected elementary abelian p-group of finite Morley rank.*

(a) *If x is a p-element and $\langle x\rangle$ acts on V definably, then $[V, x]$ is a proper subgroup of V. In particular, if $\operatorname{rk}(V) = 1$, then the action is trivial.*

(b) *If P is a p-torus which acts on V definably, then the action is trivial.*

Proof. (a) We will work in $\operatorname{End} V$. Let $x \in \operatorname{End} V$ of order p^k. Since $(x-1)^{p^k} = x^{p^k} - 1 = 0$, we get a descending chain of definable subgroups

$$V \geqslant V(x-1) \geqslant V(x-1)^2 \geqslant \cdots$$

which reaches 0 in at most p^k steps. Thus, the chain does not become stationary before it reaches 0. Therefore, $V(x-1) = [V, x]$ is a proper subgroup in V.

(b) Since V has finite Morley rank, for any p-element x acting definably on V the above chain reaches 0 in at most $\operatorname{rk}(V)$ steps. Therefore, if $p^k \geqslant \operatorname{rk}(V)$ then $V(x^{p^k} - 1) = V(x-1)^{p^k} = 0$. Since P is a p-torus, for any $y \in P$, there exists $x \in P$ such that $y = x^{p^k}$. Hence, $V(y-1) = V(x^{p^k} - 1) = 0$, and we are done. □

4. Preliminary results

Throughout this section, we assume G and V are groups of finite Morley rank, V is a connected elementary abelian p-group of Morley rank n, where p is an odd prime, and G acts on V definably and faithfully.

Lemma 4.1. *Let H a definable connected subgroup of G, and $q \neq p$ a prime number. Then H does not contain any definable connected q-groups of bounded exponent. In particular, if H has an involution, then the connected component of any of its Sylow 2-subgroups is a 2-torus.*

Proof. Combine Facts 2.12, 2.3 and 2.4. □

4A. ***Groups of p-unipotent type.*** Following [Borovik et al. 2007b], we shall call a group K a *group of p-unipotent type*, if every definable connected solvable subgroup in K is a nilpotent p-group of bounded exponent. We still work under the assumptions of this section.

Proposition 4.2. *Let K be a definable subgroup in G which contains no good tori. Then K is a torsion group of p-unipotent type. In addition, K does not contain nontrivial definable divisible abelian subgroups.*

Proof. First note that by Fact 3.3, K contains no nontrivial p-tori. Therefore, every connected definable solvable $p^{\perp}$-subgroup in K is trivial by Fact 2.11.

Now it is easy to see that very definable divisible abelian subgroup in K is trivial. Indeed, if such a subgroup, say A, contains a nontrivial p-element, then it contains a nontrivial p-torus, which is impossible by the above paragraph. Hence A is a $p^{\perp}$-group and is trivial again by above.

Next, notice that every element in K is of finite order. Indeed, if $x \in K$ is of infinite order, then the connected component $d(x)^{\circ}$ of the definable closure of $\langle x \rangle$ is a divisible abelian group, which contradicts the above paragraph.

By Fact 2.5, $M = BD$, where B and D are connected, B is of bounded exponent and D is divisible. However, $D = 1$ by above, and B is a p-group by Lemma 4.1. Therefore, every definable connected nilpotent subgroup M in K is a p-group of bounded exponent.

By Fact 2.6(b), if M is a connected solvable subgroup in K, then $M = [M, M]C$ where C is a connected nilpotent subgroup.

Finally, we will prove that K is of p-unipotent type. Let M be a definable connected solvable subgroup of K. Then by above, $M = [M, M]C$, where C is a connected nilpotent subgroup. By Fact 2.6(a), $[M, M]$ is also connected and nilpotent. Hence both subgroups are p-groups of bounded exponent by above; therefore, so is M. Now the nilpotency of M follows from Fact 2.6(c). □

4B. ***Basis of induction.*** Connected groups acting faithfully and definably on abelian groups of Morley rank $n \leqslant 3$ are well understood. To prove these special cases of our theorem, the following results will be used.

Fact 4.3 [Deloro 2009]. *Let G be a connected nonsolvable group acting faithfully on a connected abelian group V. If $\mathrm{rk}(V) = 2$, then there exists an algebraically closed field K such that the action $G \curvearrowright V$ is equivalent to $\mathrm{GL}_2(K) \curvearrowright K^2$ or $\mathrm{SL}_2(K) \curvearrowright K^2$.*

Fact 4.4 [Borovik and Deloro 2016; Frécon 2018]. *Let G be a connected nonsolvable group acting faithfully and minimally on an abelian group V. If $\mathrm{rk}(V) = 3$ then there exists an algebraically closed field K such that $V = K^3$ and G is isomorphic to either $\mathrm{PSL}_2(K) \times Z(G)$ or $\mathrm{SL}_3(K) * Z(G)$. The action is the adjoint action in the former case, and the natural action in the latter case.*

4C. ***Throwback to pseudoreflection actions.*** To exclude the case when G in our Theorem 1.1 is not connected, we will need a result which uses concepts from one of our earlier papers [Berkman and Borovik 2012]. A special case of this result, when G is connected, was stated as [Berkman and Borovik 2012, Corollary 1.3].

Proposition 4.5. *Let G be a group of finite Morley rank acting definably and faithfully on an elementary abelian p-group V of Morley rank n, where p is an* *odd prime. Assume that G contains a definable subgroup $G^\sharp \simeq \mathrm{GL}_n(F)$ for an algebraically closed field F of characteristic p. Assume also that V is definably isomorphic to the additive group of the F-vector space F^n and $G^\sharp$ acts on V as on its canonical module. Then $G^\sharp = G$.*

Proof. Observe first that $\mathrm{rk}\, F^n = n$ implies $\mathrm{rk}\, F = 1$. *Pseudoreflection subgroups* in the sense of [Berkman and Borovik 2012] are connected definable abelian subgroups $R < G^\sharp$ such that $V = [V, R] \oplus C_V(R)$ and R acts transitively on the nonzero elements of $[V, R]$. By Fact 2.9, one can immediately conclude that $R \simeq F^*$ and $[V, R] \simeq F^+$. Therefore $\mathrm{rk}\, R = 1 = \mathrm{rk}[V, R]$ in our case.

It is easy to see that pseudoreflection subgroups in $G^\sharp = \mathrm{GL}_n(F)$ are one-dimensional (in the sense of the theory of algebraic groups) tori of the form, in a suitable coordinate system in F^n,

$$R = \{\mathrm{diag}(x, 1, \dots, 1) \mid x \in F, x \neq 0\},$$

and all pseudoreflection subgroups in $G^\sharp$ are conjugate in $G^\sharp$.

If R is a pseudoreflection subgroup in $G^\sharp$, consider the subgroup $\langle R^G \rangle$ generated in G by all G-conjugates of R, which is a normal definable subgroup generated by pseudoreflection subgroups. In view of [Berkman and Borovik 2012, Theorem 1.2], $G^\sharp = \langle R^G \rangle$ is normal in G.

We will use induction on $n \geqslant 1$. When $n = 1$, $G^\sharp = R \simeq F^*$ and $V = [V, R] \simeq F^+$. The subring generated by R in the ring of definable endomorphisms of V is a field by Schur's lemma, which we will denote by E. Since G normalises R, G acts as a group of field automorphisms on E. Hence, by Fact 2.13, $G = C_G(R)$. Thus, G acts linearly on $V \simeq F^+$, and therefore, $G = F^* = R = G^\sharp$.

Now assume $n \geqslant 2$. By the Frattini argument, $G = G^\sharp N_G(R)$. Write $H = N_G(R)$ and $H^\sharp = N_{G^\sharp}(R)$. It is well known that $H^\sharp = R \times L$, where $L \simeq \mathrm{GL}_{n-1}(F)$ centralises $[V, R]$ and acts on $C_V(R) \simeq F^{n-1}$ as on a canonical module. Obviously, H leaves $[V, R]$ and $C_V(R)$ invariant. Note that the action of H/R on $C_V(R)$ is faithful. Indeed, if K/R is the kernel, then K acts faithfully on $[V, R]$ with a normal subgroup $R \cong F^*$. This brings us to the base of induction, which was discussed above. Hence $K = R$. Thus, H/R contains a definable subgroup

$$H^\sharp/R = (R \times L)/R \simeq \mathrm{GL}_{n-1}(F);$$

by the inductive assumption, $H/R = H^\sharp/R$, and hence $H = H^\sharp$. Therefore, $G = G^\sharp H = G^\sharp H^\sharp = G^\sharp$. □

5. Proof of Theorem 1.1

We present a complete proof of Theorem 1.1 in this section. Therefore, we work under the following assumptions.

We have a group of finite Morley rank G acting definably, faithfully, and generically m-transitively on a connected elementary abelian p-group V of Morley rank n, with p an odd prime and $m \geqslant n$.

First note that V is connected by Fact 2.1. Another crucial observation is that G° also acts definably, faithfully, and generically m-transitively on V by Fact 2.2. Therefore, in view of Proposition 4.5, it will suffice to prove Theorem 1.1 in the special case when $G = G^\circ$ is connected.

Therefore, from now on we assume that G is connected.

5A. ***The core configuration.*** We will focus now on a group-theoretic configuration at the heart of Theorem 1.1.

The generic m-transitivity of G on V means that there is a generic subset A of V^m on which G acts transitively. We fix an element $\bar{a} = (a_1, \ldots, a_m) \in A$, and write

$$V_0 = \langle a_1, \ldots, a_m \rangle.$$

From now on, we denote by K the pointwise stabilizer, and by H the setwise stabilizer of $\{\pm a_1, \ldots, \pm a_m\}$ in G.

In $\overline{H} = H/K$ we have m involutions $\bar{e}_i$ for $i = 1, \ldots, m$ defined by their action on $a_1, \ldots, a_m$:

$$a_j^{\bar{e}_i} = \begin{cases} -a_j & \text{if } i = j, \\ a_j & \text{otherwise.} \end{cases}$$

Lemma 5.1. *The group $\bar{E}_m = \langle \bar{e}_1, \ldots, \bar{e}_m \rangle$ is an elementary abelian group of order 2^m and $H/K \simeq \Sigma_m = \bar{E}_m \rtimes \mathrm{Sym}_m$, where Sym_m permutes the generators $\bar{e}_1, \ldots, \bar{e}_m$ of $\bar{E}_m$.*

Notice that the group Σ_m is the *hyperoctahedral* group of degree m, which prominently features in the theory of algebraic groups as the reflection group of type BC_m. This fact is not used in this paper, but is likely to pop up in some of our future work.

Proof. Since G acts generically m-transitively on V, the proof of [Berkman and Borovik 2018, Lemma 3.1] can be repeated in this context as well, and we obtain the desired result. □

5B. ***Essential subgroups and ample subgroups.*** Let D be the full preimage in H of the subgroup $\bar{E}_m < H/K$. At this point, we temporarily forget about the ambient group G and generic transitivity and focus on the group H and its subgroups D and K.

For a subgroup $X \leqslant H$, we write $X_D = X \cap D$ and $X_K = X \cap K$.

We shall call a definable subgroup $X \leqslant H$ *ample* if $KX = H$.

A definable subgroup $X \leqslant D$ is *essential* if $KX = D$. Equivalently, a definable subgroup $X < G$ is essential if

- X leaves invariant the set $\{\pm a_1, \dots \pm a_m\}$ (which is equivalent to $X \leqslant H$) and, consequently, the subgroup V_0;
- X_K is the pointwise stabiliser of $\bar{a}$ in X (and consequently $X_K = C_X(V_0)$), and $X/K_X \simeq \bar{E}_m$ acts on V_0 as on the canonical module $\mathbb{Z}_p^m$ for $\bar{E}_m$ and leaves invariant the subgroups

$$A_1 = \langle a_1 \rangle, \dots, A_m = \langle a_m \rangle.$$

Notice that $X = H$ is an ample subgroup. Obviously, if X is ample, then X_D is essential.

The following lemma summarises the application of representation theory of finite groups in our context.

Lemma 5.2. *If X is a finite essential subgroup and X_K is a p'-group, then $X_K = 1$. Also, in that case, $m = n$.*

Proof. Since X_K is a p'-group, X is also a p'-group because $p \neq 2$ and X/X_K is a 2-group which covers $D/K = \bar{E}_m$.

Let R be the enveloping algebra of X.

Notice that X (hence R) acts on each subgroup $A_1, \dots, A_m$ irreducibly. Moreover, each representation is different, because the A_i are cyclic groups of order p and, among the elements $\bar{e}_1, \dots, \bar{e}_m$, only $\bar{e}_i$ inverts A_i. By Theorem 3.1, $m = n$ and $V = V_1 \oplus \cdots \oplus V_n$, where in the modules V_i each simple R-submodule is isomorphic to A_i. But X_K acts trivially on each A_i, hence acts trivially on each V_i and therefore on V. This means $X_K = 1$. □

Notice that in the next lemma we do not assume that X is finite, and therefore we continue to accept the possibility that $m > n$.

Lemma 5.3. *If X is an essential subgroup then X_K is a $2^\perp$-group.*

Proof. Let S be a Sylow 2-subgroup in X. If X_K is not a $2^\perp$-group, then $X_K \cap S \neq 1$. Take a nontrivial element $s \in X_K \cap S$ and elements $s_1, \dots, s_m$ in S whose images in X/X_K generate $X/X_K \simeq \bar{E}_m$. By Fact 2.3, S is locally finite, so $s, s_1, \dots, s_m$ generate a finite 2-subgroup, say Y, in S. Obviously, $X_K Y = X$, and hence Y is essential and $s = 1$ by Lemma 5.2, a contradiction. □

We can now characterise essential subgroups.

Lemma 5.4. *Let E be a Sylow 2-subgroup in D. Then $KE = D$, $E_K = 1$, and $E \simeq \bar{E}_m$. In particular, E is an essential subgroup.*

Moreover, essential subgroups of D are exactly those definable subgroups which contain one of the Sylow 2-subgroups of D.

Proof. Since E is a Sylow 2-subgroup in D, so is KE/K in $D/K \simeq \bar{E}_m$, thus $D = KE$, that is, E is essential. By Lemma 5.3, E_K is a $2^\perp$-group, so it is trivial. Since $K/E_K \simeq \bar{E}_m$, the first statement follows. The second statement is clear. □

Now we obtain the equality $m = n$ in the general case.

Lemma 5.5. $$m = n.$$

Proof. Because H contains an elementary abelian 2-subgroup of order 2^m by Lemma 5.4, Fact 2.7 (or Theorem 3.1) gives us $m = n$. □

Lemma 5.6. *K contains no good tori.*

Proof. Assume the contrary, and let T be a maximal good torus in K. By conjugacy of maximal good tori [Altınel et al. 2008, Proposition IV.1.15], and the Frattini argument, we have $D = KN_D(T)$ and thus $N_D(T)$ is an essential subgroup and contains a Sylow 2-subgroup E of D. Let q be a prime such that T has q-torsion and Q the maximal elementary abelian q-subgroup in T. Obviously, E normalises Q and QE is an essential subgroup. By Lemma 5.2, $Q = 1$, a contradiction. □

Lemma 5.7. *K is a torsion group of p-unipotent type.*

Proof. This is an immediate consequence of Lemma 5.6 and Proposition 4.2. □

Lemma 5.8. *Let E be a Sylow 2-subgroup in D. Then $X = N_H(E)$ is an ample subgroup. Moreover, $X_K = 1$ and $X \simeq \Sigma_n$.*

Proof. By the Frattini argument, $DX = H$, and thus X is an ample subgroup. Since $X_K \lhd X$, $E \lhd X$, and $X_K \cap E = 1$ by Lemma 5.3, we have $[X_K, E] = 1$. If x is a p'-element in X_K, then the subgroup $\langle x\rangle \times E$ is essential and therefore $x = 1$ by Lemma 5.2. Hence X_K is a p-group.

Take the weight decomposition of V with respect to E:

$$V = V_1 \oplus \cdots \oplus V_n, \text{ where } i = 1, 2, \ldots, n.$$

Note that every element in X_K leaves every one-dimensional space V_i invariant. Since X_K is a p-group, Fact 3.3 is applicable, and thus we conclude that X_K fixes each V_i, and hence V, elementwise. Therefore $X_K = 1$, which, in its turn, implies $X \simeq \Sigma_n$. □

5C. *An almost final configuration: the ample subgroup $K^\circ\Sigma$.*

Lemma 5.9. *If $m \geqslant 2$ then G is not solvable.*

Proof. If G is solvable, by Fact 2.8, we know that Fact 2.10 is applicable, so $G' = 1$ and G is abelian. However, for $m \geqslant 2$, Σ_m is not abelian. □

At the heart of our proof of Theorem 1.1, there is a Core Configuration. We set it up by writing $Q = K^\circ$ and $X = QN_H(E)$ and listing the properties of X which we have established so far.

Core Configuration.

- X is a group of finite Morley rank acting definably and faithfully on an elementary abelian p-group V with p odd of Morley rank $n \geqslant 3$.
- $Q \lhd X$ is a nontrivial connected definable subgroup of p-unipotent type; notice that Q is not necessarily nilpotent. We also denote it by Q_n.
- $\Sigma \simeq \Sigma_n$ is a subgroup of X which normalises Q. It will be convenient to denote it just by Σ_n.
- Finally, to emphasise the inductive nature of our setting, we may write, if necessary, $X = X_n$.

In the next Lemma 5.10 we analyse the Core Configuration on its own, without using any further information about G, and prove that $Q = 1$.

Lemma 5.10. *Under assumptions of the Core Configuration, $Q_n = 1$ for all $n \geqslant 3$.*

Proof. We proceed by induction on n. If $n = 3$, Q_3 is nilpotent in view of Facts 4.3 and 4.4, which give us a basis of induction on $n \geqslant 3$. For the inductive step for $n > 3$, take the involution e_n and consider

$$C_{Q\Sigma}(e_n) = C^\circ_Q(e_n)C_\Sigma(e_n).$$

Obviously, this group leaves invariant the eigenspaces V_n^+ and V_n^-. Write $Q_{n-1} = C^\circ_Q(e_n)$. Observe $C_\Sigma(e_n) = \Sigma_{n-1} \times \langle e_n\rangle$, and we write $X_{n-1} = Q_{n-1}\Sigma_{n-1}$.

For the inductive assumption, we have $Q_{n-1} = 1$. Then $C^\circ_Q(e_n) = Q_{n-1} = 1$ and Q_n is an abelian p-group. Assume $Q_n \neq 1$. Then $[V, Q_n]$ is a nontrivial proper connected subgroup of V and is Σ_n-invariant, which contradicts the minimality of the action of Σ_n on V, Fact 2.8. The contradiction shows $Q_n = 1$. □

We can now return to the main proof.

Proposition 5.11. $K^\circ = 1.$

Proof. If $n \leqslant 3$, we know everything about G from Facts 4.3, 4.4, and Lemma 5.9, and in these cases, $K^\circ = 1$. If $n \geqslant 3$, the proposition follows from Lemma 5.10. □

Corollary 5.12. *If X is an ample subgroup, then X_K is a finite group without involutions and $X = X_K \rtimes \Sigma$.*

5D. *The final case: K is finite.*

Lemma 5.13. $K = 1$ *and* $H = \Sigma.$

Proof. Let $Q \neq 1$ be a Sylow q-subgroup of K for a prime $q \neq p$. Then by the Frattini argument, $H = KN_H(Q)$. Write $X = N_H(Q)$. Then X is an ample subgroup. We can assume without loss of generality that $E < X$. Applying Lemma 5.2 to the essential group QE, we see that $Q = 1$. Hence if $K \neq 1$ then K is a p-group.

Consider the ample group $K\Sigma$; because of the minimality of the action of Σ on V (Fact 2.8), we have $K = 1$ by Fact 3.3(a). Hence $H = \Sigma$. □

This completes the proof of Theorem 1.1.

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Worst-case expansions of complete theories

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Given a complete theory T and a subset $Y \subseteq X^k$, we precisely determine the *worst-case complexity*, with respect to further monadic expansions, of an expansion (M, Y) by Y of a model M of T with universe X. In particular, although by definition monadically stable/NIP theories are robust under arbitrary monadic expansions, we show that monadically NFCP (equivalently, mutually algebraic) theories are the largest class that is robust under anything beyond monadic expansions. We also exhibit a paradigmatic structure for the failure of each of monadic NFCP/stable/NIP and prove each of these paradigms definably embeds into a monadic expansion of a sufficiently saturated model of any theory without the corresponding property.

1. Introduction

The idea of measuring the complexity of a first order theory by determining the worst-case complexity of its models under expansions by arbitrarily many unary (monadic) predicates was introduced by Baldwin and Shelah [1985]. For example, the theory ACF of algebraically closed fields is maximally complex with respect to this measure, even though it is classically very simple and has many well-studied tame monadic expansions. One way to see this complexity is to first name an infinite linearly independent set by a unary predicate A; then any graph G with vertex set A is definable in the further expansion by the unary predicate $B_G = \{g + h : g, h \in A \text{ and } (g, h) \text{ is an edge in } G\}$. As any structure in a finite language is definable in a monadic expansion of a graph (e.g., by the construction in [Hodges 1993, Theorem 5.5.1]), we may for example define models of ZFC in monadic expansions of models of ACF.

In contrast to ACF, some theories such as $\mathrm{Th}(\mathbb{Q}, <)$ are monadically NIP, i.e., no monadic expansion has the independence property. (The definitions of NIP, as well as stability and NFCP, are recalled in the next section.) If a theory is not monadically NIP then it can define arbitrary graphs in unary expansions of its models, as ACF does, and thus is also maximally complex by our measure. Similarly, there exist

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monadically stable theories, such as the theory of an equivalence relation with infinitely many infinite classes, and monadically NFCP theories (which coincide with the mutually algebraic theories of [Laskowski 2013]), such as $\mathrm{Th}(\mathbb{Z}, \mathrm{succ})$.

Our first result shows that the random graph, $(\mathbb{Q}, <)$, and the equivalence relation with infinitely many infinite classes are paradigms of structures that respectively are not monadically NIP/stable/NFCP, in the sense that we may define these paradigms on singletons in a monadic expansion of any sufficiently saturated model without the corresponding property (Theorem 3.2).

For our main result, recall that while monadically NIP and monadically stable theories are closed under monadic expansions by definition, the monadically NFCP theories satisfy a stronger closure property: if T is monadically NFCP and $M \models T$, then any expansion of M by arbitrarily many relations definable in monadically NFCP structures with the same universe as M remains monadically NFCP [Laskowski 2013]. Our main result proves that any attempt to extend these closure statements to larger classes of relations fails spectacularly, producing expansions of models defining arbitrary graphs.

Before stating the result, we must introduce an extremely simple class of theories.

Definition 1.1. A complete theory T is *purely monadic* if, for every model $M \models T$ with universe λ, every definable (with parameters) $Y \subseteq \lambda^k$ is definable in a monadic structure $(\lambda, U_1, \ldots, U_n)$.

Theorem 1.2. *Suppose a complete theory T is not purely monadic and $Y \subseteq \lambda^k$ is not definable in a purely monadic structure, where $|\lambda| \geq |T|$.*

If either T is not monadically NFCP or Y is not definable in a monadically NFCP structure, then there is $M \models T$ with universe λ such that the expansion (M, Y) is not monadically NIP.

Otherwise, if T is monadically NFCP and Y is definable in a monadically NFCP structure, then for every $M \models T$ with universe λ, the expansion (M, Y) is monadically NFCP.

The cases ruled out by the hypotheses of this theorem are straightforward, and are handled by Fact 2.3.

Section 3 is dedicated to the result on paradigmatic failures of monadic properties mentioned above, while in Section 4 we find a canonical configuration present in any structure that is monadically NFCP but not purely monadic. In Section 5, Theorem 1.2 is then proved in cases, by suitably overlaying the available configurations to monadically define arbitrary graphs.

2. Preliminaries

We recall the following standard conditions on a partitioned formula $\phi(\overline{x}, \overline{y})$, when we are working in a sufficiently saturated model $\mathfrak{C}$ of a complete theory T:

- $\phi(\bar{x}, \bar{y})$ has the *finite-cover property* (*FCP*) if, for arbitrarily large n, there are $\langle \bar{a}_i : i < n \rangle$ in $\mathfrak{C}$ such that

$$\mathfrak{C} \models \neg \exists \bar{x} \big(\bigwedge_{i<n} \phi(\bar{x}, \bar{a}_i) \big) \wedge \bigwedge_{\ell<n} \exists \bar{x} \big(\bigwedge_{i<n, i \neq \ell} \phi(\bar{x}, \bar{a}_i) \big).$$

- $\phi(\bar{x}, \bar{y})$ has the *order property* if, for each n, there are $\langle \bar{a}_i : i < n \rangle$ in $\mathfrak{C}$ such that, for each $k < n$,

$$\mathfrak{C} \models \bigwedge_{k<n} \big[\exists \bar{x} \big(\bigwedge_{i<k} \phi(\bar{x}, \bar{a}_i) \wedge \bigwedge_{k \leq i < n} \neg \phi(\bar{x}, \bar{a}_i) \big) \big].$$

- $\phi(\bar{x}, \bar{y})$ has the *independence property* if, for each n, there are $\langle \bar{a}_i : i < n \rangle$ in $\mathfrak{C}$ such that

$$\mathfrak{C} \models \bigwedge_{s \subseteq [n]} \big[\exists \bar{x} \big(\bigwedge_{i \in s} \phi(\bar{x}, \bar{a}_i) \wedge \bigwedge_{i \in n \setminus s} \neg \phi(\bar{x}, \bar{a}_i) \big) \big].$$

A complete theory T is *NFCP* if no partitioned formula $\phi(\bar{x}, \bar{y})$ has the FCP, T is *stable* if no partitioned formula $\phi(\bar{x}, \bar{y})$ has the order property, and T is *NIP* if no partitioned formula $\phi(\bar{x}, \bar{y})$ has the independence property.

It is well known that for complete theories, NFCP $\implies$ stable $\implies$ NIP, and as purely monadic theories are NFCP (e.g., by the comment after Fact 4.2), we have the following implications for a complete theory T:

$$\text{purely monadic} \implies \text{mon. NFCP} \implies \text{mon. stable} \implies \text{mon. NIP}.$$

We now introduce some definitions for convenience.

Definition 2.1. Given a complete theory T, a cardinal λ, a subset $Y \subseteq \lambda^k$ for some $k \geq 1$, and a property P of theories (we will be particularly interested in monadic NIP), we say (T, Y) *is always* P if $\mathrm{Th}(M, Y)$ has P for all models M of T with universe λ.

Definition 2.2. A subset $Y \subseteq \lambda^k$ is *monadically definable* if it is definable in some monadic structure $(N, U_1, \ldots, U_n)$.

$Y \subseteq \lambda^k$ is *monadically NFCP definable* if it is definable in some monadically NFCP structure N. Analogously, Y is *monadically stable/monadically NIP definable* if it is definable in some monadically stable/monadically NIP structure N.

Equivalently, a subset $Y \subseteq \lambda^k$ is monadically definable (respectively, monadically NFCP/stable/NIP definable) if and only if the structure $N = (\lambda, Y)$ in a language with a single k-ary predicate symbol is purely monadic (respectively, monadically NFCP/stable/NIP).

Thus, we have the following implications for $Y \subseteq \lambda^k$:

$$\text{mon. definable} \implies \text{mon. NFCP def} \implies \text{mon. stable def} \implies \text{mon. NIP def.}$$

The hypotheses of Theorem 1.2 ruled out the cases where T is purely monadic or Y is monadically definable. The following fact is immediate from unpacking definitions, but we include it for completeness.

Fact 2.3. *Let T be a complete theory, $Y \subseteq \lambda^k$, and*

$P \in \{$*purely monadic, monadically NFCP, monadically stable, monadically NIP*$\}$.

(1) *If T is purely monadic and Y is P definable then (T, Y) is always P.*

(2) *If T is P and $Y \subseteq \lambda^k$ is monadically definable then (T, Y) is always P.*

There are many equivalents to monadic NFCP (e.g., see [Laskowski 2009; 2013; Braunfeld and Laskowski 2022]), monadic stability (see [Baldwin and Shelah 1985; Anderson 1990]), and monadic NIP (see [Baldwin and Shelah 1985; Shelah 1986; Braunfeld and Laskowski 2021]). What we use is encapsulated in the rest of this section.

Definition 2.4. Let T be a complete theory. T is *weakly minimal* if for any pair $M \preceq N$ of models, every nonalgebraic 1-type $p \in S_1(M)$ has a unique nonalgebraic extension $q \in S_1(N)$. T is (*forking trivial*) if whenever $\{A, B, C\}$ is pairwise forking-independent over D, then it is an independent set over D. T is *totally trivial* if for all A, B, C, D, if $A \mathrel{\unicode{x2AEB}}_D B$ and $A \mathrel{\unicode{x2AEB}}_D C$ then $A \mathrel{\unicode{x2AEB}}_D BC$. (This is obtained from the definition of triviality by removing the hypothesis that $B \mathrel{\unicode{x2AEB}}_D C$.)

Fact 2.5 [Laskowski 2013, Theorem 3.3]. *The following are equivalent for a complete theory T:*

(1) *T is monadically NFCP.*

(2) *T is mutually algebraic* (see Definition 4.1 below).

(3) *T is weakly minimal and trivial.*

Although we will not explicitly use it, "trivial" could be replaced by "totally trivial" in (3), since they are equivalent assuming weak minimality, for example, by [Goode 1991, Proposition 5].

We will make use of the following sufficient condition from [Baldwin and Shelah 1985] for monadically defining arbitrary graphs or, equivalently, by Fact 2.7, for the failure of monadic NIP.

Definition 2.6. A structure M *admits coding* if there are infinite subsets A, B, C of M^1 and a formula $\phi(x, y, z)$ whose restriction to $A \times B \times C$ is the graph of a bijection $f : A \times B \to C$. A theory T (*monadically*) *admits coding* if (some monadic expansion M^* of) some model M of T admits coding.

Fact 2.7 [Baldwin and Shelah 1985; Braunfeld and Laskowski 2021]. *The following are equivalent for a complete theory T:*

(1) *T is monadically NIP.*

(2) *T does not monadically admit coding.*

(3) *There is a graph that is not definable in any monadic expansion of any model of T.*

Fact 2.8 [Baldwin and Shelah 1985; Anderson 1990]. *The following are equivalent for a stable complete theory T*:

(1) *T is monadically stable.*

(2) *T is monadically NIP.*

(3) *T does not admit coding.*

(4) *T is totally trivial and forking is transitive on singletons*, *i.e.*, *for all D*, *if $a \not\mathop{\smile}\!\!\!\!\!\mid_D b$ and $b \not\mathop{\smile}\!\!\!\!\!\mid_D c$ then $a \not\mathop{\smile}\!\!\!\!\!\mid_D c$.*

Proof. (1) $\Longrightarrow$ (2) is clear, (2) $\Longrightarrow$ (3) follows from Fact 2.7, and (3) $\Longrightarrow$ (4) is [Baldwin and Shelah 1985, Lemma 4.2.6]. Finally, (4) $\Longrightarrow$ (1) is essentially contained in Theorems 3.2.4 and 4.2.17 of [Baldwin and Shelah 1985], but verifying this involves tracing through several other results. The implication is more cleanly stated in Theorems 2.17 and 2.21 of [Anderson 1990], noting that what [Anderson 1990, Definition 2.5] calls *forking-triviality* is equivalent to the two conditions in (4) by some basic forking-calculus manipulations. □

Lemma 2.9. *If T is monadically stable* (*equivalently*, *stable and monadically NIP*) *but not monadically NFCP*, *then T is not weakly minimal.*

Proof. Fact 2.8 shows the parenthetical equivalence, and also shows that if T is monadically stable then it is (totally) trivial. So by Fact 2.5, if T is not monadically NFCP then it cannot be weakly minimal. □

3. Finding paradigms of nonmonadically NFCP theories

In this section, we show the following classical structures will always witness the failure of monadic NIP/stability/NFCP in a suitable monadic expansion.

• The random graph, sometimes called the Rado graph, $\mathcal{R} = (A, E)$ is the standard example of a structure whose theory has the independence property. In particular, its theory is not monadically NIP.

• Dense linear order (DLO), the theory of $(\mathbb{Q}, \leq)$, is one of the simplest nonstable theories as $\leq$ visibly witnesses the order property. Thus, DLO is not monadically stable, but it is monadically NIP (e.g., see [Simon 2015, Proposition A.2]).

• Let $\mathcal{E} = (X, E)$, where $X = \omega \times \omega$ (so each element of X can be uniquely written as $(a, b) \in \omega^2$) and $E((a_1, b_1), (a_2, b_2))$ holds if and only if $a_1 = a_2$. Thus, $\mathcal{E}$ is the (unique) model of the ω-categorical theory of an equivalence relation with infinitely many classes, with each class infinite. The theory $\mathrm{Th}(\mathcal{E})$ is monadically stable, but it is not monadically NFCP. To see the former, one can check it satisfies the conditions in Fact 2.8(4). To see the latter, one can add a single unary predicate whose interpretation contains exactly n elements from the n-th E-class. This expanded structure is a paradigm of a stable structure with the finite-cover property.

We next show that these paradigms all *definably embed* into a monadic expansion of any model of its class. It is crucial to consider structures defined in M^1 rather than in a cartesian power, as this will allow us to name substructures in unary expansions.

Definition 3.1. We say a structure $\mathcal{A}$ *definably embeds* into another structure M (possibly in a different language) if $\mathcal{A}$ is definable on singletons in M.

Explicitly, let $\mathcal{A} = (A, R)$ be any structure in a language with a binary relation, and let M be an L-structure in some arbitrary language. We say $\mathcal{A}$ *definably embeds* into M if there are L-definable $X \subseteq M^1$ and $R' \subseteq X^2$ and a bijection $f : A \to X$ such that for all $a, b \in A$, $\mathcal{A} \models R(a, b)$ if and only if $M \models R'(f(a), f(b))$. (Informally, (X, R') is an "isomorphic copy of $\mathcal{A}$".)

A definable embedding $f : (A, R) \to (X, R')$ is *type-respecting* if, in addition, for any tuples $\overline{a}, \overline{a}' \in A^n$, if $\mathrm{qftp}_{\mathcal{A}}(\overline{a}) = \mathrm{qftp}_{\mathcal{A}}(\overline{a}')$, then $\mathrm{tp}_M(f(\overline{a})) = \mathrm{tp}_M(f(\overline{a}'))$.

Theorem 3.2. *Let T be a complete L-theory.*

(1) *If T is not monadically NIP, then the random graph $\mathcal{R}$ definably embeds into some monadic expansion M^* of a model M of T.*

(2) *If T is not monadically stable, then there is a definable, type-respecting embedding of $(\mathbb{Q}, \leq)$ into some monadic expansion M^* of a model M of T.*

(3) *If T is monadically stable but not monadically NFCP, then there is a definable, type-respecting embedding of $\mathcal{E}$ into some monadic expansion M^* of a model M of T.*

(4) *If T is not monadically NFCP, $\mathcal{E}$ definably embeds into some monadic expansion M^* of a model M of T.*

Proof. (1) Assume T is not monadically NIP. By either [Baldwin and Shelah 1985] or [Braunfeld and Laskowski 2021], there is a monadic expansion M^* of a model of T that admits coding, i.e., there are infinite sets A, B, C and a 3-ary L^*-formula $\phi(x, y, z)$ coding the graph of a bijection from $A \times B$ to C. By adding more unary predicates, we may assume each of A, B, C are definable in M^* and are countably infinite, and by replacing ϕ by $\phi(x, y, z) \wedge A(x) \wedge B(y) \wedge C(z)$, the graph of ϕ is precisely the bijection. Now add a unary predicate $D \subseteq C$ so that for every $a_1 \neq a_2 \in A$, there is a unique $b \in B$ such that $M^* \models \exists(d_1, d_2 \in D)(\phi(a_1, b, d_1) \wedge \phi(a_2, b, d_2))$. Thus, in this expansion, one can think of B as coding (symmetric) edges of A via this formula. For the whole of D, we get a complete graph on A, but for any predetermined graph $\mathcal{G}$ with universe A, one can add a single unary predicate $E \subseteq D$ so that for any $a_1 \neq a_2 \in A$, the following formula holds if and only if a_1 and a_2 are edge-related in $\mathcal{G}$:

$$\exists y \, \exists z_1 \, \exists z_2 (E(z_1) \wedge E(z_2) \wedge \phi(a_1, y, z_1) \wedge \phi(a_2, y, z_2)).$$

In particular, we get a definable embedding of $\mathcal{R}$ into this expansion of M^*.

(2) By passing to a monadic expansion, we may assume T itself is unstable. (In fact, any monadically NIP, nonmonadically stable theory must itself be unstable, but we don't need this.) By [Simon 2021], after adding parameters, there is a formula $\phi(x, y)$ with the order property, where x and y are both singletons. Thus, by adding an additional unary predicate for each of the parameters c (with interpretation $\{c\}$) there is a monadic expansion M^* of a model of T with a 0-definable L^*-formula $\psi(x, y)$ with the order property.

By Ramsey and compactness and by passing to an L^*-elementary extension, we may assume that there are order-indiscernible subsets $A = \{a_i : i \in \mathbb{Q}\}$ and $B = \{b_j : j \in \mathbb{Q}\}$ of M^* such that $M^* \models \psi(a_i, b_j)$ if and only if $i \leq j$. By replacing M^* by a monadic expansion of itself, we may additionally assume there are predicates for A and B. But now, the ordering $a_i \leq' a_j$ is definable on A via the 0-definable L^*-formula $(\forall b \in B)[\psi(a_j, b) \to \psi(a_i, b)]$. Then $(A, \leq')$ witnesses that there is a type-respecting, definable embedding of $(\mathbb{Q}, \leq)$ into M^*.

(3) By Lemma 2.9, T is not weakly minimal, so the following will suffice.

Fact 3.3. *If T is stable but not weakly minimal, then, working in a large, saturated model $\mathfrak{C}$ of T, there is a model $M \preceq \mathfrak{C}$ and singletons a and b such that* $\mathrm{tp}(a/Mb)$ *is not algebraic, but forks over M.*

Proof of fact. As T is not weakly minimal, there are $M_0 \preceq N$ and $p \in S_1(M_0)$ that has two nonalgebraic extensions to $S_1(N)$. As p is stationary, this implies there is a nonalgebraic $q \in S_1(N)$ that forks over M_0. Let a be any realization of q, and choose Y to be maximal such that $M_0 \subseteq Y \subseteq N$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_{M_0} Y$. As $\mathrm{tp}(a/N)$ forks over M_0, $Y \neq N$, so choose any singleton $b \in N \setminus Y$. By the maximality of Y, $a \not\mathop{\smile\!\!\!\!\!\!\mid}_Y b$. To complete the proof, choose a model $M \supseteq Y$ with $M \mathop{\smile\!\!\!\!\!\!\mid}_Y ab$. It follows by symmetry and transitivity of nonforking that $a \not\mathop{\smile\!\!\!\!\!\!\mid}_M b$. Also, since $\mathrm{tp}(a/N)$ is nonalgebraic, so is $\mathrm{tp}(a/Yb)$. But, as $\mathrm{tp}(a/M)$ does not fork over Yb, $\mathrm{tp}(a/M)$ is nonalgebraic as well. $\square$

Fix a, b, M as in Fact 3.3 and choose a formula $\phi(x, y) \in \mathrm{tp}(ab/M)$ (with parameters from M) that witnesses the forking over M.

Let $r = \mathrm{tp}(b/M)$ and choose a Morley sequence $B = \{b_n : n \in \omega\}$ in r. Let $q = \mathrm{stp}(a/Mb)$, and for each n, let q_{b_n} be the strong type over Mb_n conjugate to q. Recursively construct sets $\{I_n : n \in \omega\}$ where each $I_n = \{a_{n,m} : m \in \omega\}$ is a Morley sequence of realizations of the nonforking extension $q^*_{b_n}$ of q_{b_n} to $M \cup B \cup \bigcup\{I_k : k < n\}$. It follows by symmetry and transitivity of nonforking that each I_n is independent and fully indiscernible over $MB \cup \bigcup\{I_k : k \neq n\}$.

Let $A = \{a_{n,m} : n, m \in \omega\}$. Now, any permutation $\sigma \in \mathrm{Sym}(B)$ is L_M-elementary and, in fact, induces an L_M-elementary permutation $\sigma^* \in \mathrm{Sym}(AB)$. Let $L^* = L \cup \{A, B, C_1, \ldots, C_n\}$ and let $\mathfrak{C}^*$ be the natural monadic expansion of $\mathfrak{C}$ formed by interpreting A and B as above, and interpreting each C_i as $\{c_i\}$, where $\{c_1, \ldots, c_n\}$

are the parameters occurring in ϕ. (We silently replace $\phi(x, y)$ by the natural 0-definable L^*-formula formed by replacing each c_i by C_i.) Finally, define an L^*-definable binary relation E on A^2 by

$$E(a, a') \iff (\exists b \in B)[\phi(a, b) \wedge \phi(a', b)].$$

It is easily checked that E is an equivalence relation, whose classes are precisely $\{I_n : n \in \omega\}$. Thus, (A, E) is the image of a type-respecting, definable embedding of $\mathcal{E}$ into $\mathfrak{C}^*$.

(4) We prove this by cases. If T is not monadically NIP, then $\mathcal{R}$ definably embeds in a unary expansion, and expanding by a further unary predicate naming infinitely many infinite cliques with no edges between them definably embeds $\mathcal{E}$, so assume T is monadically NIP. If T is also monadically stable, we are done by (3), so assume T is not monadically stable. Then, by (2), there is a type-respecting, definable embedding of $(\mathbb{Q}, \leq)$ into some monadic expansion M^* of a model of T. Thus, it suffices to prove that $\mathcal{E}$ definably embeds into some monadic expansion of $(\mathbb{Q}, \leq)$. But this is easy. Let $A = \mathbb{Q} \setminus \mathbb{Z}$. Then A is 0-definable in the monadic expansion $(\mathbb{Q}, \leq, A)$, as is the relation $E \subseteq A^2$ given by

$$E(a, a') \iff \forall x([a < x < a' \vee a' < x < a] \to A(x)).$$

It is easily checked that (A, E) is isomorphic to $\mathcal{E}$. $\square$

We close this section by stating one "improvement" of Theorem 3.2(4) that will be used in Section 5. Whereas Theorem 3.2 speaks about a definable embedding of $\mathcal{E}$ into some monadic expansion of some model of T, we isolate the following corollary, which describes a weaker configuration that can be found in arbitrary models of T in the original language.

Corollary 3.4. *Suppose T is a complete L-theory that is monadically NIP, but not monadically NFCP. Then there is an L-formula $\phi(x, y, \overline{z})$ such that, for every model N of T and every $n \geq 1$, there is $\overline{d}_n$ and disjoint sets $B_n = \{b^n_i : i < n\}$, $A_n = \{a^n_{i,j} : i, j < n\}$ that are without repetition such that:*

(1) *The sets $\{A_n, B_m : n, m \in \omega\}$ are pairwise disjoint.*

(2) *For all n and all $i, j, k < n$, one of the following holds:*
   (a) *T is stable and $N \models \phi(b^n_k, a^n_{i,j})$ if and only if $k = i$.*
   (b) *T is unstable and $N \models \phi(b^n_k, a^n_{i,j})$ if and only if $k \leq i$.*

Moreover, we may additionally assume that the set $X = N \setminus \bigcup_{n \geq 1}(A_n \cup B_n)$ is infinite.

Proof. As in the proof of parts (2) and (3) of Theorem 3.2, we split into cases depending on whether or not T is stable. If T is unstable, as in the proof of Theorem 3.2(2), choose an L-formula $\phi(x, y, \overline{z})$ witnessing the order property in large, sufficiently saturated models of T. Now, choose any N such that $N \models T$. As there is some sufficiently saturated $N' \succeq N$ in which $\phi(x, y, \overline{d})$ codes the order property, it follows from elementarity that, for any fixed n, there are $\overline{d}_n \in N^{\lg(\overline{z})}$ and disjoint sets $\{b_i : i < n\}$ and $\{a_{i,j} : i, j < n\}$ such that for all $k, i, j < n$, $N \models \phi(b_k, a_{i,j}, \overline{d}_n)$ if and only if $k \leq i$.

To get the pairwise disjointness, note that if $\{b_i : i < n\}$ and $\{a_{i,j} : i, j < n\}$ work for n, then for any subset $s \subseteq n$, the subsets $\{b_i : i \in s\}$ and $\{a_{i,j} : i, j \in s\}$ work for $n' = |s|$. Thus, given any fixed finite set F to avoid, given any n, by choosing $m \geq n$ large enough and choosing an appropriate $s \subseteq m$, we can find disjoint sets $\{b_i : i < n\}$ and $\{a_{i,j} : i, j < n\}$ each of which are disjoint from F.

Using this, we can recursively define sequences $\overline{d}_n$ and pairwise disjoint families $B_n = \{b_i^n : i < n\}$ and $A_n = \{a_{i,j}^n : i, j < n\}$ such that for all $k, i, j < n$, $N \models \phi(b_k, a_{i,j}, \overline{d}_n)$ if and only if $k \leq i$. By passing to an infinite subsequence, using the remarks above, and reindexing, we can shrink any family $\{B_n, A_n : n \in \omega\}$ to one satisfying the "Moreover ..." clause.

If T is stable, then T is not weakly minimal, by Lemma 2.9. Thus, as in the proof of Theorem 3.2(3), there is a sufficiently saturated elementary extension $N' \succeq N$ and a formula $\phi(x, y, \overline{z})$ that witnesses forking, such that in N' there are $\{b_i : i \in \omega\}$, $\{a_{i,j} : i, j \in \omega\}$, and $\overline{d}$ such that for all $i, j, k \in \mathbb{Z}$, $N' \models \phi(b_k, a_{i,j}, \overline{d})$ if and only if $k = i$.

Using this configuration, the methods used in the unstable case apply here also. $\square$

4. Sets definable in purely monadic and monadically NFCP structures

Fact 2.5 asserts that a theory is monadically NFCP if and only if it is mutually algebraic, so we recall what is known about sets definable in a mutually algebraic structure. Throughout this section, fix an infinite cardinal λ and think of the set $\lambda = \{\alpha : \alpha \in \lambda\}$ as being the universe of a structure.

Definition 4.1. Fix any infinite cardinal λ and any integer $k \geq 1$.

- A subset $Y \subseteq \lambda^k$ is *mutually algebraic* if there is some integer m so that for every $a \in \lambda$, $\{\overline{a} \in Y : a \in \overline{a}\}$ has size at most m.
- A subset $Y^* \subseteq \lambda^{k+\ell}$ is *padded mutually algebraic* if, for some permutation $\sigma \in \mathrm{Sym}(k + \ell)$ of the coordinates, there is a mutually algebraic $Y \subseteq \lambda^k$ and $Y^* = \sigma(Y \times \lambda^\ell)$.
- A model M with universe λ is *mutually algebraic* if, for every n, every definable (with parameters) $D \subseteq \lambda^n$ is a boolean combination of definable (with parameters) padded mutually algebraic sets.
- A complete theory T is *mutually algebraic* if some (equivalently, all) models of T are mutually algebraic.

Trivially, every unary subset $Y \subseteq \lambda^1$ is mutually algebraic.

Fact 4.2 [Laskowski and Terry 2020, Theorem 2.1]. *An L-structure M is mutually algebraic if and only if every atomic L-formula $\alpha(x_1, \ldots, x_n)$ is equivalent to a boolean combination of quantifier-free definable (with parameters) padded mutually algebraic sets.*

It follows immediately that any purely monadic structure is mutually algebraic.

In this section, our goal is to obtain a particular configuration, described in Lemma 4.5, appearing in any mutually algebraic structure whose theory is not purely monadic. This will be used in the proof of Theorem 1.2, when a nonmonadically definable Y induces a mutually algebraic structure.

We begin by characterizing which mutually algebraic sets $Y \subseteq \lambda^k$ are monadically definable. Obviously, every $Y \subseteq \lambda^1$ is monadically definable, so we concentrate on $k \geq 2$. Let $\Delta_k = \{(a, a, \ldots, a) \in \lambda^k : a \in \lambda\}$ denote the set of constant k-tuples.

Lemma 4.3. *Fix any infinite cardinal λ and any integer $k \geq 2$. A mutually algebraic subset $Y \subseteq \lambda^k$ is monadically definable if and only if $Y \setminus \Delta_k$ is finite.*

Proof. First, suppose $Y \setminus \Delta_k$ is finite. Let $F = \bigcup(Y \setminus \Delta_k) = \{a_1, \ldots, a_n\} \subseteq \lambda$ and let $Z = \{a \in \lambda : (a, a, \ldots, a) \in Y\}$. Let $N = (\lambda, U_1, \ldots, U_n, U_{n+1})$ be the structure in which U_i is interpreted as $\{a_i\}$ for each $i \leq n$ and U_{n+1} is interpreted as Z. Then Y is definable in N, so Y is monadically definable.

Conversely, suppose Y is mutually algebraic and definable in some monadic $N = (\lambda, U_1, \ldots, U_n)$. It is easily seen that N admits elimination of quantifiers. Collectively, the unary predicates U_i color each element $a \in \lambda$ into one of 2^n colors. Some of these 2^n colors will have infinitely many elements of λ, while other colorings have only finitely many elements. Let F be the set

$$\left\{a \in \lambda : \text{there are only finitely many } b \in \lambda \text{ such that } N \models \textstyle\bigwedge_{i=1}^{n} U_i(a) \leftrightarrow U_i(b)\right\}.$$

Clearly, F is finite. Now, the elements of $\lambda \setminus F$ are partitioned into finitely many infinite chunks, each of which is fully indiscernible over its complement. Thus, it follows that $F = \mathrm{acl}_N(\varnothing)$ and for any $a \in \lambda$, $\mathrm{acl}_N(a) = F \cup \{a\}$. To show $Y \setminus \Delta_k$ finite, it suffices to prove the following.

Claim. $$Y \subseteq F^k \cup \Delta_k.$$

Proof of claim. Choose any $\overline{a} \in \lambda^k \setminus (F^k \cup \Delta_k)$. Since $\overline{a} \notin F^k$, choose a coordinate $a^* \in \overline{a}$ with $a^* \notin F$. Since the k-tuple $\overline{a}$ is not constant, choose $b \in \overline{a}$ with $b \neq a^*$. Now, by way of contradiction, suppose $\overline{a} \in Y$. As Y is mutually algebraic, $a^* \in \mathrm{acl}_N(b) = F \cup \{b\}$, which it isn't. □

Lemma 4.4. *Suppose M is a mutually algebraic structure with universe λ such that $\mathrm{Th}(M)$ is not purely monadic. Then, for some $k \geq 2$, there is some L_M-definable, mutually algebraic $Y \subseteq \lambda^k$ with $Y \setminus \Delta_k$ infinite.*

Proof. Fix such an M and assume that no such L_M-definable, mutually algebraic set existed. By Lemma 4.3 we would have that for every k, every L_M-definable, mutually algebraic subset of λ^k is monadically definable. From this, it follows easily that every L_M-definable, padded mutually algebraic set would be monadically definable, as would every boolean combination of these. As M is mutually algebraic, it follows that every L_M-definable set is monadically definable, contradicting $\mathrm{Th}(M)$ not being purely monadic. □

We now obtain our desired configuration.

Lemma 4.5. *Suppose M is a mutually algebraic structure with universe λ whose theory is not purely monadic. Then there is some $k \geq 2$, some L_M-definable $Y \subseteq \lambda^k$ and an infinite set $\mathcal{F} = \{\overline{a}_n : n \in \omega\} \subseteq Y \setminus \Delta_k$ such that*

(1) *for each $n \in \omega$, $(\overline{a}_n)_1 \neq (\overline{a}_n)_2$ (the first two coordinates differ), and*

(2) *$\overline{a}_n \cap \overline{a}_m = \varnothing$ for distinct $n, m \in \omega$.*

In particular, if $F = \bigcup \mathcal{F}$, then for every $a \in F$ there is exactly one $\overline{a} \in Y$ with $\overline{a} \subseteq F$ (and hence $(\overline{a})_1 \neq (\overline{a}_2)$).

Proof. By Lemma 4.4, choose $k \geq 2$ and an L_M-definable, mutually algebraic $Y \subseteq \lambda^k$ such that $X := Y \setminus \Delta_k$ is infinite. By mutual algebraicity, choose an integer K such that for every $a \in \lambda$, there are at most K k-tuples $\overline{a} \in Y$ with $a \in \overline{a}$. As each element of X is a nonconstant k-tuple, by the pigeonhole principle we can find an infinite $X' \subseteq X$ and $i \neq j \in [k]$ such that $(\overline{a})_i \neq (\overline{a})_j$ for each $\overline{a} \in X'$. By applying a permutation $\sigma \in \mathrm{Sym}([k])$ to Y, we may assume $i = 1$ and $j = 2$, so after this transformation, (1) holds for any $\overline{a} \in X'$. But now, as $X' \subseteq Y$ is infinite, while every element $a \in \lambda$ occurs in only finitely many $\overline{a} \in X'$, it is easy to recursively construct $\mathcal{F} = \{\overline{a}_n : n \in \omega\} \subseteq X'$. □

5. Monadically stable and monadically NIP are aptly named

In this section, we prove Theorem 1.2. The positive part, that (T, Y) is always monadically NFCP whenever both T is and $Y \subseteq \lambda^k$ is monadically NFCP definable, is immediate from the following.

Lemma 5.1. *Suppose N_1 and N_2 are structures, both with universe λ, in disjoint languages L_1 and L_2. If both N_1 and N_2 are monadically NFCP (= mutually algebraic) then the expansion $N^* = (N_1, N_2)$ is monadically NFCP as well.*

Proof. By replacing each function and constant symbol by its graph, we may assume both L_1 and L_2 only have relation symbols. As the languages are disjoint, this implies that every $L_1 \cup L_2$-atomic formula is either L_1-atomic or L_2-atomic. Thus, every atomic formula in N^* is either equivalent to a boolean combination of either L_1-definable or L_2-definable padded, mutually algebraic formulas. As the notion of a set $Y \subseteq \lambda^k$ being padded mutually algebraic is independent of any structure, the result follows by applying Fact 4.2. □

The negative directions are more involved. To efficiently handle the various cases, we first prove two propositions, from which all of the negative results follow in Theorem 5.4.

For the following proposition, first note that a structure with two cross-cutting equivalence relations admits coding. We will essentially encode this configuration, but since we don't want to assume that either N_1 or N_2 is saturated for our eventual application, we must work with the finitary approximations to an equivalence relation with infinitely many infinite classes provided by Corollary 3.4.

Proposition 5.2. *Suppose L_1 and L_2 are disjoint languages, $\lambda \geq \|L_1 \cup L_2\|$ is a cardinal, N_1 is an L_1-structure with universe λ, and N_2 is an L_2-structure with universe λ. If both $\mathrm{Th}(N_1)$ and $\mathrm{Th}(N_2)$ are not monadically NFCP, then there is a permutation $\sigma \in \mathrm{Sym}(\lambda)$ such that the $L_1 \cup L_2$-structure $(N_1, \sigma(N_2))$ has a theory that is not monadically NIP.*

Proof. We may assume $\mathrm{Th}(N_1)$ and $\mathrm{Th}(N_2)$ are monadically NIP, since we are finished otherwise. Apply Corollary 3.4 to both N_1 and N_2. This gives an L_1-formula $\phi(x, y, \overline{z})$ and, for each n, pairwise disjoint sets $A_n = \{\alpha^n_{i,j} : i, j < n\}$, $B_n = \{\beta^n_i : i < n\}$ and $\overline{r}_n$ as there, with exceptional set $X = \lambda \setminus \bigcup_{n \geq 1}(A_n \cup B_n)$. Note that as each A_n, B_n is finite, $|X| = \lambda$. On the L_2-side, choose an L_2-formula $\psi(x, y, \overline{w})$ such that, for all $n \geq 1$, there is $\overline{s}_n \in \lambda^{\lg(\overline{w})}$ and pairwise disjoint sets $C_n = \{\gamma^n_{i,j} : i, j < n\}$ and $D_n = \{\delta^n_i : i < n\}$ as there.

Now choose $\sigma \in \mathrm{Sym}(\lambda)$ to be any permutation such that for all $n \geq 1$,

(1) $\sigma(D_n) \subseteq X$, and

(2) σ maps C_n bijectively onto A_n via $\sigma(\gamma^n_{i,j}) = \alpha^n_{j,i}$.

Note that there are many permutations σ satisfying these constraints. Choose one, and let $\sigma(N_2)$ be the unique L_2-structure with universe λ so that σ is an L_2-isomorphism.

Claim. *The $L_1 \cup L_2$-theory $\mathrm{Th}(N_1, \sigma(N_2))$ is not monadically NIP.*

Proof of claim. We will produce M^*, a monadic expansion of an $L_1 \cup L_2$-elementary extension $\overline{M} \succeq (N_1, \sigma(N_2))$ that admits coding, which suffices. To do this, first note that by compactness, there is an $L_1 \cup L_2$-elementary extension $\overline{M} \succeq (N_1, \sigma(N_2))$ that contains disjoint sets $A = \{a_{i,j} : i, j \in \mathbb{Z}\}$, $B = \{b_i : i \in \mathbb{Z}\}$, $D = \{d_j : j \in \mathbb{Z}\}$, and tuples $\overline{r}$ and $\overline{s}$ such that, for all $k, i, j \in \mathbb{Z}$, either (if $\mathrm{Th}(N_1)$ is unstable) $\overline{M} \models \phi(b_k, a_{i,j}, \overline{r})$ if and only if $k \leq i$, or (if $\mathrm{Th}(N_1)$ is stable) $\overline{M} \models \phi(b_k, a_{i,j}, \overline{r})$ if and only if $k = i$; and dually, either (if $\mathrm{Th}(N_2)$ is unstable) $\overline{M} \models \psi(d_k, a_{i,j}, \overline{s})$ if and only if $k \leq j$, or (if $\mathrm{Th}(N_2)$ is stable) $\overline{M} \models \psi(d_k, a_{i,j}, \overline{s})$ if and only if $k = j$.

Now, given $\overline{M}$, let $L^* = L_1 \cup L_2 \cup \{A, B, D\}$ and let M^* be the natural monadic expansion of $\overline{M}$ described by A, B, D above. To show that M^* admits coding, we need to rectify the ambiguity between the stable and unstable cases. Specifically, we claim that there is an L^*-formula $\phi^*(x, y, \overline{z})$ such that for all $b_i \in B$, the solution set $\phi^*(b_i, M^*, \overline{r})$ is $\{a_{i,j} : j \in \mathbb{Z}\}$. If $\mathrm{Th}(N_1)$ is stable, this is easy: just take $\phi^*(x, y, \overline{z}) := A(y) \wedge \phi(x, y, \overline{z})$. However, when $\mathrm{Th}(N_1)$ is unstable, we need some more L^*-definability in M^*. Specifically, note that in this case, the natural ordering on B is L^*-definable via

$$b_i \leq b_j \quad \text{if and only if} \quad \forall y[(A(y) \wedge \phi(b_j, y, \overline{r})) \rightarrow \phi(b_i, y, \overline{r})].$$

As the ordering on B is discrete, every element $b \in B$ has a unique successor, $S(b)$, and this operation is L^*-definable since $\leq$ is. Using this, the L^*-formula

$$\phi^*(x, y, \overline{z}) := B(x) \wedge A(y) \wedge \phi(x, y, \overline{z}) \wedge \neg\phi(S(x), y, \overline{z})$$

is as desired.

Arguing similarly, there is an L^*-formula $\psi^*(x, y, \overline{w})$ such that for all $d_j \in D$, the solution set $\psi^*(d_j, M^*, \overline{s})$ is $\{a_{i,j} \in A : i \in \mathbb{Z}\}$. Putting these together, let $\theta(u, v, y, \overline{z}, \overline{w})$ be the L^*-formula

$$B(u) \wedge D(v) \wedge A(y) \wedge \phi^*(u, y, \overline{z}) \wedge \psi^*(v, y, \overline{w}).$$

Then the solution set of $\theta(u, v, y, \overline{r}, \overline{s})$ is precisely the graph of a bijection from $B \times D$ onto A. Thus, M^* admits coding, which suffices. □

The proof of the next proposition is in many ways similar. Here our ideal infinitary configuration consists of an equivalence relation with infinitely many infinite classes, with each tuple from the configuration in Lemma 4.5 pairing two classes by intersecting them. But again, instead of our ideal equivalence relation, we must restrict ourselves to the finitary approximations from Corollary 3.4.

Proposition 5.3. *Suppose L_1 and L_2 are disjoint languages, $\lambda \geq \|L_1 \cup L_2\|$ is a cardinal, N_1 is an L_1-structure with universe λ, and N_2 is an L_2-structure with universe λ. If $\mathrm{Th}(N_1)$ is not monadically NFCP, and if $\mathrm{Th}(N_2)$ is monadically NFCP but not purely monadic, then there is a permutation $\sigma \in \mathrm{Sym}(\lambda)$ such that the $L_1 \cup L_2$-structure $(N_1, \sigma(N_2))$ has a theory that is not monadically NIP.*

Proof. We may assume $\mathrm{Th}(N_1)$ is monadically NIP, since we are finished otherwise. Apply Corollary 3.4 to N_1, obtaining an L_1-formula $\phi(x, y, \overline{z})$ and, for each n, pairwise disjoint sets $A_n = \{\alpha^n_{i,j} : i, j < n\}$, $B_n = \{\beta^n_i : i < n\}$ and $\overline{r}_n$ as there, with exceptional set $X = \lambda \setminus \bigcup_{n\geq 1}(A_n \cup B_n)$. Note that as each A_n, B_n is finite, $|X| = \lambda$. For the N_2 side, apply Lemma 4.5, getting an N_2-definable $Y \subseteq \lambda^k$ and a distinguished set $\mathcal{F} = \{\overline{e}_\ell : \ell \in \omega\} \subseteq Y$ as there. Say Y is defined using parameters $\{c_1, \ldots, c_n\}$. Let $L_2^V = L_2 \cup \{V, C_1, \ldots, C_n\}$ and let N_2^V be the monadic expansion of N_2, interpreting V as $F = \bigcup \mathcal{F}$ and each C_i as $\{c_i\}$. Note that in N_2^V, the subsets $F_1 = \{(\overline{e})_1 : \overline{e} \in \mathcal{F}\}$ and $F_2 = \{(\overline{e})_2 : \overline{e} \in \mathcal{F}\}$ of F are L_2^V-definable (without parameters), along with the bijection $f : F_1 \to F_2$ given by $f(x) = (\overline{e})_2$, where $\overline{e}$ is the unique element of $\mathcal{F}$ containing x. Fix an enumeration $\{\gamma_\ell : \ell \in \omega\}$ of $F_1 \subseteq \lambda$.

We now choose a permutation $\sigma \in \mathrm{Sym}(\lambda)$ that satisfies:

- For all $n \geq 1$ and all distinct $i < j < n$, there is some (in fact, a unique) $\ell \in \omega$ such that $\sigma(\gamma_\ell) = \alpha^n_{i,j}$ and $\sigma(f(\gamma_\ell)) = \alpha^n_{j,i}$.

Let $\sigma(N_2^V)$ be the L_2^V-structure with universe λ so that σ is an L_2^V-isomorphism. Let $M_0^V = (N_1, \sigma(N_2^V))$ be the expansion of N_1 to an $L_1 \cup L_2^V$-structure. So M_0^V has universe λ and satisfies:

- For all $n \geq 1$ and $i < j < n$, $f(\alpha^n_{i,j}) = \alpha^n_{j,i}$.
- The relationships given by N_1 hold for M_0^V.

Let M_0 be the $L_1 \cup L_2$-reduct of M_0^V.

Claim. *The $L_1 \cup L_2$-theory of M_0 is not monadically NIP.*

Proof of claim. We show that the $L_1 \cup L_2^V$-theory of M_0^V is not monadically NIP, which suffices. For this, the strategy is similar to the proof of Proposition 5.2. We will find an $L_1 \cup L_2^V$-elementary extension $\overline{M}$ of M_0^V and then find a monadic expansion M^* of $\overline{M}$ that admits coding. Specifically, choose an $L_1 \cup L_2 \cup \{V\}$-elementary extension $\overline{M}$ for which there are sets $B = \{b_i : i \in \mathbb{Z}\}$ and $A = \{a_{i,j} : i \neq j \in \mathbb{Z}\}$ such that:

(1) For all $i < j$ from $\mathbb{Z}$, $f(a_{i,j}) = a_{j,i}$.
(2) One of the following holds:
  (a) $\mathrm{Th}(N_1)$ is unstable, and $\overline{M} \models \phi(b_k, a_{i,j}, \overline{r})$ if and only if $k \leq i$.
  (b) $\mathrm{Th}(N_1)$ is stable, and $\overline{M} \models \phi(b_k, a_{i,j}, \overline{r})$ if and only if $k = i$.

Given such an $\overline{M}$, let $L^* = L_1 \cup L_2^V \cup \{A, B\}$, and let M^* be the expansion of $\overline{M}$ interpreting A and B as themselves. Exactly as in the proof of Proposition 5.2, find an L^*-formula $\phi^*(x, y, \overline{z})$ such that for all $b_i \in B$, the solution set $\phi^*(b_i, M^*, \overline{r})$ is $\{a_{i,j} : j \in \mathbb{Z}, j \neq i\}$. Finally, let $L^+ = L^* \cup \{B^-, B^+, A^*\}$ with $B^- = \{b_i : i \in \mathbb{Z}^{<0}\}$, $B^+ = \{b_i : i \in \mathbb{Z}^{>0}\}$, and $A^* = \{a_{i,j} : i \in Z^{<0}, j \in \mathbb{Z}^{>0}\}$. Let $\theta(u, v, y, \overline{z})$ be the L^+-formula

$$B^-(u) \wedge B^+(v) \wedge A^*(y) \wedge \phi^*(u, y, \overline{z}) \wedge \phi^*(v, f(y), \overline{z}).$$

Then the formula $\theta(u, v, y, \overline{r})$ is the graph of a bijection from $B^- \times B^+ \to A^*$, which suffices. $\square$

Using Propositions 5.2 and 5.3 we are now able to prove the negative portions of Theorem 1.2. As the positive portion was proved in Lemma 5.1, this suffices.

Theorem 5.4. *Suppose T is a complete L-theory and $Y \subseteq \lambda^k$ with $\lambda \geq \|L\|$. Then*:

(1) *If T is not monadically NFCP and Y is not monadically definable*, *then (T, Y) is not always monadically NIP.*

(2) *If T is not purely monadic and Y is not monadically NFCP definable*, *then (T, Y) is not always monadically NIP.*

Proof. (1) Choose $N_1 \models T$ with universe λ, and let $N_2 = (\lambda, Y)$ be the structure in the language $L_2 = \{Y\}$ with the obvious interpretation. Now, depending on whether $\mathrm{Th}(N_2)$ is monadically NFCP or not, apply either Proposition 5.2 or Proposition 5.3 to get a permutation $\sigma \in \mathrm{Sym}(\lambda)$ such that $\mathrm{Th}(N_1, \sigma(N_2))$ is not monadically NIP. Of course, Y need not be preserved here, so apply σ^{-1}. That is, let $(\sigma^{-1}(N_1), Y)$ be the $L\cup\{Y\}$-structure so that σ^{-1} is an $L\cup\{Y\}$-isomorphism. As $\sigma(N_1) \models T$, this structure witnesses that (T, Y) is not always monadically NIP.

(2) Let $N_1 = (\lambda, Y)$ and let N_2 be any model of T with universe λ. Again, by either Proposition 5.2 or Proposition 5.3 (depending on $\mathrm{Th}(N_2)$), we get a permutation $\sigma \in \mathrm{Sym}(\lambda)$ such that $(N_1, \sigma(N_2))$ has a nonmonadically NIP theory. But this structure is precisely $(\sigma(N_2), Y)$ and $\sigma(N_2) \models T$, so again (T, Y) is not always monadically NIP. □

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CM-trivial structures without the canonical base property

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Based on Hrushovski, Palacín and Pillay's example (*Selecta Mathematica* **19**:4 (2013), 865–877), we produce a new structure without the canonical base property, which is interpretable in Baudisch's group. Said structure is, in particular, CM-trivial, and thus at the lowest possible level of the ample hierarchy.

1. Introduction

In geometric stability, one is often interested in quantifying the complexity of forking in a given theory. An important example is one-basedness: a stable theory is one-based if for any tuples a and b, the canonical base $\mathrm{Cb}(\mathrm{stp}(a/b))$ is algebraic over a. This has very strong structural consequences, for example, on definable groups, which must be abelian-by-finite.

This is only the first step of a strictly increasing hierarchy of complexity: the ample hierarchy, introduced by Pillay [2000]. A theory can be n-ample for any $n \in \mathbb{N}$, and is 1-ample if and only if it is not one-based. This was motivated by Hrushovski's construction [1993] of a new strongly minimal set that is 1-ample, but not 2-ample, which was the first such example. Because algebraically closed fields are n-ample for all n [Pillay 2000, Proposition 3.13] this also provided a counterexample to Zilber's trichotomy: a non-one-based strongly minimal set not interpreting an algebraically closed field.

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Not being 2-ample is also called CM-triviality in the literature, and is defined as follows: a theory T is CM-trivial if whenever $A \subset B$ are parameters and c is a tuple satisfying $\mathrm{acl}^{\mathrm{eq}}(c, A) \cap \mathrm{acl}^{\mathrm{eq}}(B) = \mathrm{acl}^{\mathrm{eq}}(A)$, then $\mathrm{Cb}(\mathrm{stp}(c/A))$ is algebraic over $\mathrm{Cb}(\mathrm{stp}(c/B))$.

Another way to generalize one-basedness is to introduce *internality* in the definition. Recall that if $\mathcal{P}$ is a family of types, a stationary type $p \in S(A)$ is $\mathcal{P}$-internal (resp. almost $\mathcal{P}$-internal) if there is a set of parameters C such that for any realization $a \models p$, there is a tuple e of realizations of types in $\mathcal{P}$, each based over C, such that $a \in \mathrm{dcl}(e, C)$ (resp. $a \in \mathrm{acl}(e, C)$). Internality is essential to the understanding of superstable theories of finite rank, via the machinery of analyzability: any type can be constructed as an iterated fibration, with $\mathcal{P}$-internal fibers at each step, where $\mathcal{P}$ is the family of Lascar rank one types.

A relative version of one-basedness, inspired by the model theory of compact complex spaces, is the *canonical base property*, which was implicitly studied in [Pillay 2001] and [Pillay and Ziegler 2003], and first formally defined in [Moosa and Pillay 2008]. A superstable theory has the canonical base property (CBP) if (possibly working over some parameters) for any tuples a and b, if $\mathrm{stp}(a)$ has finite Lascar rank and $b = \mathrm{Cb}(\mathrm{stp}(a/b))$, then $\mathrm{stp}(b/a)$ is almost $\mathcal{P}$-internal, where $\mathcal{P}$ is the family of Lascar rank one types. One observes that the canonical base property is obtained by replacing "algebraic" with "almost $\mathcal{P}$-internal" in the definition of one-basedness.

It was at first conjectured that all superstable structures of finite Lascar rank had the CBP, until Hrushovski, Palacín and Pillay produced a counterexample [Hrushovski et al. 2013], which is interpretable in (and interprets) an algebraically closed field of characteristic zero. More recently, Loesch [2021] has produced new structures without the CBP, which are conjectured to not be interpretable in the first one.

Nevertheless, all the known examples interpret an algebraically closed field, and it is a natural extension of Zilber's trichotomy to ask if it is always the case. Moreover, the interaction between the CBP and the ample hierarchy, which are both based on generalizing one-basedness, is so far unknown. In the present article, we make progress in both of these directions by producing a CM-trivial structure that does not have the CBP. Thus, a structure without the CBP can exist at the lowest possible level of the ample hierarchy, and does not have to interpret an algebraically closed field.

Our structure is interpretable in Baudisch's uncountably categorical group [1996], but our methods are based on his second account of his construction [2009] (itself inspired by methods developed in [Baudisch et al. 2006; 2007]). Said group is constructed via a Hrushovski–Fraissé amalgamation (with collapse) of finite-dimensional 2-nilpotent Lie algebras over a finite field, and was the first example of a CM-trivial superstable group. As a matter of fact, Baudisch's group is obtained from the amalgamated Lie algebra (which we will call Baudisch's Lie algebra), and we will work with the Lie algebra rather than the group.

Our proof consists of a formal transposition of the techniques used by Hrushovski, Palacín and Pillay in [Hrushovski et al. 2013] to interpret a structure without the CBP in an algebraically closed field of characteristic zero. In said article, the authors carefully pick a cover of the complex numbers by their additive group to ensure that its automorphisms are given by derivations of the field. This in turn gives them enough flexibility to produce a configuration contradicting the CBP.

Here, we will mimic their proof by first constructing derivations of Baudisch's Lie algebra. This is the technical heart of our article, and requires some elementary, but tedious, bilinear algebra. The rest of the proof is mostly routine, and a direct transposition of [Hrushovski et al. 2013]: we pick a similar cover, and show that derivations of Baudisch's Lie algebra give rise to automorphisms of this structure. By using a criteria for the CBP first noticed in [Pillay and Ziegler 2003], we can copy the proof given in [Hrushovski et al. 2013] to prove that our structure does not have the CBP.

There are still open questions regarding the canonical base property. First, there is the tantalizing conjecture of Hrushovski, Palacín and Pillay that any structure interpretable in an algebraically closed field of positive characteristic has the canonical base property. At present, the authors do not see any reason to confirm or infirm this, except that the known cover constructions do not transpose to this case. Second, there is the existence of a structure without the CBP, and not interpreting a group. Geometric stability theory considerations show that such a structure cannot be $\aleph_1$-categorical, but there is no other known obstruction to its existence. Perhaps it could be interpretable in Hrushovski's original strongly minimal set.

The article is organized as follows: In Section 2, we give necessary preliminaries on both 2-nilpotent Lie algebras and Baudisch's construction. In Section 3, we construct many derivations on Baudisch's Lie algebra, using bilinear algebra considerations, and properties of Baudisch's construction. Finally, everything comes together in Section 4, where said derivations are used to produce a cover of Baudisch's Lie algebra without the canonical base property.

2. Preliminaries

From now on, we denote by $\langle A\rangle$ the vector subspace generated by a subset A in a vector space.

2A. ***2-nilpotent graded Lie algebras.*** Before we start our journey into Baudisch's work [1996; 2009], we will remind the reader of a few elementary definitions and results on 2-nilpotent Lie algebras, which will be central to our construction.

Definition 2.1. Let $\mathbb{K}$ be a field. A $\mathbb{K}$-Lie algebra is a $\mathbb{K}$-vector space $\mathfrak{g}$ equipped with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, that is

- alternative, i.e., $[x, x] = 0$ for all $x \in \mathfrak{g}$,
- anticommutative, i.e., $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$,
- satisfies the Jacobi identity, i.e., $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in \mathfrak{g}$.

In Baudisch's construction, one considers graded 2-nilpotent Lie algebras.

Definition 2.2. A 2-nilpotent graded Lie algebra is a Lie algebra $\mathfrak{g}$, which is graded as a vector space, meaning $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, and satisfies

- $\langle [\mathfrak{g}_1, \mathfrak{g}_1] \rangle = \mathfrak{g}_2$,
- $[\mathfrak{g}, \mathfrak{g}_2] = \{0\}$.

A Lie algebra homomorphism is a $\mathbb{K}$-linear map preserving the Lie bracket. Just as for commutative rings, kernels of homomorphisms will be exactly ideals:

Definition 2.3. Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra. A vector space $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra if it is preserved by the Lie bracket. It is an ideal if it moreover satisfies $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$.

Given an ideal $\mathfrak{h} \subseteq \mathfrak{g}$, one can form the quotient Lie algebra $\mathfrak{g}/\mathfrak{h}$.

Remark 2.4. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a 2-nilpotent graded Lie algebra:

- Any vector subspace $\mathfrak{g}_1'$ of $\mathfrak{g}_1$ generates a *subgraded* algebra $\mathfrak{g}' = \mathfrak{g}_1' \oplus \mathfrak{g}_2'$, i.e., a 2-nilpotent graded subalgebra $\mathfrak{g}'$ such that $(\mathfrak{g}')_i = \mathfrak{g}_i \cap \mathfrak{g}'$.
- Any vector subspace of $\mathfrak{g}_2$ is an ideal.

In fact, 2-nilpotent graded Lie algebras correspond exactly to quotients of exterior squares:

Definition 2.5. Let V be a vector space over a field $\mathbb{K}$. The exterior algebra $\bigwedge V$ of V is defined as the quotient of the tensor algebra $T(V)$ by the (two-sided) ideal generated by $\{x \otimes x : x \in V\}$. We denote by $x \wedge y$ the product in $\bigwedge V$, and call it the wedge product of x and y.

The exterior square $\bigwedge^2 V$ is the vector space generated by $\{x \wedge y : x, y \in V\}$.

We will frequently state that a family of wedge products is free. When we do, it is a consequence of the following:

Fact 2.6. Let $\{a_i : i \in I\}$ be a basis of V, and fix an ordering $<$ of I. Then $\{a_i \wedge a_j : i < j\}$ is a basis of $\bigwedge^2 V$.

From the exterior square, we can construct 2-nilpotent Lie algebras:

Observation 2.7. *The vector space $V \bigoplus \bigwedge^2 V$ can be equipped with a Lie algebra structure by setting $[x, y] = x \wedge y$ for all $x, y \in V$, and $[x, [y, z]] = 0$ for all $y, z \in V$ and $x \in V \bigoplus \bigwedge^2 V$. This Lie algebra is 2-nilpotent graded by construction. It is the free 2-nilpotent Lie algebra over V, denoted by $F_2(V)$.*

A trivial but essential remark, which is used heavily by Baudisch [2009], is the following:

Remark 2.8. Consider a 2-nilpotent graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The identity from $V = \mathfrak{g}_1$ to $\mathfrak{g}_1$ extends canonically to a Lie algebra morphism φ from $F_2(V)$ onto $\mathfrak{g}$, which induces an Lie isomorphism between $\mathfrak{g}$ and $F_2(V)/N_{\mathfrak{g}}$, where $N_{\mathfrak{g}}$ is the kernel of φ. Note that

$$N_{\mathfrak{g}} = \left\{ \sum_{i=1}^{n} \lambda_i x_i \wedge y_i : x_i, y_i \in V,\ \lambda_i \in \mathbb{K} \text{ such that } \sum_{i=1}^{n} \lambda_i [x_i, y_i] = 0 \right\}.$$

Thus, one can identify 2-nilpotent graded Lie algebras with Lie algebras of the form $V \oplus \left(\bigwedge^2 V\right)/N$, where V is a $\mathbb{K}$-vector space and N a $\mathbb{K}$-vector subspace of $\bigwedge^2 V$. When there is no ambiguity, we denote by $N(V)$ the considered ideal N of $F_2(V)$.

Let $\mathfrak{g} = A \oplus \left(\bigwedge^2 A\right)/N(A)$ be a 2-nilpotent graded Lie algebra and B be a vector subspace of A. The subalgebra $\mathfrak{g}'$ generated by B is isomorphic to $B \oplus \left(\bigwedge^2 B\right)/N(B)$, where $N(B) = N(A) \cap \bigwedge^2 B$.

From linear maps on the first components of 2-nilpotent graded Lie algebras, one can obtain Lie algebra morphisms:

Remark 2.9. Let $\mathfrak{g} = U \oplus \left(\bigwedge^2 U\right)/N(U)$ and $\mathfrak{h} = V \oplus \left(\bigwedge^2 V\right)/N(V)$ be 2-nilpotent graded Lie algebras.

- Any linear map $\sigma : U \to F_2(V)$ extends uniquely to a Lie algebra morphism $\sigma : F_2(U) \to F_2(V)$.
- Moreover, if $\sigma(N(U)) \subseteq N(V)$, it induces by quotients a unique Lie algebra morphism $\sigma : \mathfrak{g} \to \mathfrak{h}$.

In the rest of the paper, we will consider only *graded* Lie algebra morphisms, which are morphisms of the form $\sigma : \mathfrak{g} \to \mathfrak{h}$ such that $\sigma(U) \subseteq V$ or equivalently linear maps $\sigma : U \to V$ such that $\sigma(N(U)) \subseteq N(V)$.

Derivations play a key role in the study of Lie algebras, and will feature prominently in our construction:

Definition 2.10. Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra.

A derivation $\delta : \mathfrak{g} \to \mathfrak{g}$ is a $\mathbb{K}$-linear map satisfying the Leibniz law, that is, $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathfrak{g}$.

A partially defined derivation δ is a $\mathbb{K}$-linear map from a subalgebra $\mathfrak{g}'$ to $\mathfrak{g}$ satisfying the Leibniz law (on elements in $\mathfrak{g}'$).

When $\mathfrak{g}$ is a 2-nilpotent graded and $\mathfrak{g}'$ a graded Lie subalgebra, we say that a partially defined derivation $\delta : \mathfrak{g}' \to \mathfrak{g}$ is *graded* if $\delta(\mathfrak{g}'_1) \subseteq \mathfrak{g}_1$.

Note that for any Lie algebra $\mathfrak{g}$ and $g \in \mathfrak{g}$, the application $\delta_g : x \to [x, g]$ is a derivation. However, in the case of a graded 2-nilpotent Lie algebra, this derivation is *not* graded. One can construct graded derivations just as graded morphisms were previously constructed:

Remark 2.11. Let $\mathfrak{g} = A \oplus \left(\bigwedge^2 A\right)/N(A)$ be a 2-nilpotent graded Lie algebra, B be a vector subspace of A, and $\mathfrak{g}'$ be the subalgebra generated by B.

- A linear map $f : B \to F_2(A)$ induces a unique partially defined derivation $\tilde{f} : F_2(B) \to F_2(A)$ such that $\tilde{f}|_B = f$.
- A partially defined derivation $\delta : \mathfrak{g}' \to \mathfrak{g}$ is uniquely determined by the linear map $\delta|_B : B \to \mathfrak{g}$. Moreover, we have $\tilde{\delta}|_B(N(B)) \subseteq N(A)$.
- Reciprocally, a linear map $f : B \to F_2(A)$ such that $\tilde{f}(N(B)) \subseteq N(A)$ induces by quotients a unique partially defined derivation $\delta : \mathfrak{g}' \to \mathfrak{g}$.
- If $f(B) \subseteq A$, the corresponding partially defined derivation is graded.

When there is no ambiguity, we will use the same notation for the linear map and the partially defined derivations. In such a setting, we will say that $f : B \to A$ is a partially defined derivation if f is a linear map from B to A such that $f(N(B)) \subseteq N(A)$ (where f denotes $\tilde{f}$ in the last inclusion).

2B. *Baudisch's group.* Our final structure will be based on Baudisch's group (or rather, the Lie algebra associated to it). In this section, entirely due to Baudisch, we recall how this algebra is constructed, and state results that we will use. This group was first constructed in [Baudisch 1996], but we found the later article [Baudisch 2009] easier to use for our purposes. We will freely refer to results and definitions from said article, and we encourage the reader to have it within reach.

Fix a finite field $\mathbb{F}_q$ with $q > 2$. Baudisch [2009] constructs an ω-stable 2-nilpotent graded $\mathbb{F}_q$-Lie algebra M as the Hrushovski–Fraissé limit of a class of finite 2-nilpotent graded $\mathbb{F}_q$-Lie algebras. He then constructs a CM-trivial 2-nilpotent graded $\mathbb{F}_q$-Lie algebra of Morley rank 2 as a collapse M_μ, which depends on a certain function μ.

To see 2-nilpotent graded $\mathbb{F}_q$-Lie algebras as first-order structures, we will use an expansion of the language of $\mathbb{F}_q$-vector spaces by a binary function symbol $[\cdot, \cdot]$ for the Lie bracket, as well as two unary predicates for the degree-one and degree-two components. We denote by L this language. We reserve the notation $\wedge$ for the Lie bracket in free 2-nilpotent Lie algebras:

Notation. If $A \oplus \left(\bigwedge^2 A\right)/N(A)$ is a 2-nilpotent Lie algebra, we will often work in the free 2-nilpotent Lie algebra $F_2(A)$. When we do, we will always use the wedge product notation. Thus, an equality of the form $\sum_{i,j} [a_i, b_j] = 0$ in the Lie algebra $A \oplus \left(\bigwedge^2 A\right)/N(A)$ is equivalent to $\sum_{i,j} a_i \wedge b_j \in N(A) \subseteq \bigwedge^2 A$.

We will say that $A \oplus \left(\bigwedge^2 A\right)/N(A)$ satisfies $\sum_{i,j} a_i \wedge b_j \in N(A)$, even though this is not, strictly speaking, a formula in L.

Let us recall how Baudisch's class of Lie algebras, as well as their predimension function, are defined.

Definition 2.12 (predimension). For any finite 2-nilpotent graded Lie algebra $\mathfrak{g} = A \oplus \left(\bigwedge^2 A\right)/N(A)$, we let

$$\delta(\mathfrak{g}) = \mathrm{ldim}(A) - \mathrm{ldim}(N(A)),$$

where ldim denotes the $\mathbb{F}_q$-linear dimension.

When we are in an ambient 2-nilpotent Lie algebra $V \oplus \left(\bigwedge^2 V\right)/N(V)$, for any finite vector subspace B of V, we simply denote by $\delta(B)$ the predimension of the subalgebra generated by B.

Note that $\delta(B) = \mathrm{ldim}(B) - \mathrm{ldim}(N(B))$, where $N(B) = N(V) \cap \bigwedge^2 B$.

More generally, the relative predimension is defined for vector subspaces $C \subseteq B$ of V such that B is finitely generated over C by

$$\delta(B/C) = \mathrm{ldim}(B/C) - \mathrm{ldim}(N(B)/N(C)).$$

Observation 2.13 (submodularity). *The predimension is submodular: for two vector subspaces A and B such that A is finitely generated over $A \cap B$, we have*

$$\delta(A + B/B) \leq \delta(A/A \cap B).$$

Definition 2.14 (strongness). When working in an ambient 2-nilpotent Lie algebra $V \oplus \left(\bigwedge^2 V\right)/N(V)$, for any vector subspaces $B \subseteq A$ of V, we say that B is self-sufficient or strong in A (denoted by $B \leq A$) if for any vector subspace $B \subseteq C \subseteq A$, finitely generated over B, we have $\delta(C/B) \geq 0$.

Note that because of submodularity, the intersection of two subspaces that are strong in V is also strong. Thus, given $B \subset V$, there is a smallest strong subspace of V that contains B, namely the intersection all strong subspaces of V containing B. We call it the self-sufficient closure of B.

We say that $A \supsetneq B$ is *a minimal strong extension* of B if $B \leq A$ and there is no vector subspace $B \subsetneq C \subsetneq A$ such that $C \leq A$.

Definition 2.15 (class $\mathcal{K}$). The class $\mathcal{K}$ is made of all 2-nilpotent Lie algebras $V \oplus \left(\bigwedge^2 V\right)/N(V)$ such that

- $[v, w] \neq 0$ for any linearly independent v and w in V;
- $\mathbb{F}_q \cdot v \leq V$ for all v in V.

Note that the second assumption is equivalent to $\delta(A) \geq 1$ for any finite nontrivial vector subspace of V.

In the class $\mathcal{K}$, there are three possibilities for minimal strong extensions:

- Transcendental: In this case, $\mathrm{ldim}(A) = \mathrm{ldim}(B) + 1$ and $\delta(A) = \delta(B + 1)$.
- Algebraic: In this case, $\mathrm{ldim}(A) = \mathrm{ldim}(B) + 1$ and $\delta(A) = \delta(B)$. This forces the existence of $a \in A \setminus B$ such that $A = B \oplus \langle a \rangle$ and $N(A) = N(B) \oplus \langle a \wedge b + c \rangle$ for $b \in B$ and $c \in \bigwedge^2 B$.
- Prealgebraic: This contains all other cases, meaning that $\mathrm{ldim}(A) > \mathrm{ldim}(B) + 1$ and $\delta(A) = \delta(B)$.

Example 2.16. Let $B = \langle b_0, b_1, b_2 \rangle$ be a 3-dimensional $\mathbb{F}_q$-vector space, and consider the Lie graded 2-nilpotent algebra $F_2(B)$. We construct an extension of B of each type, using linearly independent elements a_0 and a_1 over B.

- Let $A_{\mathrm{tr}} = B \oplus \langle a_0 \rangle$, and consider the Lie algebra $F_2(A_{\mathrm{tr}})$. This is a transcendental extension of B.
- Let $A_{\mathrm{alg}} = B \oplus \langle a_0 \rangle$, and consider the Lie algebra $F_2(A_{\mathrm{alg}})/N(A_{\mathrm{alg}})$, with $N(A_{\mathrm{alg}}) = \langle a_0 \wedge b_0 + b_1 \wedge b_2 \rangle$. This is an algebraic extension of B.
- Let $A_{\mathrm{pr}} = B \oplus \langle a_0, a_1 \rangle$, and consider the Lie algebra $F_2(A_{\mathrm{pr}})/N(A_{\mathrm{pr}})$, with $N(A_{\mathrm{pr}}) = \langle a_0 \wedge b_0 + a_1 \wedge b_1, a_0 \wedge a_1 + b_1 \wedge b_2 \rangle$. This is a prealgebraic extension of B.

A key property is:

Theorem 2.17 [Baudisch 2009, Theorems 8.3 and 8.4]. *The subclass $\mathcal{K}_{\mathrm{fin}}$ of finite L-structures in $\mathcal{K}$ has the amalgamation property with respect to strong embeddings. Therefore, there is a unique up to isomorphism countable L-structure $M = V \oplus \left(\bigwedge^2 V\right)/N(V)$ in $\mathcal{K}$, constructed by Hrushovski–Fraissé amalgamation, that is rich, meaning: if $B \leq V$ is a finite self-sufficient subspace of V and $A \geq B$ is a finite strong extension of B in $\mathcal{K}$, there is a strong embedding f of A in V over B.*

Note that because we imposed, for any $A \in \mathcal{K}$, that $[a, b] \neq 0$ for any linearly independent $a, b \in A$, any algebraic extension in V is algebraic in the model-theoretic sense. However, this is not the case for prealgebraic extensions, which are of Morley rank one over their basis. This is why V and M have, respectively, Morley rank ω and $\omega \cdot 2$: we can take successive prealgebraic extensions, and reach any finite Morley rank in V.

We will denote by T the theory of the L-structure M. Note that M is the countable saturated model of T [Baudisch 2009, Theorem 8.6].

Because we want a finite-rank structure, what is needed is to *collapse* this structure, meaning force the prealgebraic extensions to become algebraic. As is classical in Hrushovski constructions, this is done via a set of *codes* for prealgebraic extensions. More precisely, Baudisch [2009] considers a set of *good codes*, which is a set $(\varphi_\alpha(\bar{x}, y))_{\alpha \in \mathcal{C}}$ of L^{eq}-formulas with, in particular, the following properties (and many others useful for the construction):

- $\bar{x}$ is an n-tuple of variables in the first predicate of L (that is, a tuple of elements of degree one) and y is an imaginary variable.
- For $b \in M^{\mathrm{eq}}$, either the formula $\varphi_\alpha(V^n, b)$ is empty or b is the canonical parameter of $\varphi_\alpha(\bar{x}, b)$ in T.
- There exist n terms

$$\Phi_i(\bar{x}, \bar{y}_i, z_i) = \sum_{j<k<n} \lambda_{ijk} x_j \wedge x_k + \sum_{j<n} y_{ij} \wedge x_j + z_i,$$

where $\bar{y}_i$ is a tuple of variables in the first predicate and z_i a variable of the second predicate, which describe up to isomorphisms a prealgebraic extension in the following sense: if $b \in \mathrm{dcl}^{\mathrm{eq}}(B)$ for a vector subspace B of V (and $\varphi_\alpha(V^n, b) \neq \varnothing$) there exist $c_{ij} \in B$ and $\psi_i \in \bigwedge^2 B$ such that for any $\bar{a} \in \varphi_\alpha(V^n, b)$,

- $\Phi_i(\bar{a}, \bar{c}_i, \psi_i) \in N(V)$,
- if $\bar{a}$ is *w-generic* over B, that is, if $\bar{a}$ is linearly independent over B and $\delta(\langle \bar{a}, B\rangle / B) = 0$, then $\langle \bar{a}, B\rangle$ is a prealgebraic minimal extension of B, where

$$N(\langle \bar{a}, B\rangle) = N(B) \oplus \langle \Phi_i(\bar{a}, \bar{c}_i, \psi_i) : i < n\rangle.$$

Then using a notion of *difference sequences* for good codes, a subclass $\mathcal{K}^\mu$ of $\mathcal{K}$ is defined for any *good map* $\mu : \mathcal{C} \to \mathbb{N}$, so that:

Theorem 2.18 [Baudisch 2009; 1996]. *The subclass $\mathcal{K}^\mu_{\mathrm{fin}}$ of finite L-structures in $\mathcal{K}^\mu$ has the amalgamation property with respect to strong embeddings and the theory T^μ of the countable rich structure $M_\mu = V_\mu \oplus \left(\bigwedge^2 V_\mu\right)/N(V_\mu)$ of $\mathcal{K}^\mu$ is uncountably categorical of Morley rank* 2, *with V_μ being strongly minimal. Moreover*, *T^μ is CM-trivial.*

The *bounds for difference sequences* [Baudisch 2009, Section 5] is central in the above construction. In particular there is a characterization of minimal extensions which do not belong in $\mathcal{K}^\mu$ [Baudisch 2009, Corollary 5.3]. In order to extend derivations, we will need only the following fact that one can deduce directly from this characterization:

Fact 2.19. Let D be in $\mathcal{K}_\mu$ and $D \leq D'$ be a minimal extension in $\mathcal{K} \setminus \mathcal{K}_\mu$. Then the extension D' is prealgebraic, and there is a good code $\alpha \in \mathcal{C}$ such that one of the following holds:

(a) $D' = D + \langle \bar{e}\rangle$ for a w-generic realization $\bar{e}$ over D of $\varphi_\alpha(\bar{e}, b)$, where $b \in \mathrm{dcl}^{\mathrm{eq}}(D)$, and there is at least one realization $\bar{e}'$ in D of $\varphi_\alpha(\bar{x}, b)$.

(b) There is a vector subspace $D \subseteq E \subseteq D'$ and $b \in \mathrm{dcl}^{\mathrm{eq}}(E)$ with at least two realizations $\bar{e}$ and $\bar{e}'$ in D' of $\varphi_\alpha(\bar{x}, b)$, where $\bar{e}$ is w-generic over E and $\bar{e}'$ is w-generic over $E + \langle \bar{e}\rangle$.

3. Constructing derivations

To construct derivations on M_μ, we will proceed by induction, namely, extending derivations step-by-step. We will consider only partially defined graded derivations on finite 2-nilpotent Lie algebras, or equivalently, in an ambient 2-nilpotent Lie algebra $V \oplus \left(\bigwedge^2 V\right)/N(V)$, we consider partially defined derivations $f : B \to V$, where B is a finite vector subspace of V and $f : B \to V$ a linear map such that $f(N(B)) \subseteq N(B + f(B))$ (see Remark 2.11).

Definition 3.1. A *derivation extension problem* is the data of two finite vector subspaces $B \leq A \leq V_\mu$ and a partially defined graded derivation $f : B \to V_\mu$ with $A + f(B) \leq V_\mu$. We denote it by $(B \leq A, f)$.

For any derivation extension problem $(B \leq A, f)$, we want to extend the derivation to A and obtain a partially defined graded derivation $f : A \to V_\mu$ with $A + f(A) \leq V_\mu$.

Let us fix such a derivation extension problem $(B \leq A, f)$ for now.

First we construct what we will call a free pseudosolution. The Lie algebra generated by $A + f(A)$ for this free pseudosolution will not be in general in $\mathcal{K}_\mu$, and finding a solution realized in M_μ will be the technical heart of the proof.

Construction 3.2. *Consider an* ***abstract*** *vector space U over $A + f(B)$ such that* $\operatorname{ldim}(U/A + f(B)) = \operatorname{ldim}(A/B)$. *Extend the linear map $f : B \to B + f(B)$ to a linear map $f : A \to U$ which sends a (any) basis of A over B onto a basis of U over $A + f(B)$. From now on, we set $U = A + f(A)$.*

By Remark 2.11, this map canonically gives us a partially defined derivation $f : F_2(A) \to F_2(A + f(A))$.

Now we define $N(A + f(A)) = N(A + f(B)) + f(N(A))$ and we consider the 2-nilpotent graded algebra $\mathfrak{g} = (A + f(A)) \oplus \left(\bigwedge^2(A + f(A))\right)/N(A + f(A))$. Then we have to check that $N(A+f(B)) = N(A+f(A)) \cap \bigwedge^2(A+f(B))$ in order to prove that the Lie subalgebra generated by $A + f(B)$ in M_μ is also a subgraded algebra of $\mathfrak{g}$. We will use a particular (and simpler) case of the following lemma for $A' = B$.

Lemma 3.3. *For any $B \subseteq A' \subseteq A$, we have $f^{-1}\left(\bigwedge^2(A + f(A'))\right) = \bigwedge^2 A'$.*

Thus, if the family $(e_1, \dots, e_t)$ of vectors in $\bigwedge^2 A$ is free over $\bigwedge^2 A'$, then the family $(f(e_1), \dots, f(e_t))$ is free over $\bigwedge^2(A + f(A'))$.

Proof. Consider a basis $(a_1, \dots, a_n)$ of A over B such that $A' = \langle B, a_1, \dots, a_s\rangle$.

Letting $e \in \bigwedge^2 A$, we can write

$$e = \sum_{i,j} \lambda_{i,j}(a_i \wedge b_j) + \sum_{k<\ell} \beta_{k,\ell}(a_k \wedge a_\ell) + c$$

with $\lambda_{i,j}$, $\beta_{k,\ell} \in \mathbb{F}_q$, $c \in \bigwedge^2 B$ and $b_1, \dots, b_m$ linearly independent vectors of B. Then

$$\begin{aligned} f(e) &= \sum_{i,j} \lambda_{i,j}(f(a_i) \wedge b_j + a_i \wedge f(b_j)) + \sum_{k<\ell} \beta_{k,\ell}(f(a_k) \wedge a_\ell + a_k \wedge f(a_\ell)) + f(c) \\ &= \sum_{i,j} \lambda_{i,j}(f(a_i) \wedge b_j) - \sum_{\ell<k} \beta_{\ell,k}(a_k \wedge f(a_\ell)) + \sum_{k<\ell} \beta_{k,\ell}(a_k \wedge f(a_\ell)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \sum_{i,j} \lambda_{i,j}(a_i \wedge f(b_j)) + f(c). \end{aligned}$$

The last two terms of that sum belong to $\bigwedge^2(A + f(B))$. Moreover, the family $\{f(a_i) \wedge b_j : s < i \le n, 1 \le j \le m\} \cup \{a_k \wedge f(a_\ell) : 1 \le k \le n, s < \ell \le n\}$ is linearly independent over $\bigwedge^2(A + f(A')) = \bigwedge^2\big(A + f(B) \oplus \langle f(a_1), \dots, f(a_s)\rangle\big)$. Thus if $f(e) \in \bigwedge^2(A + f(A'))$, we must have $\lambda_{i,j} = 0$ for all $i > s$ and $\beta_{k,\ell} = 0$ for all $\ell > s$, implying that $e \in \bigwedge^2 A'$.

Thus $f^{-1}\big(\bigwedge^2(A + f(A'))\big) \subseteq \bigwedge^2 A'$, and the other inclusion is immediate. □

Corollary 3.4. $N(A + f(B)) = N(A + f(A)) \cap \bigwedge^2(A + f(B))$. *Thus, the Lie subalgebra generated by* $A + f(B)$ *in* M_μ *is also a subgraded algebra of* $\mathfrak{g} = (A + f(A)) \oplus \big(\bigwedge^2(A + f(A))\big)/N(A + f(A))$, *and the linear map* $f : A \to A + f(A)$ *induces a partially defined graded derivation* $f : A \oplus \big(\bigwedge^2 A\big)/N(A) \to \mathfrak{g}$.

Proof. Recall $N(A + f(A)) = N(A + f(B)) + f(N(A))$. Let $e = e_1 + f(e_2)$ with $e \in \bigwedge^2(A + f(B))$, $e_1 \in N(A + f(B))$ and $e_2 \in N(A)$. Then $f(e_2) \in \bigwedge^2(A + f(B))$, and by the previous lemma, $e_2 \in \bigwedge^2 B$. Since $N(B) = B \cap N(A)$ and $f : B \to B + f(B)$ is a derivation, we obtain that $f(e_2) \in N(B + f(B))$ and $e \in N(A + f(B))$.

By definition of $N(A + f(A))$, we have $f(N(A)) \subseteq N(A + f(A))$, and then, by Remark 2.11, the linear map $f : A \to A + f(A)$ induces a partially graded defined derivation $f : A \oplus \big(\bigwedge^2 A\big)/N(A) \to \mathfrak{g}$, which extends the partially graded defined derivation $f : B \oplus \big(\bigwedge^2 B\big)/N(B) \to (B + f(B)) \oplus \big(\bigwedge^2(B + f(B))\big)/N(B + f(B))$. □

We call the above extension the *free pseudosolution* of the derivation extension problem (it is unique up to isomorphisms). When there is no ambiguity, we will call $A + f(A)$ the free pseudosolution of $(B \le A, f)$.

The following lemma will be useful in understanding this derivation:

Lemma 3.5. *Let* $A + f(A)$ *be the free pseudosolution of a derivation extension problem* $(B \le A, f)$ *and* A' *a vector subspace of* A *containing* B. *Then*

$$\delta(A + f(A')/A + f(B)) = \delta(A'/B).$$

More precisely, the family $(f(e_1), \dots, f(e_t))$ *is a basis of* $N(A + f(A'))$ *over* $N(A + f(B))$ *for any basis* $(e_1, \dots, e_t)$ *of* $N(A')$ *over* $N(B)$.

Proof. By construction, we know that $N(A+f(A')) \subset N(A+f(B)) + f(N(A))$. As in the previous proof, let $\eta = \eta_1 + f(\eta_2)$ be in $N(A+f(A'))$, decomposed along this sum, i.e., $\eta_1 \in N(A+f(B))$, $\eta_2 \in N(A)$ and $f(\eta_2) \in N(A+f(A'))$. By Lemma 3.3 we obtain $\eta_2 \in N(A) \bigcap \bigwedge^2 A' = N(A')$. Thus $N(A+f(A')) = N(A+f(B)) + f(N(A'))$.

Consider a basis $(e_1, \dots, e_t)$ of $N(A')$ over $N(B)$. The family $(e_1, \dots, e_t)$ is free over $\bigwedge^2 B$ because $N(B) = N(A') \cap \bigwedge^2 B$, and by Lemma 3.3, the family $(f(e_1), \dots, f(e_t))$ is free over $\bigwedge^2(A+f(B))$. Moreover, $f(N(A'))$ is generated by $(f(e_1), \dots, f(e_t))$ over $f(N(B)) \subseteq N(B+f(B))$ (since $f : B \to B+f(B)$ is a derivation). Thus $(f(e_1), \dots, f(e_t))$ is a basis of $N(A+f(B)) + f(N(A')) = N(A+f(A'))$ over $N(A+f(B))$, and $\operatorname{ldim}(N(A+f(A'))/N(A+f(B)) = \operatorname{ldim}(N(A')/N(B))$.

Finally, by construction, $\operatorname{ldim}(A+f(A')/A+f(B)) = \operatorname{ldim}(A'/B)$, and therefore $\delta(A+f(A')) = \delta(A'/B)$. □

We will consider *minimal* derivation extension problems $(B \leq A, f)$, i.e., derivation extension problems such that A is a minimal strong extension of B.

Corollary 3.6. *The free pseudosolution $A+f(A)$ of a (minimal) derivation extension problem $(B \leq A, f)$ is a (minimal) strong extension of $A+f(B)$.*

Proof. Consider a vector subspace C such that $A+f(B) \subseteq C \subseteq A+f(A)$. By linearity, $C = A+f(A')$, where $A' = f^{-1}(C)$. Note that $B \subseteq A' \subseteq A$. By the previous lemma, $\delta(C/A+f(B)) = \delta(A'/B) \geq 0$ since $B \leq A$. One deduces that $A+f(A)$ is a strong extension of $A+f(B)$.

Suppose now that A is a minimal strong extension of B. If $\operatorname{ldim}(A/B) = 1$, then $\operatorname{ldim}(A+f(A)/B+f(A)) = 1$ and $A+f(A)$ is minimal over $A+f(B)$. Otherwise, A is prealgebraic over B and, in particular, $\delta(A/B) = 0$. In this case, if $A+f(B) \subsetneq C \subsetneq A+f(A)$, we have $\delta(C/A+f(B)) = \delta(A'/B) > 0 = \delta(A/B) = \delta(A+f(A)/A+f(B))$, so C is not strong in $A+f(A)$. □

Our goal is to use the Lie algebra just constructed to extend a derivation from $B \leq V_\mu$ to $B \leq A \leq V_\mu$. It is not guaranteed that the free pseudosolution will belong to $\mathcal{K}_\mu$. There are multiple cases to consider:

(Case A) The free pseudosolution does not belong to $\mathcal{K}$.

(Case B) The free pseudosolution belongs to $\mathcal{K}$, but not to $\mathcal{K}_\mu$.

(Case C) The free pseudosolution belongs to $\mathcal{K}_\mu$. In that case, we will be able to extend the derivation without further work, by using richness of M_μ.

Let us start dealing with Case A. We first notice:

Lemma 3.7. *Let $A+f(A)$ be the free pseudosolution of a derivation extension problem $(B \leq A, f)$. For any nonzero vector subspace C of $A+f(A)$, we have $\delta(C) \geq 1$.*

Proof. Let $C \subseteq A + f(A)$ be a nonzero vector subspace.

If $C \cap (A + f(B))$ is nonzero, then, by submodularity,

$$\delta(C) \geq \delta(C \cap (A + f(B))) + \delta(C + A + f(B)/A + f(B)) \geq 1 + 0,$$

since $A + f(B) \in \mathcal{K}$ and $A + f(B) \leq A + f(A)$.

Otherwise, consider a basis $(f(a_1) + a'_1, \ldots, f(a_s) + a'_s)$ of C. Since we have $C \cap (A + f(B)) = \{0\}$, the family $(f(a_1), \ldots, f(a_s))$ is free over $A + f(B)$, and the basis $\big((f(a_i)+a'_i) \wedge (f(a_j)+a'_j) : i < j\big)$ of $\bigwedge^2 C \subseteq \bigwedge^2(A + f(A))$ is free over $\big\langle \alpha \wedge \alpha', f(\beta) \wedge f(\beta'), \alpha \wedge f(\alpha') : \alpha, \alpha' \in A, \beta, \beta' \in B \big\rangle$.

By construction,

$$N(A + f(A)) \subseteq \big\langle \alpha \wedge \alpha', f(\beta) \wedge f(\beta'), \alpha \wedge f(\alpha') : \alpha, \alpha' \in A, \beta, \beta' \in B \big\rangle,$$

and thus $N(C) = N(A + f(A)) \bigcap \bigwedge^2 C = \{0\}$. Then $\delta(C) = s > 0$. □

Therefore, the only way for the free pseudosolution of $(B \leq A, f)$ to not belong to $\mathcal{K}$ is for $F_2(A + f(A))$ to contain linearly independent vectors v_0 and v_1 such that $v_0 \wedge v_1 \in N(A + f(A))$ or, equivalently, $[v_0, v_1] = 0$ in $A + f(A) \oplus \big(\bigwedge^2(A + f(A))\big)/N(A + f(A))$. This possibility cannot be eliminated in general, as the following shows:

Example 3.8. Suppose we have vectors b_0, b_1 and b_2 such that $f(b_1) = f(b_2) = 0$ and $f(b_0) = b_0$ and want to extend f to an element a satisfying $[a, b_0] + [b_1, b_2] = 0$.

Let $A + f(A)$ be the free pseudosolution of this derivation extension problem. It has to satisfy $[f(a), b_0] + [a, b_0] = 0$, which can be factorized as $[f(a) + a, b_0] = 0$. Therefore, the free pseudosolution cannot belong to $\mathcal{K}$.

An obvious workaround, in that case, is to consider the linear map g with $g(b_i) = f(b_i)$ for $i = 0, 1, 2$, and $g(a) = -a$, and extend it into a derivation of $\langle a, b_0, b_1, b_2 \rangle$. This is easily checked to quotient into a derivation extending f.

The solution presented in the previous example is the idea behind the general case:

Lemma 3.9 (Case A). *Suppose that the free pseudosolution of a minimal derivation extension problem $(B \leq A, f)$ is not in $\mathcal{K}$. Then the linear map $f : B \to B + f(B)$ can be extended to a linear map $g : A \to A + f(B)$ such that $g(N(A)) \subseteq N(A + f(B))$, which gives a solution of the extension problem.*

Proof. By the previous lemma, in such a case, the free pseudosolution $A + f(A)$ contains linearly independent vectors v_0 and v_1 such that $v_0 \wedge v_1 \in N(A + f(A))$ (i.e., such that $[v_0, v_1] = 0$).

As $N(A + f(A))$ is generated by $f(N(A))$ over $N(A + f(B))$, there are $e \in N(A)$ and $c \in N(A + f(B))$ such that $f(e) + c = v_0 \wedge v_1$.

Note first that $f(e) + c \in \langle f(a) \wedge a' : a, a' \in A \rangle + \bigwedge^2(A + f(B))$.

We can write $v_0 = f(a_0) + a'_0$ and $v_1 = f(\tilde{a}_1) + \tilde{a}'_1$ with $a_0, a'_0, \tilde{a}_1, \tilde{a}'_1 \in A$. Since $M_\mu \in \mathcal{K}$, at least one v_i is not in $A + f(B)$. We may assume that $a_0 \in A \setminus B$.

Note that $f(\tilde{a}_1) \in \langle A, f(a_0)\rangle$: otherwise,

$$f(a_0) \wedge f(\tilde{a}_1) \notin \langle f(a) \wedge a' \colon a, a' \in A\rangle + \bigwedge^2 (A + f(B)),$$

which contradicts the equality

$$v_0 \wedge v_1 = f(a_0) \wedge f(\tilde{a}_1) + f(a_0) \wedge \tilde{a}'_1 + a'_0 \wedge f(\tilde{a}_1) + a'_0 \wedge \tilde{a}'_1 = f(e) + c.$$

Thus, there is $\alpha \in \mathbb{F}_q$ and $a'_1 \in A$ such that $v_1 = \alpha f(a_0) + a'_1$, and in fact

$$v_0 \wedge v_1 = f(a_0) \wedge (a'_1 - \alpha a'_0) + a'_0 \wedge a'_1.$$

Now, complete a_0 to a basis $(a_0, \dots, a_n)$ of A over B. We can write

$$e = \sum_{i<j\le n} \lambda_{i,j} a_i \wedge a_j + \sum_{i \le n} a_i \wedge b_i + d$$

with $\lambda_{i,j} \in \mathbb{F}_q$, $b_i \in B$ and $d \in \bigwedge^2 B$.

Then

$$f(e) = \sum_{i<j\le n} \lambda_{i,j} f(a_i) \wedge a_j - \sum_{j<i\le n} \lambda_{j,i} f(a_i) \wedge a_j + c'$$

with $c' \in \langle f(a) \wedge b \colon a \in A,\, b \in B\rangle + \bigwedge^2 (A + f(B))$.

But the family $\{f(a_i) \wedge a_j \colon i \neq 0,\ j \neq i\}$ is free over

$$\langle f(a_0) \wedge a' \colon a' \in A\rangle + \langle f(a) \wedge b \colon a \in A,\, b \in B\rangle + \bigwedge^2 (A + f(B)).$$

Hence the equality $f(e) + c = v_0 \wedge v_1$ yields that $\lambda_{j,i} = 0$ for all $j < i \le n$ and thus

$$e = \sum_{i \le n} a_i \wedge b_i + d.$$

This yields

$$f(e) = \sum_{i \le n} f(a_i) \wedge b_i + c^*$$

with $b_i \in B$ and $c^* \in \bigwedge^2 (A + f(B))$.

Note that if $(a^*_0, \dots, a^*_n)$ is a family of vectors of A, then the family

$$\{f(a_i) \wedge a^*_i \colon a^*_i \neq 0, i = 0, \dots, n\}$$

is free over $\bigwedge^2 (A + f(B))$. Therefore, the equality $f(e) + c = v_0 \wedge v_1$ yields that $b_0 = a'_1 - \alpha a'_0$ and $b_i = 0$ for all $i > 0$.

We have $e = a_0 \wedge b_0 + d$, where $a_0 \in A \setminus B$, $b_0 \in B$ and $d \in \bigwedge^2 B$. Since v_0 and v_1 are linearly independent, we have $b_0 = a'_1 - \alpha a'_0 \neq 0$. Therefore, $e \in N(A) \setminus N(B)$, and $B \oplus \langle a_0\rangle$ defines an algebraic extension of B. By minimality of $B \le A$, we deduce that $A = \langle B, a_0\rangle$.

Since $a_1' = \alpha a_0' + b_0$, we have $v_0 \wedge v_1 = f(a_0) \wedge b_0 + a_0' \wedge a_1' = f(a_0) \wedge b_0 + a_0' \wedge b_0$. Then the equality $f(e) + c = v_0 \wedge v_1$ yields the following equality in $\bigwedge^2(A + f(B))$:

$$a_0 \wedge f(b_0) + f(d) + c = a_0' \wedge b_0.$$

Let us consider the linear map $g : A \to A + f(B)$ such that $g|_B = f$ and $g(a_0) = -a_0'$. We claim that g is a solution to the derivation extension problem $(B \leq A, f)$.

By Remark 2.11, it is enough to show that $g(e) \in N(A + f(B))$. We have

$$\begin{aligned} g(e) &= g(a_0 \wedge b_0 + d) \\ &= g(a_0) \wedge b_0 + a_0 \wedge g(b_0) + g(d) \\ &= -a_0' \wedge b_0 + a_0 \wedge f(b_0) + f(d) \\ &= -c \in N(A + f(B)). \end{aligned}$$

This g is thus the solution we were looking for. □

Now consider Case B:

Lemma 3.10 (Case B). *Suppose the free pseudosolution of a minimal derivation extension problem $(B \leq A, f)$ belongs to $\mathcal{K}$, but does not belong to $\mathcal{K}_\mu$. Then the linear map $f : B \to B + f(B)$ can be extended again to a linear map $g : A \to A + f(B)$ such that $g(N(A)) \subseteq N(A + f(B))$, which gives a solution of the extension problem.*

Proof. By Corollary 3.6, $A + f(A)$ is a minimal strong extension of $A + f(B)$, and by Fact 2.19, this can happen for two different reasons:

(a) There is a good code $\alpha \in \mathcal{C}$ and $b \in \mathrm{dcl}^{\mathrm{eq}}(A + f(B))$ such that $A + f(A) = \langle A + f(B), \bar{e}\rangle$ with $\bar{e}$ w-generic in $\varphi_\alpha(\bar{x}, b)$ over $A + f(B)$. Moreover, there is $\bar{e}' \in A + f(B)$ which realizes $\varphi_\alpha(\bar{x}, b)$ (in this case $\bar{x}$ is an n-tuple, where $n = \mathrm{ldim}(A/B)$).

(b) There exist a vector subspace $A + f(B) \subseteq E \subseteq A + f(A)$, a good code $\alpha \in \mathcal{C}$, an imaginary $b \in \mathrm{dcl}^{\mathrm{eq}}(E)$, and realizations $\bar{e}$ and $\bar{e}'$ of $\varphi_\alpha(\bar{x}, b)$ such that $\bar{e}$ is w-generic over E and $\bar{e}'$ is w-generic over $\langle E, \bar{e}\rangle$ (in this case $\bar{x}$ is an m-tuple with $m < \mathrm{ldim}(A/B)$).

Let us take care of subcase (a) first. In this case, there are n terms $(\Phi_i)_{i<n}$:

$$\Phi_i(x_0, \ldots, x_{n-1}) = \sum_{j<k<n} \lambda_{ijk} x_j \wedge x_k + \sum_{j<n} c_{ij} \wedge x_j + \psi_i,$$

where $\lambda_{ijk} \in \mathbb{F}_p$, $c_{ij} \in A + f(B)$ and $\psi_i \in \bigwedge^2(A + f(B))$, such that

$$\Phi_i(\bar{e}') \in N(A + f(B)) \quad \text{for } i < n$$

and

$$N(A + f(A)) = N(A + f(B)) \oplus \langle \Phi_i(\bar{e})\rangle_{i<n}.$$

Consider π the linear map $A + f(A) \to A + f(B)$ such that $\pi|_{A+f(B)} = \mathrm{Id}$ and $\pi(\bar{e}) = \bar{e}'$.

Claim 3.11. *The linear map π induces a Lie algebra morphism from $A + f(A)$ onto $A + f(B)$.*

Proof. Remember that π extends canonically to a Lie algebra morphism from $F_2(A + f(A))$ to $F_2(A + f(B))$. Then $\pi(\Phi_i(\bar{e})) = \Phi_i(\bar{e}')$, so $\pi(N(A + f(A)) = N(A + f(B))$. By Remark 2.9, π induces a Lie algebra morphism from $A + f(A)$ to $A + f(B)$. □

Now, consider the linear map $g = \pi \circ f : A \to A + f(B)$.

Claim 3.12. *The linear map g induces a partially defined graded derivation from A to $A + f(B)$.*

Proof. By Remark 2.11 we have to check that $g(N(A)) \subseteq N(A + f(B))$, where g is the canonically partially defined derivation from $F_2(A)$ to $F_2(A + f(B))$. Note that g on $F_2(A)$ is equal to the composition of the Lie algebra morphism π on $F_2(A + f(A))$ with the derivation f on $F_2(A)$. Indeed, if $x, y \in A$, we obtain

$$\begin{aligned}\pi(f(x \wedge y)) &= \pi(f(x) \wedge y + x \wedge f(y)) = \pi(f(x)) \wedge \pi(y) + \pi(x) \wedge \pi(f(y)) \\ &= \pi(f(x)) \wedge y + x \wedge \pi(f(y)) = g(x) \wedge y + x \wedge g(y) \\ &= g(x \wedge y).\end{aligned}$$

Thus,

$$g(N(A)) = \pi\big(f(N(A))\big) \subseteq \pi\big(N(A + f(A))\big) = N(A + f(B)). \qquad \square$$

Since the partially defined derivation $g : A \to A + f(B)$ extends the derivation $f|_B : B \to B + f(B)$, the previous claim gives a solution in $\mathcal{K}_\mu$ to our derivation extension problem.

We are now going to show that in the specific case of derivation extension problems, subcase (b) cannot happen.

By way of contradiction, suppose that $A + f(B) \leq A + f(A)$ is of type (b). In this case,

$$\mathrm{ldim}(E + \langle \bar{e}, \bar{e}' \rangle / E) = \mathrm{ldim}\big(N(E + \langle \bar{e}, \bar{e}' \rangle)/N(E)\big) = 2m,$$

and the linear map which sends $\bar{e}$ onto $\bar{e}'$ over E induces a Lie algebra isomorphism over E between $E + \langle \bar{e} \rangle$ and $E + \langle \bar{e}' \rangle$.

Thus, there are m terms $(\Phi_i)_{i<m}$:

$$\Phi_i(x_0, \dots, x_{m-1}) = \sum_{j<k<m} \lambda_{ijk} x_j \wedge x_k + \sum_{j<m} c_{ij} \wedge x_j + \psi_i,$$

where $\lambda_{ijk} \in \mathbb{F}_p$, $c_{ij} \in E$ and $\psi_i \in \bigwedge^2 E$, such that

$$N(E + \langle \bar{e}, \bar{e}' \rangle) = N(E) \oplus \langle \Phi_i(\bar{e}), \Phi_i(\bar{e}') \rangle_{i<n} \subseteq N(A + f(A)).$$

Recall that

$$N(A+f(A)) = N(A+f(B)) + f(N(A)) \subseteq \bigwedge^2(A+f(B)) + \langle f(a) \wedge a' \colon a, a' \in A \rangle.$$

Since $A + f(B) \subseteq E \subseteq A + f(A)$ and $\bar{e}$ is a tuple of m linearly independent vectors in $A + f(A)$ over E, there exist a vector subspace $B \subseteq A_0 \subset A$ such that $E = A + f(A_0)$ and linearly independent vectors $a_0, \ldots, a_{m-1} \in A$ over A_0 such that $\bar{e} = (f(a_0) + v_0, \ldots, f(a_{m-1}) + v_{m-1})$, where $v_j \in A + f(B)$. If $(c_l)_{l<m}$ is a family of vectors in $E \setminus (A + f(B))$, the family

$$(f(a_j) \wedge f(a_k), c_l \wedge f(a_l) \colon j < k < m, l < m)$$

is free over $\bigwedge^2 E + \langle f(a) \wedge v \colon a \in A, v \in A + f(B) \rangle$. It follows that $\lambda_{ij} = 0$ and $c_{ij} \in A + f(B)$ for all i and j.

The difference between the equations $\Phi_i(\bar{e})$ and $\Phi_i(\bar{e}')$ gives us m linearly independent equations in $N(A + f(A))$ over $N(A + f(B))$:

$$\sum_{j<m} c_{ij} \wedge (e_j - e'_j) \in N(A + f(A)).$$

So $\delta\big(A+f(B)+\langle e_j - e'_j \colon j < m \rangle / A+f(B)\big) = 0$, contradicting the minimality of the extension $A+f(A)$ over $A+f(B)$ $\big($since $m < n = \operatorname{ldim}(A+f(A))/A+f(B))\big)$.

Therefore, subcase (b) cannot happen. □

Finally, in Case C, we conclude by using richness of M_μ:

Proposition 3.13. *Every minimal derivation extension problem* $(B \leq A, f)$ *has a solution*: *i.e.*, *there is a partially defined derivation* $g : A \to A + g(A)$ *extending* f *such that* $A + g(A) \leq V_\mu$.

Proof. For any such extension problem, if the free pseudosolution $A + f(A)$ is in $\mathcal{K}_\mu$ (Case C), then by richness of M_μ, there is a strong embedding h of $A + f(A)$ in V_μ over $A + f(B)$, and we can take $g = h \circ f$. Otherwise, we can extend f by a partially defined derivation $g : A \to A + f(B)$ (Lemmas 3.9 and 3.10). □

Let us show how we can use this to construct derivations on M_μ by reducing every configuration to a minimal derivation extension problem. Given a partially defined derivation f on B and a strong extension A of B, there is no reason a priori for $A + f(B)$ to be strong in V_μ, and we cannot directly apply the previous proposition in order to extend the derivation. In that case, we will extend in several steps to the self-sufficient closure of $A + f(B)$:

Lemma 3.14. *Let $B \leq V_\mu$ be a finite strong subspace and $f : B \to B + f(B)$ a partially defined graded derivation with $B + f(B) \leq V_\mu$.*

If $B \leq A \leq V_\mu$ is a minimal strong extension of B in V_μ such that $A + f(B)$ is not self-sufficient in V_μ, then

- $\delta(A/B) = 1$, *and*
- *the partially defined derivation f can be extended to a partially defined graded derivation on $\tilde{A}$, where $\tilde{A}$ is the self-sufficient closure of $A + f(B)$ in V_μ.*

Proof. Since $B \leq A \leq V_\mu$ and $B + f(B) \leq V_\mu$, by submodularity of δ, we have

$$B \leq A \cap (B + f(B)) \leq A.$$

By minimality of $B \leq A$, there are two cases, either $A \cap (B + f(B)) = A$ or $A \cap (B + f(B)) = B$. In the first case, $A \subseteq B + f(B)$, and since $B \subseteq A$, we get $A + f(B) = B + f(B) \leq V_\mu$, a contradiction.

Thus, $A \cap (B + f(B)) = B$. Again, by submodularity,

$$0 \leq \delta\big(A + f(B)/B + f(B)\big) \leq \delta(A/B) \leq 1.$$

Since $A + f(B)$ is not self-sufficient in V_μ, we have necessarily

$$\delta\big(A + f(B)/B + f(B)\big) = \delta(A/B) = 1.$$

Because $B \leq B + f(B)$, we decompose it into a tower of minimal extensions $B = B_0 \leq \cdots \leq B_n = B + f(B)$. Since $B \subset B_1 \subset B + f(B)$, we have that $B_1 + f(B) = B + f(B) \leq V_\mu$, and we can extend f to B_1 using Proposition 3.13. Then $B_2 + f(B_1) = B_1 + f(B_1)$, and iteratively, we extend f to $B + f(B)$. Let $\tilde{A}$ be the self-sufficient closure of $A + f(B)$. Because $A + f(B)$ is not self-sufficient in V_μ and $\delta\big(A + f(B)/B + f(B)\big) = 1$, we obtain $\delta(\tilde{A}/B + f(B)) = 0$.

There is a sequence of minimal extensions $B + f(B) = C_0 \leq \cdots \leq C_n = \tilde{A}$. Moreover, as $\delta(\tilde{A}/A + f(B)) = 0$, all these extensions are either prealgebraic or algebraic. In particular, we can iteratively extend f to each of them using Proposition 3.13, since $\delta(C_{i+1}/C_i) = 0$ imposes at each step that $C_{i+1} + f(C_i) \leq V_\mu$. □

Theorem 3.15. *Let $B \leq V_\mu$ be finite and $f : B \to B + f(B)$ be a partially defined graded derivation, with $B + f(B) \leq V_\mu$. Let $a \in V_\mu$. There exists a finite $B \leq A$ with $a \in A$ and a graded extension $f : A \to A + f(A)$ to A with $A \leq V_\mu$ and $A + f(A) \leq V_\mu$.*

Proof. Let A' be the self-sufficient closure of $\langle B, a\rangle$. There is a sequence $B \leq A'_0 \leq \cdots \leq A'_n = A'$ of minimal extensions. Moreover, we know that $\delta(A'/B) \leq 1$, so at most one of these extensions is transcendental.

We extend iteratively f to A'_i. While $A'_{i+1} + f(A'_i) \leq V_\mu$, we apply Proposition 3.13. If it is the case for all $i < n$, we obtain an extension f to $A = A' \supseteq \langle B, a\rangle$.

Otherwise, there is $i_0 < n$ such that f is extended to A'_{i_0} and $A'_{i_0+1} + f(A'_{i_0})$ is not self-sufficient in V_μ. By Lemma 3.14, $\delta(A'_{i_0+1}/A'_{i_0}) = 1$ and we can extend f to A_{i_0+1}, where A_{i_0+1} is the self-sufficient closure of $A'_{i_0+1} + f(A'_{i_0})$.

Now define $A_i = A'_i + A_{i_0+1}$ for $i > i_0 + 1$. Since $\delta(A'/B) \leq 1$, we have $\delta(A'_{i+1}/A'_i) = 0$ for all $i > i_0$.

This implies that $A_i \leq V_\mu$ for all $i > i_0$. Indeed, if $A_i \subset D$ for some D, then

$$\begin{aligned}
\delta(D/A_i) &= \delta(D) - \delta(A_{i_0+1} + A'_i) \\
&\geq \delta(D) - \delta(A_{i_0+1}) - \delta(A'_i) + \delta(A_{i_0+1} \cap A'_i) \quad \text{(submodularity)} \\
&= \delta(D/A_{i_0+1}) - \delta(A'_i/A_{i_0+1} \cap A'_i) \\
&\geq -\delta(A'_i/A_{i_0+1} \cap A'_i) \text{ as } A_{i_0+1} \leq V_\mu \\
&\geq 0,
\end{aligned}$$

where the last line is a consequence of the fact that A'_i is a strong extension of A'_{i_0+1} with $\delta(A'_i/A'_{i_0+1}) = 0$, and $A'_{i_0+1} \subset A_{i_0+1} \cap A'_i \subset A'_i$.

This implies that $\delta(A_{i+1}/A_i) = 0$ for all $i > i_0$, as

$$\begin{aligned}
\delta(A_{i+1}/A_i) &= \delta(A_i + A'_{i+1}/A_i) \\
&\leq \delta(A'_{i+1}/A_i \cap A'_{i+1}) \quad \text{(submodularity)} \\
&\leq 0,
\end{aligned}$$

where we obtain the last line by similarly considering the strong extension $A'_i \leq A'_{i+1}$ and $A'_i \subset A_i \cap A'_{i+1} \subset A'_{i+1}$. We get $\delta(A_{i+1}/A_i) = 0$ by strongness.

Again by Lemma 3.14, for $i > i_0$, we have iteratively $A_{i+1} + f(A_i) \leq V_\mu$, and we can extend f to A_{i+1} by Proposition 3.13. Thus, we obtain an extension f to $A = A_n \supseteq A' \supseteq \langle B, a\rangle$. □

By a direct induction, any *finite* partially defined graded derivation can be extended to M_μ:

Corollary 3.16. *Any partially defined graded derivation $f : B \to B + f(B)$ with $B \leq B + f(B) \leq V_\mu$ and B finite can be extended to a graded derivation f on M_μ (i.e., $f(V_\mu) \subseteq V_\mu$).*

4. A cover without the CBP

We are now ready to define a cover of the sort V_μ in M_μ that will not have the canonical base property. It will closely mirror the example constructed in [Hrushovski et al. 2013].

Let us first recall a few definitions, valid in any superstable theory.

Definition 4.1. Let $\mathcal{P}$ be a family of partial types. A stationary type $p \in S(A)$ is $\mathcal{P}$-internal (resp. almost $\mathcal{P}$-internal) if there is a set of parameters C such that for any realization $a \models p$ independent from C over A, there is a tuple e of realizations of types in $\mathcal{P}$, each based over C, such that $a \in \operatorname{dcl}(e, C)$ (resp $a \in \operatorname{acl}(e, C)$).

Fact 4.2. If all partial types in $\mathcal{P}$ are over some parameters A and $p \in S(A)$, the parameters C can be picked as a Morley sequence $a_1, \ldots, a_n$ in p, called a fundamental system of solutions. See the proof of [Pillay 1996, Chapter 7, Lemma 4.2] for the $\mathcal{P}$-internal case. It immediately adapts to the almost internal case.

More generally, a stationary type $p \in S(A)$ is said to be $\mathcal{P}$-analyzable if there is $a \models q$ and $a = a_n, a_{n-1}, \ldots, a_1$ such that $\operatorname{tp}(a_1/A)$ is almost $\mathcal{P}$-internal, and for all $n > i \geq 1$ we have that $\operatorname{tp}(a_{i+1}/Aa_i)$ is stationary, $\mathcal{P}$-internal and $a_i \in \operatorname{dcl}(a_{i+1}A)$.

Definition 4.3. A theory has the canonical base property (CBP) if (possibly working over some parameters) for any tuples a and b, if $\operatorname{tp}(a)$ has finite Lascar rank and $b = \operatorname{Cb}(\operatorname{stp}(a/b))$, then $\operatorname{stp}(b/a)$ is almost $\mathcal{P}$-internal, where $\mathcal{P}$ is the family of Lascar rank one types that are not locally modular.

In this last section, we construct a structure without the CBP that is interpretable in M_μ with its 2-nilpotent graded Lie algebra structure. As a consequence, this structure will be CM-trivial:

Definition 4.4. A stable theory T is CM-trivial if whenever $A \subset B$ are parameters and c is a tuple satisfying $\operatorname{acl}^{\mathrm{eq}}(c, A) \cap \operatorname{acl}^{\mathrm{eq}}(B) = \operatorname{acl}^{\mathrm{eq}}(A)$, then $\operatorname{Cb}(\operatorname{stp}(c/A))$ is algebraic over $\operatorname{Cb}(\operatorname{stp}(c/B))$.

In particular, our structure will not interpret a field, by a result of Pillay [2000]. Let us now define it.

Let Q be a sort, given by M_μ, and let $P = V_\mu$. The second sort S is given (as a set) by V_μ^2. We equip $Q = M_\mu$ with all its L-structure, i.e., its structure of 2-nilpotent graded Lie algebra. Moreover, we equip S with

- the projection $\pi : S \to P$ with $\pi((a, u)) = a$;
- the group action $(P, +) \times S \to S$ given by $b * (a, u) = (a, u + b)$;
- the group law $+ : S^2 \to S$ given by the addition on V_μ;
- for any $\varnothing$-definable set W in V_μ^{2n} given by $\sum_{i=1}^n [x_i, y_i] = 0$, a relation T_W on S given by
$$\sum_{i=1}^n [x_i, y_i] = 0 \quad \text{and} \quad \sum_{i=1}^n ([u_i, y_i] + [x_i, v_i]) = 0$$
for (x_i, u_i) and (y_i, v_i) in S.

We denote this two-sorted structure N. Note that the relations T_W are, formally, defined exactly like tangent bundles in an algebraically closed field.

Before proving that N does not have the canonical base property, allow us to explain why it is CM-trivial. First, the Lie algebra M_μ is CM-trivial, as proven by

Baudisch [1996] in his construction, and thus so is M_μ^{eq}. Moreover, our structure N is simply a reduct of M_μ^{eq} (in fact, of (M_μ, V_μ^2)). By a result of Nübling [2005], any reduct of a finite Morley rank CM-trivial theory is also CM-trivial.

The CBP refers to (almost) internality to the family of strongly minimal types that are not locally modular. Thus we need to identify this family in N. Note that by construction, this structure is 2-analyzable in $P = V_\mu$, which is strongly minimal and not locally modular. Thus N is $\aleph_1$-categorical, and P is, up to nonorthogonality, the only strongly minimal set that is not locally modular, and we will identify (almost) internality to strongly minimal types that are not locally modular and (almost) internality to P. To prove that N does not have the CBP, the key observation is:

Lemma 4.5. *For any graded derivation f of M_μ (i.e., $f(V_\mu) \subseteq V_\mu$), the map*

$$\sigma_f : N \to N, \quad (a, u) \in S \to (a, u + f(a)), \quad b \in Q \to b$$

is an automorphism of N, fixing Q pointwise by definition.

Proof. Let f be a graded derivation and σ_f be the associated map. It is obviously a bijective map, so we only have to prove that it preserves the functions and relations of N. As it is the identity on Q, it preserves its relations and functions. Preservation of the projection, group law and group action are immediate.

In order to show that the definable sets T_W are preserved, consider a tuple

$$((a_1, u_1), \ldots, (a_n, u_n), (b_1, v_n), \ldots, (b_n, v_n))$$

in T_w, with W given by $\sum_{i=1}^n [x_i, y_i] = 0$. We have

$$\begin{aligned}\sum_{i=1}^n ([u_i + f(a_i), b_i] + [a_i, v_i + f(b_i)]) &= \sum_{i=1}^n ([u_i, b_i] + [a_i, v_i]) + \sum_{i=1}^n f([a_i, b_i]) \\ &= \sum_{i=1}^n ([u_i, b_i] + [a_i, v_i]) + f\Bigl(\sum_{i=1}^n [a_i, b_i]\Bigr) \\ &= \sum_{i=1}^n ([u_i, b_i] + [a_i, v_i]) \\ &= 0\end{aligned}$$

and thus

$$\bigl((a_1, u_1 + f(a_1)), \ldots, (a_n, u_n + f(a_n)), (b_1, v_n + f(b_1)), \ldots, (b_n, v_n + f(b_n))\bigr)$$

is in T_W. □

Our previous work allowed us to construct many derivations on M_μ, which will thus give rise to automorphisms of N fixing Q. The key consequence of this is:

Corollary 4.6. *For any $(a, u) \in S$, the type* $\mathrm{tp}(a, u)$ *is not almost P-internal.*

Proof. First note that the structure N is a countable saturated model of its theory because M_μ is a countable saturated model of T_μ.

Thus, almost P-internality of a type over $\varnothing$ should be witnessed in N since we are only considering type-definable sets over finite sets of parameters.

Namely, if $\mathrm{tp}(a, u)$ was almost P-internal, there would exist, in N, a sequence $(b_1, v_1), \ldots, (b_n, v_n)$ of realizations of $\mathrm{tp}(a, u)$, independent over (a, u), and a tuple $\bar{c} \in P$ such that $(a, u) \in \mathrm{acl}((b_1, v_1), \ldots, (b_n, v_n), \bar{c})$ (we are using Fact 4.2 here). Let us suppose so, and try to derive a contradiction.

A consequence of independence is that $a, b_1, \ldots, b_n$ are independent, as points in M_μ. Thus we have $\langle a, b_1, \ldots, b_n\rangle \leq V_\mu$, and they do not satisfy any Lie algebra equations, meaning $N(\langle a, b_1, \ldots, b_n\rangle) = \{0\}$. Thus, if e is any nonzero point of V_μ, independent from $a, b_1, \ldots, b_n$, we have $\langle a, b_1, \ldots, b_n, e\rangle \leq V_\mu$ and the linear map

$$f : \langle a, b_1, \ldots, b_n\rangle \to \langle a, b_1, \ldots, b_n, e\rangle, \quad a \to e, \quad b_i \to 0$$

gives a partially defined graded derivation. Applying Corollary 3.16, we extend f to a graded derivation of M_μ, which yields an automorphism σ_f of N, fixing Q pointwise. Moreover, this automorphism fixes (b_i, v_i) for all i, and $\sigma_f((a, u)) = (a, u+e)$. As we can find such an automorphism for any $e \in V_\mu$ independent from $a, b_1, \ldots, b_n$, the orbit of (a, u) under automorphisms fixing $(b_1, v_1), \ldots, (b_n, v_n), \bar{c}$ is infinite, which is a contradiction. □

To prove that our structure does not have the CBP, we will use what is called the "group version" of the CBP, observed by Pillay and Ziegler in [2003]:

Proposition 4.7 [Hrushovski et al. 2013, Fact 1.3]. *Assume T has the CBP. Let G be a definable group, let $a \in G$, and assume that $p = \mathrm{stp}(a/A)$ has finite stabilizer. Then p is almost internal to the family of strongly minimal types that are not locally modular.*

We can now prove:

Theorem 4.8. *The structure N does not have the CBP.*

Proof. Using Proposition 4.7, we need to find a tuple $((a_1, u_1), \ldots, (a_n, u_n)) \in S^n$ such that $\mathrm{stp}((a_1, u_1), \ldots, (a_n, u_n))$ has finite stabilizer for the group law of S. Indeed, such a type is never P-internal, by Corollary 4.6.

To do so, we will, mutatis mutandis, use the proof of [Hrushovski et al. 2013, Theorem 3.7]. For the convenience of the reader, we repeat the proof of Hrushovski, Palacín and Pillay here.

Fix (a, u), (b, v), (c, r), (d, s), generic points of T_W, where W is the definable set given by $\{(x, y, z, w) \in P^4 : [x, z] + [y, w] = 0\}$. Let

$$q = \mathrm{stp}((a, u), (b, v), (c, r), (d, s)).$$

Concretely, this means that we have the equalities

$$[a, c] + [b, d] = 0 \quad \text{and} \quad [u, c] + [a, r] + [v, d] + [b, s] = 0.$$

We will prove that $\mathrm{Stab}(q)$ is trivial. First, we show that $\mathrm{Stab}(\mathrm{stp}(a, b, c, d))$ is trivial in $(V_\mu, +)^4$. Suppose that $e_1, e_2, e_3, e_4 \in \mathrm{Stab}(\mathrm{stp}(a, b, c, d))$ are independent from a, b, c, d. Then we have

$$[a + e_1, c + e_3] + [b + e_2, d + e_4] = 0,$$

which simplifies into

$$[a, e_3] + [e_1, c] + [e_1, e_3] + [b, e_4] + [e_2, d] + [e_2, e_4] = 0,$$

which contradicts the independence assumption, unless $e_i = 0$ for $i = 1, \ldots, 4$. Thus $\mathrm{Stab}(e_1, e_2, e_3, e_4))$ is trivial.

Hence any element in the stabilizer of q is of the form $(0, x)$, $(0, y)$, $(0, w)$, $(0, z)$. Pick such a tuple, independent from (a, u), (b, v), (c, r), (d, s). Then we have

$$[u + x, c] + [a, r + w] + [v + y, d] + [b, s + z] = 0,$$

giving us

$$[x, c] + [a, w] + [y, d] + [b, z] = 0.$$

If x, y, z, w were elements of V_μ, independent over a, b, c, d, we could directly conclude that $x = y = z = w = 0$. However, this is not the case, as in the structure N, only the tuples $(0, x)$, $(0, y)$, $(0, z)$, $(0, w)$ exist. We will now find elements x'', y'', z'', $w'' \in V_\mu$ satisfying the same equation and independence.

Let $(0, x')$, $(0, y')$, $(0, w')$, $(0, z')$ be another tuple in the stabilizer of q, this time independent from both (a, u), (b, v), (c, r), (d, s) and $(0, x)$, $(0, y)$, $(0, w)$, $(0, z)$. We similarly obtain

$$[x', c] + [a, w'] + [y', d] + [b, z'] = 0.$$

Pick x'', y'', w'', $z'' \in P$ such that $x'' * (0, x) = (0, x')$ (i.e., $x'' = x' - x$ in the full structure), and similarly for y, w and z. Again we have

$$[x'', c] + [a, w''] + [y'', d] + [b, z''] = 0.$$

A quick forking computation yields that x'', y'', w'', z'' are independent from a, b, c, d in $Q = M_\mu$. This yields $x'' = y'' = w'' = z'' = 0$, and thus $(x, y, w, z) = (x', y', w', z')$. As $(0, x)$, $(0, y)$, $(0, w)$, $(0, z)$ and $(0, x')$, $(0, y')$, $(0, w')$, $(0, z')$ are independent over the empty set, this implies that $\mathrm{Stab}(q)$ is trivial. □

Remark 4.9. It is unclear how many different, up to interpretability, new theories without the CBP this construction yields. Indeed, for any $q > 2$ and good map μ, we obtain a structure $N = N_{q,\mu}$. There are countably many choices for q, and uncountably many for μ, once q is fixed. It remains to be seen if any $N_{q,\mu}$ can be interpreted in another.

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Kim-independence in positive logic

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An important dividing line in the class of unstable theories is being NSOP_1, which is more general than being simple. In NSOP_1 theories forking independence may not be as well behaved as in stable or simple theories, so it is replaced by another independence notion, called Kim-independence. We generalise Kim-independence over models in NSOP_1 theories to positive logic — a proper generalisation of full first-order logic where negation is not built in, but can be added as desired. For example, an important application is that we can add hyperimaginary sorts to a positive theory to get another positive theory, preserving NSOP_1 and various other properties. We prove that, in a thick positive NSOP_1 theory, Kim-independence over existentially closed models has all the nice properties that it is known to have in an NSOP_1 theory in full first-order logic. We also provide a Kim–Pillay style theorem, characterising which thick positive theories are NSOP_1 by the existence of a certain independence relation. Furthermore, this independence relation must then be the same as Kim-independence. Thickness is the mild assumption that being an indiscernible sequence is type-definable.

In full first-order logic Kim-independence is defined in terms of Morley sequences in global invariant types. These may not exist in thick positive theories. We solve this by working with Morley sequences in global Lascar-invariant types, which do exist in thick positive theories. We also simplify certain tree constructions that were used in the study of Kim-independence in full first-order logic. In particular, we only work with trees of finite height.

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1. Introduction

The study of (ternary) independence relations in model theory goes back to Shelah's notion of forking independence, which is an abstract generalisation of classical independence notions such as linear independence in vector spaces and algebraic independence in algebraically closed fields. Forking independence was initially used to study stable theories, in which it enjoys particularly nice properties. It was later discovered that forking independence can be useful in studying the broader class of simple theories, as it retains most of its features in that class [Kim 1998; Kim and Pillay 1997]. Moreover, the fundamental properties of forking independence in simple theories, such as transitivity, symmetry, and local character, fail in *all* nonsimple theories, which suggested that forking independence might not be so useful in studying any broader class of theories. On the other hand, some natural examples of nonsimple theories admitting useful notions of independence have been known, including the theories of infinite-dimensional vector spaces with a generic bilinear form [Granger 1999], ω-free PAC fields [Chatzidakis 2002; 2008], and random parametrised equivalence relations. Inspired by some ideas of Kim [2009], and building on [Chernikov and Ramsey 2016], Kaplan and Ramsey [2020] defined the notion of Kim-independence (denoted by $\mathop{\smile\!\!\!\!\!\!\mid}^K$), and they have proved that in NSOP_1 theories — a class containing all simple theories and, among many others, the three nonsimple theories mentioned above [Chernikov and Ramsey 2016, Section 6] — it satisfies over models all the main properties of forking independence in simple theories except base-monotonicity.

The goal of this paper is to generalise the theory of Kim-independence in NSOP_1 theories to the class of thick positive theories. Positive model theory, introduced in [Ben-Yaacov 2003a; Ben Yaacov and Poizat 2007] (with some ideas in a similar direction present also in [Hrushovski 1998] and [Pillay 2000]), provides a framework generalising that of full first-order logic and allows the study of a wider range of objects using model-theoretic techniques. An important class of such objects, which motivated the work undertaken in [Ben-Yaacov 2003a], is that of the hyperimaginary extensions T^{heq} of theories T in full first-order logic. In the context of NSOP_1 theories, elimination of hyperimaginaries has been assumed in [Kim 2021] in order to carry out a construction of weak canonical bases. It was asked there (in the discussion following Definition 4.1) whether T^{heq} satisfies the existence axiom for forking independence provided that T does. We observe that this is indeed true (Theorem 10.20), which might be helpful in eliminating the assumption of elimination of hyperimaginaries in [Kim 2021] by working with Kim-independence in T^{heq}.

Haykazyan and Kirby [2021] studied the theory ECEF of existentially closed exponential fields, and working with an arbitrary JEP-refinement (which, intuitively, corresponds to a completion of an incomplete theory in full first-order logic), they have found an invariant ternary relation satisfying over models the following properties: strong finite character, existence, monotonicity, symmetry, and independence theorem. They have also proved that, for any positive theory, the existence of such a relation implies NSOP_1, so in particular the JEP-refinements of ECEF are NSOP_1. As in the full first-order setting, a natural question whether every positive NSOP_1 theory admits a ternary relation satisfying these properties arises.

Another class of examples of nonsimple NSOP_1 theories in positive logic comes from a recent work [d'Elbée et al. 2021], where d'Elbée, Kaplan and Neuhauser show that for any integral domain R all JEP-refinements of the theory $F_{R\text{-module}}$ of fields with a generic R-submodule are NSOP_1 but not simple. In particular, this applies to the theory of algebraically closed fields of characteristic zero with a generic additive subgroup.

We work under the mild assumption that the theory is thick. This means that being an indiscernible sequence is type-definable. Theories in full first-order logic, and their hyperimaginary extensions, are always thick. The theories ECEF and $F_{R\text{-module}}$ mentioned above are also thick.

Main results. The main results of our paper state that in every thick NSOP_1 theory, Kim-independence satisfies: symmetry (Theorem 6.5), the (strong) independence theorem (Theorems 7.7 and 7.15), transitivity (Theorem 8.4) and local character (Corollary 9.7), as well as invariance under automorphisms, existence, extension, monotonicity and (strong) finite character. Moreover, we prove a Kim–Pillay style theorem: in any thick positive theory T, if there exists a ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfying all the above properties, then T is NSOP_1 and $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^K$ (Theorem 9.1).

Challenges. In contrast to the full first-order setting, in a positive theory, a type over an existentially closed model may fail to have an invariant global extension. This is a fundamental obstacle to generalising Kim-independence to the positive setting, as the original definition of it relies on existence of invariant extensions in the full first-order setting. We show, however, that in a thick theory any type over an existentially closed model M extends to a global M-Lascar-invariant type. We define Kim-independence in an arbitrary thick positive theory replacing the use of invariant types by Lascar-invariant types.

One of the difficulties in adapting the results of [Kaplan and Ramsey 2020; 2021] to the positive setting is that the tree-modelling property [Kim et al. 2014, Theorem 4.3], on which most of the constructions there rely, is not available in the positive setting. This forced us in particular to work only with trees of finite height, which turns out to be enough due to compactness and a careful choice of the global types with which we work. Consequently, we substitute the notion of a tree Morley sequence used in [Kaplan and Ramsey 2020] with a weaker notion of a parallel-Morley sequence. In particular, we do not have a counterpart of the chain condition [Kaplan and Ramsey 2020, Corollary 5.15] for parallel-Morley sequences, which causes some additional technical difficulties in our proof of the strong independence theorem.

Our proofs yield in particular alternative proofs of the results in full first-order logic on Kim-independence not using any combinatorial tools other than the Ramsey theorem: while we do use the Erdős–Rado theorem to extract indiscernible sequences, in the full first-order setting this can be always replaced by the standard use of the Ramsey theorem; the technique of extracting strongly indiscernible trees from s-indiscernible trees from [Kaplan and Ramsey 2020] relying on the Erdős–Rado theorem is not used by us.

Overview. In Section 2 we review some basic terminology and facts about positive logic and NSOP_1 theories, and we make some observations which are used throughout the paper. In Section 3 we define a notion of a Morley sequence in a global Lascar-invariant type, and we prove some basic properties of these. In Section 4 we define Kim-dividing in an arbitrary thick NSOP_1 theory, we give several characterisations of Kim-dividing and we establish some basic properties of Kim-independence. In Section 5 we develop some tools which we later use in certain tree constructions: the EM-modelling property, which is a weak version of the modelling property used in [Kaplan and Ramsey 2020], parallel-Morley sequences, which serve as our substitute for the notion of a tree Morley sequence from [Kaplan and Ramsey 2020], and q-spread-outness, which is a variant of the spread-outness used in [Kaplan and Ramsey 2020]. Sections 6, 7 and 8 contain the proofs of the main properties of Kim-independence in thick positive theories: symmetry, independence theorem and transitivity, and Section 9 is dedicated to proving a Kim–Pillay-style characterisation of the NSOP_1 property among thick positive theories by existence of an abstract independence relation satisfying certain properties, and the characterisation of Kim-independence in NSOP_1 theories as the only relation satisfying them. In Section 10 we describe in detail some examples of thick NSOP_1 theories: Poizat's example of a thick non-semi-Hausdorff theory, (JEP refinements of) the positive theory of existentially closed exponential fields studied in [Haykazyan and Kirby 2021], and the hyperimaginary extensions of NSOP_1 theories.

2. Preliminaries

In this section we recall the basics of positive logic that we need in this paper. For a more extensive treatment we refer to [Ben-Yaacov 2003a; Poizat and Yeshkeyev 2018].

Throughout the paper variables will be of arbitrary (possibly infinite) length, unless stated otherwise.

Definition 2.1. Fix a signature $\mathcal{L}$. A *positive existential formula* in $\mathcal{L}$ is one that is obtained from combining atomic formulas using $\wedge$, $\vee$, $\top$, $\bot$ and $\exists$. An *h-inductive sentence* is a sentence of the form $\forall x(\varphi(x) \to \psi(x))$, where $\varphi(x)$ and $\psi(x)$ are positive existential formulas. A *positive theory* is a set of h-inductive sentences.

Note that every positive existential formula $\varphi(x)$ is equivalent to one of the form $\exists y \psi(x, y)$, where $\psi(x, y)$ is positive quantifier-free. Positive existential sentences and their negations can be used as axioms in a positive theory, since $\forall x \varphi(x)$ and $\forall x \neg\varphi(x)$ are equivalent to $\forall x(\top \to \varphi(x))$ and $\forall x(\varphi(x) \to \bot)$ respectively.

As in full first-order logic, we will assume that $\mathcal{L}$ contains a symbol $=$ interpreted in every $\mathcal{L}$-structure as equality.

Remark 2.2. We can study full first-order logic as a special case of positive logic. This is done through a process called *Morleyisation*. For this we add a relation symbol $R_\varphi(x)$ to our language for every formula $\varphi(x)$ in full first-order logic. Then we have our theory (inductively) express that $R_\varphi(x)$ and $\varphi(x)$ are equivalent. This way every formula in full first-order logic is (equivalent to) a relation symbol, and thus in particular to a positive existential formula.

Many definitions later in this section simplify in this case to familiar concepts. Every homomorphism will be an elementary embedding, and thus in particular an immersion. So every model will be an e.c. model. A theory has JEP if and only if it is complete, and the JEP-refinements correspond to completions.

Since we will only be considering full first-order logic as a special case of positive logic, we will make the following convention.

Convention 2.3. Whenever we say "formula" or "theory" we will mean "positive existential formula" and "positive theory" respectively, unless explicitly stated otherwise. This also means that every formula and theory we consider will be implicitly assumed to be positive (existential).

In full first-order logic we consider elementary embeddings because they preserve and reflect truth of full first-order formulas. Since we do not have negation in positive logic, there is a difference between preserving and reflecting truth of positive existential formulas.

Definition 2.4. A function $f : M \to N$ between $\mathcal{L}$-structures is called a *homomorphism* if for every $\varphi(x)$ and every $a \in M$ we have

$$M \models \varphi(a) \implies N \models \varphi(f(a)).$$

We call f an *immersion* if additionally the converse implication holds for all $\varphi(x)$ and all $a \in M$.

In positive model theory we study the existentially closed models.

Definition 2.5. We call a model M of T an *existentially closed model* or an *e.c. model* if the following equivalent conditions hold:

(i) Every homomorphism $f : M \to N$ with $N \models T$ is an immersion.

(ii) For every $a \in M$ and $\varphi(x)$ such that there is a homomorphism $f : M \to N$ with $N \models T$ and $N \models \varphi(f(a))$, we have that $M \models \varphi(a)$.

(iii) For every $a \in M$ and $\varphi(x)$ such that $M \not\models \varphi(a)$ there is $\psi(x)$ with $T \models \neg\exists x(\varphi(x) \wedge \psi(x))$ and $M \models \psi(a)$.

Fact 2.6. *Let T be some theory.*

(i) *Unions: The union of a chain of (e.c.) models is an (e.c.) model.*

(ii) *Amalgamation: If one of $M_1 \leftarrow M \to M_2$ is an immersion then there are $M_1 \to N \leftarrow M_2$ making the relevant square commute. In particular, every e.c. model is an amalgamation base.*

(iii) *Existential completion: For every $M \models T$ there is a homomorphism $f : M \to N$, where N is an e.c. model of T.*

(iv) *Compactness: Let $\Sigma(x)$ be a set of positive existential formulas and suppose that for every finite $\Sigma_0(x) \subseteq \Sigma(x)$ there is $M \models T$ with $a \in M$ such that $M \models \Sigma_0(a)$. Then there is an e.c. model N of T with $a \in N$ such that $N \models \Sigma(a)$.*

In the statement of compactness, Fact 2.6(iv), we have explicitly mentioned positive existential formulas because it is crucial that we cannot use all formulas from full first-order logic in $\Sigma(x)$. This is actually one of the big obstacles in this paper. We provide two examples to indicate how full compactness can fail.

Example 2.7. Consider the theory T with a symbol for inequality and ω many disjoint unary predicates $P_n(x)$. Then e.c. models of T are precisely those which consist of ω-many disjoint infinite sets, one for each predicate. If we had full compactness then the set

$$\Sigma(x) = \{\neg P_n(x) : n < \omega\}$$

would have a realisation in some e.c. model, which is impossible.

Example 2.8. It could happen that there is a definable set that is infinite and bounded. This does not contradict compactness: it just means that inequality is not positively definable on that set. Such situations might arise when adding hyperimaginaries as real elements, which can be done in positive logic (see Section 10C).

Definition 2.9. We say that a theory T has the *joint embedding property* or *JEP* if the following equivalent conditions hold:

(i) For any two models M_1 and M_2 there are homomorphisms $M_1 \to N \leftarrow M_2$.

(ii) If $T \models \neg\varphi \vee \neg\psi$ then $T \models \neg\varphi$ or $T \models \neg\psi$.

For a theory T we call an extension T' of T a *JEP-refinement* of T if it has JEP and every e.c. model of T' is also an e.c. model of T.

As suggested in Remark 2.2, having JEP is like requiring the theory to be complete. We can always find a JEP-refinement (a "completion") by taking the set of h-inductive sentences that are true in some e.c. model.

Fix a sufficiently large cardinal $\bar{\kappa}$. We will say a set is *small* if it is of cardinality smaller than $\bar{\kappa}$.

Convention 2.10. We will assume our theory T has JEP so we can work in a *monster model* $\mathfrak{M}$ (sometimes also called a universal domain), that is, a model which is:

- Existentially closed: $\mathfrak{M}$ is an e.c. model.
- Very homogeneous: Any partial immersion $f : \mathfrak{M} \to \mathfrak{M}$ with small domain and codomain extends to an automorphism on all of $\mathfrak{M}$.
- Very saturated: Any finitely satisfiable small set of formulas over $\mathfrak{M}$ is satisfiable in $\mathfrak{M}$.

We will assume all parameter sets considered to be small, except when we consider the monster model as a parameter set. We will use lowercase Latin letters $a, b, \ldots$ for (possibly small infinite) tuples inside the monster model and uppercase Latin letters $A, B, \ldots$ for (small) parameter sets inside the monster model. We will use letters M and N when these sets are e.c. models.

As is common, we use the notation $\models \varphi(a)$ to abbreviate $\mathfrak{M} \models \varphi(a)$.

The above also means that the right notion of a type in positive model theory is that of a positive existential type. That is, we write $\mathrm{tp}(a/B)$ for the set of all positive existential formulas over B satisfied by a. So we have $\mathrm{tp}(a/B) = \mathrm{tp}(a'/B)$ if and only if there is an automorphism $f : \mathfrak{M} \to \mathfrak{M}$ fixing B such that $f(a) = a'$. We also write $a \equiv_B a'$ in this case. By a type (over A) in T we will always mean a maximal consistent with T set of positive existential formulas (over A). By a partial type (over A) in T we will mean any consistent set of positive existential formulas (over A).

There are some subtle differences in possible definitions of saturatedness; see for example [Poizat and Yeshkeyev 2018, Section 2.4]. We are only interested in e.c. models, so for us it will mean the following. Constructing models of a certain level of saturation is then standard.

Definition 2.11. Let M be an e.c. model of some theory T. We say that M is *κ-saturated* if for every $A \subseteq M$ with $|A| < \kappa$ we have that a set $\Sigma(x)$ of formulas over A is satisfiable in M if and only if it is finitely satisfiable in M.

Fact 2.12. *For any $\kappa \geq |A| + |T|$ there is a κ^+-saturated $N \supseteq A$ with $|N| \leq 2^\kappa$.*

The following definitions are taken from [Ben-Yaacov 2003a; 2003c]. We added the notion of being Boolean.

Definition 2.13. Let T be a theory and work in a monster model. We call T

- *Boolean* if every formula in full first-order logic is equivalent to a positive existential formula, modulo T;
- *Hausdorff* if for any two distinct types $p(x)$ and $q(x)$ there are $\varphi(x) \notin p(x)$ and $\psi(x) \notin q(x)$ such that $\models \forall x(\varphi(x) \vee \psi(x))$;
- *semi-Hausdorff* if equality of types is type-definable, so there is a partial type $\Omega(x, y)$ such that $\mathrm{tp}(a) = \mathrm{tp}(b)$ if and only if $\models \Omega(a, b)$;
- *thick* if being an indiscernible sequence is type-definable, so there is a partial type $\Theta((x_i)_{i<\omega})$ such that $(a_i)_{i<\omega}$ is indiscernible if and only if $\models \Theta((a_i)_{i<\omega})$.

Remark 2.14. The reason for the name Hausdorff is that this corresponds to the type spaces being Hausdorff, where formulas correspond to closed sets. The name thick is based on the notion of thick formulas, which were originally defined in the setting of full first-order logic (see also [Ben-Yaacov 2003c]).

The name Boolean comes from the fact that the Lindenbaum–Tarski algebra of positive existential formulas forms a Boolean algebra, and this is in fact an equivalent assertion. In [Haykazyan 2019] these theories are called "positively model complete", but we think this name is more descriptive.

Through Morleyisation, Boolean theories are essentially the same as theories in full first-order logic, and so we will treat them as the same. The list of properties in Definition 2.13 is really a hierarchy, so Boolean implies Hausdorff implies semi-Hausdorff implies thick.

Definition 2.15. Let a and a' be two tuples, and let B be any parameter set. We write $\mathrm{d}_B(a, a') \leq n$ if there are $a = a_0, a_1, \ldots, a_n = a'$ such that a_i and a_{i+1} are on a B-indiscernible sequence for all $0 \leq i < n$.

Fact 2.16 [Ben-Yaacov 2003c, Proposition 1.5]. *A theory is thick if and only if the property "$\mathrm{d}_B(x, x') \leq n$" is type-definable over B for all B and $n < \omega$.*

The following appears as [Pillay 2000, Lemma 3.1] and [Ben-Yaacov 2003b, Lemma 1.2].

Lemma 2.17. *Let A be any parameter set, κ any cardinal, and let $\lambda = \beth_{(2^{|T|+|A|+\kappa})^+}$. Then for any sequence $(a_i)_{i<\lambda}$ of κ-tuples there is an A-indiscernible sequence $(b_i)_{i<\omega}$ such that for all $n < \omega$ there are $i_1 < \cdots < i_n < \lambda$ with $b_1 \cdots b_n \equiv_A a_{i_1} \cdots a_{i_n}$.*

Definition 2.18. In the notation of Lemma 2.17 we say that $(b_i)_{i<\omega}$ is *based on* the sequence $(a_i)_{i<\lambda}$ (over A).

Often the parameter set A will be clear from the context (it will be the set that the new sequence is indiscernible over), so we may leave out the "over A".

Definition 2.19. We write $\lambda_\kappa := \beth_{(2^\kappa)^+}$ for any cardinal κ and $\lambda_T := \lambda_{|T|}$.

Lemma 2.20. *Let M be a λ_T-saturated e.c. model of a thick theory. Then $a \equiv_M b$ implies* $\mathrm{d}_M(a, b) \leq 2$.

Proof. By thickness, $\mathrm{d}_M(x, y) \leq 1$ is M-type-definable. Let $\varphi(x, y)$ be a finite conjunction of formulas in $\mathrm{d}_M(x, y) \leq 1$. It is enough to show that $\varphi(x, a) \wedge \varphi(x, b)$ is satisfiable, because then the partial type "$\mathrm{d}_M(x, a) \leq 1$ and $\mathrm{d}_M(x, b) \leq 1$" is finitely satisfiable.

Since φ is just a formula, we may as well assume a and b to be finite. Let m denote the (finite) part of M that appears in φ. By λ_T-saturatedness of M there is a sequence $(a_i)_{i<\lambda_T}$ in M such that $a_i(a_j)_{j<i} \equiv_m a(a_j)_{j<i}$ for all $i < \lambda_T$. Using Lemma 2.17 we then find m-indiscernible $(a'_i)_{i<\omega}$ based on $(a_i)_{i<\lambda_T}$. So $\models \varphi(a'_0, a'_1)$, and thus there are $i_0 < i_1 < \lambda_T$ such that $M \models \varphi(a_{i_0}, a_{i_1})$. By construction we have $a_{i_1}a_{i_0} \equiv_m aa_{i_0}$, so $\models \varphi(a_{i_0}, a)$. Since $a \equiv_M b$ and $a_{i_0} \in M$ we also have $\models \varphi(a_{i_0}, b)$. ☐

Lemma 2.21. *Let T be a thick theory. Let $B \supseteq A$ and κ any cardinal, and set $\lambda = \lambda_{|T|+|B|+\kappa}$. Then for any A-indiscernible sequence $(a_i)_{i<\lambda}$ of κ-tuples, there is B-indiscernible $(a'_i)_{i<\lambda}$ based on $(a_i)_{i<\lambda}$ such that* $\mathrm{d}_A((a_i)_{i<\lambda}, (a'_i)_{i<\lambda}) \leq 1$.

Proof. By Lemma 2.17 there is B-indiscernible $(b_i)_{i<\omega}$ based on $(a_i)_{i<\lambda}$. Extend this to B-indiscernible $(b_i)_{i<\lambda}$. Define

$$\Sigma((x_i)_{i<\lambda}) = \mathrm{tp}((b_i)_{i<\lambda}/B) \cup \text{“} \mathrm{d}_A((x_i)_{i<\lambda}, (a_i)_{i<\lambda}) \leq 1\text{”},$$

and let $\Sigma_0(x_{i_1}, \ldots, x_{i_n}) \subseteq \Sigma((x_i)_{i<\lambda})$ be finite, only mentioning parameters in B and $a_{i_1}, \ldots, a_{i_n}$. Let $j_1 < \cdots < j_n < \lambda$ be such that $a_{j_1} \cdots a_{j_n} \equiv_B b_1 \cdots b_n \equiv_B b_{i_0} \cdots b_{i_n}$. It follows from the proof of Lemma 2.17 that we may choose j_1 to be arbitrarily large below λ, so we may assume $j_1 > i_n$. Then $a_{j_1} \cdots a_{j_n}$ realises Σ_0. By compactness we find the required $(a'_i)_{i<\lambda}$ as a realisation of Σ. ☐

The definition of dividing in positive theories is the same as in full first-order logic [Pillay 2000; Ben-Yaacov 2003b]. Following [Pillay 2000] we have to adjust forking to allow infinite disjunctions because compactness can no longer guarantee disjunctions to be finite.

Definition 2.22. We say that a partial type $\Sigma(x, b)$ *divides over C* if there is a C-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 \equiv_C b$ such that $\bigcup_{i<\omega} \Sigma(x, b_i)$ is inconsistent.

We say $\Sigma(x, b)$ *forks over C* if there is a (possibly infinite) set of formulas $\Phi(x)$ with parameters, each of which divides over C, such that $\Sigma(x, b)$ implies $\bigvee \Phi(x)$.

We write $a \mathop{\smile\!\!\!\!\!\!\,|}^d_C b$ (or $a \mathop{\smile\!\!\!\!\!\!\,|}^f_C b$) if $\mathrm{tp}(a/Cb)$ does not divide (fork) over C.

Remark 2.23. We have that $\mathrm{tp}(a/Cb)$ divides over C if and only if there is a formula $\varphi(x, b) \in \mathrm{tp}(a/Cb)$ that divides over C. This follows directly from compactness. Note that for forking this is no longer necessarily true, because the disjunction may be infinite so we cannot apply compactness.

For a type p over a set B and a subset $A \subseteq B$, the restriction of p to A is a type over A which we denote by $p|_A$. We recall the notions of an heir and a coheir, which also make sense in positive logic.

Definition 2.24. Let $M \subseteq B$, and let $p = \mathrm{tp}(a/B)$ be a type over B. We say that p is a *coheir* of $p|_M$, and write $a \mathop{\smile\!\!\!\!\!\!\mid}^u_M B$, if p is finitely satisfiable in M. We say that p is an *heir* of $p|_M$ if for every formula $\varphi(x, y)$, with parameters in M, and every $b \in B$ such that $\varphi(x, b) \in p$ there is some $b' \in M$ such that $\varphi(x, b') \in p$. In this case we write $a \mathop{\smile\!\!\!\!\!\!\mid}^h_M B$.

Remark 2.25. As in full first-order logic, we have $A \mathop{\smile\!\!\!\!\!\!\mid}^u_M B$ if and only if $B \mathop{\smile\!\!\!\!\!\!\mid}^h_M A$.

In Proposition 3.13 we compare the above notions of independence further.

We recall that $2^{<\omega}$ is the set of all finite sequences of zeroes and ones. For $\eta, \nu \in 2^{<\omega}$ we write $\eta \preceq \nu$ if ν continues the sequence η. We write $\eta^\frown\nu$ for concatenation, so for example $\eta^\frown 0$ is the sequence η with a 0 concatenated to it.

Definition 2.26. Let T be a theory, and let $\varphi(x, y)$ be a formula. We say that $\varphi(x, y)$ has SOP_1 if there are $\psi(y_1, y_2)$ and $(a_\eta)_{\eta \in 2^{<\omega}}$ such that:

(i) For every $\sigma \in 2^\omega$ the set $\{\varphi(x, a_{\sigma|n}) : n < \omega\}$ is consistent.

(ii) $\psi(y_1, y_2)$ implies that $\varphi(x, y_1) \wedge \varphi(x, y_2)$ is inconsistent, that is,

$$T \models \forall y_1 y_2 \neg[\psi(y_1, y_2) \wedge \exists x(\varphi(x, y_1) \wedge \varphi(x, y_2))].$$

(iii) For every $\eta, \nu \in 2^{<\omega}$ such that $\eta^\frown 0 \preceq \nu$ we have $\models \psi(a_{\eta^\frown 1}, a_\nu)$.

We say that T is NSOP_1 if no formula has SOP_1.

Remark 2.27. The idea of introducing the inconsistency witness $\psi(y_1, y_2)$ is due to Haykazyan and Kirby [2021]. In full first-order logic we can just take $\psi(y_1, y_2)$ to be $\neg\exists x(\varphi(x, y_1) \wedge \varphi(x, y_2))$, so we see that the definitions coincide there. The point of having ψ is that the inconsistency in (iii) is again definable by a single formula for all relevant η and ν. This enables us to apply compactness to make the tree $(a_\eta)_{\eta \in 2^{<\omega}}$ as big as we wish.

The following lemma, or rather its contrapositive, is what will actually be useful to us. If, in an NSOP_1 theory, we have two sequences that are "parallel to each other" in a certain way then we can transfer consistency for a formula along one sequence to the other. We will therefore give it the name "parallel sequences lemma".

Lemma 2.28 (parallel sequences lemma). *Suppose that* $\varphi(x, y)$ *is a formula, and* $(\bar{c}_i) = (c_{i,0}, c_{i,1})_{i \in I}$ *is an infinite indiscernible sequence satisfying*

(i) $c_{i,0} \equiv_{\bar{c}_{<i}} c_{i,1}$ *for all* $i \in I$;

(ii) $\{\varphi(x; c_{i,0}) : i \in I\}$ *is consistent*;

(iii) $\{\varphi(x; c_{i,1}) : i \in I\}$ *is inconsistent.*

Then T *has* SOP_1.

Proof. This is the same as [Kaplan and Ramsey 2020, Lemma 2.3] and that proof mostly goes through. We sketch a few small changes that are needed. Obviously we already start with an indiscernible sequence and by compactness we can freely change the order type of I preserving properties (i)–(iii). Then in the claim in that proof we need to make the array $(a_{i,0}, a_{i,1})$ sufficiently long. This can easily be done by elongating the original indiscernible sequence $(\bar{c}_i)$. Then we can find an indiscernible sequence based on $(\bar{a}_i) = (a_{i,0}, a_{i,1})$. Note that properties (1)–(3) in that claim are preserved by this operation. The reason for all this is because we need to start with an indiscernible sequence in [Kaplan and Ramsey 2020, Lemma 2.2] as well. Then the rest of that proof goes through. Finally, inconsistency of $\{\varphi(x, c_{l,1}), \chi(x, d_{l',0})\}$ should be witnessed by some formula (similarly for [Kaplan and Ramsey 2020, Lemma 2.2]), but the existence of such a witness easily follows from the construction of χ. □

3. Global Lascar-invariant types

The definition of Lascar strong types from the first-order setting easily generalises to (thick) positive logic; see [Pillay 2000, Definition 3.13, Lemma 3.15] and [Ben-Yaacov 2003b, Lemma 1.38].[1]

Definition 3.1. We say a and b have the same *Lascar strong type over* A, and write $a \equiv^{\mathrm{Ls}}_A b$, if the following equivalent conditions hold:

(i) $\mathrm{d}_A(a, b) \leq n$ for some $n < \omega$.

(ii) For each bounded A-invariant equivalence relation $E(x, y)$ we have $E(a, b)$.

(iii) There are λ_T-saturated e.c. models $M_1, \ldots, M_n$, each containing A, and $a = a_0, \ldots, a_n = b$ such that $a_i \equiv_{M_{i+1}} a_{i+1}$ for all $0 \leq i < n$.

We write $\mathrm{Lstp}(a/A)$ for the $\equiv^{\mathrm{Ls}}_A$-equivalence class of a.

Lemma 3.2. *The conditions in Definition 3.1 are equivalent in a thick theory.*

[1] Simplicity is assumed in [Ben-Yaacov 2003b, Lemma 1.38] but not used in the equivalence of the properties we mention. It is used for what is (iii) there.

Proof. The equivalence of (i) and (ii) is proved in both [Pillay 2000, Lemma 3.15] and [Ben-Yaacov 2003b, Lemma 1.38]. So we prove (i) $\iff$ (iii).

(i) $\implies$ (iii): Let $a = a_0, \dots, a_n = b$ such that a_i and a_{i+1} are on an A-indiscernible sequence. Let $0 \le i < n$, let $(a'_j)_{j<\omega}$ be an A-indiscernible sequence with $a'_0 a'_1 = a_i a_{i+1}$, and let $M \supseteq A$ be some λ_T-saturated model. By Lemma 2.17 and an automorphism there is $M_{i+1} \equiv_A M$ such that $(a'_j)_{j<\omega}$ is M_{i+1}-indiscernible. So in particular $a_i \equiv_{M_{i+1}} a_{i+1}$, as required.

(iii) $\implies$ (i): By Lemma 2.20 $a_i \equiv_{M_i} a_{i+1}$ implies that $\mathrm{d}_{M_i}(a_i, a_{i+1}) \le 2$ and as $A \subseteq M_i$ we are done. □

Definition 3.1(iii) allows for the following definition.

Definition 3.3. Let $\mathrm{Aut}_f(\mathfrak{M}/A)$ be the group generated by

$$\bigcup \{\mathrm{Aut}(\mathfrak{M}/M) : M \text{ is a } \lambda_T\text{-saturated model and } A \subseteq M\}.$$

We call its elements *Lascar strong automorphisms*. It is clear that in a thick theory $a \equiv^{\mathrm{Ls}}_A b$ precisely when there is $f \in \mathrm{Aut}_f(\mathfrak{M}/A)$ such that $f(a) = b$.

Remark 3.4. If T is semi-Hausdorff we may replace "λ_T-saturated model" by "e.c. model" in Definition 3.1 and Lemma 3.2; see [Ben-Yaacov 2003c, Proposition 3.13].

Convention 3.5. Recall that a *global type* is a type over the monster model $\mathfrak{M}$. Building on Convention 2.10 about the monster model, we will use lowercase Greek letters $\alpha, \beta, \dots$ for realisations of global types (in a bigger monster).

Definition 3.6. A global type q is called *A-Ls-invariant*, short for *A-Lascar-invariant*, if for a realisation $\alpha \models q$ we have that $b \equiv^{\mathrm{Ls}}_A b'$ implies $\alpha b \equiv^{\mathrm{Ls}}_A \alpha b'$.

Note that this definition does not depend on the choice of α. If α' is any other realisation of q, then $\alpha \equiv_{\mathfrak{M}} \alpha'$. So there is an automorphism f of the bigger monster over $\mathfrak{M}$ with $f(\alpha) = \alpha'$. So if $b \equiv^{\mathrm{Ls}}_A b'$ then $\alpha b \equiv^{\mathrm{Ls}}_A \alpha b'$ and therefore $f(\alpha) f(b) \equiv^{\mathrm{Ls}}_{f(A)} f(\alpha) f(b')$, which is just $\alpha' b \equiv^{\mathrm{Ls}}_A \alpha' b'$, since f fixes $\mathfrak{M}$.

Remark 3.7. Let q be any global type in a thick theory, let $\alpha \models q$, and let A be any (small) parameter set. Then there is $a \in \mathfrak{M}$ with $a \equiv^{\mathrm{Ls}}_A \alpha$. To see this, let $M \supseteq A$ be a λ_T-saturated model, and take any $a \models q|_M$.

Lemma 3.8. *Suppose that q is a global A-Ls-invariant type in a thick theory. Then*:

(i) *For any $f \in \mathrm{Aut}(\mathfrak{M}/A)$ the type $f(q)$ is A-Ls-invariant.*

(ii) *For any $B \supseteq A$, q is also B-Ls-invariant.*

Proof. Point (i) is straightforward. We prove (ii). Let $\alpha \models q$ and $b \equiv^{\mathrm{Ls}}_B b'$. Then there are λ_T-saturated models $M_1, \dots, M_n$, all containing B, and $b = b_0, \dots, b_n = b'$ such that $b_i \equiv_{M_{i+1}} b_{i+1}$ for all $0 \le i < n$. Letting $0 \le i < n$, it is enough to show $\alpha b_i \equiv_{M_{i+1}} \alpha b_{i+1}$. We have $b_i M_{i+1} \equiv^{\mathrm{Ls}}_A b_{i+1} M_{i+1}$, so by A-Ls-invariance, $\alpha b_i M_{i+1} \equiv^{\mathrm{Ls}}_A \alpha b_{i+1} M_{i+1}$, which implies the desired result. $\square$

Lemma 3.9. *Let T be thick, and let $p = \mathrm{tp}(a/B)$ be a coheir over $M \subseteq B$. Then there is a global M-Ls-invariant type extending p.*

Proof. Define

$$\Gamma(x) = p(x) \cup \bigcup \{\mathrm{d}_M(xc, xc') \leq 1 : c, c' \in \mathfrak{M} \text{ with } \mathrm{d}_M(c, c') \leq 1\}.$$

We claim that $\Gamma(x)$ is consistent. For finite $p_0(x) \subseteq p(x)$ there is $d \in M$ such that $d \models p_0$. Then for any c and c' with $\mathrm{d}_M(c, c') \leq 1$ we have that $\mathrm{d}_M(dc, dc') \leq 1$ because d is in M. Any maximal extension of $\Gamma(x)$ will be a desired global M-Ls-invariant type. $\square$

Definition 3.10. For $A \subseteq B$ we say that $\mathrm{Lstp}(c/B)$ *extends* $\mathrm{Lstp}(c'/A)$ if $c \equiv^{\mathrm{Ls}}_A c'$.

Corollary 3.11. *In a thick theory we have that* $\mathrm{Lstp}(a/M)$ *extends to a global M-Ls-invariant type for any a and M.*

Proof. By Lemma 3.9 we have that $p = \mathrm{tp}(a/M)$ extends to some global M-Ls-invariant type q. For $\alpha \models q$ let $a' \equiv^{\mathrm{Ls}}_M \alpha$. Then there is $f \in \mathrm{Aut}(\mathfrak{M}/M)$ such that $f(a') = a$. So by Lemma 3.8(i), $f(q)$ is global M-Ls-invariant and is exactly what we need. $\square$

Definition 3.12. For a type $p = \mathrm{tp}(a/Cb)$ write $a \mathop{\smile\!\!\!\!\!\mid}^{i\mathrm{Ls}}_C b$ if there is a global C-Ls-invariant extension of p.

Proposition 3.13. *In any thick theory T we have*

$$a \mathop{\smile\!\!\!\!\!\mid}^{u}_C b \implies a \mathop{\smile\!\!\!\!\!\mid}^{i\mathrm{Ls}}_C b \implies a \mathop{\smile\!\!\!\!\!\mid}^{f}_C b \implies a \mathop{\smile\!\!\!\!\!\mid}^{d}_C b.$$

Proof. This is standard, but we write out the arguments to check they hold with the slightly changed definitions for positive logic. The first implication is precisely Lemma 3.9, while the last implication is direct from the definition of dividing and forking.

We prove the middle implication. Assume $a \mathop{\smile\!\!\!\!\!\mid}^{i\mathrm{Ls}}_C b$ and suppose for a contradiction that $p(x) = \mathrm{tp}(a/Cb)$ forks over C. Let $\Phi(x)$ be a set of formulas that all divide over C such that $p(x)$ implies $\bigvee \Phi(x)$. Let q be a global C-Ls-invariant extension of p, and let $\alpha \models q$. Then there must be $\varphi(x, d) \in \Phi(x)$ such that $\models \varphi(\alpha, d)$. Let $(d_i)_{i<\omega}$ be C-indiscernible with $d_0 = d$. For all $i < \omega$ we have $d \equiv^{\mathrm{Ls}}_C d_i$ and thus $\alpha d \equiv^{\mathrm{Ls}}_C \alpha d_i$. So in particular $\alpha \models \{\varphi(x, d_i) : i < \omega\}$, which contradicts that $\varphi(x, d)$ divides over C. $\square$

In the remainder of this section we will develop tensoring of global Ls-invariant types. This comes down to verifying that the usual constructions for global invariant types (see, e.g., [Simon 2015, Section 2.2.1]) work when we carefully replace types by Lascar strong types everywhere.

Lemma 3.14. *Suppose that T is thick, q is a global A-Ls-invariant type and $p = \mathrm{Lstp}(a^*/A)$. Then, for $\beta \models q$, the set*

$$R_{p,q}(A) = \{(a, b) \in \mathfrak{M} : a \equiv^{\mathrm{Ls}}_A a^* \text{ and } b \equiv^{\mathrm{Ls}}_{Aa} \beta\}$$

is (the set of realisations of) a Lascar strong type over A.

Proof. Clearly this does not depend on the choice of a^* or β. The set is nonempty, as for any $b \equiv^{\mathrm{Ls}}_{Aa^*} \beta$ we have $(a^*, b) \in R_{p,q}(A)$.

Let $(a, b), (a', b') \in R_{p,q}(A)$. Then $a \equiv^{\mathrm{Ls}}_A a^* \equiv^{\mathrm{Ls}}_A a'$, so by A-Ls-invariance $ab \equiv^{\mathrm{Ls}}_A a\beta \equiv^{\mathrm{Ls}}_A a'\beta \equiv^{\mathrm{Ls}}_A a'b'$. Conversely, suppose $(a, b) \in R_{p,q}(A)$ and $ab \equiv^{\mathrm{Ls}}_A a'b'$. Then $a' \equiv^{\mathrm{Ls}}_A a \equiv^{\mathrm{Ls}}_A a^*$. Furthermore, by A-Ls-invariance $\beta ab \equiv^{\mathrm{Ls}}_A \beta a'b'$, so applying an automorphism to $b \equiv^{\mathrm{Ls}}_{Aa} \beta$ we get $b' \equiv^{\mathrm{Ls}}_{Aa'} \beta$ and thus $(a', b') \in R_{p,q}(A)$. □

Theorem 3.15. *Suppose T is thick with global A-Ls-invariant types q and r. Then there is a unique global A-Ls-invariant type $q \otimes r$ such that for any $\alpha \models q$, $\beta \models r$ and $(\alpha', \beta') \models q \otimes r$, the following are equivalent for all $B \supseteq A$ and all a and b:*

(i) $ab \equiv^{\mathrm{Ls}}_B \alpha'\beta'$.

(ii) $a \equiv^{\mathrm{Ls}}_B \alpha$ *and* $b \equiv^{\mathrm{Ls}}_{Ba} \beta$.

In particular, this implies that also $\alpha' \models q$ and $\beta' \models r$.

Proof. Throughout, let $\alpha \models q$ and $\beta \models r$. For $B \supseteq A$, denote by q_B the Lascar strong type $\mathrm{Lstp}(\alpha/B)$. By Lemma 3.8(ii) and Lemma 3.14, we have a well-defined Lascar strong type $R_{q_B,r}(B)$.

Claim. *For $A \subseteq B \subseteq C$ we have $R_{q_C,r}(C) \subseteq R_{q_B,r}(B)$.*

Proof of claim. Let $(a, b) \in R_{q_C,r}(C)$. Then $a \equiv^{\mathrm{Ls}}_C \alpha$ and $b \equiv^{\mathrm{Ls}}_{Ca} \beta$. Hence $a \equiv^{\mathrm{Ls}}_B \alpha$ and $b \equiv^{\mathrm{Ls}}_{Ba} \beta$, so $(a, b) \in R_{q_B,r}(B)$. □

For $M \supseteq A$ a λ_T-saturated model $R_{q_M,r}(M)$ corresponds to the usual syntactic type over M. So viewing $R_{q_M,r}(M)$ as a set of formulas over M, we get, by the claim, that the following is a well-defined global type:

$$q \otimes r := \bigcup \{R_{q_M,r}(M) : M \text{ is a } \lambda_T\text{-saturated model and } A \subseteq M\}.$$

First we verify that $q \otimes r$ satisfies the universal property we claimed. So let $(\alpha', \beta') \models q \otimes r$ and $B \supseteq A$. Let $M \supseteq B$ be a λ_T-saturated model and pick $a'b' \equiv^{\mathrm{Ls}}_M \alpha'\beta'$. Then by construction $(a', b') \in R_{q_M,r}(M)$ and so by the claim $(a', b') \in R_{q_B,r}(B)$. So for any a and b we have $ab \equiv^{\mathrm{Ls}}_B \alpha'\beta'$ if and only if $ab \equiv^{\mathrm{Ls}}_B a'b'$ if and only if $(a, b) \in R_{q_B,r}(B)$ if and only if $a \equiv^{\mathrm{Ls}}_B \alpha$ and $b \equiv^{\mathrm{Ls}}_{Ba} \beta$.

Uniqueness follows because any global type satisfying this universal property must restrict to $R_{q_M,r}(M) = (q \otimes r)|_M$ for all λ_T-saturated $M \supseteq A$.

Finally we prove A-Ls-invariance. Let $d \equiv^{\text{Ls}}_A d'$, and pick a and b in $\mathfrak{M}$ such that $ab \equiv^{\text{Ls}}_{Add'} \alpha'\beta'$. So $a \equiv^{\text{Ls}}_{Add'} \alpha'$ and thus, by A-Ls-invariance of q,

$$ad \equiv^{\text{Ls}}_A \alpha' d \equiv^{\text{Ls}}_A \alpha' d' \equiv^{\text{Ls}}_A ad'.$$

Then A-Ls-invariance of r gives us $\beta' ad \equiv^{\text{Ls}}_A \beta' ad'$. From the universal property we get $b \equiv^{\text{Ls}}_{Add'a} \beta'$, so $abd \equiv^{\text{Ls}}_A abd'$. Because, by assumption, $ab \equiv^{\text{Ls}}_{Add'} \alpha'\beta'$, we conclude that $\alpha'\beta' d \equiv^{\text{Ls}}_A \alpha'\beta' d'$ and we are done. □

Lemma 3.16. *For any global A-Ls-invariant types p, q, r in a thick theory we have*:

(i) Associativity: $(p \otimes q) \otimes r = p \otimes (q \otimes r)$.

(ii) Monotonicity: *For any $q'(x_0) = q(x_0, x_1)|_{x_0} \subseteq q(x_0, x_1)$ and any $r'(y_0) = r(y_0, y_1)|_{y_0} \subseteq r(y_0, y_1)$, we have $q' \otimes r' \subseteq q \otimes r$.*

Proof. (i) Let $(\alpha, \beta, \gamma) \models (p \otimes q) \otimes r$ and $(\alpha', \beta', \gamma') \models p \otimes (q \otimes r)$. We will prove that $\alpha\beta\gamma \equiv^{\text{Ls}}_B \alpha'\beta'\gamma'$ for all $B \supseteq A$. Let $abc \equiv^{\text{Ls}}_B \alpha\beta\gamma$. Then $b \equiv^{\text{Ls}}_{Ba} \beta$ and $c \equiv^{\text{Ls}}_{Bab} \gamma$. So we have $bc \equiv^{\text{Ls}}_{Ba} \beta'\gamma'$. Since also $a \equiv^{\text{Ls}}_B \alpha$ we thus conclude that $abc \equiv^{\text{Ls}}_B \alpha'\beta'\gamma'$.

(ii) Let $(\alpha, \beta) = ((\alpha_0, \alpha_1), (\beta_0, \beta_1)) \models q \otimes r$, and let $ab \equiv^{\text{Ls}}_B \alpha\beta$, where $B \supseteq A$ is arbitrary. Then in particular $a_0 \equiv^{\text{Ls}}_B \alpha_0$ and $b_0 \equiv^{\text{Ls}}_{Ba_0} \beta_0$. So if we let $(\alpha', \beta') \models q' \otimes r'$ then $\alpha_0\beta_0 \equiv^{\text{Ls}}_B a_0 b_0 \equiv^{\text{Ls}}_B \alpha'\beta'$. So $(\alpha_0, \beta_0) \models q' \otimes r'$ and we are done. □

Definition 3.17. For a global A-Ls-invariant type, we define $q^{\otimes\delta}$ for an ordinal $\delta \geq 1$ by induction as follows:

- $q^{\otimes 1} = q$,
- $q^{\otimes \delta+1} = q^{\otimes\delta} \otimes q$,
- $q^{\otimes\delta} = \bigcup_{\gamma<\delta} q^{\otimes\gamma}$ when δ is a limit.

A *Morley sequence in q (over A)* is a sequence $(a_i)_{i<\delta}$ such that $(a_i)_{i<\delta} \equiv^{\text{Ls}}_A (\alpha_i)_{i<\delta}$, where $(\alpha_i)_{i<\delta} \models q^{\otimes\delta}$.

Note that we define Morley sequences in terms of Lascar strong types here. So saying that $(a_i)_{i<\omega}$ is a Morley sequence in q over A is generally a stronger statement than just saying $(a_i)_{i<\omega} \models q^{\otimes\omega}|_A$. Of course, if A is a λ_T-saturated model in a thick theory then the two coincide.

Lemma 3.18. *Suppose that q is a global A-Ls-invariant type, and let $(\alpha_i)_{i<\delta} \models q^{\otimes\delta}$. Then for any strictly increasing sequence $(i_\eta)_{\eta<\gamma}$ in δ we have that $(\alpha_{i_\eta})_{\eta<\gamma} \models q^{\otimes\gamma}$.*

Proof. From the construction of $q^{\otimes\delta}$ it is clear that for $\gamma < \delta$ and $(\alpha_i)_{i<\delta} \models q^{\otimes\delta}$ we have $(\alpha_i)_{i<\gamma} \models q^{\otimes\gamma}$.

We prove the lemma by induction to γ. The base case and the limit step are easy, so we prove the successor step. So suppose that $(\alpha_{i_\eta})_{\eta<\gamma} \models q^{\otimes\gamma}$. We will prove that $(\alpha_{i_\eta})_{\eta<\gamma}\alpha_{i_\gamma} \equiv^{\text{Ls}}_B \alpha_{<\gamma}\alpha_\gamma$ for all $B \supseteq A$. Let $a_{\leq i_\gamma} \equiv^{\text{Ls}}_B \alpha_{\leq i_\gamma}$. Then in particular $(a_{i_\eta})_{\eta<\gamma} \equiv^{\text{Ls}}_B (\alpha_{i_\eta})_{\eta<\gamma}$ and $a_{i_\gamma} \equiv^{\text{Ls}}_{B(a_{i_\eta})_{\eta<\gamma}} \alpha_{i_\gamma}$. By the induction hypothesis and the universal property this means $(a_{i_\eta})_{\eta<\gamma} a_{i_\gamma} \equiv^{\mathrm{Ls}}_B \alpha_{<\gamma}\alpha_\gamma$, which concludes the successor step. $\square$

By Lemma 3.18, $(a_i)_{i<\delta} \models q^{\otimes\delta}|_A$ if and only if $(a_{i_1}, \dots, a_{i_n}) \models q^{\otimes n}|_A$ for all $i_1 < \cdots < i_n < \delta$. From this perspective it makes sense to make the following convention, even though we technically have not defined $q^{\otimes I}$ for arbitrary linear orders I.

Convention 3.19. Let I be any linear order, and let q be a global A-Ls-invariant type. Then by $(a_i)_{i\in I} \models q^{\otimes I}|_A$ we mean that for any $i_1 < \cdots < i_n$ in I we have $(a_{i_1}, \dots, a_{i_n}) \models q^{\otimes n}|_A$.

Proposition 3.20. *For any Morley sequence $(a_i)_{i<\delta}$ in a global A-Ls-invariant type q the following hold*:

(i) *For all $i < \delta$, $a_i \equiv^{\mathrm{Ls}}_{Aa_{<i}} \alpha$, where $\alpha \models q$.*

(ii) *$(a_i)_{i<\delta}$ is A-indiscernible.*

Proof. We first prove (i). Let $(\alpha_i)_{i<\delta} \models q^{\otimes\delta}$ and $i < \delta$. Then $a_{<i}a_i \equiv^{\mathrm{Ls}}_A \alpha_{<i}\alpha_i$. As $\alpha_{<i}\alpha_i \models q^{\otimes i} \otimes q$, the universal property yields $a_i \equiv^{\mathrm{Ls}}_{Aa_{<i}} \alpha_i$, as required.

For (ii), consider any $i_1 < \cdots < i_n < \delta$. By Lemma 3.18, $\alpha_{i_1} \cdots \alpha_{i_n} \equiv_{\mathfrak{M}} \alpha_1 \cdots \alpha_n$, so in particular $\alpha_{i_1} \cdots \alpha_{i_n} \equiv^{\mathrm{Ls}}_A \alpha_1 \cdots \alpha_n$. As $(a_i)_{i<\delta} \equiv_A (\alpha_i)_{i<\delta}$, we conclude that $a_{i_1} \cdots a_{i_n} \equiv_A a_1 \cdots a_n$. $\square$

4. Kim-dividing

The idea of Kim-dividing is to restrict dividing witnesses to nonforking Morley sequences. Proving the existence of such sequences over arbitrary sets turns out to be difficult, and is in fact an open problem for NSOP_1 theories in full first-order logic; see [Dobrowolski et al. 2022, Remark 2.6, Question 6.6]. In [Kaplan and Ramsey 2020] this is solved by using Morley sequences in some global invariant type. In full first-order logic any type over a model extends to a global invariant type. In positive logic we need to assume the theory to be semi-Hausdorff to find global invariant extensions [Ben-Yaacov 2003c, Lemma 3.11], because they may not exist otherwise (see Section 10A). In the more general setting of thick positive theories we can always find global Ls-invariant extensions and the notion of a Morley sequence makes sense in such a global Ls-invariant type; see Section 3. Since we can generally only extend types over e.c. models to global Ls-invariant types, we will consider Kim-dividing only over e.c. models (compare Question 10.21).

Definition 4.1. Let $\Sigma(x, b)$ be a partial type in a thick theory, possibly with parameters in M, and let q be a global M-Ls-invariant extension of $\mathrm{tp}(b/M)$. We say that $\Sigma(x, b)$ *q-divides over M* if for any (equivalently, some) Morley sequence $(b_i)_{i<\omega}$ in q (over M) the set $\bigcup_{i<\omega} \Sigma(x, b_i)$ is inconsistent.

By compactness q-dividing does not depend on the length of the Morley sequence, as long as it is infinite.

Proposition 4.2. *Let T be thick, let q be a global M-Ls-invariant extension of* $\operatorname{tp}(b/M)$ *and write $p(x, y) = \operatorname{tp}(ab/M)$. Then the following are equivalent*:

(i) *The type $p(x, b)$ does not q-divide.*

(ii) *For any $f \in \operatorname{Aut}(\mathfrak{M}/M)$ the type $p(x, b)$ does not $f(q)$-divide.*

(iii) *For any (equivalently, some) $(b_i)_{i<\omega} \models q^{\otimes\omega}|_M$ the set $\bigcup_{i<\omega} p(x, b_i)$ is consistent.*

(iv) *There is an Ma-indiscernible sequence $(b_i)_{i<\omega} \models q^{\otimes\omega}|_M$ with $b_0 = b$.*

Proof. (i) $\iff$ (ii) $\iff$ (iii): This follows because consistency of $\bigcup_{i<\omega} p(x, b_i)$ only depends on $\operatorname{tp}((b_i)_{i<\omega}/M)$, together with the fact that given a Morley sequence $(b_i)_{i<\omega}$ in q we have that $(f(b_i))_{i<\omega}$ is a Morley sequence in $f(q)$.

(i) $\implies$ (iv): Let $(b_i)_{i<\lambda}$ be a Morley sequence in q for big enough λ. Let a^* realise $\bigcup_{i<\lambda} p(x, b_i)$, and let $(b'_i)_{i<\omega}$ be Ma^*-indiscernible, based on $(b_i)_{i<\lambda}$. So there is $i < \lambda$ such that $a^* b'_0 \equiv_M a^* b_i \equiv_M ab$. Let $(b''_i)_{i<\omega}$ with $b''_0 = b$ be such that $a(b''_i)_{i<\omega} \equiv_M a^*(b'_i)_{i<\omega}$. Then $(b''_i)_{i<\omega}$ is Ma-indiscernible. Furthermore, since $(b_i)_{i<\lambda}$ was already M-indiscernible, we have $(b''_i)_{i<\omega} \equiv_M (b'_i)_{i<\omega} \equiv_M (b_i)_{i<\omega}$, so $(b''_i)_{i<\omega} \models q^{\otimes\omega}|_M$.

(iv) $\implies$ (iii): For such an Ma-indiscernible sequence $(b_i)_{i<\omega}$ we have, for all $i < \omega$, $ab = ab_0 \equiv_M ab_i$. So a realises $\bigcup_{i<\omega} p(x, b_i)$. $\square$

Proposition 4.3. *Let T be thick, let $\Sigma(x, b)$ be a partial type with parameters in M, and let q be a global M-Ls-invariant extension of* $\operatorname{tp}(b/M)$. *If $\Sigma(x, b)$ does not q-divide over M then there is a complete $p(x, b) \supseteq \Sigma(x, b)$ that does not q-divide over M.*

Proof. Let $(b_i)_{i<\lambda} \models q^{\otimes\lambda}|_M$ with $b_0 = b$. Then there is some $a \models \bigcup_{i<\lambda} \Sigma(x, b_i)$. Then, assuming we chose λ large enough, there is some $i_0 < \lambda$ such that for infinitely many $i < \lambda$ we have $ab_i \equiv_M ab_{i_0}$. Set $p(x, y) = \operatorname{tp}(ab_{i_0}/M)$. Then $p(x, b_{i_0})$ does not q-divide, while also $\Sigma(x, b_{i_0}) \subseteq p(x, b_{i_0})$. By invariance $p(x, b)$ does not q-divide. $\square$

The following lemma is the core of the connection between Kim-dividing and NSOP_1 theories. It tells us that q-dividing does not depend on the global Lascar-invariant type q. More discussion on the origins of this lemma can be found in [Kaplan and Ramsey 2020]. Briefly put, Kim [1998, Proposition 2.1] proved that in simple theories a formula divides with respect to every Morley sequence if and only if it divides with respect to some Morley sequence. The lemma below is an analogue of that for NSOP_1 theories.

Proposition 4.4 (Kim's lemma). *If T is thick* NSOP_1, *then q-dividing does not depend on q. That is, if q and r are global M-invariant types extending* $\mathrm{tp}(b/M)$ *then a partial type $\Sigma(x, b)$ q-divides if and only if it r-divides.*

Proof. This is essentially the proof of [Kaplan and Ramsey 2020, Proposition 3.15], adapted to the thick positive logic setting. By Proposition 4.2(ii) we may assume that q and r extend $\mathrm{Lstp}(b/M)$. Suppose that $\Sigma(x, b)$ does not q-divide, while it r-divides. We will prove that T has SOP_1. Let $(\bar{b}_i)_{i<\omega} = (b_{i,0}, b_{i,1})_{i<\omega}$ be a Morley sequence in $q \otimes r$. By Lemma 3.16(ii) and induction, $(b_{i,0})_{i<\omega}$ and $(b_{i,1})_{i<\omega}$ are Morley sequences in q and r respectively.

Since $\Sigma(x, b)$ r-divides, the set $\bigcup_{i<\omega} \Sigma(x, b_{i,1})$ is inconsistent. So by compactness there is an M-formula $\varphi(x, y) \in \Sigma(x, y)$ such that $\{\varphi(x, b_{i,1}) : i < \omega\}$ is inconsistent. Because $\Sigma(x, b)$ does not q-divide we have that $\{\varphi(x, b_{i,0}) : i < \omega\}$ is consistent.

We wish to apply the parallel sequences lemma (Lemma 2.28) to $\varphi(x, y)$ and $(\bar{b}_i)_{i<\omega^{\mathrm{op}}}$, where ω^{op} carries the opposite order of ω. So we are left to prove that $b_{i,0} \equiv_{M\bar{b}_{>i}} b_{i,1}$ for all $i < \omega$. We do so by proving that $b_{i,0}(\bar{b}_i)_{i<j<n} \equiv_M b_{i,1}(\bar{b}_i)_{i<j<n}$ for all $i < n < \omega$. Let $(\bar{\beta}_i)_{i<\omega} \models (q \otimes r)^{\otimes \omega}$. By Lemma 3.16(i) we have $(q \otimes r)^{\otimes n} = (q \otimes r)^{\otimes i+1} \otimes (q \otimes r)^{\otimes n-i-1}$. So we have $\bar{\beta}_{<n} \models (q \otimes r)^{\otimes i+1} \otimes (q \otimes r)^{\otimes n-i-1}$ and because $\bar{b}_{<n} \equiv^{\mathrm{Ls}}_M \bar{\beta}_{<n}$ we have $(\bar{b}_j)_{i<j<n} \equiv^{\mathrm{Ls}}_{M\bar{b}_{\leq i}} (\bar{\beta}_j)_{i<j<n}$. As $b_{i,0} \equiv^{\mathrm{Ls}}_M b \equiv^{\mathrm{Ls}}_M b_{i,1}$, we get, by M-Ls-invariance, that $b_{i,0}(\bar{\beta}_j)_{i<j<n} \equiv^{\mathrm{Ls}}_M b_{i,1}(\bar{\beta}_j)_{i<j<n}$. Putting the two together yields the required result. □

Definition 4.5. We say $\Sigma(x, b)$ *Kim-divides* (*over M*) if it q-divides for some global M-Ls-invariant q that extends $\mathrm{tp}(b/M)$. We write $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$ when $\mathrm{tp}(a/Mb)$ does not Kim-divide over M and call this *Kim-independence*.

Remark 4.6. By Lemma 3.9 we can extend any type over an e.c. model M in a thick theory to a global M-Ls-invariant type. So assuming NSOP_1, we have by Proposition 4.4 that $\mathrm{tp}(a/Mb)$ Kim-divides if and only if it q-divides for *any* global M-invariant extension q of $\mathrm{tp}(b/M)$.

In some constructions it will be necessary to stay within the same Lascar strong type. For this we introduce the technical tool of q-Ls-dividing.

Definition 4.7. Let T be thick, and let q be a global M-Ls-invariant extension of $\mathrm{Lstp}(b/M)$. We say that $\mathrm{Lstp}(a/Mb)$ *does not q-Ls-divide* (*over M*) if there is a Morley sequence $(b_i)_{i<\omega}$ in q with $b_0 = b$ that is Ma-indiscernible.

Remark 4.8. The length of the Morley sequence does not matter in Definition 4.7, as long as it is infinite. However, the argument here takes a little more care than for q-dividing.

One direction is clear: if there is an Ma-indiscernible Morley sequence $(b_i)_{i<\delta}$ in q for some $\delta \geq \omega$, then we can just take an initial segment. For the other direction we let $N \supseteq M$ be λ_T-saturated and $(b_i)_{i<\omega} \models q^{\otimes\omega}|_N$. Then $(b_i)_{i<\omega}$ is a Morley sequence in q. Applying a Lascar strong automorphism we find $a'b_0 \equiv^{\mathrm{Ls}}_M ab$ such that $(b_i)_{i<\omega}$ is Ma'-indiscernible. Let n be such that $\mathrm{d}_M(a'b_0, ab) \leq n$. Consider the set of formulas

$$q^{\otimes\delta}|_N((y_i)_{i<\delta}) \cup \text{“}(xy_i)_{i<\delta} \text{ is } M\text{-indiscernible”} \cup \mathrm{d}_M(xy_0, ab) \leq n.$$

This set is finitely satisfiable, and hence it has a realisation. So we find an Ma''-indiscernible Morley sequence $(b'_i)_{i<\delta}$ in q with $a''b'_0 \equiv^{\mathrm{Ls}}_M ab$. The result follows by applying a Lascar strong automorphism.

Lemma 4.9. *Let T be thick, and let q be a global M-Ls-invariant extension of* $\mathrm{Lstp}(b/M)$. *A type $p = \mathrm{tp}(a/Mb)$ does not q-divide if and only if there is a realisation $a' \models p$ such that* $\mathrm{Lstp}(a'/Mb)$ *does not q-Ls-divide.*

Proof. The right-to-left direction is clear by Proposition 4.2(iv). For the other direction we let $(b'_i)_{i<\omega}$ be a Morley sequence in q with $b'_0 = b$. By Proposition 4.2(iv) there is $(b_i)_{i<\omega} \models q^{\otimes\omega}|_M$ that is Ma-indiscernible with $b_0 = b$. Pick a' such that $a'(b'_i)_{i<\omega} \equiv_M a(b_i)_{i<\omega}$ and we are done. □

Corollary 4.10. *Let T be thick, and let q be a global M-Ls-invariant extension of* $\mathrm{Lstp}(b/M)$. *Suppose that there is $M \subseteq N \subseteq b$ such that N is λ_T-saturated. Then* $\mathrm{tp}(a/Mb)$ *does not q-divide if and only if* $\mathrm{Lstp}(a/Mb)$ *does not q-Ls-divide.*

Proof. By Lemma 4.9 we only need to prove the left-to-right direction. So suppose that $\mathrm{tp}(a/Mb)$ does not q-divide. Then there is a' with $a' \equiv_{Mb} a$ such that $\mathrm{Lstp}(a'/Mb)$ does not q-Ls-divide. In particular, we have that $a'b \equiv_N ab$, so $a'b \equiv^{\mathrm{Ls}}_M ab$. It follows that $\mathrm{Lstp}(a/Mb)$ does not q-Ls-divide. □

Proposition 4.11. *In a thick* NSOP_1 *theory Kim-independence always satisfies the following properties*:

(i) Strong finite character: *If $a \not\mathrel{\perp\!\!\!\perp}^K_M b$, then there is a formula $\varphi(x, b, m)$ in* $\mathrm{tp}(a/Mb)$ *such that for any $a' \models \varphi(x, b, m)$ we have $a' \not\mathrel{\perp\!\!\!\perp}^K_M b$.*

(ii) Existence over models: *$a \mathrel{\perp\!\!\!\perp}^K_M M$.*

(iii) Monotonicity: *$aa' \mathrel{\perp\!\!\!\perp}^K_M bb' \Longrightarrow a \mathrel{\perp\!\!\!\perp}^K_M b$.*

Proof. All follow directly from the definitions, using compactness for (i). □

Remark 4.12. Let T be a thick theory. Then Kim-dividing implies dividing because any Morley sequence in some q is in particular an indiscernible sequence. So by Proposition 3.13,

$$a \mathrel{\perp\!\!\!\perp}^u_M b \;\Longrightarrow\; a \mathrel{\perp\!\!\!\perp}^{i\mathrm{Ls}}_M b \;\Longrightarrow\; a \mathrel{\perp\!\!\!\perp}^f_M b \;\Longrightarrow\; a \mathrel{\perp\!\!\!\perp}^d_M b \;\Longrightarrow\; a \mathrel{\perp\!\!\!\perp}^K_M b.$$

Proposition 4.13. *Let T be a thick theory and M an e.c. model of T, and let a, b, c be tuples. Let also $q(x, y)$ be a global M-Ls-invariant extension of* $\mathrm{Lstp}(bc/M)$

and write $r(x) = q|_x$. *If* $\operatorname{Lstp}(a/Mb)$ *does not* r*-Ls-divide then there is* $c^*b \equiv^{\mathrm{Ls}}_M cb$ *such that* $\operatorname{Lstp}(a/Mbc^*)$ *does not* q*-Ls-divide.*

Proof. Let $(b_i, c_i)_{i<\lambda}$ be a Morley sequence over M in q for some big enough λ. Since $(b_i)_{i<\lambda}$ is a Morley sequence over M in r and $\operatorname{Lstp}(a/Mb)$ does not r-divide there is a' with $a'b_0 \equiv^{\mathrm{Ls}}_M ab$ such that $(b_i)_{i<\lambda}$ is Ma'-indiscernible.

Let $f \in \operatorname{Aut}_f(\mathfrak{M}/M)$ be such that $f(a'b_0) = ab$ and put $(b'_i, c'_i) = (f(b_i), f(c_i))$. Then $b'_0 = b$, $(b'_i)_{i<\lambda}$ is Ma-indiscernible and $(b'_i, c'_i)_{i<\lambda}$ is a Morley sequence over M in q.

Let $M' \supseteq Ma$ be λ_T-saturated and use Lemma 2.21 to find M'-indiscernible $(b''_i, c''_i)_{i<\lambda}$ based on $(b'_i, c'_i)_{i<\lambda}$ and such that $\mathrm{d}_M((b''_i, c''_i)_{i<\lambda}, (b'_i, c'_i)_{i<\lambda}) \leq 1$. In particular, $(b''_i, c''_i)_{i<\lambda}$ is a Morley sequence over M in q. Let $i < \lambda$ be such that $b''_0 \equiv_{M'} b'_i$. Then $b''_0 \equiv^{\mathrm{Ls}}_{Ma} b'_i \equiv^{\mathrm{Ls}}_{Ma} b'_0 = b$. So there is $g \in \operatorname{Aut}_f(\mathfrak{M}/Ma)$ such that $g(b''_0) = b$. Set $c^* = g(c''_0)$. So $bc^* \equiv^{\mathrm{Ls}}_M b''_0 c''_0 \equiv^{\mathrm{Ls}}_M b'_0 c'_0 \equiv^{\mathrm{Ls}}_M b_0 c_0 \equiv^{\mathrm{Ls}}_M bc$. Finally, since $(g(b''_i), g(c''_i))_{i<\lambda}$ is a Morley sequence over M in q starting with bc^* that is Ma-indiscernible, we conclude that $\operatorname{Lstp}(a/Mbc^*)$ does not q-Ls-divide. □

Corollary 4.14 (extension). *In a thick* NSOP_1 *theory we have that if* $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$ *then for any* c *there is* $c' \equiv^{\mathrm{Ls}}_{Mb} c$ *such that* $a \mathop{\smile\!\!\!\!\!\!|}^K_M bc'$.

Proof. We first prove a weaker version where we conclude $c' \equiv_{Mb} c$ instead of $c' \equiv^{\mathrm{Ls}}_{Mb} c$.

Let $q(x, y)$ be an M-Ls-invariant extension of $\operatorname{Lstp}(bc/M)$ and write $r(x) = q|_x$, where x matches b. Since $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$ there is $a'b \equiv_M ab$ such that $\operatorname{Lstp}(a'/Mb)$ does not r-Ls-divide. By Proposition 4.13 we thus find $bc^* \equiv^{\mathrm{Ls}}_M bc$ such that $\operatorname{Lstp}(a'/Mbc^*)$ does not q-Ls-divide. Letting c' be such that $abc' \equiv_M a'bc^*$, then c' satisfies $a \mathop{\smile\!\!\!\!\!\!|}^K_M bc'$ and furthermore we have $bc' \equiv_M bc^* \equiv_M bc$.

Now we use the weaker version to prove the full version. Let $N \supseteq Mb$ be some λ_T-saturated model. By the above we can find $N' \equiv_{Mb} N$ such that $a \mathop{\smile\!\!\!\!\!\!|}^K_M N'$. Then using the above again we find $c' \equiv_{N'} c$ such that $a \mathop{\smile\!\!\!\!\!\!|}^K_M N'c'$. Since $Mb \subseteq N'$ we thus get $c' \equiv^{\mathrm{Ls}}_{Mb} c$ and $a \mathop{\smile\!\!\!\!\!\!|}^K_M bc'$, as required. □

5. EM-modelling and parallel-Morley sequences

In this section we will introduce some tools which will be useful later in certain tree constructions.

Definition 5.1 [Kim et al. 2014, Defintion 2.1]. The *Shelah language*

$$L_s = \{\trianglelefteq, \wedge, <_{\mathrm{lex}}, (P_\alpha)_{\alpha<\omega}\}$$

consists of binary relation symbols $\trianglelefteq$ and $<_{\mathrm{lex}}$, a binary function symbol $\wedge$, and unary relation symbols P_α. We will consider a tree $\omega^{\leq k}$ (with $k < \omega$) as an L_s-structure, where $\trianglelefteq$ is interpreted as the containment relation, $<_{\mathrm{lex}}$ as the lexicographic order, $\wedge$ as the meet function and P_α as the α-th level of the tree.

Definition 5.2 [Kim et al. 2014, Definition 3.7]. Let I be an arbitrary index structure and C an arbitrary set of parameters. The EM-*type* of a tuple $A = (a_i)_{i \in I}$ over C is the partial type in variables $(x_i)_{i \in I}$, consisting of all the formulas of the form $\varphi(x_{\bar{i}})$ over C (where $\bar{i}$ is a tuple in I) satisfying the following property: $\models \varphi(a_{\bar{j}})$ holds whenever $\bar{j}$ is a tuple in I with $\mathrm{qftp}_I(\bar{j}) = \mathrm{qftp}_I(\bar{i})$. We let $\mathrm{EM}_I(A/C)$ denote this partial type.

In particular, we write $\mathrm{EM}_s(A/C)$ (respectively, $\mathrm{EM}_<(A/C)$) for $\mathrm{EM}_I(A/C)$, where I is considered as an L_s-structure (respectively, a $\{<\}$-structure).

Definition 5.3. Let I be an index structure, and let $A = (a_i)_{i \in I}$ and $B = (b_i)_{i \in I}$ be I-indexed tuples of compatible parameters. We will say that A is EM_I*-based on* B *over* C if $\mathrm{EM}_I(A/C) \supseteq \mathrm{EM}_I(B/C)$.

Corollary 5.4. *If A is any set of parameters, then for any compatible sequence $(a_i)_{i<\omega}$ there is an A-indiscernible sequence $(b_i)_{i<\omega}$ which is* $\mathrm{EM}_<$*-based on $(a_i)_{i<\omega}$ over A.*

Proof. By compactness there is a sequence $(a'_i)_{i<\lambda_{|T|+|A|+|a_0|}}$ which is $\mathrm{EM}_<$-based on $(a_i)_{i<\omega}$ over A. Then by Lemma 2.17 there is an A-indiscernible sequence $(b_i)_{i<\omega}$ which is $\mathrm{EM}_<$-based on $(a'_i)_{i<\lambda_{|T|+|A|+|a_0|}}$ over A, hence $\mathrm{EM}_<$-based on $(a_i)_{i<\omega}$ over A. $\square$

In what follows we consider $\omega^{\leq k}$ as an L_s-structure (see Definition 5.1). We will only work with trees of width ω, as we will only need those, but everything naturally works for arbitrary (infinite) widths.

Definition 5.5. We call a tree $(a_\eta)_{\eta \in \omega^{\leq k}}$ *s-indiscernible over C* if for any $\bar{\eta}, \bar{\nu} \subseteq \omega^{\leq k}$ such that $\bar{\eta} \equiv_{\mathrm{qf}} \bar{\nu}$ we have that $a_{\bar{\eta}} \equiv_C a_{\bar{\nu}}$.

Lemma 5.6. *Suppose $\bar{\eta} = (\eta_0, \dots, \eta_{n-1}) \equiv_{\mathrm{qf}} \bar{\nu} = (\nu_0, \dots, \nu_{n-1})$ are tuples of elements of $\omega^{\leq k}$ for some $k < \omega$. Then there exists a sequence I of n-tuples of elements of $\omega^{\leq k}$ such that $\bar{\eta} \frown I$ and $\bar{\nu} \frown I$ are qf-indiscernible sequences in $\omega^{\leq k}$.*

Proof. Let $l < \omega$ be such that $\bar{\eta}, \bar{\nu} \subseteq \{\varnothing\} \cup \left\{\xi \in \omega^{\leq k} \setminus \{\varnothing\} : \xi(0) < l\right\}$. For every $0 < m < \omega$ choose a tuple $\bar{\chi}^m \subseteq \{\varnothing\} \cup \left\{\xi \in \omega^{\leq k} \setminus \{\varnothing\} : ml \leq \xi(0) < (m+1)l\right\}$ such that $\bar{\chi}^m \equiv_{\mathrm{qf}} \bar{\eta} \equiv_{\mathrm{qf}} \bar{\nu}$ (for example, for every $n' < n$ put $\chi^m_{n'}(0) = \eta_{n'}(0) + ml$ and $\chi^m_{n'}(i) = \eta_{n'}(i)$ for every $0 < i \leq k$). Finally, put $I = (\bar{\chi}^m)_{0<m<\omega}$. $\square$

Corollary 5.7. *If T is thick then s-indiscernibility is type-definable, i.e., for every $k < \omega$ and a tuple of variables y there is a partial type $\pi((x_\eta)_{\eta \in \omega^{\leq k}}, y)$ over $\varnothing$ such that for all D with $|D| = |y|$, $((a_\eta)_{\eta \in \omega^{\leq k}}, D) \models \pi$ if and only if $(a_\eta)_{\eta \in \omega^{\leq k}}$ is s-indiscernible over D.*

More specifically, we can take $\pi((x_\eta)_{\eta\in\omega^{\leq k}}, y)$ to be the partial type that expresses that for any $(\eta_0, \dots, \eta_{n-1}) \equiv_{\mathrm{qf}} (\nu_0, \dots, \nu_{n-1})$ the Lascar distance of $(x_{\eta_0}, \dots, x_{\eta_{n-1}})$ and $(x_{\nu_0}, \dots, x_{\nu_{n-1}})$ over y is at most 2.

Proof. Let π be as above. Consider arbitrary $(a_\eta)_{\eta\in\omega^{\leq k}}$ and D. If $((a_\eta)_{\eta\in\omega^{\leq k}}, D) \models \pi$ then $(a_\eta)_{\eta\in\omega^{\leq k}}$ is indiscernible over D, as being at Lascar distance at most 2 over D implies equality of types over D.

Conversely, if $((a_\eta)_{\eta\in\omega^{\leq k}}, D)$ is s-indiscernible over D and

$$\bar{\eta} = (\eta_0, \dots, \eta_{n-1}) \equiv_{\mathrm{qf}} \bar{\nu} = (\nu_0, \dots, \nu_{n-1}),$$

then with $I = (\bar{\chi}^m)_{0<m<\omega}$ given by Lemma 5.6 we have that $a_{\bar{\eta}} \frown (a_{\bar{\chi}^m})_{0<m<\omega}$ and $a_{\bar{\nu}} \frown (a_{\bar{\chi}^m})_{0<m<\omega}$ are both indiscernible sequences over D, so $a_{\bar{\eta}}$ and $a_{\bar{\nu}}$ are at Lascar distance at most 2 over D. □

We now adapt the proof of [Kim et al. 2014, Theorem 4.3] to obtain the EM_s-modelling property for positive logic.

Proposition 5.8. *Suppose T is thick and consider an arbitrary set of parameters D and $k < \omega$. Then for any tree $A = (a_\eta)_{\eta\in\omega^{\leq k}}$ of compatible tuples there is an s-indiscernible over D tree $C = (c_\eta)_{\eta\in\omega^{\leq k}}$ which is EM_s-based on A over D.*

Proof. We proceed by induction on k. The case $k = 0$ is trivial. Suppose the assertion holds for some k and consider any $A = (a_\eta)_{\eta\in\omega^{\leq k+1}}$. For any $i < \omega$ consider an $\omega^{\leq k}$-indexed tree $A_i := (a_{i\frown\eta})_{\eta\in\omega^{\leq k}}$. Using the inductive hypothesis we choose inductively for each $i < \omega$ a tree $B_i = (b^i_\eta)_{\eta\in\omega^{\leq k}}$ which is s-indiscernible over $Da_\varnothing B_{<i} A_{>i}$ and EM_s-based on A_i over $Da_\varnothing B_{<i} A_{>i}$. Let $B = (b_\eta)_{\eta\in\omega^{\leq k+1}}$, where $b_\varnothing = a_\varnothing$ and $b_{i\frown\xi} = b^i_\xi$ for every $i < \omega$ and $\xi \in \omega^{\leq k}$.

Claim. *B_i is s-indiscernible over $Db_\varnothing B_{\neq i}$ for every $i < \omega$.*

Proof of claim. Fix $i < \omega$. We will show by induction on j that B_i is s-indiscernible over $Db_\varnothing B_{<i} B_{i+1} \cdots B_{j-1} A_{\geq j}$ for every $j > i$, which is enough by Corollary 5.7. For $j = i + 1$ this follows directly from the choice of B_i. Now suppose the assertion holds for some $j > i$. By Corollary 5.7 there is a type $\pi((x_\eta)_{\eta\in\omega^{\leq k}}, \bar{y})$ over $D' := Db_\varnothing B_{<i} B_{i+1} \cdots B_{j-1} A_{>j}$, where $\bar{y} = (y_\eta)_{\eta\in\omega^{\leq k}}$, expressing that $(x_\eta)_{\eta\in\omega^{\leq k}}$ is s-indiscernible over $D'\bar{y}$. Then $B_i A_j \models \pi$. Note that the type $\pi(B_i, \bar{y})$ is invariant under all permutations of $\bar{y}$, and therefore if $\varphi(y_{\eta_0}, \dots, y_{\eta_{n-1}}) \in \pi(B_i, \bar{y})$ then $\varphi(y_{\nu_0}, \dots, y_{\nu_{n-1}}) \in \mathrm{tp}(A_j/D'B_i)$ for all $\nu_0, \dots, \nu_{n-1} \in \omega^{\leq k}$. In particular, $\pi(B_i, \bar{y}) \subseteq \mathrm{EM}_s(A_j/D'B_i)$. Thus, by the choice of B_j, we have that $\pi(B_i, \bar{y}) \subseteq \mathrm{EM}_s(B_j/D'B_i)$, so in particular $B_i B_j \models \pi$. Hence B_i is indiscernible over $D'B_j = Db_\varnothing B_{<i} B_{i+1} \cdots B_j A_{\geq j+1}$, as required. □

Claim. *B is EM_s-based on A over D.*

Proof of claim. Consider any $i < \omega$ and the trees $E = (e_\eta)_{\eta\in\omega^{\leq k+1}}$ and $F = (f_\eta)_{\eta\in\omega^{\leq k+1}}$ given by

$$e_\varnothing = f_\varnothing = a_\varnothing, \quad e_{j\frown\eta} = \begin{cases} b_{j\frown\eta} & \text{for } j < i, \\ a_{j\frown\eta} & \text{for } j \geq i, \end{cases} \quad \text{and} \quad f_{j\frown\eta} = \begin{cases} b_{j\frown\eta} & \text{for } j \leq i, \\ a_{j\frown\eta} & \text{for } j > i. \end{cases}$$

We will prove that $\pi_0 := \mathrm{EM}_s(E/D) \subseteq \mathrm{EM}_s(F/D) =: \pi_1$, which clearly is sufficient to prove the claim. Let $\bar{x} = (x_\eta)_{\eta\in\omega^{\leq k+1}}$ be a tuple of variables compatible with the a_η's. We naturally view π_0 and π_1 as partial types in the variable $\bar{x}$. Consider any formula

$$\varphi(x_{\eta_0}, \dots, x_{\eta_l}, x_{\eta_{l+1}}, \dots, x_{\eta_{l'}}) \in \pi_0$$

over D with

$$\eta_0, \dots, \eta_l \in K_i := \{i \frown \xi : \xi \in \omega^{\leq k}\} \quad \text{and} \quad \eta_{l+1}, \dots, \eta_{l'} \in \omega^{\leq k+1} \setminus K_i.$$

We will be done if we show

$$\models \varphi(f_{\eta_0}, \dots, f_{\eta_{l'}}).$$

Write $\eta_t = i \frown \xi_t$ for $t = 0, 1, \dots, l$. For any $\xi'_0, \dots, \xi'_l \in \omega^{\leq k}$ with

$$\mathrm{qftp}_{L_s}(\xi'_0, \dots, \xi'_l) = \mathrm{qftp}_{L_s}(\xi_0, \dots, \xi_l),$$

we have

$$\begin{aligned} \mathrm{qftp}_{L_s}(\eta_0, \dots, \eta_{l'}) &= \mathrm{qftp}_{L_s}(i \frown \xi_0, \dots, i \frown \xi_l, \eta_{l+1}, \dots, \eta_{l'}) \\ &= \mathrm{qftp}_{L_s}(i \frown \xi'_0, \dots, i \frown \xi'_l, \eta_{l+1}, \dots, \eta_{l'}), \end{aligned}$$

so, as $\varphi \in \pi_0$, we get that $\models \varphi(e_{i\frown\xi'_0}, \dots, e_{i\frown\xi'_l}, e_{\eta_{l+1}}, \dots, e_{\eta_{l'}})$. This shows that

$$\varphi(y_{\xi_0}, \dots, y_{\xi_l}, e_{\eta_{l+1}}, \dots, e_{\eta_{l'}}) \in \mathrm{EM}_s(A_i/a_\varnothing A_{<i} B_{>i}),$$

where A_i is naturally indexed by $\omega^{\leq k}$, so, by the choice of B_i, we get that

$$\models \varphi(b^i_{\xi_0}, \dots, b^i_{\xi_l}, e_{\eta_{l+1}}, \dots, e_{\eta_{l'}}).$$

Because

$$(b^i_{\xi_0}, \dots, b^i_{\xi_l}, e_{\eta_{l+1}}, \dots, e_{\eta_{l'}}) = (f_{i\frown\xi_0}, \dots, f_{i\frown\xi_l}, f_{\eta_{l+1}}, \dots, f_{\eta_{l'}}) = (f_{\eta_0}, \dots, f_{\eta_{l'}}),$$

this means that $\models \varphi(f_{\eta_0}, \dots, f_{\eta_{l'}})$, as required. □

By Corollary 5.4 we find a sequence $(C_i)_{i<\omega} = ((c^i_\eta)_{\eta\in\omega^{\leq k}})_{i<\omega}$ which is $\mathrm{EM}_<$-based on $(B_i)_{i<\omega}$ over $Db_\varnothing$ and indiscernible over $Db_\varnothing$. Let $C = (c_\eta)_{\eta\in\omega^{\leq k+1}}$ be given by $c_\varnothing = b_\varnothing$ and $c_{i\frown\xi} = c^i_\xi$ for any $\xi \in \omega^{\leq k}$ and $i < \omega$. By the first claim on page 76 and Corollary 5.7 we get that C_i is s-indiscernible over $C_{\neq i} D c_\varnothing$ for every $i < \omega$, which, together with $Dc_\varnothing$-indiscernibility of $(C_i)_{i<\omega}$, easily gives that C is s-indiscernible over D (as in [Kim et al. 2014]). It is left to prove:

Claim. *C is EM_s-based on B (and hence on A) over D.*

Proof of claim. Consider any formula $\varphi(x_{i_1\frown\xi_1}, \dots, x_{i_l\frown\xi_l}, x_\varnothing) \in \mathrm{EM}_s(B/D)$ with $i_1, \dots, i_l \in \omega$ and $\xi_1, \dots, \xi_l \in \omega^{\leq k}$. Then for every $j_1, \dots, j_l \in \omega$ with $\mathrm{qftp}_{\{<\}}(j_1, \dots, j_l) = \mathrm{qftp}_{\{<\}}(i_1, \dots, i_l)$, we have that

$$\mathrm{qftp}_{L_s}(j_1 \frown \xi_1, \dots, j_l \frown \xi_l, \varnothing) = \mathrm{qftp}_{L_s}(i_1 \frown \xi_1, \dots, i_l \frown \xi_l, \varnothing),$$

so $\models \varphi(b_{j_1\frown\xi_1}, \dots, b_{j_l\frown\xi_l}, b_\varnothing)$. This means that

$$\varphi(x_{i_1\frown\xi_1}, \dots, x_{i_l\frown\xi_l}, b_\varnothing) \in \mathrm{EM}_<((B_i)_{i<\omega}/b_\varnothing D),$$

and therefore, by the choice of C, we have that $\models \varphi(c_{i_1\frown\xi_1}, \dots, c_{i_l\frown\xi_l}, c_\varnothing)$, and thus $\varphi(x_{i_1\frown\xi_1}, \dots, x_{i_l\frown\xi_l}, x_\varnothing) \in \mathrm{EM}_s(C/D)$, as required. □

Definition 5.9. Let I be a linearly ordered set. For a global M-Ls-invariant type q, we will call a sequence $(a_i)_{i\in I}$ a *parallel-Morley sequence in q over M* if there is some $(b_i)_{i\in I} \models q^{\otimes I}|_M$ such that the pair (a_i, b_i) starts an $Ma_{>i}b_{>i}$-indiscernible sequence for every $i \in I$. We will say that $(a_i)_{i\in I}$ is a parallel-Morley sequence in $\mathrm{tp}(a/M)$ if it is a parallel-Morley sequence in some global M-Ls-invariant type $q \supseteq \mathrm{tp}(a/M)$.

In the semi-Hausdorff case we can replace the condition "(a_i, b_i) starts an $Ma_{>i}b_{>i}$-indiscernible sequence" by "$a_i \equiv_{Ma_{>i}b_{>i}} b_i$". The reason for which we need the stronger condition in thick theories is that equality of types is not necessarily type-definable there, so some of the compactness arguments below would not work with the weaker condition.

Note that a parallel-Morley sequence is *not* required to be indiscernible. The reason for the name "parallel-Morley sequence" is because such a sequence is parallel to a Morley sequence, in the sense of the parallel sequences lemma (Lemma 2.28). We make this precise in Corollary 5.11, for which we first slightly reformulate the parallel sequences lemma.

Lemma 5.10. *Let T be thick and suppose $\varphi(x, y)$ is a formula and $(c_{i,0}, c_{i,1})_{i\in I}$ is an infinite sequence of pairs with $(c_{i,1})_{i\in I}$ indiscernible, such that*

(i) *for every $i \in I$, the pair $(c_{i,0}, c_{i,1})$ starts a $c_{>i,0}c_{>i,1}$-indiscernible sequence*;

(ii) *$\{\varphi(x; c_{i,0}) : i \in I\}$ is consistent*;

(iii) *$\{\varphi(x; c_{i,1}) : i \in I\}$ is inconsistent.*

Then T has SOP_1.

Proof. We may assume the tuples $c_{i,0}$ and $c_{i,1}$ to be finite. As $(c_{i,1})_{i\in I}$ is indiscernible and $\{\varphi(x, c_{i,1}) : i \in I\}$ is inconsistent, there is some $\psi(y_1, \dots, y_k)$ that implies $\neg\exists x(\varphi(x, y_1) \wedge \dots \wedge \varphi(x, y_k))$ such that for any $i_1 < \dots < i_k \in I$ we have $\models \psi(c_{i_1,1}, \dots, c_{i_k,1})$. Call this ψ-inconsistent. By compactness there is a sequence of pairs $(\bar{c'}_i)_{i<\lambda_T} = (c'_{i,0}, c'_{i,1})_{i<\lambda_T}$ such that $(c'_{i,0}, c'_{i,1})$ starts a $\bar{c'}_{>i}$-indiscernible

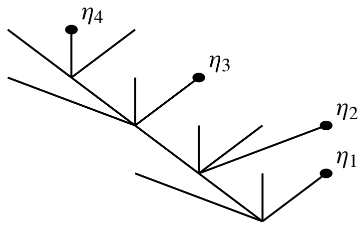

Figure 1. An example of η_i's from Definition 5.12.

sequence for every $i < \lambda_T$, $\{\varphi(x, c'_{i,0}) : i < \lambda_T\}$ is consistent and $\{\varphi(x, c'_{i,1}) : i < \lambda_T\}$ is ψ-inconsistent. Then an indiscernible sequence based on $(\bar{c}'_i)_{i<\lambda_T}$ will satisfy the assumptions of Lemma 2.28, so T has SOP_1. $\square$

By Kim's lemma (Proposition 4.4) and Lemma 5.10 we easily get the following.

Corollary 5.11. *Suppose T is thick* NSOP_1 *with an e.c. model M, $\Sigma(x, b)$ is a partial type, I is an infinite linearly ordered set, and $(b_i)_{i\in I}$ a parallel-Morley sequence in* $\mathrm{tp}(b/M)$. *If $\bigcup\{\Sigma(x, b_i) : i \in I\}$ is consistent then $\Sigma(x, b)$ does not Kim-divide over M. If $(b_i)_{i\in I}$ is indiscernible over M, then the converse also holds.*

Definition 5.12. Let M be an e.c. model and q a global M-Ls-invariant type.

(i) We say that a tree $(c_\eta)_{\eta\in\omega^{\le k}}$ is *q-spread-out over M* if for any $\eta_1 \in \omega^1$, $\eta_2 \in \omega^2, \ldots, \eta_k \in \omega^k$ such that

$$\eta_1 >_{\mathrm{lex}} \eta_2 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} \eta_k \quad \text{and} \quad (\forall l < l' \le k)(\eta_{l'} \wedge \eta_l \in \omega^{l-1}),$$

we have that $(c_{\eta_k}, \ldots, c_{\eta_1})$ is a Morley sequence in q over M.

(ii) We say that $(c_\eta)_{\eta\in\omega^{\le k}}$ is *weakly q-spread-out over M* if $(c_{\eta_k}, \ldots, c_{\eta_1}) \models q^{\otimes k}|_M$ for η_i's as in (i).

Clearly q-spread-outness implies weak q-spread-outness. We will freely use the above definition for trees of parameters indexed by trees naturally isomorphic to trees of the form $\omega'^{\le k'}$, e.g., subtrees of $\omega^{\le k}$ consisting of all nodes extending a fixed node.

The point of the conditions on the η_i's in Definition 5.12 is that this is quantifier-free definable by an L_s-formula. This is useful for preservation when EM_s-basing trees on one another, as we do in the following lemma.

Lemma 5.13. *Let k be a natural number, M an e.c. model and q a global M-Ls-invariant type.*

(i) *If $((c_{i\frown\eta})_{\eta\in\omega^{\le k-1}})_{i<\omega}$ is a Morley sequence in a global M-Ls-invariant type $r(x, z) \supseteq q(x)$ over M, where x corresponds to the elements c_i and where $(c_{0\frown\eta})_{\eta\in\omega^{\le k-1}}$ is q-spread-out over M then also $(c_\eta)_{\eta\in\omega^{\le k}}$ is q-spread-out over M for any choice of root $c_\varnothing$.*

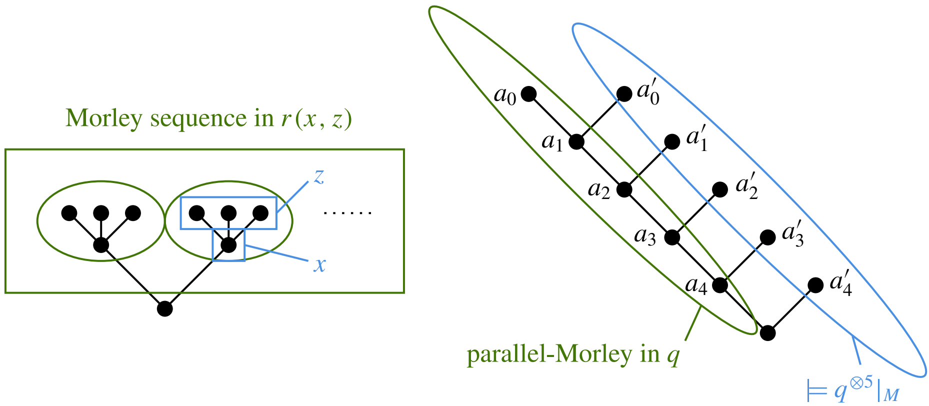

Figure 2. Lemma 5.13(i), left, and Lemma 5.13(iii), right.

(ii) *If* $(c_\eta)_{\eta\in\omega^{\leq k}}$ *is weakly* q*-spread-out over* M *and*

$$(c'_\eta)_{\eta\in\omega^{\leq k}} \models \mathrm{EM}_s((c_\eta)_{\eta\in\omega^{\leq k}}/M),$$

then also $(c'_\eta)_{\eta\in\omega^{\leq k}}$ *is weakly* q*-spread-out over* M.

(iii) *If* $(c_\eta)_{\eta\in\omega^{\leq k}}$ *is weakly* q*-spread-out over* M *and* s*-indiscernible over* M, *then for* $a_i = c_{0^{k-i}}$ *we have that* $(a_i)_{i<k}$ *is a parallel-Morley sequence in* q *over* M.

Proof. (i) Let $\eta_k \in \omega^k, \ldots, \eta_1 \in \omega^1$ be such that

$$\eta_1 >_{\text{lex}} \cdots >_{\text{lex}} \eta_k \quad \text{and} \quad (\forall l < l' \leq k)(\eta_{l'} \wedge \eta_l \in \omega^{l-1}).$$

We will prove that $(c_{\eta_k}, \ldots, c_{\eta_1})$ is a Morley sequence in q. For each $\ell \geq 2$, let $\beta_\ell \in \omega^1$ be such that $\eta_\ell \trianglerighteq \beta_\ell$. For every $\ell > 2$, we have by assumption that $\eta_2 \wedge \eta_\ell = \eta_2|_1 = \beta_2$, and hence $\beta_\ell = \beta_2 =: \beta$ (and $\eta_1 >_{\text{lex}} \beta$ as $\eta_1 >_{\text{lex}} \eta_2$). In particular, $(c_{\eta_k}, \ldots, c_{\eta_2})$ is contained in $(c_{\beta\frown\eta})_{\eta\in\omega^{\leq k-1}}$, which has the same Lascar strong type over M as $(c_{0\frown\eta})_{\eta\in\omega^{\leq k-1}}$. So, as $(c_{0\frown\eta})_{\eta\in\omega^{\leq k-1}}$ is q-spread-out by assumption, $(c_{\eta_k}, \ldots, c_{\eta_2})$ is a Morley sequence in q. As $((c_{i\frown\eta})_{\eta\in\omega^{\leq k-1}})_{i<\omega}$ is a Morley sequence in r, we have that $(c_{\eta_1\frown\eta})_{\eta\in\omega^{\leq k-1}}$, which contains c_{η_1}, has the same Lascar strong type over $M(c_{\beta\frown\eta})_{\eta\in\omega^{\leq k-1}}$, which contains $Mc_{\eta_k}\cdots c_{\eta_2}$, as some realisation of r. Since $q(x) = r|_x$ we see that c_{η_1} has the same Lascar strong type over $Mc_{\eta_k}, \ldots, c_{\eta_2}$ as some realisation of q. So we conclude that $(c_{\eta_k}, \ldots, c_{\eta_1})$ is indeed a Morley sequence in q.

(ii) This holds because the condition on $(\eta_1, \ldots, \eta_k)$ in the definition of weak q-spread-outness is expressible by a quantifier-free L_s-formula.

(iii) Put $a'_i := c_{0^{k-i-1}\frown 1}$ for $i < k$. Then $(a'_i)_{i<k} \models q^{\otimes k}|_M$ by weak q-spread-outness, and (a_i, a'_i) starts an $M_{a_{>i}a'_{>i}}$-indiscernible sequence for each $i < k$ by s-indiscernibility. $\square$

6. Symmetry

Lemma 6.1 (chain condition). *Let T be a thick* NSOP_1 *theory, and let M be an e.c. model. Let $(b_i)_{i<\kappa}$ be a Morley sequence in some global M-Ls-invariant $q(x)$. If $(b_i)_{i<\kappa}$ is Ma-indiscernible then $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M (b_i)_{i<\kappa}$.*

Proof. We will prove that $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b_{i_1} \cdots b_{i_k}$ for all $i_1 < \cdots < i_k < \kappa$. This is indeed enough by finite character. By Ma-indiscernibility of $(b_i)_{i<\kappa}$ we may assume $\{i_1, \ldots, i_k\} = \{0, \ldots, k-1\}$.

We have $(b_i)_{i<\omega} \equiv^{\mathrm{Ls}}_M (\beta_i)_{i<\omega}$ for some $(\beta_i)_{i<\omega} \models q^{\otimes\omega}$. Define the tuple $\gamma_i = (\beta_{ik}, \beta_{ik+1}, \ldots, \beta_{ik+k-1})$ for all $i < \omega$. Then $(\gamma_i)_{i<\omega} \models (q^{\otimes k})^{\otimes\omega}$ by associativity of tensoring (Lemma 3.16). We let $c_i = (b_{ik}, b_{ik+1}, \ldots, b_{ik+k-1})$ for all $i < \omega$. Then $(c_i)_{i<\omega} \equiv^{\mathrm{Ls}}_M (\gamma_i)_{i<\omega}$. So $(c_i)_{i<\omega}$ is a Morley sequence in $q^{\otimes k}$ over M and $(c_i)_{i<\omega}$ is Ma-indiscernible. So $\mathrm{tp}(a/Mc_0) = \mathrm{tp}(a/Mb_0 \cdots b_{k-1})$ does not $q^{\otimes k}$-divide, and thus $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b_0 \cdots b_{k-1}$, as required. $\square$

Definition 6.2. Suppose M is an e.c. model, q a global type extending $\mathrm{Lstp}(a/M)$ and λ a cardinal. We will say that the extension $q \supseteq \mathrm{Lstp}(a/M)$ satisfies $(*)_\lambda$ if for every c with $|c| \leq \lambda$ there is a global M-Ls-invariant type $r(x, y) \supseteq \mathrm{Lstp}(ac/M)$ extending $q(x)$ (in particular, q is M-Ls-invariant).

Lemma 6.3. *For any e.c. model M, tuple a and cardinal λ there is $q \supseteq \mathrm{Lstp}(a/M)$ satisfying $(*)_\lambda$.*

Proof. Let M, a and λ be as in the statement. Choose a small tuple d such that for any c with $|c| \leq \lambda$ there is some $d' \subseteq d$ with $\mathrm{Lstp}(ad'/M) = \mathrm{Lstp}(ac/M)$ (this is possible as the number of Lascar types of tuples of fixed length over M is bounded by Lemma 2.20). Now take a global M-Ls-invariant extension $r(x, y)$ of $\mathrm{Lstp}(ad/M)$, where x corresponds to a. Then $q := r|_x$ is an extension of $\mathrm{Lstp}(a/M)$ satisfying $(*)_\lambda$. $\square$

Remark 6.4. If $q \supseteq \mathrm{Lstp}(a/M)$ is finitely satisfiable in M then it satisfies $(*)_\lambda$ for any cardinal λ [Mennuni 2020, Lemma 3.4]. However, finitely satisfiable extensions may not exist in thick theories.

Theorem 6.5 (symmetry). *In a thick* NSOP_1 *theory, $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$ implies $b \mathop{\smile\!\!\!\!\!\!\mid}^K_M a$.*

Proof. We may assume that b enumerates a λ_T-saturated model containing M. If this is not the case, let $N \supseteq Mb$ be a λ_T-saturated model. By extension, Corollary 4.14, we find $N' \equiv_{Mb} N$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M N'$. Now we replace b by N' and we continue the proof.

Set $\lambda = |ab|$. By Lemma 6.3 we can choose a global extension $q \supseteq \mathrm{Lstp}(a/M)$ satisfying $(*)_\lambda$. Let $p(y, a) = \mathrm{tp}(b/Ma)$. We will show that there is a parallel-Morley sequence $(a_i)_{i<\omega}$ in q over M such that $\bigcup_{i<\omega} p(y, a_i)$ is consistent, which is enough by Corollary 5.11. All the properties we wish $(a_i)_{i<\omega}$ to have are type-definable. It is thus enough to find such a sequence of length k for every $k<\omega$.

So fix any $k<\omega$. By backward induction on $k'=k+1,k,\dots,1$ we will define trees $(c_\eta)_{\eta\in S_{k'}}$, where $S_{k'}=\{\xi\in\omega^{\leq k+1}:0^{k'-1}\trianglelefteq\xi\}$. We will write $S^*_{k'}$ for $S_{k'}$ without the root, so $S^*_{k'}=S_{k'}-\{0^{k'-1}\}$. For each k' the tree $(c_\eta)_{\eta\in S_{k'}}$ will satisfy the following conditions:

$(\mathrm{A1})_{k'}$ $c_\eta c_\nu\equiv^{\mathrm{Ls}}_M ab$ for all $\nu\triangleright\eta\in S_{k'}$ with $\nu\in\omega^{k+1}$ and $\eta\in\omega^{\leq k}$.

$(\mathrm{A2})_{k'}$ $(c_\eta)_{\eta\in S_{k'}\cap\omega^{\leq k}}$ is q-spread-out over M.

$(\mathrm{A3})_{k'}$ We have $c_{0^{k'-1}}\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M (c_\eta)_{\eta\in S^*_{k'}}$ (the root is independent from the rest).

For $k'=k+1$ we let t be a global M-Ls-invariant extension of $\mathrm{Lstp}(b/M)$. Since $a\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M b$ we have that $\mathrm{tp}(a/Mb)$ does not t-divide. By Corollary 4.10 and our assumption on b, this means that $\mathrm{Lstp}(a/Mb)$ does not t-Ls-divide. So we find an Ma-indiscernible Morley sequence $(c_{0^k\frown\alpha})_{\alpha<\omega}$ in t with $c_{0^{k+1}}=b$. By Lemma 6.1, we have that $a\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M (c_{0^k\frown\alpha})_{\alpha<\omega}$. So we pick $c_{0^k}=a$ and directly satisfy $(\mathrm{A3})_{k'}$. Condition $(\mathrm{A2})_{k'}$ is vacuous and $(\mathrm{A1})_{k'}$ follows directly from Ma-indiscernibility of $(c_{0^k\frown\alpha})_{\alpha<\omega}$ and the fact that $c_{0^{k+1}}=b$.

For the inductive step, suppose we have constructed $(c_\eta)_{\eta\in S_{k'}}$. By $(\mathrm{A1})_{k'}$ there is a tuple d such that $c_{0^{k'-1}}(c_\eta)_{\eta\in S^*_{k'}}\equiv^{\mathrm{Ls}}_M ad$. So, by $(*)_\lambda$, there is a global M-Ls-invariant type $r(x,z)\supseteq q(x)$ extending $\mathrm{Lstp}(c_{0^{k'-1}}(c_\eta)_{\eta\in S^*_{k'}}/M)$. By $(\mathrm{A3})_{k'}$ we have that $c_{0^{k'-1}}\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M(c_\eta)_{\eta\in S^*_{k'}}$. So since $b\subseteq(c_\eta)_{\eta\in S^*_{k'}}$ and using our assumption on b we have by Corollary 4.10 that $\mathrm{Lstp}(c_{0^{k'-1}}/M(c_\eta)_{\eta\in S^*_{k'}})$ does not $r|_z$-Ls-divide. By extension for Ls-dividing, Proposition 4.13, we find c such that $c(c_\eta)_{\eta\in S^*_{k'}}\equiv^{\mathrm{Ls}}_M c_{0^{k'-1}}(c_\eta)_{\eta\in S^*_{k'}}$ and $\mathrm{Lstp}(c/M(c_\eta)_{\eta\in S_{k'}})$ does not r-Ls-divide. So there is an Mc-indiscernible Morley sequence $((d_{\eta,i})_{\eta\in S_{k'}})_{i<\omega}$ in r such that $(d_{\eta,0})_{\eta\in S_{k'}}=(c_\eta)_{\eta\in S_{k'}}$. We set $c_{0^{k'-2}}=c$ and $c_{0^{k'-2}\frown i\frown\zeta}=d_{0^{k'-1}\frown\zeta,i}$. Again, using Lemma 6.1 we directly get $(\mathrm{A3})_{k'-1}$.

Now $(\mathrm{A2})_{k'-1}$ follows from Lemma 5.13(i). We verify $(\mathrm{A1})_{k'-1}$. Everything above the root consists of copies (via a Lascar strong automorphism over M) of $(c_\eta)_{\eta\in S_{k'}}$, so we only need to check that $c_{0^{k'-2}}c_\nu\equiv^{\mathrm{Ls}}_M ab$ for all $\nu\in S_{k'-1}\cap\omega^{k+1}$. By indiscernibility we may assume $\nu\in S_{k'}\cap\omega^{k+1}$. Then $(\mathrm{A1})_{k'-1}$ follows from $(\mathrm{A1})_{k'}$ and the fact that $c_{0^{k'-2}}(c_\eta)_{\eta\in S^*_{k'}}\equiv^{\mathrm{Ls}}_M c_{0^{k'-1}}(c_\eta)_{\eta\in S^*_{k'}}$.

Thus the inductive step, and hence the construction of the tree $(c_\eta)_{\eta\in\omega^{k+1}}=(c_\eta)_{\eta\in S_1}$, is completed.

Consider the following condition:

$(\mathrm{A1}')_1$ $c_\eta c_\nu\equiv_M ab$ for all $\nu\triangleright\eta$ with $\nu\in\omega^{k+1}$ and $\eta\in\omega^{\leq k}$.

This condition is clearly implied by $(\mathrm{A1})_1$ as it is seen by the EM_s-type of $(c_\eta)_{\eta\in\omega^{\leq k+1}}$ over M. Let $(c'_\eta)_{\eta\in\omega^{\leq k+1}}$ be an s-indiscernible tree that is EM_s-based on $(c_\eta)_{\eta\in\omega^{\leq k+1}}$ over M, and we get that $(c'_\eta)_{\eta\in\omega^{\leq k+1}}$ satisfies $(\mathrm{A1}')_1$, and $(c'_\eta)_{\eta\in\omega^{\leq k}}$ is weakly q-spread-out over M by Lemma 5.13(ii).

Put $a_i = c'_{0^{k+1-i}}$. Then $(a_1, \dots, a_k)$ is a parallel-Morley sequence in q over M by Lemma 5.13(iii), and by $(\mathrm{A}1')_1$ we have that $\bigcup_{1 \le i \le k} p(y, a_i)$ is consistent because it is realised by $c'_{0^{k+1}}$. This completes the proof. $\square$

Lemma 6.6. *Let T be a thick theory. Suppose that $\varphi(x, y)$ has SOP_1, witnessed by $\psi(y_1, y_2)$. Then there is an e.c. model M and b_1, b_2, c_1, c_2 such that $c_1 \mathop{\smile\!\!\!\!\!\!\perp}^u_M c_2$, $c_1 \mathop{\smile\!\!\!\!\!\!\perp}^u_M b_1$, $c_2 \mathop{\smile\!\!\!\!\!\!\perp}^u_M b_2$ and $b_1 c_1 \equiv^{\mathrm{Ls}}_M b_2 c_2$ and $\models \varphi(b_1, c_1) \wedge \varphi(b_2, c_2) \wedge \psi(c_1, c_2)$.*

Proof. The proof is mostly the same as that of [Haykazyan and Kirby 2021, Proposition A.7] but we have to adjust a few things throughout to get equality of Lascar strong types rather than just equality of types. As in that proof, we will use a Skolemisation technique for positive logic [Haykazyan and Kirby 2021, Lemma A.6]. In such a Skolemised theory the positively definable closure of any set is an e.c. model and the reduct of an e.c. model (to the original language) is an e.c. model (of the original theory). It is not directly clear whether this Skolemisation construction preserves thickness, but that is not a problem. Ultimately we are interested in Lascar strong types in our original theory. So even though we technically work in a Skolemised theory the (type-definable) predicate $\mathrm{d}(x, y) \le 1$ should be taken as in our original theory.

Let κ be any cardinal. By compactness we find parameters $(a_\eta)_{\eta \in 2^{<\kappa}}$ such that

(i) for every $\sigma \in 2^\kappa$ the set $\{\varphi(x, a_{\sigma|i}) : i < \kappa\}$ is consistent,

(ii) for every $\eta, \nu \in 2^{<\kappa}$ such that $\eta^\frown 0 \preceq \nu$, we have $\models \psi(a_{\eta^\frown 1}, a_\nu)$.

For a big enough cardinal λ, we construct by induction a sequence $(\eta_i, \nu_i)_{i<\lambda}$ with $\eta_i, \nu_i \in 2^{<\kappa}$ such that

(1) $\eta_i \trianglelefteq \eta_j$ and $\eta_i \trianglelefteq \nu_j$ for all $i < j < \lambda$,

(2) $\eta_i \trianglerighteq (\eta_i \wedge \nu_i) \frown 0$, $\nu_i = (\eta_i \wedge \nu_i) \frown 1$, and (a_{η_i}, a_{ν_i}) starts an $a_{\eta_{<i}} a_{\nu_{<i}}$-indiscernible sequence for every $i < \lambda$.

Assume $(\eta_j, \nu_j)_{j<i}$ has been constructed and set $\eta = \bigcup_{j<i} \eta_j$. Assuming we chose κ large enough, then, by applying Lemma 2.17 to $(a_{\eta^\frown 0^\alpha {}^\frown 1})_{\alpha > 0}$, it follows that there are $0 < \alpha < \beta < \kappa$ such that $(a_{\eta^\frown 0^\alpha {}^\frown 1}, a_{\eta^\frown 0^\beta {}^\frown 1})$ starts an $\{\eta_j, \nu_j : j < i\}$-indiscernible sequence. We set $\nu_i = \eta^\frown 0^{\alpha\frown} 1$ and $\eta_i = \eta^\frown 0^{\beta\frown} 1$.

By (i) and (1), there is b_2 realising $\{\varphi(x, a_{\eta_i}) : i < \lambda\}$. Now let $(e_i, d_i)_{i<\omega+2}$ be indiscernible over b_2 based on $(a_{\eta_i}, a_{\nu_i})_{i<\lambda}$.

Let M be the positively definable closure of $\{e_i, d_i : i < \omega\}$. As discussed, we may assume M to be an e.c. model. Set $c_1 = d_\omega$ and $c_2 = e_{\omega+1}$. Then $c_1 \mathop{\smile\!\!\!\!\!\!\perp}^u_{\{e_i, d_i : i<\omega\}} c_2$ and $c_2 \mathop{\smile\!\!\!\!\!\!\perp}^u_{\{e_i, d_i : i<\omega\}} b_2$ by indiscernibility. So $c_1 \mathop{\smile\!\!\!\!\!\!\perp}^u_M c_2$, $c_2 \mathop{\smile\!\!\!\!\!\!\perp}^u_M b_2$ and $\models \varphi(b_2, c_2)$. By construction $c_1 c_2 = d_\omega e_{\omega+1} \equiv a_{\nu_{i_0}} a_{\eta_{i_1}}$ for some $i_0 < i_1 < \lambda$ and thus $\models \psi(c_1, c_2)$ by (ii), (1), and (2).

To find b_1 we first claim that $\mathrm{d}_M(e_\omega, d_\omega) \le 1$. By compactness it suffices to prove that $\mathrm{d}_A(e_\omega, d_\omega) \le 1$ for all finite $A \subseteq M$. By how we constructed M it then suffices to prove that (e_ω, d_ω) starts an indiscernible sequence over $\{e_i, d_i : i < n\}$ for all $n < \omega$. To prove this last statement we let $i_0 < \cdots < i_{n+1} < \lambda$ be such that

$$e_0 d_0 \cdots e_n d_n e_\omega d_\omega \equiv a_{\eta_{i_0}} a_{\nu_{i_0}} \cdots a_{\eta_{i_n}} a_{\nu_{i_n}} a_{\eta_{i_{n+1}}} a_{\nu_{i_{n+1}}}.$$

By how we constructed $(\eta_i, \nu_i)_{i<\lambda}$ we have $(a_{\eta_{i_{n+1}}}, a_{\nu_{i_{n+1}}})$ starts an indiscernible sequence over $\{a_{\eta_{i_0}} a_{\nu_{i_0}} \cdots a_{\eta_{i_n}} a_{\nu_{i_n}}\}$. So the claim follows after applying the automorphism.

Now we leave the Skolemised theory and work in the original theory, in which $\mathrm{d}(x, y) \leq 1$ corresponds to actually having Lascar distance one. We have that $c_2 = e_{\omega+1} \equiv^{\mathrm{Ls}}_M e_\omega \equiv^{\mathrm{Ls}}_M d_\omega = c_1$, so there is $f \in \mathrm{Aut}_f(\mathfrak{M}/M)$ such that $f(c_2) = c_1$. Let $b_1 = f(b_2)$. Then $c_2 b_2 \equiv^{\mathrm{Ls}}_M c_1 b_1$, and thus also $\models \varphi(b_1, c_1)$ and $c_1 \mathop{\smile\!\!\!\!\!\!|}\,^u_M b_1$, as required. □

Theorem 6.7. *Let T be a thick theory. The following are equivalent*:

(i) *T is* NSOP_1.

(ii) *Symmetry: $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M b$ implies $b \mathop{\smile\!\!\!\!\!\!|}\,^K_M a$.*

(iii) *Weak symmetry: $a \mathop{\smile\!\!\!\!\!\!|}\,^{i\mathrm{Ls}}_M b$ implies $b \mathop{\smile\!\!\!\!\!\!|}\,^K_M a$.*

Proof. Theorem 6.5 is precisely (i) $\Longrightarrow$ (ii). For (ii) $\Longrightarrow$ (iii) we just note that $a \mathop{\smile\!\!\!\!\!\!|}\,^{i\mathrm{Ls}}_M b$ implies $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M b$. Finally, for (iii) $\Longrightarrow$ (i) we proceed as in [Kaplan and Ramsey 2020, Proposition 3.22] replacing their reference to [Chernikov and Ramsey 2016] by Lemma 6.6 and being careful about using global Ls-invariant types instead of just global invariant types.

We prove the contrapositive, so assume that T has SOP_1. Then, by Lemma 6.6, there is an e.c. model M and b_1, b_2, c_1, c_2 such that $c_1 \mathop{\smile\!\!\!\!\!\!|}\,^u_M c_2$, $c_1 \mathop{\smile\!\!\!\!\!\!|}\,^u_M b_1$, $c_2 \mathop{\smile\!\!\!\!\!\!|}\,^u_M b_2$ and $b_1 c_1 \equiv^{\mathrm{Ls}}_M b_2 c_2$. Furthermore, for $p(x, c_1) = \mathrm{tp}(b_1 c_1 / M)$, we have that $p(x, c_1) \cup p(x, c_2)$ is inconsistent. In particular, we have that $\mathrm{Lstp}(c_1/Mc_2)$ extends to a global M-Ls-invariant q. Then as $c_1 \equiv^{\mathrm{Ls}}_M c_2$ there is a Morley sequence $(d_i)_{i<\omega}$ in q with $d_0 d_1 = c_2 c_1$. We thus have that $\bigcup\{p(x, d_i) : i < \omega\}$ is inconsistent. So $b_2 \not\mathop{\smile\!\!\!\!\!\!|}\,^K_M c_2$. Since also $c_2 \mathop{\smile\!\!\!\!\!\!|}\,^u_M b_2$ and therefore $c_2 \mathop{\smile\!\!\!\!\!\!|}\,^{i\mathrm{Ls}}_M b_2$, we see that weak symmetry fails and this concludes our proof. □

7. Independence theorem

We recall the following facts. The first is the same as [Kaplan and Ramsey 2020, Lemma 7.4] and the second is the same as the claim in [Dobrowolski et al. 2022, Lemma 5.3]. Their proofs work in our setting as well.

Fact 7.1. *The following hold in any thick* NSOP_1 *theory.*

(i) *If $a \mathop{\smile\!\!\!\!\!\!|}\,^d_M bc$ and $b \mathop{\smile\!\!\!\!\!\!|}\,^K_M c$ then $ab \mathop{\smile\!\!\!\!\!\!|}\,^K_M c$.*

(ii) *If $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M b$ and $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M c$ then there is c' with $ac' \equiv_M ac$ such that $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M bc'$.*

For the following lemma we borrow a trick from [Dobrowolski et al. 2022, Lemma 5.4].

Lemma 7.2. *Let T be thick* NSOP_1, *and let $a \equiv^{\mathrm{Ls}}_M a'$, $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$ and $a' \mathop{\smile\!\!\!\!\!\!|}^K_M c$. Then there is c' such that $ac' \equiv^{\mathrm{Ls}}_M a'c$ and $a \mathop{\smile\!\!\!\!\!\!|}^K_M bc'$.*

Proof. Let c^* be such that $ac^* \equiv^{\mathrm{Ls}}_M a'c$, so $a \mathop{\smile\!\!\!\!\!\!|}^K_M c^*$. Let $N' \supseteq M$ be λ_T-saturated, and let q be a global M-Ls-invariant extension of $\mathrm{Lstp}(N'/M)$. Let N realise $q|_{Mabc^*}$, so we have $N \mathop{\smile\!\!\!\!\!\!|}^{i\mathrm{Ls}}_M abc^*$. By Fact 7.1(i), we then have $Na \mathop{\smile\!\!\!\!\!\!|}^K_M b$ and $Na \mathop{\smile\!\!\!\!\!\!|}^K_M c^*$. So by fact Fact 7.1(ii), we find c' with $Nac' \equiv_M Nac^*$ and $Na \mathop{\smile\!\!\!\!\!\!|}^K_M bc'$. We thus have $ac' \equiv^{\mathrm{Ls}}_M ac^* \equiv^{\mathrm{Ls}}_M a'c$, as required. □

Definition 7.3. We write $b \mathop{\smile\!\!\!\!\!\!|}^*_M c$ to mean that $\mathrm{Lstp}(b/Mc)$ extends to a global M-Ls-invariant type $\mathrm{tp}(N/\mathfrak{M})$ for some $\beth_\omega(\lambda_T + |Mbc|)$-saturated model $N \supseteq M$. Extending $\mathrm{Lstp}(b/Mc)$ here means that there is some $\beta \in N$ with $\beta \equiv^{\mathrm{Ls}}_{Mc} b$.

The point of the enormous cardinal $\beth_\omega(\lambda_T + |Mbc|)$ is that we will want to find a λ_T-saturated model M' containing M and a copy of b in N, and then again some λ_T-saturated $M'' \supseteq M'$ inside N. By Fact 2.12 we can choose these λ_T-saturated models small enough so that this process can be repeated any finite number of times.

We easily see that $\mathop{\smile\!\!\!\!\!\!|}^*$ is invariant under automorphisms and, assuming thickness, that $b \mathop{\smile\!\!\!\!\!\!|}^*_M M$ for all M.

Lemma 7.4. *We have that $\mathop{\smile\!\!\!\!\!\!|}^*$ satisfies the following extension properties.*

(i) *Left extension: If $b \mathop{\smile\!\!\!\!\!\!|}^*_M c$ and $|d| < \beth_\omega(\lambda_T + |Mbc|)$, then there is $d' \equiv^{\mathrm{Ls}}_{Mb} d$ such that $bd' \mathop{\smile\!\!\!\!\!\!|}^*_M c$.*

(ii) *Right extension: If $b \mathop{\smile\!\!\!\!\!\!|}^*_M c$ and $|d| < \beth_\omega(\lambda_T + |Mbc|)$, then there is $d' \equiv^{\mathrm{Ls}}_{Mc} d$ such that $b \mathop{\smile\!\!\!\!\!\!|}^*_M cd'$.*

Proof. In both cases we assume $b \mathop{\smile\!\!\!\!\!\!|}^*_M c$. So let $q = \mathrm{tp}(N/\mathfrak{M})$ be a global M-Ls-invariant extension of $\mathrm{Lstp}(b/Mc)$ for some $\beth_\omega(\lambda_T + |Mbc|)$-saturated $N \supseteq M$.

We first prove left extension. Let $N' \equiv^{\mathrm{Ls}}_{Mc} N$ be in $\mathfrak{M}$. By moving things by a Lascar strong automorphism over Mc we may assume $b \in N'$. By Fact 2.12 there is $Mb \subseteq M' \subseteq N'$ where M' is λ_T-saturated and of cardinality $\leq 2^{\lambda_T + |Mb|}$. Let d' realise $\mathrm{tp}(d/M')$ in N'. Therefore, $d' \equiv^{\mathrm{Ls}}_{Mb} d$ while q also extends $\mathrm{Lstp}(bd'/Mc)$, so indeed $bd' \mathop{\smile\!\!\!\!\!\!|}^*_M c$.

Now we prove right extension. Let $\beta \in N$ be such that $\beta \equiv^{\mathrm{Ls}}_{Mc} b$. Pick $b' \in \mathfrak{M}$ such that $b' \equiv^{\mathrm{Ls}}_{Mcd} \beta$. Then clearly $b' \mathop{\smile\!\!\!\!\!\!|}^*_M cd$. We finish the proof by picking d' such that $bd' \equiv^{\mathrm{Ls}}_{Mc} b'd$. □

Proposition 7.5 (weak independence theorem). *Let T be thick* NSOP_1. *Suppose that $a \equiv^{\mathrm{Ls}}_M a'$, $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$, $a' \mathop{\smile\!\!\!\!\!\!|}^K_M c$ and $b \mathop{\smile\!\!\!\!\!\!|}^*_M c$. Then there is a'' with $a'' \equiv^{\mathrm{Ls}}_{Mb} a$ and $a'' \equiv^{\mathrm{Ls}}_{Mc} a'$ such that $a'' \mathop{\smile\!\!\!\!\!\!|}^K_M bc$.*

Proof. We may assume that b and c both enumerate a λ_T-saturated model containing M. If this is not the case, let $N \supseteq Mb$ be λ_T-saturated and such that $|N| < \beth_\omega(\lambda_T + |Mbc|)$. By left extension from Lemma 7.4 we then find $N' \equiv^{\mathrm{Ls}}_{Mb} N$ with $N' \mathop{\smile\!\!\!\!\!\!|}\,^*_M c$. By Corollary 4.14 we find a_0 with $a_0 \equiv^{\mathrm{Ls}}_{Mb} a$ and $a_0 \mathop{\smile\!\!\!\!\!\!|}\,^K_M N'$. Now we can replace a by a_0 and b by N' and continue the proof. The case for c is analogous.

By Lemma 7.2 there is c' such that $ac' \equiv^{\mathrm{Ls}}_M a'c$ and $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M bc'$. Apply left extension from Lemma 7.4 to $b \mathop{\smile\!\!\!\!\!\!|}\,^*_M c$ and c' to find $c'' \equiv^{\mathrm{Ls}}_{Mb} c$ with $bc' \mathop{\smile\!\!\!\!\!\!|}\,^*_M c''$. Let b^* be such that $b^*c'' \equiv^{\mathrm{Ls}}_M bc'$ and apply right extension from Lemma 7.4 to $bc' \mathop{\smile\!\!\!\!\!\!|}\,^*_M c''$ and b^* to find $b'' \equiv^{\mathrm{Ls}}_{Mc''} b^*$ with $bc' \mathop{\smile\!\!\!\!\!\!|}\,^*_M b''c''$. In particular, $b''c'' \equiv^{\mathrm{Ls}}_M bc'$, and $\mathrm{Lstp}(bc'/Mb''c'')$ extends to a global M-Ls-invariant type q. So there is a Morley sequence $(b_ic_i)_{i<\omega}$ in q with $(b_0, c_0) = (b'', c'')$ and $(b_1, c_1) = (b, c')$. As $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M bc'$, we can find a^* with $a^*b''c'' \equiv_M abc'$ such that $(b_ic_i)_{i<\omega}$ is Ma^*-indiscernible. By construction we had $c'' \equiv^{\mathrm{Ls}}_{Mb} c$, so there is a Lascar strong automorphism σ over Mb such that $\sigma(c'') = c$. Setting $a'' = \sigma(a^*)$, we check that this is indeed the a'' we are looking for.

By the chain condition (Lemma 6.1), $a^* \mathop{\smile\!\!\!\!\!\!|}\,^K_M (b_ic_i)_{i<\omega}$, so we have $a^* \mathop{\smile\!\!\!\!\!\!|}\,^K_M bc''$, and $a'' \mathop{\smile\!\!\!\!\!\!|}\,^K_M bc$ then follows by invariance. By Ma^*-indiscernibility we have $a''b \equiv_M a^*b \equiv_M a^*b'' \equiv_M ab$. We assumed b to enumerate a λ_T-saturated model, so indeed $a'' \equiv^{\mathrm{Ls}}_{Mb} a$. By construction of c' we have $a''c \equiv_M a^*c'' \equiv_M ac' \equiv_M a'c$. We assumed c to enumerate a λ_T-saturated model, so indeed $a'' \equiv^{\mathrm{Ls}}_{Mc} a'$, which concludes the proof. $\square$

Fact 7.6. *In a thick theory, if $N \supseteq M$ is $(2^{|M|+\lambda_T})^+$-saturated and q and r are global M-Ls-invariant types with $q|_N = r|_N$, then $q = r$.*

Proof. By Fact 2.12 there is $M \subseteq M' \subseteq N$ where M' is a λ_T-saturated model and $|M'| < (2^{|M|+\lambda_T})^+$. Let $\varphi(x, b)$ be any formula with parameters b. Let $b' \in N$ realise $\mathrm{tp}(b/M')$. Then $b \equiv^{\mathrm{Ls}}_M b'$. By M-Ls-invariance and $q|_N = r|_N$, we have

$$\varphi(x, b) \in q \iff \varphi(x, b') \in q \iff \varphi(x, b') \in r \iff \varphi(x, b) \in r,$$

which concludes the proof. $\square$

Theorem 7.7 (independence theorem). *Let T be a thick* NSOP_1 *theory. Suppose that $a \equiv^{\mathrm{Ls}}_M a'$, $a \mathop{\smile\!\!\!\!\!\!|}\,^K_M b$, $a' \mathop{\smile\!\!\!\!\!\!|}\,^K_M c$ and $b \mathop{\smile\!\!\!\!\!\!|}\,^K_M c$. Then there is a'' with $a'' \equiv^{\mathrm{Ls}}_{Mb} a$, $a'' \equiv^{\mathrm{Ls}}_{Mc} a'$ and $a'' \mathop{\smile\!\!\!\!\!\!|}\,^K_M bc$.*

Proof. We may assume that b and c both enumerate a λ_T-saturated model containing M. If this is not the case, let $N \supseteq Mb$ be λ_T-saturated. By extension (Corollary 4.14) and symmetry, then find $N' \equiv^{\mathrm{Ls}}_{Mb} N$ with $N' \mathop{\smile\!\!\!\!\!\!|}\,^K_M c$. Applying extension again we find a_0 with $a_0 \equiv^{\mathrm{Ls}}_{Mb} a$ and $a_0 \mathop{\smile\!\!\!\!\!\!|}\,^K_M N'$. Now we can replace a by a_0 and b by N' and continue the proof. The case for c is analogous.

Let $N_0 \supseteq M$ be $(2^{|M|+\lambda_T})^+$-saturated, and let κ be a big enough cardinal (depending only on $|N_0 bc|$). Pick some global M-Ls-invariant type $q(y, z)$ extending $\text{Lstp}(bc/M)$ such that q also extends to a global M-Ls-invariant type $\text{tp}(N/\mathfrak{M})$ for some saturated enough $N \supseteq M$ (depending only on κ). So there is β realising $q|_y$ with $\beta \equiv^{\text{Ls}}_M b$. Let $(b_i c_i)_{i<\kappa}$ be a Morley sequence in q with $b_0 = b$, and let $b_\kappa \equiv^{\text{Ls}}_{M(b_i c_i)_{i<\kappa}} \beta$. Then we have $b_i c_i \mathop{\smile\!\!\!\!\!\!\mid}^*_M b_{<i} c_{<i}$ for all $i < \kappa$ and $b_\kappa \mathop{\smile\!\!\!\!\!\!\mid}^*_M (b_i c_i)_{i<\kappa}$.

We will inductively construct a sequence $(b'_i)_{i\leq\kappa}$ with $b'_0 = b$ such that at step i,

(i) $c \mathop{\smile\!\!\!\!\!\!\mid}^K_M b'_{\leq i}$,

(ii) $cb'_i \equiv^{\text{Ls}}_M cb$,

(iii) $b'_{\leq i} \equiv^{\text{Ls}}_M b_{\leq i}$.

The base case is already fixed: $b'_0 = b$. So suppose we have constructed $b'_{\leq i}$. By induction hypothesis (iii) we can find $b^* b'_{\leq i} \equiv^{\text{Ls}}_M b_{i+1} b_{\leq i}$. So $b^* \mathop{\smile\!\!\!\!\!\!\mid}^*_M b'_{\leq i}$. Let c^* be such that $c^* b^* \equiv^{\text{Ls}}_M cb$, so $c^* \mathop{\smile\!\!\!\!\!\!\mid}^K_M b^*$. Therefore, also using (i) from the induction hypothesis, we can apply the weak independence theorem (Proposition 7.5) to find c' such that $c' \mathop{\smile\!\!\!\!\!\!\mid}^K_M b'_{\leq i} b^*$, $c' \equiv^{\text{Ls}}_{Mb^*} c^*$ and $c' \equiv^{\text{Ls}}_{Mb'_{\leq i}} c$. We now pick b'_{i+1} to be such that $cb'_{i+1} \equiv^{\text{Ls}}_{Mb'_{\leq i}} c'b^*$. Then indeed $c \mathop{\smile\!\!\!\!\!\!\mid}^K_M b'_{\leq i+1}$. We also have

$$b'_{\leq i} b'_{i+1} \equiv^{\text{Ls}}_M b'_{\leq i} b^* \equiv^{\text{Ls}}_M b_{\leq i} b_{i+1}.$$

Finally,

$$cb'_{i+1} \equiv^{\text{Ls}}_M c'b^* \equiv^{\text{Ls}}_M c^* b^* \equiv^{\text{Ls}}_M cb.$$

This concludes the successor step. For the limit stage we assume we have constructed $b'_{<i}$. We then have $c \mathop{\smile\!\!\!\!\!\!\mid}^K_M b'_{<i}$ by finite character. We also have $b'_{\leq j} \equiv^{\text{Ls}}_M b_{\leq j}$ for all $j < i$. So we have $b'_{<i} \equiv_M b_{<i}$. We assumed b to enumerate a λ_T-saturated model containing M, so because $b'_0 = b = b_0$ we do in fact have $b'_{<i} \equiv^{\text{Ls}}_M b_{<i}$. We then construct b'_i in an analogous way to the successor step.

We let $(c'_i)_{i<\kappa}$ be such that $b'_\kappa (b'_i c'_i)_{i<\kappa} \equiv^{\text{Ls}}_M b_\kappa (b_i c_i)_{i<\kappa}$. So by M-Ls-invariance of $q|_y$ we have $\beta b'_\kappa (b'_i c'_i)_{i<\kappa} \equiv^{\text{Ls}}_M \beta b_\kappa (b_i c_i)_{i<\kappa}$ and thus by how we chose b_κ we have $b'_\kappa \equiv^{\text{Ls}}_{M(b'_i c'_i)_{i<\kappa}} \beta$.

Because $q \subseteq \text{tp}(N/\mathfrak{M})$ for some saturated enough N, we can find

$$\beta\gamma(\beta_i, \gamma_i)_{i<\kappa} \equiv^{\text{Ls}}_M b'_\kappa c(b'_i, c'_i)_{i<\kappa}$$

in N, where $\beta\gamma \models q$. Here we used that $b'_\kappa c \equiv^{\text{Ls}}_M bc$. Set

$$q'((y_i, z_i)_{i<\kappa}, y, z) = \text{tp}((\beta_i, \gamma_i)_{i<\kappa} \beta\gamma/\mathfrak{M}).$$

Then q' is global M-Ls-invariant because $\text{tp}(N/\mathfrak{M})$ is global M-Ls-invariant. By Fact 7.6 and our choice of κ, we get that some global M-Ls-invariant type $q'|_{y_i z_i yz}$ occurs for κ many i (modulo identifying the variables for different i's). We now focus on a subsequence of length ω such that (after relabelling) $q'|_{y_i z_i yz}$ does not depend on i, and we forget about κ. We also relabel b'_κ to b'.

Claim 1. *In summary*:

(i) *We constructed a Morley sequence* $(b_i'c_i')_{i<\omega}$ *in* q, *where* q *is a global* M*-Ls-invariant extension of* $\mathrm{Lstp}(bc/M)$.

(ii) *For every* $i<\omega$, *we have* $b_i'c \equiv_M^{\mathrm{Ls}} b'c \equiv_M^{\mathrm{Ls}} bc$.

(iii) *Let* $\beta \models q|_y$. *Then* $b' \equiv^{\mathrm{Ls}}_{M(b_i'c_i')_{i<\omega}} \beta$.

(iv) $q(y,z) \subseteq q'((y_i,z_i)_{i<\omega},y,z)$ *and* q' *is global* M*-Ls-invariant and extends* $\mathrm{Lstp}((b_i',c_i')_{i<\omega}b'c/M)$.

(v) *There is some sufficiently saturated* N *such that* $q' \subseteq \mathrm{tp}(N/\mathfrak{M})$ *and* $\mathrm{tp}(N/\mathfrak{M})$ *is* M*-Ls-invariant.*

(vi) *The type* $q'|_{y_iz_iyz}$ *does not depend on* i, *modulo identifying variables for different* i*'s.*

Claim 2. *For every* $k<\omega$, *there are*

$$g_0h_0g_1h_1\cdots g_{k-1}h_{k-1}g_k, \quad g_0'h_0'g_1'h_1'\cdots g_{k-1}'h_{k-1}' \quad \text{and} \quad h_0''g_1''h_1''\cdots g_{k-1}''h_{k-1}''g_k''$$

such that

(i) $(g_i'h_i')_{i<k} \models (q'|_{y_0,z})^{\otimes k}|_M$,

(ii) $(h_i''g_{i+1}'')_{i<k} \models (q'|_{z_0,y})^{\otimes k}|_M$,

(iii) $(g_ih_i, g_i'h_i')$ *starts an* $Mg_{>i}h_{>i}g'_{>i}h'_{>i}$*-indiscernible sequence for every* $i<k$,

(iv) $(h_ig_{i+1}, h_i''g_{i+1}'')$ *starts an* $Mh_{>i}g_{>i+1}h''_{>i}g''_{>i+1}$*-indiscernible sequence for every* $i<k$.

We first prove that the theorem follows from Claim 2. We set $p_0(x,y)=\mathrm{tp}(ab/M)$ and $p_1(x,z)=\mathrm{tp}(a'c/M)$. We will prove that $p_0(x,b)\cup p_1(x,c)$ does not Kim-divide over M. This is enough, because by Proposition 4.3 we can then extend it to a complete type that does not Kim-divide over M. Since we assumed b and c to enumerate λ_T-saturated models containing M, any realisation a'' of that complete type is then what we needed to construct.

By compactness, we can find M-indiscernible $(g_ih_ig_i'h_i'g_i''h_i'')_{i\in\mathbb{Z}}$ such that $(g_i'h_i')_{i\in\mathbb{Z}} \models (q'|_{y_0,z})^{\otimes\mathbb{Z}}|_M$ and $(h_i''g_{i+1}'')_{i\in\mathbb{Z}} \models (q'|_{z_0,y})^{\otimes\mathbb{Z}}|_M$. Furthermore, we can make it so that for every $i\in\mathbb{Z}$ we have that

$$g_ih_i \equiv_{Mg_{>i}h_{>i}g'_{>i}h'_{>i}} g_i'h_i' \quad \text{and} \quad h_ig_{i+1} \equiv_{Mh_{>i}g_{>i+1}h''_{>i}g''_{>i+1}} h_i''g_{i+1}''.$$

We have $q'|_{y,z_0} \supseteq \mathrm{tp}(b'c_0'/M)$, by Claim 1(iv). So, by parts (iii) and (v) of Claim 1, we have that $b' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^*_M c_0'$. Then by Proposition 7.5 we have that $p_0(x,g_1'')\cup p_1(x,h_0'')$ does not Kim-divide. Then because $(h_i''g_{i+1}'')_{i\geq n} \models (q'|_{z_0,y})^{\otimes\omega}|_M$ for all $n\in\mathbb{Z}$, we get that $\bigcup_{i\in\mathbb{Z}} p_0(x,g_{i+1}'')\cup p_1(x,h_i'')$ is consistent. By the parallel sequences lemma (Lemma 2.28) we thus have that $\bigcup_{i\in\mathbb{Z}} p_0(x,g_{i+1})\cup p_1(x,h_i)$ is consistent. This is the same set as $\bigcup_{i\in\mathbb{Z}} p_0(x,g_i)\cup p_1(x,h_i)$. So again by the parallel sequences lemma we get that $\bigcup_{i\in\mathbb{Z}} p_0(x, g'_i) \cup p_1(x, h'_i)$ is consistent. By parts (ii) and (iii) of Claim 1, we have that $q'|_{y_0,z}$ extends $\mathrm{Lstp}(bc/M)$. So we conclude that $p_0(x, b) \cup p_1(x, c)$ does not Kim-divide over M, as required.

We are left to verify Claim 2. We fix k and by backwards induction on $k' = 2k, 2k-1, \dots, 1$ we will define trees $(d_\eta e_\eta)_{\eta\in S_{k'}}$ where $S_{k'} = \{\xi \in \omega^{\leq 2k+1} : 0^{k'-1} \trianglelefteq \xi\}$ such that for each k' the tree $(d_\eta e_\eta)_{\eta\in S_{k'}}$ satisfies the following condition:

$(\mathbf{P})_{k'}$ For every $\eta \in \omega^{\leq 2k-1}$ and $i < \omega$ such that $\eta \frown i \in S_{k'}$ we have that

$$(d_{\eta\frown i\frown j} e_{\eta\frown i\frown j})_{j<\omega} d_{\eta\frown i} e_{\eta\frown i} \equiv^{\mathrm{Ls}}_{M(d_{\trianglerighteq \eta\frown i'} e_{\trianglerighteq \eta\frown i'})_{i'<i}} (\beta_j \gamma_j)_{j<\omega} \beta\gamma.$$

Recall that $q' = \mathrm{tp}((\beta_j\gamma_j)_{j<\omega}\beta\gamma/\mathfrak{M})$. So in particular

$$(d_{\eta\frown j} e_{\eta\frown j})_{j<\omega} d_\eta e_\eta \equiv^{\mathrm{Ls}}_M (\beta_j\gamma_j)_{j<\omega}\beta\gamma \quad \text{for all } \eta \in \omega^{\leq 2k} \cap S_{k'}.$$

For $k' = 2k$ we let $(d_\eta e_\eta)_{\eta\in S_{2k}}$ just be $(b'_i c'_i)_{i<\omega} b' c$. Suppose now that we have constructed $(d_\eta e_\eta)_{\eta\in S_{k'}}$. We have $(d_{0^{k'-1}\frown i} e_{0^{k'-1}\frown i})_{i<\omega} d_{0^{k'-1}} e_{0^{k'-1}} \equiv^{\mathrm{Ls}}_M (\beta_i\gamma_i)_{i<\omega}\beta\gamma$, by $(\mathbf{P})_{k'}$. So by Claim 1(v) there is a global M-Ls-invariant $r \supseteq q'$ such that r also extends $\mathrm{Lstp}((d_\eta e_\eta)_{\eta\in S_{k'}}/M)$. Here we match $(d_{0^{k'-1}\frown i} e_{0^{k'-1}\frown i})_{i<\omega} d_{0^{k'-1}} e_{0^{k'-1}}$ with the variables in q'. Let $((d_{\eta,i} e_{\eta,i})_{\eta\in S_{k'}})_{i<\omega}$ be a Morley sequence in r with $(d_{\eta,0} e_{\eta,0})_{\eta\in S_{k'}} = (d_\eta e_\eta)_{\eta\in S_{k'}}$. We set

$$d_{0^{k'-2}\frown i\frown\xi} e_{0^{k'-2}\frown i\frown\xi} = d_{0^{k'-1}\frown\xi,i} e_{0^{k'-1}\frown\xi,i} \quad \text{for all } i < \omega \text{ and } \xi \in \omega^{\leq 2k+2-k'}.$$

We directly get $(\mathbf{P})_{k'-1}$ for $\eta \in S_{k'} - \{0^{k'-2}\}$ by virtue of $((d_{\eta,i} e_{\eta,i})_{\eta\in S_{k'}})_{i<\omega}$ being a Morley sequence. By Claim 1(iv), we have that $(d_{0^{k'-2}\frown i} e_{0^{k'-2}\frown i})_{i<\omega}$ is a Morley sequence in q. So we can find $d_{0^{k'-2}} e_{0^{k'-2}}$ such that

$$(d_{0^{k'-2}\frown i} e_{0^{k'-2}\frown i})_{i<\omega} d_{0^{k'-2}} e_{0^{k'-2}} \equiv^{\mathrm{Ls}}_M (\beta_i\gamma_i)_{i<\omega}\beta\gamma,$$

and that concludes the construction of $(d_\eta e_\eta)_{\eta\in S_{k'-1}}$.

Similarly as in the proof of Lemma 5.13, we will now show by induction on $n \leq k$ that the following holds.

$(\mathbf{Q})_n$ Let $\eta_{2k-2m} \in \omega^{2k-2m}$ and $\nu_{2k-2m+1} \in \omega^{2k-2m+1}$ for $0 \leq m \leq n$. Suppose that $\eta_{2k-2m} \triangleleft \nu_{2k-2m+1}$ for all $0 \leq m \leq n$, $\eta_{2k} >_{\mathrm{lex}} \eta_{2k-2} >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} \eta_{2k-2n}$ and for all $0 \leq m' < m \leq n$ we have that $\eta_{2k-2m} \wedge \eta_{2k-2m'} \in \omega^{2k-2m-1}$. Then $(d_{\nu_{2k-2m+1}} e_{\nu_{2k-2m+1}} d_{\eta_{2k-2m}} e_{\eta_{2k-2m}})_{m\leq n}$ is a Morley sequence in $q'|_{y_0 z_0 y z}$.

For $n = 0$ this follows immediately from $(\mathbf{P})_1$ and Claim 1(vi). So suppose $(\mathbf{Q})_n$ holds for some $n < k$, and let $\eta_{2k-2m} \in \omega^{2k-2m}$ and $\nu_{2k-2m+1} \in \omega^{2k-2m+1}$ for $0 \leq m \leq n+1$ be as in the statement of $(\mathbf{Q})_{n+1}$. For any $m < n$ we have that $\eta_{2k-2m} \wedge \eta_{2k-2n-2} = \eta_{2k-2n-2}|_{2k-2n-3}$. Therefore we can write $\eta_{2k-2n-2} = \xi \frown i$ for some $\xi \in \omega^{2k-2n-3}$ and $i < \omega$. We then have $\eta_{2k-2m} \trianglerighteq \xi \frown i'$ for some $i' < i$ for all $m \leq n$. So it follows from $(\mathbf{P})_1$, Claim 1(vi) and the induction hypothesis that $(d_{\nu_{2k-2m+1}} e_{\nu_{2k-2m+1}} d_{\eta_{2k-2m}} e_{\eta_{2k-2m}})_{m\leq n+1}$ is a Morley sequence in $q'|_{y_0 z_0 y z}$.

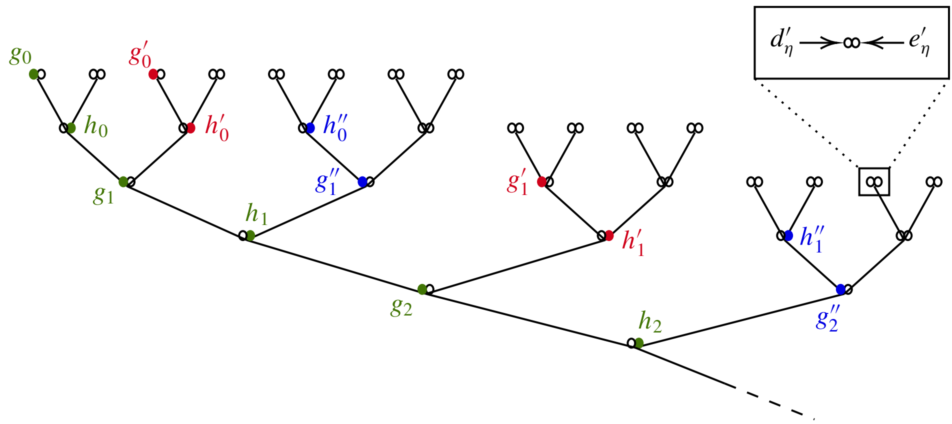

Figure 3. Choice of the $g_i h_i g'_i h'_i g''_i h''_i$.

By exactly the same argument we also have the following condition. It differs from $(\mathbf{Q})_n$ in that the levels have been shifted by one (therefore we only consider it for $n < k$).

$(\mathbf{Q}')_n$ Let $\eta_{2k-2m-1} \in \omega^{2k-2m-1}$ and $\nu_{2k-2m} \in \omega^{2k-2m}$ for $0 \leq m \leq n$. Suppose that $\eta_{2k-2m-1} \lhd \nu_{2k-2m}$ for all $0 \leq m \leq n$, $\eta_{2k-1} >_{\text{lex}} \eta_{2k-3} >_{\text{lex}} \cdots >_{\text{lex}} \eta_{2k-2n-1}$ and for all $0 \leq m' < m \leq n$ we have that $\eta_{2k-2m-1} \wedge \eta_{2k-2m'-1} \in \omega^{2k-2m-2}$. Then $(d_{\nu_{2k-2m}} e_{\nu_{2k-2m}} d_{\eta_{2k-2m-1}} e_{\eta_{2k-2m-1}})_{m \leq n}$ is a Morley sequence in $q'|_{y_0 z_0 y z}$.

Now let $(d'_\eta e'_\eta)_{\eta \in \omega^{2k+1}}$ be an s-indiscernible over M tree which is EM_s-based on $(d_\eta e_\eta)_{\eta \in \omega^{2k+1}}$ over M. We put $g_i = d'_{0^{2(k-i)+1}}$ for $i \leq k$, and for $i < k$ we put $h_i = e'_{0^{2(k-i)}}$, $g'_i = d'_{0^{2(k-i)-1} \frown 1 \frown 0}$, $h'_i = e'_{0^{2(k-i)-1} \frown 1}$, $g''_{i+1} = d'_{0^{2(k-i-1)} \frown 1}$ and $h''_i = e'_{0^{2(k-i-1)} \frown 1 \frown 0}$; see Figure 3. Then conditions (i) and (ii) from Claim 2 follow from $(\mathbf{Q})_k$ and $(\mathbf{Q}')_{k-1}$, while conditions (iii) and (iv) follow from s-indiscernibility. $\square$

Now that we have proved the independence theorem, we first note some useful immediate consequences in Corollary 7.10. After that, the rest of this section will be devoted to proving a stronger version of the independence theorem, Theorem 7.15.

Definition 7.8. Let I be a linear order. We say that $(a_i)_{i \in I}$ is a $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M$-*independent sequence* if $a_i \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M a_{<i}$ for every $i \in I$. We say that $(a_i)_{i \in I}$ is $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M$-*Morley* if it is $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M$-independent and M-indiscernible.

Lemma 7.9. *Let T be thick* NSOP_1 *with an e.c. model M, and let a, b, c be any tuples of parameters and x a tuple of variables. Then there exists a (partial) type $\Sigma(x, y)$ over Mab such that for any x and y we have that*

$$\models \Sigma(x, y) \iff (y \equiv_{Mb} c) \wedge (xa \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M yb).$$

In particular, taking $y = \varnothing$, we get that the condition $xa \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M b$ is type-definable over Mab in the variable x.

Proof. Let $q(y,z)$ be a global M-Ls-invariant type extending $\mathrm{tp}(cb/M)$. Then, by Kim's lemma, for any $y \equiv_{Mb} c$ and any x, the condition $xa \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M yb$ is equivalent to

$$\exists (y_i z_i)_{i<\omega}\big(q^{\otimes\omega}|_M((y_i z_i)_{i<\omega}) \text{ and } y_0 z_0 = yb \text{ and } (y_i z_i)_{i<\omega} \text{ is } Max\text{-indiscernible}\big),$$

which is clearly a type-definable over Mab condition by thickness. $\square$

In particular, we get that being an $\mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M$-independent sequence in a fixed type over M is type-definable over M in thick NSOP_1 theories. That is, for a linear order I, we can use the type

$$\bigcup_{i\in I} \Sigma(x_{<i}, x_i),$$

where Σ is as in Lemma 7.9. Then, by symmetry, Theorem 6.5, this (partial) type expresses exactly what we wanted.

Corollary 7.10. *Suppose T is thick* NSOP_1 *with an e.c. model M.*

(i) *If $a \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M b$ and $a \equiv^{\mathrm{Ls}}_M b$ then there exists an infinite M-indiscernible sequence starting with (a,b).*

(ii) *If $a \equiv^{\mathrm{Ls}}_M b$ then a and b are at Lascar distance at most 2 over M. In particular, Lascar equivalence over e.c. models is type-definable.*

(iii) *Generalised independence theorem: Let $(a_i)_{i<\kappa}$ be an $\mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M$-independent sequence. Suppose $b_i \equiv^{\mathrm{Ls}}_M b$ and $b_i \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M a_i$ for every $i<\kappa$. Then there exists b' such that $b'a_i \equiv^{\mathrm{Ls}}_M b_i a_i$ for every $i<\kappa$ and $b' \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M (a_i)_{i<\kappa}$.*

Proof. (i) We can inductively find a sequence $(c_i)_{i<\omega}$ such that $c_0c_1 = ab$, $c_i \equiv^{\mathrm{Ls}}_M b$, $c_i \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M c_{<i}$ and $c_ic_j \equiv_M ab$ for all $i<j<\omega$: indeed, if we have constructed $c_{\le i}$ then by the independence theorem we can choose c_{i+1} such that $c_{i+1} \equiv^{\mathrm{Ls}}_{Mc_{<i}} c_i$, $c_ic_{i+1} \equiv^{\mathrm{Ls}}_M ab$ and $c_{i+1} \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M c_{\le i}$.

By compactness we can find a sequence $(c'_i)_{i<\lambda_{|T|+|Ma|}}$ with $c'_ic'_j \equiv_M ab$ for all $i<j<\lambda_{|T|+|Ma|}$. Choose an M-indiscernible sequence $(d_i)_{i<\omega}$ based on $(c'_i)_{i<\lambda_{|T|+|Ma|}}$ over M. Then $d_0d_1 \equiv_M ab$, so we conclude that the pair (a,b) starts an M-indiscernible sequence.

(ii) By extension (Corollary 4.14) we can choose $c \equiv^{\mathrm{Ls}}_M a$ with $c \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M ab$. By (i) we get that (a,c) and (b,c) both start M-indiscernible sequences.

(iii) We choose inductively a sequence $(b'_j)_{j\le\kappa}$ such that $b'_ja_i \equiv^{\mathrm{Ls}}_M b_ia_i$ for every $i<j$ and $b'_j \mathop{\smile\!\!\!\!\!\mid}\nolimits^K_M (a_i)_{i<j}$, so that we can put $b' := b_\kappa$. The successor step follows directly by the independence theorem, and the limit step follows by type-definability of Lascar equivalence over M, Lemma 7.9 and compactness. $\square$

Definition 7.11. We will say that a tree $(c_\eta)_{\eta\in\omega^{\le k}}$ is *spread-out over M* if $(c_{\trianglerighteq \eta^\frown i})_{i<\omega}$ is a Morley sequence in some global M-Ls-invariant type for every $\eta \in \omega^{\le k-1}$.

There are two differences between being spread-out over M and being q-spread-out over M (see Definition 5.12 for the latter). In the latter the global M-Ls-invariant type involved has to be q, while the former just requires some global M-Ls-invariant type. The second difference is in the sequence in the tree that is required to be a Morley sequence. In the former we consider a sequence of subtrees above some fixed node, all at the same level. In the latter we consider a sequence of nodes in the tree, one in every level (except for the root), as pictured in Figure 1.

The following lemma follows from the independence theorem exactly as in [Kaplan and Ramsey 2020, Lemma 6.2/Remark 6.3], so we omit the proof.

Fact 7.12. *Suppose that T is thick* NSOP$_1$, *M is an e.c. model, $a \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M b$, $(b_\eta)_{\eta\in\omega^{\le k}}$ (with $k<\omega$) is a spread-out over M tree such that $b_\eta \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M b_{\rhd\eta}$ and $b_\eta \equiv^{\mathrm{Ls}}_M b$ for every $\eta\in\omega^{\le k}$. Then, writing $p(x,b)=tp(a/Mb)$, there exists $a' \models \bigcup_{\eta\in\omega^{\le k}} p(x,b_\eta)$ with $a' \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M (b_\eta)_{\eta\in\omega^{\le k}}$ and $a' \equiv^{\mathrm{Ls}}_M a$.*

Lemma 7.13. *Suppose that T is thick* NSOP$_1$, *M is an e.c. model, $b \equiv^{\mathrm{Ls}}_M b'$, $b \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M b'$ and I is a linear order with two distinct elements* 0 *and* 1. *Then there is a $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M$-Morley parallel-Morley in* $\mathrm{tp}(b/M)$ *sequence $(b_i)_{i\in I}$ with $b_0=b$ and $b_1=b'$.*

Proof. By extension (Corollary 4.14) there is a λ_T-saturated model $N \supseteq Mb$ with $N \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M b'$. Then there is a λ_T-saturated model $N' \supseteq Mb'$ with $N' \equiv^{\mathrm{Ls}}_M N$. Hence, again by extension, we can find $N'' \equiv^{\mathrm{Ls}}_{Mb'} N'$ with $N \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M N''$. So replacing b and b' by N and N'' we may assume without loss of generality that b and b' are λ_T-saturated models containing M. Put $\lambda=|b|$ and (using Lemma 6.3) choose a global M-Ls-invariant extension q of $\mathrm{Lstp}(b'/M)$ satisfying $(*)_\lambda$.

We claim that it is enough to show that for any $1<k<\omega$ there is a $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M$-independent parallel-Morley sequence $(a_i)_{i<k}$ in q over M with $a_i \equiv^{\mathrm{Ls}}_M b'$ and $a_i a_j \equiv_M bb'$ for all $i<j<k$: indeed, if we show this, then, as all these conditions are type-definable by Lemma 7.9 and Corollary 7.10(ii), we can find by compactness a $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K$-independent over M parallel-Morley sequence $(a_i)_{i<\lambda_{|T|+|b|}}$ in q over M with $a_i a_j \equiv_M bb'$ for each $i<j$, and then taking an M-indiscernible sequence indexed by I which is based on $(a_i)_{i<\lambda_{|T|+|Mb|}}$ over M and moving it by an automorphism to guarantee that $b_0b_1=bb'$ (note this may change q) will do the job.

So fix any $1<k<\omega$ and put $p=tp(b'/Mb)$. By backward induction on $k'=k+1,k,\dots,1$ we will define trees $(c_\eta)_{\eta\in S_{k'}}$ where $S_{k'}:=\{\xi\in\omega^{\le k}:0^{k'-1}\trianglelefteq\xi\}$ such that for each k' the tree $(c_\eta)_{\eta\in S_{k'}}$ is spread-out over M and satisfies the following conditions:

(A1)$_{k'}$ $c_\eta c_\nu \equiv_M bb'$ for any $\nu,\eta\in S_{k'}$ with $\nu\vartriangleleft\eta$ and $c_\eta \equiv^{\mathrm{Ls}}_M b'$ for any $\eta\in S_{k'}$.

(A2)$_{k'}$ $(c_\eta)_{\eta\in S_{k'}}$ is q-spread-out over M.

(A3)$_{k'}$ $c_\eta \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_M c_{\rhd\eta}$ for every $\eta\in S_{k'}$.

For $k' = k+1$ putting $c_{0^k} = b'$ works. Now suppose we are done for some $k' \leq k+1$. By Fact 7.12 we can find $c' \models \bigcup_{\eta \in S_{k'}} p(x, c_\eta)$ with $c' \equiv^{\mathrm{Ls}}_M b'$ and $c' \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M (c_\eta)_{\eta \in S_{k'}}$. By $(\mathrm{A1})_{k'}$ there is a tuple d such $c_{0^{k'-1}}(c_\eta)_{\eta \in S^*_{k'}} \equiv^{\mathrm{Ls}}_M b'd$. Now, by $(*)_\lambda$ there is some global M-Ls-invariant type $r(x, z) \supseteq q(x)$ which extends $\mathrm{Lstp}(b'd/M) = \mathrm{Lstp}(c_{0^{k'-1}}(c_\eta)_{\eta \in S^*_{k'}}/M)$. Also, as $c' \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M (c_\eta)_{\eta \in S_{k'}}$ and c_η's are λ_T-saturated models (as b' is), we get by Corollary 4.10 that $\mathrm{Lstp}(c'/M(c_\eta)_{\eta \in S_{k'}})$ does not $r(x, z)$-Ls-divide over M. Hence, there is an Mc'-indiscernible Morley sequence $I := ((c_{\eta,i})_{\eta \in S_{k'}})_{i<\omega}$ in $r(x, z)$ over M with $c_{\eta,0} = c_\eta$ for each $\eta \in S_{k'}$. By the chain condition (Lemma 6.1) we have that $c' \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M I$. Thus, putting $c_{0^{k'-2} \frown i \frown \zeta} := c_{0^{k'-1} \frown \zeta, i}$ for all $i < \omega$, $\zeta \in \omega^{\leq k+1-k'}$, and $c_{0^{k'-2}} := c'$, we immediately get that the tree $(c_\eta)_{\eta \in S_{k'-1}}$ satisfies $(\mathrm{A3})_{k'-1}$. $(\mathrm{A1})_{k'-1}$ follows from $(\mathrm{A1})_{k'}$, the choice of c' and Mc' indiscernibility of I. $(\mathrm{A2})_{k'-1}$ follows from $(\mathrm{A2})_{k'}$ and Lemma 5.13(i). This completes the inductive construction.

Letting $(c'_\eta)_{\eta \in \omega^{\leq k}}$ be an s-indiscernible over M tree which is EM_s-based on $(c_\eta)_{\eta \in \omega^{\leq k}}$ over Mb', we get that $(c'_\eta)_{\eta \in \omega^{\leq k}}$ satisfies $(\mathrm{A1})_1$ and $(\mathrm{A3})_1$ (by Lemma 7.9 and Corollary 7.10(ii)) and is weakly q-spread-out over M by Lemma 5.13(ii).

Put $a_i := c'_{0^{k-i}}$ for $i < k$. Then by Lemma 5.13(iii) we have that $(a_i)_{i<k}$ is parallel-Morley in q over M. Also, $a_i a_j \equiv_M bb'$ for all $i < j < k$ by $(\mathrm{A1})_1$, and $(a_i)_{i<k}$ is $\mathop{\smile\!\!\!\!\!\!\mid}\,^K_M$-independent over M by $(\mathrm{A3})_1$. This completes the proof. □

Lemma 7.14 (chain condition for $\mathop{\smile\!\!\!\!\!\!\mid}\,^K$-Morley sequences). *Suppose T is thick* NSOP$_1$ *with an e.c. model M, $(d_i)_{i \in I}$ is an infinite $\mathop{\smile\!\!\!\!\!\!\mid}\,^K_M$-Morley sequence and $a \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M d_{i_0}$ for some $i_0 \in I$. Then there exists $a^* d_{i_0} \equiv^{\mathrm{Ls}}_M a d_{i_0}$ such that $(d_i)_{i \in I}$ is indiscernible over Ma^* and $a^* \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M (d_i)_{i \in I}$.*

Proof. By compactness, there exists a $\mathop{\smile\!\!\!\!\!\!\mid}\,^K_M$-Morley sequence $(d''_i)_{i<\lambda}$ such that $(d_i)_{i \in I} \frown (d''_i)_{i<\lambda}$ is M-indiscernible, where $\lambda = \lambda_{|T|+|Mad_0|+|I|}$. As $d_{i_0} \equiv^{\mathrm{Ls}}_M d''_0$, $a \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M d_{i_0}$ and $(d''_i)_{i<\lambda}$ is $\mathop{\smile\!\!\!\!\!\!\mid}\,^K$-independent over M, we get by Corollary 7.10(iii) that there exists a' with $a' d''_i \equiv^{\mathrm{Ls}}_M a d_{i_0}$ for every $i < \lambda$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M (d''_i)_{i<\lambda}$. Let $(d'_i)_{i \in I}$ be an Ma'-indiscernible sequence based on $(d''_i)_{i<\lambda}$ over $Maa'(d_i)_{i \in I}$. Therefore (by finite character and invariance of $\mathop{\smile\!\!\!\!\!\!\mid}\,^K$), $a' \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M (d'_i)_{i \in I}$, $(d'_i)_{i \in I} \equiv^{\mathrm{Ls}}_M (d_i)_{i \in I}$ (as $(d_i)_{i \in I} \frown (d'_i)_{i \in I}$ is indiscernible over M), and $a' d'_{i_0} \equiv^{\mathrm{Ls}}_M a d_{i_0}$. Hence, letting f be a Lascar strong automorphism over M sending $(d'_i)_{i \in I}$ to $(d_i)_{i \in I}$ and putting $a^* = f(a')$ we get that $a^* \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M (d_i)_{i \in I}$ and $(d_i)_{i \in I}$ is Ma^*-indiscernible. Also $a^* d_{i_0} \equiv^{\mathrm{Ls}}_M a' d'_{i_0} \equiv^{\mathrm{Ls}}_M a d_{i_0}$, as required. □

Theorem 7.15 (strong independence theorem). *Suppose T is thick* NSOP$_1$ *with an e.c. model M, $a_0 \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M b$, $a_1 \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M c$, $b \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M c$, and $a_0 \equiv^{\mathrm{Ls}}_M a_1$. Then there exists an a such that $a \equiv^{\mathrm{Ls}}_{Mb} a_0$, $a \equiv^{\mathrm{Ls}}_{Mc} a_1$, $a \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M bc$, $b \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M ac$ and $c \mathop{\smile\!\!\!\!\!\!\mid}\,^K_M ab$.*

Proof. By a similar trick as at the start of the proof of Theorem 7.7 we may assume that b and c enumerate λ_T-saturated models containing M.

By the independence theorem there is a_2 with $a_2 \equiv^{\mathrm{Ls}}_{Mb} a_0$, $a_2 \equiv^{\mathrm{Ls}}_{Mc} a_1$ and $a_2 \mathop{\smile\!\!\!\!\!\mid}^K_M bc$. By extension (Corollary 4.14) there is $b' \equiv^{\mathrm{Ls}}_{Mc} b$ such that $b \mathop{\smile\!\!\!\!\!\mid}^K_M b'c$, and thus $b'c \mathop{\smile\!\!\!\!\!\mid}^K_M b$ by symmetry. By extension again, there is $c' \equiv^{\mathrm{Ls}}_{Mb} c$ with $b'c \mathop{\smile\!\!\!\!\!\mid}^K_M bc'$. As $b'c \equiv^{\mathrm{Ls}}_M bc \equiv^{\mathrm{Ls}}_M bc'$, we get by Lemma 7.13 that there is a $\mathop{\smile\!\!\!\!\!\mid}^K_M$-Morley parallel-Morley in $\mathrm{tp}(bc/M)$ sequence $I = (b_i, c_i)_{i\in\mathbb{Z}}$ with $b_0c_0 = bc'$ and $b_1c_1 = b'c$. As $a_2 \mathop{\smile\!\!\!\!\!\mid}^K_M bc$, we get by Lemma 7.14 that there is some a such that $abc' \equiv^{\mathrm{Ls}}_M a_2bc$, I is Ma-indiscernible and $a \mathop{\smile\!\!\!\!\!\mid}^K_M I$.

Then, by monotonicity, $a \mathop{\smile\!\!\!\!\!\mid}^K_M bc$. We also have $ab \equiv^{\mathrm{Ls}}_M a_2b \equiv^{\mathrm{Ls}}_M a_0b$, and by indiscernibility, $ac \equiv^{\mathrm{Ls}}_M ac' \equiv^{\mathrm{Ls}}_M a_2c \equiv^{\mathrm{Ls}}_M a_1c$. Since b and c were assumed to enumerate λ_T-saturated models we get $a \equiv^{\mathrm{Ls}}_{Mb} a_0$ and $a \equiv^{\mathrm{Ls}}_{Mc} a_1$. Also, $(b_i)_{i\le 0}$ is an Mac-indiscernible parallel-Morley sequence in $\mathrm{tp}(b/M)$ with $b_0 = b$, which gives $b \mathop{\smile\!\!\!\!\!\mid}^K_M ac$ by Corollary 5.11. Similarly, as $(c_i)_{i\ge 1}$ is an Mab-indiscernible parallel-Morley sequence in $\mathrm{tp}(c/M)$ with $c_1 = c$, we get that $c \mathop{\smile\!\!\!\!\!\mid}^K_M ab$. $\square$

8. Transitivity

Lemma 8.1. *If $M \subseteq N$ are e.c. models of a thick* NSOP_1 *theory, $a \mathop{\smile\!\!\!\!\!\mid}^K_M N$, and μ is a small cardinal, then there is a parallel-Morley in* $\mathrm{tp}(a/N)$ *sequence $(a_i)_{i\in\mu}$ with $a_0 = a$ such that $a_i \mathop{\smile\!\!\!\!\!\mid}^K_M Na_{<i}$ for every $i < \mu$.*

Proof. Put $\lambda = |Na| + \aleph_0$ and (using Lemma 6.3) choose a global N-Ls-invariant extension q of $\mathrm{Lstp}(a/N)$ satisfying $(*)_\lambda$.

By Lemma 7.9, compactness, finite character of Kim-independence, and an automorphism, it is enough to find for any given $k < \omega$ a parallel-Morley sequence $(a_i)_{i<k}$ in q over N such that $a_i \mathop{\smile\!\!\!\!\!\mid}^K_M Na_{<i}$ for every $i < k$.

So fix any $k < \omega$. By backward induction on $k' = k+1, k, \ldots, 1$ we will construct trees $(c_\eta)_{\eta\in S_{k'}}$, where $S_{k'} := \{\xi \in \omega^{\le k} : 0^{k'-1} \trianglelefteq \xi\}$, such that for each k' the tree $(c_\eta)_{\eta\in S_{k'}}$ satisfies the following conditions:

$(\mathrm{A1})_{k'}$ For any $\eta \in S_{k'}$ we have $c_\eta \mathop{\smile\!\!\!\!\!\mid}^K_M Nc_{\triangleright\eta}$ and $c_\eta \equiv^{\mathrm{Ls}}_N a$.

$(\mathrm{A2})_{k'}$ $(c_\eta)_{\eta\in S_{k'}}$ is q-spread-out over N.

For $k' = k+1$ we let $c_{0^k} = a$. For the inductive step, suppose we are done for some k'. By $(\mathrm{A1})_{k'}$ we have $c_{0^{k'-1}} \equiv^{\mathrm{Ls}}_N a$, so by $(*)_\lambda$ there is a global N-Ls-invariant type $r(x, y) \supseteq q(x)$ extending $\mathrm{Lstp}(c_{0^{k'-1}}, (c_\eta)_{\eta\in S^*_{k'}}/N)$ where x corresponds to $c_{0^{k'-1}}$. Choose a Morley sequence $I := ((c_{\eta,i})_{\eta\in S_{k'}})_{i<\omega}$ in $r(x, y)$ over N with $c_{\eta,0} = c_\eta$ for each $\eta \in S_{k'}$. By extension (Corollary 4.14) there is $c' \equiv^{\mathrm{Ls}}_N a$ with $c' \mathop{\smile\!\!\!\!\!\mid}^K_M NI$. Put $c_{0^{k'-2}\frown i\frown\zeta} := c_{0^{k'-1}\frown\zeta,i}$ for all $i < \omega$, $\zeta \in \omega^{\le k+1-k'}$, and $c_{0^{k'-2}} := c'$. Then $(\mathrm{A2})_{k'-1}$ follows by Lemma 5.13(i), whereas $(\mathrm{A1})_{k'-1}$ with $\eta \in S^*_{k'-1}$ follows by invariance of Kim-independence, and $(\mathrm{A1})_{k'-1}$ with $\eta = 0^{k'-2}$ follows by the choice of $c_{0^{k'-2}} = c'$. Thus the inductive step, and hence the construction of the tree $(c_\eta)_{\eta\in\omega^{\le k}} = (c_\eta)_{\eta\in S_1}$, is completed.

Letting $(c'_\eta)_{\eta\in\omega^{\leq k}}$ be an s-indiscernible over N tree that is EM_s-based on $(c_\eta)_{\eta\in\omega^{\leq k}}$ over Na, we get that $(c'_\eta)_{\eta\in\omega^{\leq k}}$ satisfies $(\mathrm{A1})_1$ by Lemma 7.9 and Corollary 7.10(ii), and is weakly q-spread-out over N by Lemma 5.13(ii). Thus, by Lemma 5.13(iii), putting $a_i = c'_{0^{k-i}}$ for $i < k$ we get a parallel-Morley sequence $(a_i)_{i<k}$ in q over N satisfying the requirements. □

Lemma 8.2. *Suppose T is thick NSOP_1 and $M \subseteq N$ are e.c. models of T. If $a \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M N$ and $c \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M N$ then there is $c' \equiv^{\mathrm{Ls}}_N c$ such that $ac' \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M N$ and $a \mathop{\smile\!\!\!\!\!\!\;\mid}^K_N c'$.*

Proof. By Lemma 7.9 there is a type $\Gamma(x; N, a)$ that is equivalent to the condition $ax \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M N$. By Lemma 8.1 there is a parallel-Morley in $\mathrm{tp}(a/N)$ sequence $(a_i)_{i<\lambda_T}$ with $a_0 = a$ such that $a_i \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M Na_{<i}$ for every $i < \lambda_T$. Replacing $(a_i)_{i<\lambda_T}$ with an N-indiscernible sequence based on it over N and moving by an automorphism (to keep $a_0 = a$), we may assume $(a_i)_{i<\lambda_T}$ is N-indiscernible.

Claim. *$\bigcup_{i<\lambda_T} \Gamma(x; N, a_i)$ has a realisation c'' such that $c'' \equiv^{\mathrm{Ls}}_N c$.*

Proof of claim. By induction on $n < \omega$ we will find $c_n \equiv^{\mathrm{Ls}}_N c$ such that $c_n \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M Na_{<n}$ and $c_n \models \bigcup_{i<n} \Gamma(x; N, a_i)$, which is enough by compactness, N-indiscernibility of $(a_i)_{i<\lambda_T}$ and Corollary 7.10(ii). For $n = 0$ put $c_0 = c$. Assume we have found c_n and find by extension (Corollary 4.14) some $c' \equiv^{\mathrm{Ls}}_M c$ such that $c' \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M a_n$. By Theorem 7.15, there exists c_{n+1} with $c_{n+1}a_{<n} \equiv^{\mathrm{Ls}}_N c_n a_{<n}$, $c_{n+1}a_n \equiv^{\mathrm{Ls}}_M c'a_n$, $c_{n+1} \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M Na_{<n+1}$ and $a_n c_{n+1} \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M Na_{<n}$. In particular, $c_{n+1} \equiv^{\mathrm{Ls}}_N c_n \equiv^{\mathrm{Ls}}_N c$ and $c_{n+1} \models \bigcup_{i<n+1} \Gamma(x; N, a_i)$. □

Let c'' be given by the claim, and let $(a'_i)_{i<\omega}$ be an Nc''-indiscernible sequence based on $(a_i)_{i<\lambda_T}$ over $Nc''a$. Then $a'_0 \equiv^{\mathrm{Ls}}_N a$ (as $a_i \equiv^{\mathrm{Ls}}_N a$ for every $i < \lambda_T$), so there is a Lascar strong automorphism f over N sending a'_0 to $a = a_0$. Put $c' := f(c'')$. Then $(f(a'_i))_{i<\omega}$ is an Nc'-indiscernible parallel-Morley sequence in $\mathrm{tp}(a/N)$ starting with a, so $c' \mathop{\smile\!\!\!\!\!\!\;\mid}^K_N a$ by Corollary 5.11. Also, $c' \models \Gamma(x; N, a)$, so $ac' \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M N$ by the choice of Γ, and we are done. □

Lemma 8.3. *Suppose T is thick NSOP_1 with e.c. models $M \subseteq N$ and $a \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M N$. Then there is a $\mathop{\smile\!\!\!\!\!\!\;\mid}^K_N$-Morley parallel-Morley in $\mathrm{tp}(a/M)$ sequence $(a_i)_{i<\omega}$ with $a = a_0$.*

Proof. By extension (Corollary 4.14) we may assume that a is a λ_T-saturated model extending M. By Lemma 6.3 there is a global M-Ls-invariant extension $q(x) \supseteq \mathrm{tp}(a/M)$ satisfying the property $(*)_\lambda$ with $\lambda = |a| + \aleph_0$. We claim that it is enough to find for any given $k < \omega$ a parallel-Morley sequence $(a_i)_{i<k}$ in q over M such that $a_i \mathop{\smile\!\!\!\!\!\!\;\mid}^K_N a_{<i}$ and $a_i \equiv_N a$ for every $i < k$: indeed, if we prove this, then, since the condition $(a_i \equiv_N a) \wedge (a_i \mathop{\smile\!\!\!\!\!\!\;\mid}^K_N a_{<i})$ is type-definable by Lemma 7.9, we can find by compactness such a sequence of length $\lambda_{|T|+|Na|}$. Then taking an N-indiscernible sequence based on $(a_i)_{i<\lambda_{|T|+|Na|}}$ over N and moving it by an automorphism we obtain a desired sequence.

So fix any $k < \omega$. By backward induction on $k' = k+1, k, \ldots, 1$ we will define trees $(c_\eta)_{\eta \in S_{k'}}$, where $S_{k'} := \{\xi \in \omega^{\leq k} : 0^{k'-1} \trianglelefteq \xi\}$, such that for each k' the tree $(c_\eta)_{\eta \in S_{k'}}$ satisfies the following conditions:

$(\mathrm{A1})_{k'}$ For any $\eta \in S_{k'}$ we have $c_\eta \mathop{\smile\!\!\!\!\!\!\;|}\,^K_N c_{\rhd\eta}$ and $c_\eta \equiv^{\mathrm{Ls}}_N a$.

$(\mathrm{A2})_{k'}$ $(c_\eta)_{\eta \in S_{k'}}$ is q-spread-out over M.

$(\mathrm{A3})_{k'}$ $(c_\eta)_{\eta \in S_{k'}} \mathop{\smile\!\!\!\!\!\!\;|}\,^K_M N$.

For $k' = k+1$, we let $c_{0^k} = a$. For the inductive step, suppose we are done for some k'. By $(*)_\lambda$ and $(\mathrm{A1})_{k'}$ there is a global M-invariant type $r(x, y)$ extending $\mathrm{Lstp}(c_{0^{k'-1}}, (c_\eta)_{\eta \in S^*_{k'}}/M)$ and $q(x)$. As c_η's are λ_T-saturated models, we get, by $(\mathrm{A3})_{k'}$ and Corollary 4.10, that $\mathrm{Lstp}(N/(c_\eta)_{\eta \in S_{k'}})$ does not r-Ls-divide over M. Thus there is an N-indiscernible Morley sequence $I = ((c_{\eta,i})_{\eta \in S_{k'}})_{i<\omega}$ in $r(x, y)$ over M with $c_{\eta,0} = c_\eta$ for each $\eta \in S_{k'}$ and $I \mathop{\smile\!\!\!\!\!\!\;|}\,^K_M N$. By Lemma 8.2 there is $a' \equiv^{\mathrm{Ls}}_N a$ such that $a' \mathop{\smile\!\!\!\!\!\!\;|}\,^K_N I$ and $a'I \mathop{\smile\!\!\!\!\!\!\;|}\,^K_M N$. Put $c_{0^{k'-2} \frown i \frown \zeta} := c_{0^{k'-1} \frown \zeta, i}$ for all $i < \omega$, $\zeta \in \omega^{\leq k+1-k'}$, and $c_{0^{k'-2}} := a'$. Then we get $(\mathrm{A2})_{k'-1}$ by Lemma 5.13(i), we get $(\mathrm{A1})_{k'-1}$ using that $a' \mathop{\smile\!\!\!\!\!\!\;|}\,^K_N I$, and $(\mathrm{A3})_{k'-1}$ holds as $a'I \mathop{\smile\!\!\!\!\!\!\;|}\,^K_M N$. Thus the inductive step, and hence the construction of the tree $(c_\eta)_{\eta \in \omega^{\leq k}} = (c_\eta)_{\eta \in S_1}$, is completed.

Letting $(c'_\eta)_{\eta \in \omega^{\leq k}}$ be an s-indiscernible over N tree which is EM_s-based on $(c_\eta)_{\eta \in \omega^{\leq k}}$ over Na, we get that $(c'_\eta)_{\eta \in \omega^{\leq k}}$ is weakly q-spread-out over M by Lemma 5.13(ii) and satisfies $(\mathrm{A1})_1$ by Lemma 7.9 and Corollary 7.10(ii). Thus putting $a_i = c'_{0^{k-i}}$ for $i < k$ we get by Lemma 5.13(iii) a parallel-Morley sequence $(a_i)_{i<k}$ in q over M satisfying the requirements. □

Theorem 8.4 (transitivity). *Suppose T is thick* NSOP_1 *with models $M \subseteq N$. If $a \mathop{\smile\!\!\!\!\!\!\;|}\,^K_M N$ and $a \mathop{\smile\!\!\!\!\!\!\;|}\,^K_N c$, then $a \mathop{\smile\!\!\!\!\!\!\;|}\,^K_M Nc$.*

Proof. By Lemma 8.3 there is a $\mathop{\smile\!\!\!\!\!\!\;|}\,^K_N$-Morley parallel-Morley in $\mathrm{tp}(a/M)$ sequence $I = (a_i)_{i<\omega}$ with $a_0 = a$. Because $a \mathop{\smile\!\!\!\!\!\!\;|}\,^K_N c$, we get by Lemma 7.14 an Nc-indiscernible sequence $I' = (a'_i)_{i<\omega} \equiv_{Na} I$. As I' is also parallel-Morley in $\mathrm{tp}(a/M)$ and $a'_0 = a$, we get by Corollary 5.11 that $Nc \mathop{\smile\!\!\!\!\!\!\;|}\,^K_M a$, so, by symmetry, we are done. □

9. Kim–Pillay style theorem

Theorem 9.1. *Let T be a thick positive theory. Then T is* NSOP_1 *if and only if there is an automorphism invariant ternary relation $\mathop{\smile\!\!\!\!\!\!\;|}$ on small subsets of the monster model, only allowing e.c. models in the base, satisfying the following properties*:

FINITE CHARACTER *If $a \mathop{\smile\!\!\!\!\!\!\;|}_M b_0$ for all finite $b_0 \subseteq b$, then $a \mathop{\smile\!\!\!\!\!\!\;|}_M b$.*

EXISTENCE *$a \mathop{\smile\!\!\!\!\!\!\;|}_M M$ for any model M.*

MONOTONICITY *$aa' \mathop{\smile\!\!\!\!\!\!\;|}_M bb'$ implies $a \mathop{\smile\!\!\!\!\!\!\;|}_M b$.*

SYMMETRY *$a \mathop{\smile\!\!\!\!\!\!\;|}_M b$ implies $b \mathop{\smile\!\!\!\!\!\!\;|}_M a$.*

LOCAL CHARACTER *Let a be a finite tuple and $\kappa > |T|$ be regular. Then for every continuous chain $(M_i)_{i<\kappa}$ with $|M_i| < \kappa$ for all i, there is $i < \kappa$ such that $a \mathop{\smile\!\!\!\!\!\!\!\,\mid}_{M_i} M$, where $M = \bigcup_{i<\kappa} M_i$.*

INDEPENDENCE THEOREM *If $a \mathop{\smile\!\!\!\!\!\!\!\,\mid}_M b$, $a' \mathop{\smile\!\!\!\!\!\!\!\,\mid}_M c$ and $b \mathop{\smile\!\!\!\!\!\!\!\,\mid}_M c$ with $a \equiv^{\mathrm{Ls}}_M a'$, then there is a'' such that $a''b \equiv^{\mathrm{Ls}}_M ab$, $a''c \equiv^{\mathrm{Ls}}_M a'c$ and $a'' \mathop{\smile\!\!\!\!\!\!\!\,\mid}_M bc$.*

EXTENSION *If $a \mathop{\smile\!\!\!\!\!\!\!\,\mid}_M b$, then for any c there is $a' \equiv_{Mb} a$ such that $a' \mathop{\smile\!\!\!\!\!\!\!\,\mid}_M bc$.*

TRANSITIVITY *If $a \mathop{\smile\!\!\!\!\!\!\!\,\mid}_M N$ and $a \mathop{\smile\!\!\!\!\!\!\!\,\mid}_N b$ with $M \subseteq N$, then $a \mathop{\smile\!\!\!\!\!\!\!\,\mid}_M Nb$.*

Furthermore, in this case, $\mathop{\smile\!\!\!\!\!\!\!\,\mid} = \mathop{\smile\!\!\!\!\!\!\!\,\mid}^K$.

The properties in Theorem 9.1 are not as strong as they could be. For example, we actually proved the strong independence theorem for $\mathop{\smile\!\!\!\!\!\!\!\,\mid}^K$; see Theorem 7.15. The slightly simpler formulation of the properties in Theorem 9.1 is easier to verify for an arbitrary independence relation $\mathop{\smile\!\!\!\!\!\!\!\,\mid}$. Then it follows immediately from $\mathop{\smile\!\!\!\!\!\!\!\,\mid} = \mathop{\smile\!\!\!\!\!\!\!\,\mid}^K$ that such an independence relation $\mathop{\smile\!\!\!\!\!\!\!\,\mid}$ also satisfies the stronger formulations.

Remark 9.2. In the existing Kim–Pillay style theorems for full first-order logic, [Kaplan and Ramsey 2020, Theorem 9.1; 2021, Theorem 6.11] and [Chernikov et al. 2020, Theorem 5.1], there are still various properties that mention syntax. Our Theorem 9.1 is completely syntax-free. One syntax-dependent property is mentioned in all of the above theorems, and is called STRONG FINITE CHARACTER: if $a \not\mathop{\smile\!\!\!\!\!\!\!\,\mid}_M b$ then there is $\varphi(x, b, m) \in \mathrm{tp}(a/Mb)$ such that for any $a' \models \varphi(x, b, m)$ we have $a' \not\mathop{\smile\!\!\!\!\!\!\!\,\mid}_M b$.

We could replace FINITE CHARACTER and LOCAL CHARACTER in Theorem 9.1 by STRONG FINITE CHARACTER. Obviously STRONG FINITE CHARACTER implies FINITE CHARACTER and modulo the other properties it also implies LOCAL CHARACTER by Lemmas 9.5 and 9.6.

Remark 9.3. To conclude that a theory is NSOP_1 it is enough to find an independence relation with the properties STRONG FINITE CHARACTER, EXISTENCE, MONOTONICITY, SYMMETRY and INDEPENDENCE THEOREM; see [Haykazyan and Kirby 2021, Theorem 6.4]. However, that does not guarantee that the independence relation is also Kim-independence; see [Kaplan and Ramsey 2020, Remark 9.39] for an example (already in full first-order logic). We also point out that [Haykazyan and Kirby 2021, Theorem 6.4] says nothing about the properties that Kim-independence generally has in NSOP_1 theories. Finally, our proof is also different because we do not rely on the syntactic property STRONG FINITE CHARACTER.

Remark 9.4. We point out a minor difference between Theorem 7.7 and INDEPENDENCE THEOREM in Theorem 9.1. In the former we get $a'' \equiv^{\mathrm{Ls}}_{Mb} a$, which is generally stronger than the $a''b \equiv_M^{\mathrm{Ls}} ab$ in the latter (and similar for c). Again, the reason is that the latter is easier to verify. Definitely in semi-Hausdorff theories, because then $a''b \equiv_M^{\mathrm{Ls}} ab$ is equivalent to $a''b \equiv_M ab$, so we do not have to worry about Lascar strong types. For a concrete example of this, see Fact 10.3(i). The only place where INDEPENDENCE THEOREM is used, namely, to get consistency along a certain sequence, we only need this weaker version.

Lemma 9.5. *Let* $\mathop{\smile\!\!\!\!\!\mid}$ *satisfy* STRONG FINITE CHARACTER, EXISTENCE, MONOTONICITY *and* SYMMETRY. *Then* $a \mathop{\smile\!\!\!\!\!\mid}_M^u b$ *implies* $a \mathop{\smile\!\!\!\!\!\mid}_M b$.

Proof. Exactly as in [Chernikov and Ramsey 2016, Proposition 5.8]. □

Lemma 9.6. *Let* $\mathop{\smile\!\!\!\!\!\mid}$ *be as in Lemma 9.5. Then it satisfies* LOCAL CHARACTER.

Proof. By Lemma 9.5, the proof from [Kaplan et al. 2019, Theorem 3.2] applies. Our formulation of local character then follows. □

Corollary 9.7 (local character). *In a thick* NSOP_1 *theory Kim-independence satisfies* LOCAL CHARACTER.

Remark 9.8. In [Kaplan et al. 2019] there are also different formulations of LOCAL CHARACTER, for example in terms of club sets of $[M]^{|T|}$. Since their arguments apply directly, these formulations also hold for Kim-independence in any thick NSOP_1 theory.

The next definition is based on the notion of isi-dividing from [Kamsma 2020].

Definition 9.9. We say that a type $p(x, b) = \mathrm{tp}(a/Cb)$ *long divides* over C if there is μ such that for every $\lambda \geq \mu$ there is a sequence $(b_i)_{i<\lambda}$ with $b_i \equiv_C b$ for all $i < \lambda$ such that for some $\kappa < \lambda$ and every $I \subseteq \lambda$ with $|I| \geq \kappa$ we have that $\bigcup_{i \in I} p(x, b_i)$ is inconsistent. We write $a \mathop{\smile\!\!\!\!\!\mid}_C^{\mathrm{ld}} b$ if $\mathrm{tp}(a/Cb)$ does not long divide over C.

There is a close connection between long dividing and dividing. Even though we do not need this connection in our proofs, it is still interesting to explore it. Dividing implies long dividing. Given an indiscernible sequence that witnesses dividing of a type p, we can use compactness to make it as long as we wish. So we find arbitrarily long sequences where p is inconsistent along any infinite subsequence, so p long divides. The converse is not so clear to us.

Question 9.10. Does long dividing imply dividing?

At least if we assume the existence of a proper class of Ramsey cardinals then the answer is positive. To see this, suppose that p long divides, and let λ be a big enough Ramsey cardinal. Then there is some sequence $(b_i)_{i<\lambda}$ witnessing that p long divides. Since we assumed λ to be Ramsey there is a cofinal $I \subseteq \lambda$ such that $(b_i)_{i \in I}$ is indiscernible. By the definition of long dividing, $\bigcup_{i \in I} p(x, b_i)$ is then inconsistent and so we conclude that p divides.

Lemma 9.11. *We have that $a \mathop{\smile\!\!\!\!\!\!|}_C^{i\mathrm{Ls}} b$ implies $a \mathop{\smile\!\!\!\!\!\!|}_C^{\mathrm{ld}} b$.*

Proof. Let $p(x, y) = \mathrm{tp}(ab/C)$, and let λ be any regular cardinal bigger than the number of Lascar strong types over C (compatible with b). Let $(b_i)_{i<\lambda}$ be any sequence in $\mathrm{tp}(b/C)$. By choice of λ there must be $I \subseteq \lambda$ such that $b_i \equiv_C^{\mathrm{Ls}} b_j$ for all $i, j \in I$ and $|I| = \lambda$. Pick some $i_0 \in I$, and let a' be such that $a'b_{i_0} \equiv_C ab$. By assumption $a \mathop{\smile\!\!\!\!\!\!|}_C^{i\mathrm{Ls}} b$, so $a' \mathop{\smile\!\!\!\!\!\!|}_C^{i\mathrm{Ls}} b_{i_0}$. Let $q \supseteq \mathrm{tp}(a'/Cb_{i_0})$ be a global C-Ls-invariant extension, and let $\alpha \models q$. Then $\alpha b_i \equiv_C^{\mathrm{Ls}} \alpha b_{i_0}$ for all $i \in I$, so $\bigcup\{p(x, b_i) : i \in I\}$ is consistent. $\square$

Definition 9.12. Let $\mathop{\smile\!\!\!\!\!\!|}$ be some independence relation, and let $(a_i)_{i<\kappa}$ be some sequence. Suppose furthermore that there is a continuous chain $(M_i)_{i<\kappa}$ of e.c. models, with $M \subseteq M_0$, such that $a_{<i} \subseteq M_i$ and $a_i \mathop{\smile\!\!\!\!\!\!|}_M M_i$ for all $i < \kappa$. Then we call $(M_i)_{i<\kappa}$ an $\mathop{\smile\!\!\!\!\!\!|}_M$*-independence chain (for $(a_i)_{i<\kappa}$)*.

Remark 9.13. Let $\mathop{\smile\!\!\!\!\!\!|}$ be an independence relation satisfying EXISTENCE and EXTENSION, let a be any tuple, and let M be any model. Then as usual we can inductively build arbitrarily long sequences $(a_i)_{i<\kappa}$ together with an $\mathop{\smile\!\!\!\!\!\!|}_M$-independence chain $(M_i)_{i<\kappa}$, such that $a \equiv_M a_i$ for all $i < \kappa$.

The following is adapted from one half of the original Kim–Pillay theorem, and occurs in [Kamsma 2020, Theorem 1.1]. We just have to check that the use of base-monotonicity can be replaced with our more carefully formulated form of local character.

Proposition 9.14. *Let $\mathop{\smile\!\!\!\!\!\!|}$ be as in Theorem 9.1. Then $a \mathop{\smile\!\!\!\!\!\!|}_M^{\mathrm{ld}} b$ implies $a \mathop{\smile\!\!\!\!\!\!|}_M b$.*

Note that we will actually not need INDEPENDENCE THEOREM here.

Proof. It follows directly from the definition of long dividing that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ld}}$ has monotonicity on the left side. So by FINITE CHARACTER and SYMMETRY we may assume a to be finite.

By Remark 9.13 we find a sequence $(b_i)_{i<\kappa}$ with an $\mathop{\smile\!\!\!\!\!\!|}_M$-independence chain $(M_i)_{i<\kappa}$ such that $b \equiv_M b_i$ for all $i < \kappa$. Picking the right $\kappa > (|T| + |M|)^+$, there must be $I \subseteq \kappa$ with order type $(|T| + |M|)^+$ such that $\bigcup_{i \in I} p(x, b_i)$ is consistent, where $p(x, y) = \mathrm{tp}(ab/M)$. Let a' be a realisation of this set. By MONOTONICITY and downward Löwenheim–Skolem, we may assume that $(M_i)_{i \in I}$ is a continuous chain with $|M_i| \le |T| + |M|$ for all $i \in I$. Then by LOCAL CHARACTER there is $i_0 \in I$ such that $a' \mathop{\smile\!\!\!\!\!\!|}_{M_{i_0}} M_I$, where $M_I = \bigcup_{i \in I} M_i$. By MONOTONICITY we have $a' \mathop{\smile\!\!\!\!\!\!|}_{M_{i_0}} b_{i_0}$ and by construction we also have $b_{i_0} \mathop{\smile\!\!\!\!\!\!|}_M M_{i_0}$. So by SYMMETRY and TRANSITIVITY we obtain $a' \mathop{\smile\!\!\!\!\!\!|}_M b_{i_0}$. The result now follows since $a'b_{i_0} \equiv_M ab$. $\square$

We note that in the above proof it is relevant that we work with long dividing instead of dividing. This is because the application of LOCAL CHARACTER only really makes sense if the chain consists of e.c. models, as we only allow e.c. models in the base. At the same time we need those e.c. models to form an independence chain for the rest of the proof to work. If we would try to follow the same proof just for dividing then we would have to work with indiscernible sequences. Finding an indiscernible $\mathop{\smile\!\!\!\!\mid}$-independent sequence is not an issue. This can be done as usual: we first build a very long $\mathop{\smile\!\!\!\!\mid}$-independent sequence and then base an indiscernible sequence on it. This preserves being $\mathop{\smile\!\!\!\!\mid}$-independent due to FINITE CHARACTER, but it does not carry over the independence chain. In long dividing this is not an issue, because we work directly with the very long sequence we constructed. So any "decorations", such as the independence chain, are then at our disposal.

The following lemma and its proof are a weaker version of the chain condition for $\mathop{\smile\!\!\!\!\mid}^K$-Morley sequences (Lemma 7.14) that works for long enough $\mathop{\smile\!\!\!\!\mid}^K$-independent sequences.

Lemma 9.15. *Let T be a thick* NSOP$_1$ *theory. Suppose that $a \mathop{\smile\!\!\!\!\mid}^K_M b$. Let $(b_i)_{i<\kappa}$ be an $\mathop{\smile\!\!\!\!\mid}^K_M$-independent sequence, where κ is a regular cardinal larger than the number of Lascar strong types over M (compatible with b) and where $b \equiv_M b_i$ for all $i < \kappa$. Then there is $I \subseteq \kappa$ with $|I| = \kappa$ such that $\bigcup_{i \in I} p(x, b_i)$ does not Kim-divide (and is thus consistent), where $p(x, y) = \mathrm{tp}(ab/M)$.*

Proof. By the choice of κ there is $I \subseteq \kappa$ with $|I| = \kappa$ such that $b_i \equiv^{\mathrm{Ls}}_M b_j$ for all $i, j \in I$. We conclude by the generalised independence theorem (Corollary 7.10(iii)). $\square$

Proof of Theorem 9.1. We already proved that $\mathop{\smile\!\!\!\!\mid}^K$ has all the listed properties if T is NSOP$_1$. So now we assume that we have an abstract independence relation $\mathop{\smile\!\!\!\!\mid}$ satisfying the listed properties and we prove that $\mathop{\smile\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\mid}^K$ and that T is NSOP$_1$.

The direction $a \mathop{\smile\!\!\!\!\mid}_M b \Longrightarrow a \mathop{\smile\!\!\!\!\mid}^K_M b$ holds. This proof is based on the proof of the same direction in [Kaplan and Ramsey 2020, Theorem 9.1]. Let $p(x, b) = \mathrm{tp}(a/Mb)$, and let q be any global M-Ls-invariant extension of $\mathrm{tp}(b/M)$. Then a Morley sequence $(b_i)_{i<\omega}$ in q is a $\mathop{\smile\!\!\!\!\mid}_M$-Morley sequence by Lemma 9.11 and Proposition 9.14. By the standard INDEPENDENCE THEOREM argument we thus find that $\bigcup_{i<\omega} p(x, b_i)$ is consistent, and thus $a \mathop{\smile\!\!\!\!\mid}^K_M b$.

The theory T is NSOP$_1$. We prove weak symmetry as in Theorem 6.7. So suppose $a \mathop{\smile\!\!\!\!\mid}^{i\mathrm{Ls}}_M b$. Then combining Lemma 9.11 and Proposition 9.14 again we get $a \mathop{\smile\!\!\!\!\mid}_M b$. So by SYMMETRY we have $b \mathop{\smile\!\!\!\!\mid}_M a$ and then $b \mathop{\smile\!\!\!\!\mid}^K_M a$ follows from the above.

The direction $a \mathop{\smile\!\!\!\!\mid}^K_M b \Longrightarrow a \mathop{\smile\!\!\!\!\mid}_M b$ holds. This proof is based on the proof of the same direction in [Chernikov et al. 2020, Theorem 5.1]. By Remark 9.13 we obtain a long enough sequence $(b_i)_{i<\kappa}$ with an $\mathop{\smile\!\!\!\!\mid}_M$-independence chain $(M_i)_{i<\kappa}$ and $b_i \equiv_M b$ for all $i < \kappa$. By the above $(M_i)_{i<\kappa}$ is also an $\mathop{\smile\!\!\!\!\mid}^K_M$-independence chain. So by Lemma 9.15 there is $I \subseteq \kappa$ with order type κ such that $\bigcup_{i \in I} p(x, b_i)$ is consistent, where $p(x, b) = \mathrm{tp}(a/Mb)$. Let a' be a realisation of this set. By deleting an end segment, MONOTONICITY and downward Löwenheim–Skolem we may assume that $(M_i)_{i \in I}$ is a continuous chain with $|M_i| \leq |T| + |M|$ for all $i \in I$ and I has order type $(|T|+|M|)^+$. By LOCAL CHARACTER there is $i_0 \in I$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}_{M_{i_0}} M_I$, where $M_I = \bigcup_{i\in I} M_i$, and therefore $a' \mathop{\smile\!\!\!\!\!\!\mid}_{M_{i_0}} b_{i_0}$. We also have $b_{i_0} \mathop{\smile\!\!\!\!\!\!\mid}_M M_{i_0}$, and thus by SYMMETRY and TRANSITIVITY we get $a' \mathop{\smile\!\!\!\!\!\!\mid}_M b_{i_0}$, and hence $a \mathop{\smile\!\!\!\!\!\!\mid}_M b$. □

10. Examples

In this section we present some examples of thick NSOP_1 theories. First, we recall Poizat's example of a thick non-semi-Hausdorff theory (which is bounded hence NSOP_1). Next, we look at (the JEP refinements of) the positive theory of existentially closed exponential fields, which was shown to be NSOP_1 in [Haykazyan and Kirby 2021] by constructing a suitable independence relation. We deduce from the known results that this theory is Hausdorff (hence thick), and then we show that Kim-independence coincides in it with the independence relation studied in [Haykazyan and Kirby 2021]. Finally, we show that NSOP_1 is preserved under taking hyperimaginary extensions; in particular, the hyperimaginary extension of an arbitrary NSOP_1 theory in full first-order logic is a Hausdorff NSOP_1 theory.

Let us also briefly mention the class of nonsimple NSOP_1 thick theories found recently in [d'Elbée et al. 2021]. For any integral domain R, the authors consider in the language of rings enriched by a predicate P and constants for elements of R the theory $F_{R\text{-module}}$: the theory of fields together with the quantifier-free diagram of R and where P defines an R-submodule. By [d'Elbée et al. 2021, Theorem 4.2, Theorem 4.8], for any integral domain R, the theory $F_{R\text{-module}}$ is nonsimple and NSOP_1 in the sense of positive logic. Also, by [d'Elbée et al. 2021, Remark 4.9] it is thick and Kim-independence in the sense of our paper coincides there with weak independence, as defined in [d'Elbée et al. 2021, Definition 4.4]. In the particular case $R = \mathbb{Z}$ this shows that the theory of algebraically closed fields of characteristic zero with a generic additive subgroup, which is known to be noncompanionable by [d'Elbée 2021a, Remark 1.20], is nonsimple and NSOP_1 in positive logic (see also [d'Elbée 2021b, Remark 5.35]).

10A. ***A thick, non-semi-Hausdorff theory.*** The following is an example of a thick non-semi-Hausdorff theory from [Poizat 2010, Section 4]. Consider a language $L = \{P_n, R_n : n < \omega\} \cup \{r\}$ where P_n's and R_n's are unary relation symbols and r a binary relation symbol. Let $M = \{a_n, b_n : n < \omega\}$ be an L-structure with $a_0, b_0, a_1, a_2, \ldots$ pairwise distinct, in which P_n is interpreted as $\{a_n, b_n\}$, R_n as the complement of P_n, and r as the symmetric antireflexive relation $\{(a_n, b_n), (b_n, a_n) : n < \omega\}$. Let T be the h-inductive theory of the structure M. Then the models of T are bounded (in fact any e.c. extension of M adds at most two new points), so T is thick (and also NSOP_1). However, T is not semi-Hausdorff. In fact, it was observed by Rosario Mennuni that the unique nonalgebraic maximal type over M does not have any global M-invariant extensions. This shows that, in the definition of Kim-independence in thick theories, it is necessary to work with Ls-invariant types rather than just invariant types. This is also an example where having the same type over an e.c. model does not guarantee having the same Lascar strong type (over that model).

10B. ***Existentially closed exponential fields.*** In [Haykazyan and Kirby 2021] the class of existentially closed exponential fields is studied using positive logic. They prove that this is NSOP_1 by providing a nice enough independence relation. We verify that this independence relation is indeed Kim-independence.

Definition 10.1. An *exponential field* or *E-field* is a field of characteristic zero with a group homomorphism E from the additive group to the multiplicative group. We call such a field an *EA-field* if it is also an algebraically closed field. We can axiomatise EA-fields by a positive theory and call this theory $T_{\text{EA-field}}$. The existentially closed exponential fields are then the e.c. models of $T_{\text{EA-field}}$.

Our definition is slightly different from [Haykazyan and Kirby 2021] where they consider the class of e.c. models of just the theory of E-fields. However, these classes of e.c. models coincide; see [Haykazyan and Kirby 2021, Proposition 3.3] and the discussion after it.

There are also many different JEP-refinements; see [Haykazyan and Kirby 2021, Corollary 4.6]. To work in a monster model we need to fix one such JEP-refinement. This is not an issue, since everything we discuss here works in any JEP-refinement.

Definition 10.2 [Haykazyan and Kirby 2021, Definition 5.1]. For any set A write $\langle A\rangle^{\mathrm{EA}}$ for the smallest EA-subfield containing A. We define an independence notion $\mathop{\smile\!\!\!\!\!\!\mid}$ by

$$A \mathop{\smile\!\!\!\!\!\!\mid}_C B \iff \langle AC\rangle^{\mathrm{EA}} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}_{\langle C\rangle^{\mathrm{EA}}} \langle BC\rangle^{\mathrm{EA}},$$

where $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ACF}}$ is the usual independence relation in algebraically closed fields.

Note that the independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ actually makes sense over arbitrary sets. It would be interesting to compare this once Kim-independence over arbitrary sets has been developed in positive logic (see Question 10.21 below). For now we will restrict ourselves to working over e.c. models.

Fact 10.3. *We recall the following facts about* $T_{\text{EA-field}}$.

(i) *The independence relation* $\mathop{\smile\!\!\!\!\!\!\mid}$ *satisfies* STRONG FINITE CHARACTER, EXISTENCE, MONOTONICITY, SYMMETRY, INDEPENDENCE THEOREM.

(ii) *Any span* $F_1 \leftarrow F \rightarrow F_2$ *of embeddings of EA-fields can be amalgamated in such a way that, after embedding the result into the monster model,* $F_1 \mathop{\smile\!\!\!\!\!\!\mid}_F F_2$.

(iii) *For EA-fields* F_1 *and* F_2, *if* $\mathrm{qftp}(F_1) = \mathrm{qftp}(F_2)$ *then* $\mathrm{tp}(F_1) = \mathrm{tp}(F_2)$.

Proof. (i) This is [Haykazyan and Kirby 2021, Theorem 6.5]. They do not mention Lascar strong types in their formulation of INDEPENDENCE THEOREM. However, as we will see in Proposition 10.4, the theory is Hausdorff, so the types over e.c. models are Lascar strong types.

(ii) This is [Haykazyan and Kirby 2021, Theorem 4.3]. The fact that $F_1 \mathop{\smile\!\!\!\!\mid}_F F_2$ is not mentioned there, but it is direct from their proof.

(iii) This follows directly from (ii). □

To apply our theorem, Theorem 9.1, we need to verify a few more things.

Proposition 10.4. *The theory $T_{\text{EA-field}}$ is Hausdorff.*

Proof. Let T_k be the set of all h-inductive sentences that are true in all e.c. models of $T_{\text{EA-field}}$. By [Poizat and Yeshkeyev 2018, Theorem 8], being Hausdorff is equivalent to the models of T_k being amalgamation bases. By Fact 10.3(ii), the models of $T_{\text{EA-field}}$ are already amalgamation bases, so the models of T_k are in particular also amalgamation bases. So we conclude that $T_{\text{EA-field}}$ is indeed Hausdorff. □

Note that Hausdorff is the best we can get, because [Haykazyan and Kirby 2021, Corollary 3.8] tells us that $T_{\text{EA-field}}$ cannot be Boolean. They prove this by showing that in every e.c. model F of $T_{\text{EA-field}}$ we have for all $a \in F$ that

$$a \in \mathbb{Z} \iff F \models \forall x (E(x) = 1 \to E(ax) = 1),$$

so if the theory were Boolean this would contradict compactness.

Proposition 10.5. *The independence relation $\mathop{\smile\!\!\!\!\mid}$ in $T_{\text{EA-field}}$ satisfies* EXTENSION *and* TRANSITIVITY.

Proof. We first prove TRANSITIVITY. Let $A \mathop{\smile\!\!\!\!\mid}_B C$ and $A \mathop{\smile\!\!\!\!\mid}_C D$ with $B \subseteq C$. So we have $\langle AB\rangle^{\text{EA}} \mathop{\smile\!\!\!\!\mid}^{\text{ACF}}_{\langle B\rangle^{\text{EA}}} \langle BC\rangle^{\text{EA}}$, which is just

$$\langle AB\rangle^{\text{EA}} \mathop{\smile\!\!\!\!\mid}^{\text{ACF}}_{\langle B\rangle^{\text{EA}}} \langle C\rangle^{\text{EA}}.$$

We also have $\langle AC\rangle^{\text{EA}} \mathop{\smile\!\!\!\!\mid}^{\text{ACF}}_{\langle C\rangle^{\text{EA}}} \langle CD\rangle^{\text{EA}}$, and therefore, by monotonicity of ACF-independence,

$$\langle AB\rangle^{\text{EA}} \mathop{\smile\!\!\!\!\mid}^{\text{ACF}}_{\langle C\rangle^{\text{EA}}} \langle CD\rangle^{\text{EA}}.$$

Then by transitivity of ACF-independence the result follows.

Now we prove EXTENSION. Let $a \mathop{\smile\!\!\!\!\mid}_C b$ and let d be arbitrary. From the definition, we get $a \mathop{\smile\!\!\!\!\mid}_C Cb$. We apply Fact 10.3(ii) to $\langle Cab\rangle^{\text{EA}} \supseteq \langle Cb\rangle^{\text{EA}} \subseteq \langle Cbd\rangle^{\text{EA}}$, and we can embed the amalgamation in the monster in such a way that $\langle Cbd\rangle^{\text{EA}}$ remains the same. So we get some EA-field F with $\text{qftp}(F/\langle Cb\rangle^{\text{EA}}) = \text{qftp}(\langle Cab\rangle^{\text{EA}}/\langle Cb\rangle^{\text{EA}})$ and $F \mathop{\smile\!\!\!\!\mid}_{\langle Cb\rangle^{\text{EA}}} \langle Cbd\rangle^{\text{EA}}$, which simplifies to $F \mathop{\smile\!\!\!\!\mid}_{Cb} Cbd$. By Fact 10.3(iii) and restricting ourselves to the copy $a' \in F$ of a we thus have $\text{tp}(a'/Cb) = \text{tp}(a/Cb)$. So we get $a' \mathop{\smile\!\!\!\!\mid}_C Cb$ and $a' \mathop{\smile\!\!\!\!\mid}_{Cb} Cbd$, and $a' \mathop{\smile\!\!\!\!\mid}_C bd$ follows from TRANSITIVITY and MONOTONICITY. □

Corollary 10.6. *The independence relation $\mathop{\smile\!\!\!\!\!\!\!\mid}$ in $T_{\text{EA-field}}$ is the same as Kim-independence over e.c. models.*

Proof. This is a direct application of Theorem 9.1, using Remark 9.2 to replace LOCAL CHARACTER by STRONG FINITE CHARACTER. $\square$

10C. *Hyperimaginaries.* One of the main motivations for studying positive logic in [Ben-Yaacov 2003a] was to be able to add hyperimaginaries in the same way we usually add imaginaries. It is well known that by doing so we leave the framework of full first-order logic, for example because we might get a bounded infinite definable set. However, we do stay within the framework of positive logic. We show that adding hyperimaginaries as real elements does not essentially change anything. So working with hyperimaginaries in positive logic requires no special treatment.

The construction in this section is based on [Ben-Yaacov 2003a, Example 2.16], but we work things out in far greater detail. This then allows us to prove that certain properties are invariant under adding hyperimaginaries.

We fix the following things throughout the rest of this section. First, we fix a positive theory T in a signature $\mathcal{L}$ with monster model $\mathfrak{M}$. For simplicity we assume $\mathcal{L}$ is single sorted (extending this to the multisorted setting is straightforward). Let $\mathcal{E}$ be a set of partial types (over $\varnothing$) $E(x, y)$, where x and y are (possibly infinite but small) tuples of variables, such that each E defines an equivalence relation in $\mathfrak{M}$.

Definition 10.7. We define the *hyperimaginary language* $\mathcal{L}_{\mathcal{E}}$ as a multisorted extension of $\mathcal{L}$. The sort of $\mathcal{L}$ will be called the *real sort* and is denoted by S_{real}. Then for each $E \in \mathcal{E}$ we add a sort S_E, called a *hyperimaginary sort*. For a variable y of sort S_E we denote by y_r a tuple of variables of the real sort, matching the length of the representatives of the E-equivalence classes.

For all $E_1, \ldots, E_n \in \mathcal{E}$ we add a relation symbol $R_\varphi(x, y_1, \ldots, y_n)$ of sort $S_{\text{real}}^{|x|} \times S_{E_1} \times \cdots \times S_{E_n}$ for each $\mathcal{L}$-formula $\varphi(x, y_{1,r}, \ldots, y_{n,r})$.

In the above definition, not all variables in $\varphi(x, y_{1,r}, \ldots, y_{n,r})$ need to actually appear in the formula. In particular, it is not problem for the $y_{i,r}$ to be infinite tuples. Similarly, when we write something like $\exists y_r \varphi(y_r)$, then we really only quantify over the variables that actually appear in φ.

Definition 10.8. We extend $\mathfrak{M}$ to an $\mathcal{L}_{\mathcal{E}}$-structure $\mathfrak{M}^{\mathcal{E}}$ as follows. The real sort S_{real} is just $\mathfrak{M}$, and for each $E \in \mathcal{E}$ the sort S_E is $\mathfrak{M}^\alpha / E$, where α is the length of the tuples of free variables in E. From now on we will use the shorthand notation $\mathfrak{M}/E$ and not mention α. For $E_1, \ldots, E_n \in \mathcal{E}$ and $\varphi(x, y_{1,r}, \ldots, y_{n,r})$ we interpret the relation symbol R_φ as follows. We let $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(a, c_1, \ldots, c_n)$ if and only if there are representatives $b_1, \ldots, b_n$ of $c_1, \ldots, c_n$ such that $\mathfrak{M} \models \varphi(a, b_1, \ldots, b_n)$.

For a real tuple b and some $E \in \mathcal{E}$ we will write $[b]$ for the corresponding hyperimaginary in $\mathfrak{M}/E$. To prevent cluttering of notation, we will actually also use the notation $[b]$ for a tuple of hyperimaginaries. This notation leaves implicit which sort(s) $[b]$ belongs to, but that should not be a problem in what follows.

Definition 10.9. We define the $\mathcal{L}_{\mathcal{E}}$-theory $T^{\mathcal{E}}$ as the set of all h-inductive $\mathcal{L}_{\mathcal{E}}$-sentences true in $\mathfrak{M}^{\mathcal{E}}$.

In this construction, $\mathfrak{M}^{\mathcal{E}}$ will be a monster model of $T^{\mathcal{E}}$ (Theorem 10.15). Being Hausdorff/semi-Hausdorff/thick is preserved under adding hyperimaginaries (Theorem 10.17). We have that T is NSOP$_1$ if and only if $T^{\mathcal{E}}$ is NSOP$_1$ (Theorem 10.18). So in particular this means that if we start with an NSOP$_1$ theory T in full first-order logic, viewed as a positive theory, then $T^{\mathcal{E}}$ is a Hausdorff (and thus thick) NSOP$_1$ theory, and all our results apply. Finally, we also have that T satisfies the existence axiom for forking if and only if $T^{\mathcal{E}}$ satisfies the existence axiom for forking (Theorem 10.20).

We set up our construction in such a way that we can add any set $\mathcal{E}$ of hyperimaginaries. If we wish to study $\mathfrak{M}^{\mathrm{heq}}$, where we have added all hyperimaginaries, we would have to add a proper class of hyperimaginaries. We can formalise this by taking $\mathcal{E}$ to be the set of all equivalence relations $E(x, y)$ where $|x| \leq |T|$. Then, by [Ben-Yaacov 2003c, Corollary 3.3], every possible hyperimaginary is interdefinable with a set of hyperimaginaries in $\mathcal{E}$. So we can take $\mathfrak{M}^{\mathrm{heq}}$ and T^{heq} to be $\mathfrak{M}^{\mathcal{E}}$ and $T^{\mathcal{E}}$.

Lemma 10.10. *Let $\varphi(x, y)$ be an $\mathcal{L}_{\mathcal{E}}$-formula, where x is a tuple of real variables and y is a tuple of hyperimaginary variables. Then there is a set of $\mathcal{L}$-formulas $\Sigma_\varphi(x, y_r)$ such that $\mathfrak{M} \models \Sigma_\varphi(a, b)$ if and only if $\mathfrak{M}^{\mathcal{E}} \models \varphi(a, [b])$.*

Proof. We first assume that $\varphi(x, y)$ is of the form

$$\exists wz\Big(\psi(x, w) \wedge \varepsilon(y, z) \wedge \textstyle\bigwedge_{i \in I} R_{\chi_i}(x, w, y, z)\Big).$$

Here w is a tuple of real variables and z a tuple of hyperimaginary variables. The formula $\psi(x, w)$ is an $\mathcal{L}$-formula and $\varepsilon(y, z)$ is a conjunction of equalities of hyperimaginaries.

We define the partial type Γ_φ as follows. For each $i \in I$ we introduce tuples of real variables y_i and z_i matching y_r and z_r respectively. We let $E_\varepsilon(y_r, z_r)$ be the union of partial types in $\mathcal{E}$ expressing $\varepsilon([y_r], [z_r])$, and we close E_ε under conjunctions. Then we set

$$\begin{aligned}
&\Gamma_\varphi(x, y_r, w, z_r, (y_i)_{i \in I}, (z_i)_{i \in I}) \\
&\qquad = \Big\{\psi(x, w) \wedge \epsilon(y_r, z_r) \wedge \textstyle\bigwedge_{i \in I} \chi_i(x, w, y_i, z_i) : \epsilon \in E_\varepsilon\Big\} && (1)\\
&\qquad\qquad \cup \bigcup \{E_y(y_r, y_i) : i \in I\} && (2)\\
&\qquad\qquad\qquad \cup \bigcup \{E_z(z_r, z_i) : i \in I\}. && (3)
\end{aligned}$$

Here E_y and E_z are the equivalence relations corresponding to the hyperimaginary variables y and z respectively.

Let $\Sigma_\varphi(x, y_r)$ express the following:

$$\exists w z_r (y_i)_{i \in I} (z_i)_{i \in I} \Gamma_\varphi(x, y_r, w, z_r, (y_i)_{i \in I}, (z_i)_{i \in I}).$$

Now suppose that a and b are such that $\mathfrak{M} \models \Sigma_\varphi(a, b)$. Then we find realisations such that

$$\mathfrak{M} \models \Gamma_\varphi(a, b, c, d, (b_i)_{i \in I}, (d_i)_{i \in I}).$$

Then (2) and (3) tell us that $[b] = [b_i]$ and $[d] = [d_i]$ for all $i \in I$, while (1) guarantees that $\mathfrak{M}^{\mathcal{E}} \models \varphi(a, [b])$. This proves the forward direction and the converse is straightforward by just taking representatives of the hyperimaginaries that are involved.

We assumed φ to be of a particular form. Since every formula can be written as a disjunction of regular formulas (i.e., formulas built using conjunction and existential quantification), we are only left an induction step for disjunction. So let $\varphi_1(x, y)$ and $\varphi_2(x, y)$ with $\Sigma_{\varphi_1}(x, y_r)$ and $\Sigma_{\varphi_2}(x, y_r)$ be given. We define $\Sigma_{\varphi_1 \vee \varphi_2}(x, y_r)$ as

$$\{\psi_1 \vee \psi_2 : \psi_1 \in \Sigma_{\varphi_1}, \psi_2 \in \Sigma_{\varphi_2}\}.$$

One easily checks that $\mathfrak{M} \models \Sigma_{\varphi_1 \vee \varphi_2}(a, b)$ precisely when $\mathfrak{M} \models \Sigma_{\varphi_1}(a, b)$ or $\mathfrak{M} \models \Sigma_{\varphi_2}(a, b)$ or both, and the result follows. $\square$

Lemma 10.11. *Let $\Gamma(x, y)$ be a set of $\mathcal{L}_{\mathcal{E}}$-formulas, where x is a tuple of real variables and y is a tuple of hyperimaginary variables. Then there is a set of $\mathcal{L}$-formulas $\Sigma_\Gamma(x, y_r)$ such that $\mathfrak{M} \models \Sigma_\Gamma(a, b)$ if and only if $\mathfrak{M}^{\mathcal{E}} \models \Gamma(a, [b])$.*

Proof. Define

$$\Sigma_\Gamma(x, y_r) = \bigcup_{\varphi \in \Gamma} \Sigma_\varphi(x, y_r),$$

where Σ_φ is as in Lemma 10.10. $\square$

Lemma 10.12. *If* $\operatorname{tp}(a[b]) = \operatorname{tp}(a'[b'])$ *then there is b'' such that* $\operatorname{tp}(ab) = \operatorname{tp}(a'b'')$ *and* $[b'] = [b'']$.

Proof. Define

$$\Sigma(x, y) = \operatorname{tp}_{\mathcal{L}}(ab) \cup E(b', y).$$

It is enough to prove that $\Sigma(a', y)$ is finitely satisfiable. Let $\varphi(x, y) \in \operatorname{tp}_{\mathcal{L}}(ab)$. Then $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(a, [b])$, so $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(a', [b'])$. So there is $b'' \in \mathfrak{M}$ with $\mathfrak{M} \models E(b', b'')$ and $\mathfrak{M} \models \varphi(a', b'')$, as required. $\square$

Lemma 10.13. *For every tuple of hyperimaginary variables y there is a partial $\mathcal{L}_{\mathcal{E}}$-type $\Xi(y_r, y)$ such that $\mathfrak{M}^{\mathcal{E}} \models \Xi(a, [a'])$ if and only if $[a] = [a']$.*

Proof. We define

$$\Xi(y_r, y) = \{R_\varepsilon(y_r, y) : \varepsilon \in E\},$$

where E is the equivalence relation corresponding to y. The right-to-left direction is clear. For the forward direction we suppose that $\mathfrak{M}^{\mathcal{E}} \models \Xi(a, [a'])$. Consider the partial type

$$\Gamma(y_r) = E(a, y_r) \cup E(y_r, a').$$

For any $\varepsilon(a, y_r) \in E(a, y_r)$ we have $\mathfrak{M}^{\mathcal{E}} \models R_\varepsilon(a, [a'])$. So there must be $a^* \in \mathfrak{M}$ such that $[a^*] = [a']$ and $\mathfrak{M} \models \varepsilon(a, a^*)$. Therefore, $\mathfrak{M} \models \varepsilon(a, a^*) \wedge E(a^*, a')$. We thus see that Γ is finitely satisfiable, so there is a realisation a''. We conclude that $[a] = [a''] = [a']$. $\square$

Lemma 10.14. *Any automorphism $f : \mathfrak{M} \to \mathfrak{M}$ uniquely extends to an automorphism $f^{\mathcal{E}} : \mathfrak{M}^{\mathcal{E}} \to \mathfrak{M}^{\mathcal{E}}$ by setting $f^{\mathcal{E}}([b]) = [f(b)]$.*

Proof. It is straightforward to check that $f^{\mathcal{E}}$ is well defined and bijective. We need to show that $f^{\mathcal{E}}$ preserves and reflects truth of the new relation symbols in $\mathcal{L}_{\mathcal{E}}$ (preservation of equality is just saying that $f^{\mathcal{E}}$ is well defined). Suppose that $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(a, [b])$. By definition there is b' such that $[b'] = [b]$ and $\mathfrak{M} \models \varphi(a, b')$. Then $\mathfrak{M} \models \varphi(f(a), f(b'))$ and hence $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(f(a), [f(b')])$, which is just $\mathfrak{M}^{\mathcal{E}} \models R_\varphi(f^{\mathcal{E}}(a), f^{\mathcal{E}}([b]))$. The converse follows in a similar way.

Finally we check uniqueness of $f^{\mathcal{E}}$. Suppose that $g : \mathfrak{M}^{\mathcal{E}} \to \mathfrak{M}^{\mathcal{E}}$ also extends f. For $[b] \in \mathfrak{M}^{\mathcal{E}}$ we have $\mathfrak{M}^{\mathcal{E}} \models \Xi(b, [b])$ by Lemma 10.13. So if g is an automorphism we must have $\mathfrak{M}^{\mathcal{E}} \models \Xi(g(b), g([b]))$, which means that $g([b]) = [g(b)] = [f(b)]$, as required. $\square$

Theorem 10.15. *The structure $\mathfrak{M}^{\mathcal{E}}$ is a monster model of $T^{\mathcal{E}}$.*

Proof. We prove that $\mathfrak{M}^{\mathcal{E}}$ is e.c. and is just as saturated and homogeneous as $\mathfrak{M}$. So let κ be such that $\mathfrak{M}$ is κ-saturated and κ-homogeneous. Note that this means that κ is definitely bigger than the length of any tuple representing a hyperimaginary.

Existentially closed: We will use Definition 2.5(iii). Suppose that $\mathfrak{M}^{\mathcal{E}} \not\models \varphi(a, [b])$. Then $\mathfrak{M} \not\models \Sigma_\varphi(a, b)$, where Σ_φ is from Lemma 10.11. Therefore, there exists $\psi(x, y_r) \in \Sigma_\varphi(x, y_r)$ such that $\mathfrak{M} \not\models \psi(a, b)$. Because $\mathfrak{M}$ is e.c. we find $\chi(x, y_r)$ with $T \models \neg\exists x y_r(\psi(x, y_r) \wedge \chi(x, y_r))$ and $\mathfrak{M} \models \chi(a, b)$. Thus $\mathfrak{M}^{\mathcal{E}} \models R_\chi(a, [b])$. We will conclude by proving that $\mathfrak{M}^{\mathcal{E}} \models \neg\exists x y(\varphi(x, y) \wedge R_\chi(x, y))$. Suppose for a contradiction that there are a' and b' such that $\mathfrak{M}^{\mathcal{E}} \models \varphi(a', [b']) \wedge R_\chi(a', [b'])$. Then there is b'' with $[b'] = [b'']$ and $\mathfrak{M} \models \chi(a', b'')$. So $\mathfrak{M}^{\mathcal{E}} \models \varphi(a', [b''])$ and thus $\mathfrak{M} \models \Sigma_\varphi(a', b'')$. We then get that $\mathfrak{M} \models \psi(a', b'') \wedge \chi(a', b'')$, which cannot happen.

Saturation: Let $\Gamma(x, y, c, [d])$ be a finitely satisfiable partial $\mathcal{L}_{\mathcal{E}}$-type with $|c[d]| < \kappa$. Let $\Sigma_\Gamma(x, y, c, d)$ be the set of $\mathcal{L}$-formulas from Lemma 10.11. By the construction there we have

$$\Sigma_\Gamma(x, y, c, d) = \bigcup_{\varphi \in \Gamma} \Sigma_\varphi(x, y, c, d),$$

where Σ_φ is as in Lemma 10.10. So finite satisfiability of $\Gamma(x, y, c, [d])$ implies finite satisfiability of $\Sigma_\Gamma(x, y, c, d)$. We thus find $a, b \in \mathfrak{M}$ with $\mathfrak{M} \models \Sigma_\Gamma(a, b, c, d)$ and hence $\mathfrak{M}^{\mathcal{E}} \models \Gamma(a, [b], c, [d])$.

Homogeneity: If $\mathrm{tp}(a[b]) = \mathrm{tp}(a'[b'])$, then by Lemma 10.12 there is b'' such that $[b''] = [b']$ and $\mathrm{tp}(ab) = \mathrm{tp}(a'b'')$. Let $f : \mathfrak{M} \to \mathfrak{M}$ be an automorphism with $f(ab) = a'b''$. Then, by Lemma 10.14, we find $f^{\mathcal{E}} : \mathfrak{M}^{\mathcal{E}} \to \mathfrak{M}^{\mathcal{E}}$ with $f^{\mathcal{E}}(a[b]) = f(a)[f(b)] = a'[b''] = a'[b']$, as required. □

Lemma 10.16. *A sequence $(a_i[b_i])_{i \in I}$ is indiscernible if and only if there are representatives b'_i of $[b_i]$ such that $(a_i b'_i)_{i \in I}$ is indiscernible.*

Proof. We first prove the left-to-right direction. By compactness we may assume I to be long enough. We can find indiscernible $(a^*_i b^*_i)_{i \in I}$ based on $(a_i b_i)_{i \in I}$. Let $p((x_i y_{i,r})_{i \in I}) = \mathrm{tp}((a^*_i b^*_i)_{i \in I})$ and define the type

$$\Gamma = p((a_i y_{i,r})_{i \in I}) \cup \{\Xi(y_{i,r}, [b_i]) : i \in I\}.$$

Then a realisation of Γ is precisely what we need, so we prove that Γ is finitely satisfiable. That is, for $i_1 < \cdots < i_n \in I$, we will produce a realisation of Γ restricted to the variables $y_{i_1,r}, \ldots, y_{i_n,r}$ and parameters $a_{i_1}, \ldots, a_{i_n}, [b_{i_1}], \ldots, [b_{i_n}]$. By construction there are $j_1 < \cdots < j_n \in I$ such that $\mathrm{tp}(a^*_{i_1} b^*_{i_1} \cdots a^*_{i_n} b^*_{i_n}) = \mathrm{tp}(a_{j_1} b_{j_1} \cdots a_{j_n} b_{j_n})$. As $\mathrm{tp}(a_{i_1}[b_{i_1}] \cdots a_{i_n}[b_{i_n}]) = \mathrm{tp}(a_{j_1}[b_{j_1}] \cdots a_{j_n}[b_{j_n}])$, by Lemma 10.12 we can find $b'_{i_1} \cdots b'_{i_n}$ with $\mathrm{tp}(a_{i_1} b'_{i_1} \cdots a_{i_n} b'_{i_n}) = \mathrm{tp}(a_{j_1} b_{j_1} \cdots a_{j_n} b_{j_n})$ while also $[b'_{i_k}] = [b_{i_k}]$ for all $1 \le k \le n$. So $b'_{i_1} \cdots b'_{i_n}$ is the desired realisation of Γ restricted to $y_{i_1,r}, \ldots, y_{i_n,r}$ and $a_{i_1}, \ldots, a_{i_n}, [b_{i_1}], \ldots, [b_{i_n}]$.

For the right-to-left direction, we note that, for any $i_1 < \cdots < i_n \in I$ and $j_1 < \cdots < j_n \in I$, we have

$$\Sigma_{\mathrm{tp}(a_{i_1}[b_{i_1}] \cdots a_{i_n}[b_{i_n}])} \subseteq \mathrm{tp}(a_{i_1} b'_{i_1} \cdots a_{i_n} b'_{i_n}) = \mathrm{tp}(a_{j_1} b'_{j_1} \cdots a_{j_n} b'_{j_n}).$$

So $\mathrm{tp}(a_{i_1}[b_{i_1}] \cdots a_{i_n}[b_{i_n}]) \subseteq \mathrm{tp}(a_{j_1}[b_{j_1}] \cdots a_{j_n}[b_{j_n}])$, and the claim follows by maximality of types. □

Theorem 10.17. *The following properties of T are preserved when adding hyperimaginaries:*

- *Hausdorff,*
- *semi-Hausdorff,*
- *thick.*

That is, if T has the property then $T^{\mathcal{E}}$ has it as well.

Proof. Hausdorff: Let $a[b] \not\equiv a'[b']$. Then there exists $\varphi \in \mathrm{tp}(a[b])$ such that $\varphi \notin \mathrm{tp}(a'[b'])$. So there is a negation $\psi \in \mathrm{tp}(a'[b'])$ of φ. By Lemma 10.11 we have that Σ_φ and Σ_ψ are consistent while $\Sigma_\varphi \cup \Sigma_\psi$ is inconsistent.

Fix some type q of T such that $\Sigma_\psi \subseteq q$. We will produce formulas α_q and β_q such that $\Sigma_\varphi \cup \{\alpha_q\}$ is inconsistent, $\beta_q \notin q$ and $T \models \forall x y_r (\alpha_q(x, y_r) \vee \beta_q(x, y_r))$. Let $p \supseteq \Sigma_\varphi$ be a type of T. Then because T is Hausdorff there are formulas χ_p and θ_p such that $\chi_p \notin p$ and $\theta_p \notin q$, while $T \models \forall x y_r (\chi_p(x, y_r) \vee \theta_p(x, y_r))$. Then $\Sigma_\varphi \cup \{\chi_p : p \supseteq \Sigma_\varphi\}$ is inconsistent, and therefore there are $p_1, \ldots, p_n$ such that $\Sigma_\varphi \cup \{\chi_{p_1} \wedge \cdots \wedge \chi_{p_n}\}$ is inconsistent. We can now take α_q to be $\chi_{p_1} \wedge \cdots \wedge \chi_{p_n}$ and β_q to be $\theta_{p_1} \vee \cdots \vee \theta_{p_n}$.

Now $\Sigma_\psi \cup \{\beta_q : q \supseteq \Sigma_\psi\}$ is inconsistent. So there are $q_1, \ldots, q_k$ such that $\Sigma_\psi \cup \{\beta_{q_1} \wedge \cdots \wedge \beta_{q_k}\}$ is inconsistent. We set $\beta = \beta_{q_1} \wedge \cdots \wedge \beta_{q_k}$ and $\alpha = \alpha_{q_1} \vee \cdots \vee \alpha_{q_n}$. We then also have that $\Sigma_\varphi \cup \{\alpha\}$ is inconsistent and $T \models \forall x y_r (\alpha(x, y_r) \vee \beta(x, y_r))$.

Now consider the formulas $R_\alpha(x, y)$ and $R_\beta(x, y)$. By construction we have $T^{\mathcal{E}} \models \forall x y (R_\alpha(x, y) \vee R_\beta(x, y))$. We claim that $R_\alpha \notin \mathrm{tp}(a[b])$. Suppose for a contradiction that $\mathfrak{M}^{\mathcal{E}} \models R_\alpha(a, [b])$. Then there is b^* with $[b^*] = [b]$ such that $\mathfrak{M} \models \alpha(a, b^*)$. Since $\varphi \in \mathrm{tp}(a[b]) = \mathrm{tp}(a[b^*])$, we also have $\mathfrak{M} \models \Sigma_\varphi(a, b^*)$, contradicting that $\Sigma_\varphi \cup \{\alpha\}$ is inconsistent. So indeed $R_\alpha \notin \mathrm{tp}(a[b])$. Analogously we get that $R_\beta \notin \mathrm{tp}(a'[b'])$, which concludes the proof that $T^{\mathcal{E}}$ is Hausdorff.

Semi-Hausdorff: Suppose that equality of $\mathcal{L}$-types is type-definable by a partial $\mathcal{L}$-type Ω. Then for a tuple x of real variables and a tuple y of hyperimaginary variables, we consider the partial $\mathcal{L}_{\mathcal{E}}$-type $\Omega^{\mathcal{E}}(xy, x'y')$ that expresses the following:

$$\exists y_r y'_r (\Xi(y_r, y) \wedge \Xi(y'_r, y') \wedge \Omega(x y_r, x' y'_r)).$$

We claim that $\Omega^{\mathcal{E}}$ expresses equality of $\mathcal{L}_{\mathcal{E}}$-types.

If $\mathfrak{M}^{\mathcal{E}} \models \Omega^{\mathcal{E}}(a[b], a'[b'])$ then we find c and c' such that

$$\mathfrak{M}^{\mathcal{E}} \models \Xi(c, [b]) \wedge \Xi(c', [b']) \wedge \Omega(ac, a'c').$$

By Lemma 10.13, $[c] = [b]$ and $[c'] = [b']$. Therefore, $\varphi \in \mathrm{tp}(a[b]) = \mathrm{tp}(a[c])$ if and only if $\Sigma_\varphi \subseteq \mathrm{tp}(ac) = \mathrm{tp}(a'c')$ if and only if $\varphi \in \mathrm{tp}(a'[c']) = \mathrm{tp}(a'[b'])$. So $\mathrm{tp}(a[b]) = \mathrm{tp}(a'[b'])$, as required.

Conversely, if $\mathrm{tp}(a[b]) = \mathrm{tp}(a'[b'])$ then by Lemma 10.12 we find b'' such that $[b''] = [b']$ and $\mathrm{tp}(ab) = \mathrm{tp}(a'b'')$. Hence, $\models \Xi(b, [b]) \wedge \Xi(b'', [b']) \wedge \Omega(ab, a'b'')$.

Thick: Let Θ express indiscernibility of a sequence of real tuples. Then

$$\exists (y_{i,r})_{i<\omega} \Big(\Theta((x_i y_{i,r})_{i<\omega}) \wedge \bigwedge\nolimits_{i<\omega} \Xi(y_{i,r}, y_i) \Big)$$

expresses indiscernibility of $(x_i y_i)_{i<\omega}$ in $T^{\mathcal{E}}$. Here we use that a sequence in $\mathfrak{M}^{\mathcal{E}}$ is indiscernible if and only if there is an indiscernible sequence of real representatives; see Lemma 10.16. □

Theorem 10.18. *The theory T is* NSOP$_1$ *if and only if $T^{\mathcal{E}}$ is* NSOP$_1$.

The technique in the proof of Theorem 10.18 can also be applied to other combinatorial properties, such as the order property, TP, TP_2, IP, etc. Of course, to do this, one first needs to write down a proper definition of these properties for positive logic, such as Definition 2.26 for SOP_1 or [Haykazyan and Kirby 2021, Definition 6.1] for TP_2.

Proof. One direction is trivial: if T has a formula with SOP_1, then so has $T^{\mathcal{E}}$.

We prove the other direction: supposing that $T^{\mathcal{E}}$ has a formula with SOP_1, we will show that T already has a formula with SOP_1. So let $\varphi(x, y; w, z)$ be an $\mathcal{L}_{\mathcal{E}}$-formula with SOP_1. Here x and w are tuples of real variables, and y and z are tuples of hyperimaginary variables. Let $(a_\eta[b_\eta] : \eta \in 2^{<\omega})$ and $\psi(w_1, z_1; w_2, z_2)$ be witnesses of SOP_1. Let $\Sigma_\varphi(x, y_r; w, z_r)$ and $\Sigma_\psi(w_1, z_{1,r}; w_2, z_{2,r})$ be as in Lemma 10.10. Then

$$\Sigma_\psi(w_1, z_{1,r}; w_2, z_{2,r}) \cup \Sigma_\varphi(x, y_r, w_1; z_{1,r}) \cup \Sigma_\varphi(x, y_r, w_2; z_{2,r})$$

is inconsistent. Hence there are finite $\varphi' \in \Sigma_\varphi$ and $\psi' \in \Sigma_\psi$ that are inconsistent with each other. That is,

$$T \models \neg\exists x y_r w_1 z_{1,r} w_2 z_{2,r} \big(\psi'(w_1, z_{1,r}, w_2, z_{2,r}) \wedge \varphi'(x, y_r, w_1, z_{1,r}) \\ \wedge \varphi'(x, y_r, w_2, z_{2,r})\big). \quad (4)$$

As usual, any variables not actually appearing in the formulas should be ignored in the existential quantifier. We claim that φ' has SOP_1, which is witnessed by $(a_\eta b_\eta : \eta \in 2^{<\omega})$ and ψ'. We check the items in Definition 2.26.

(i) Let $\sigma \in 2^\omega$. Then $\{\varphi(x, y, a_{\sigma|_n}, [b_{\sigma|_n}]) : n < \omega\}$ is consistent. So there are c and $[d]$ such that $\models \varphi(c, [d], a_{\sigma|_n}, [b_{\sigma|_n}])$ for all $n < \omega$. That is, we have $\Sigma_\varphi(c, d, a_{\sigma|_n}, b_{\sigma|_n})$ for all $n < \omega$. In particular, $\{\varphi'(x, y_r, a_{\sigma|_n}, b_{\sigma|_n}) : n < \omega\}$ is consistent.

(ii) By construction; see (4).

(iii) Let $\eta, \nu \in \omega^{<\omega}$ such that $\eta^\frown 0 \preceq \nu$. Then $\models \psi(a_{\eta^\frown 1}, [b_{\eta^\frown 1}], a_\nu, [b_\nu])$, so $\models \Sigma_\psi(a_{\eta^\frown 1}, b_{\eta^\frown 1}, a_\nu, b_\nu)$ and in particular $\models \psi'(a_{\eta^\frown 1}, b_{\eta^\frown 1}, a_\nu, b_\nu)$. ☐

Definition 10.19. We say that a theory satisfies the *existence axiom for forking* if $\mathrm{tp}(a/B)$ does not fork over B for any a and B.

Theorem 10.20. *The theory T satisfies the existence axiom for forking if and only if $T^{\mathcal{E}}$ satisfies the existence axiom for forking.*

Proof. One direction is immediate: anything witnessing forking in T will also be in $T^{\mathcal{E}}$. We prove the other direction. So assume there is $\mathrm{tp}(a[b]/C[D])$ that forks over $C[D]$. That is, it implies a (possibly infinite) disjunction $\bigvee_{i \in I} \varphi_i(xy, e^i[f^i])$ with $\varphi_i(xy, e^i[f^i])$ dividing over $C[D]$ for each $i \in I$. For each $i \in I$ we let $(e^i_j[f^i_j])_{j \in J}$ be a long enough $C[D]$-indiscernible sequence with $e^i_0[f^i_0] = e^i[f^i]$

such that $\{\varphi_i(xy, e^i_j[f^i_j]) : j \in J\}$ is inconsistent. By Lemma 10.12 we may assume that $e^i_j f^i_j \equiv e^i f^i$ for every $j \in J$. We claim that $\Sigma_{\varphi_i}(x, y_r, e^i, f^i)$ (see Lemma 10.10) divides over CD for all $i \in I$. Note that Σ_{φ_i} may contain parameters from CD.

To prove the claim let k be such that $\{\varphi_i(xy, e^i_j[f^i_j]) : j \in J_0\}$ is inconsistent for all $J_0 \subseteq J$ with $|J_0| = k$. So $\bigcup_{j \in J_0} \Sigma_{\varphi_i}(x, y_r, e^i_j, f^i_j)$ is inconsistent for all such J_0. Let $(e_n f_n)_{n<\omega}$ be a CD-indiscernible sequence based on $(e^i_j f^i_j)_{j \in J}$ over CD. Then there are $j_1 < \cdots < j_k \in J$ such that $e_1 f_1 \cdots e_k f_k \equiv_{CD} e^i_{j_1} f^i_{j_1} \cdots e^i_{j_k} f^i_{j_k}$, and therefore $\bigcup_{n<\omega} \Sigma_{\varphi_i}(x, y_r, e_n, f_n)$ is inconsistent. We conclude that $\Sigma_{\varphi_i}(x, y_r, e^i, f^i)$ divides over CD, as claimed.

By the claim there is $\psi_i(x, y_r, e^i, f^i)$ that is implied by $\Sigma_{\varphi_i}(x, y_r, e^i, f^i)$ such that $\psi_i(x, y_r, e^i, f^i)$ divides over CD, for all $i \in I$. Let $p = \mathrm{tp}(a[b]C[D])$. Then $\Sigma_p(x, y_r, C, D)$ implies $\bigvee_{i \in I} \Sigma_{\varphi_i}(x, y_r, e^i, f^i)$. We thus have that $\Sigma_p(x, y_r, C, D)$ implies $\bigvee_{i \in I} \psi_i(x, y_r, e^i, f^i)$. So $\Sigma_p(x, y_r, C, D)$ forks over CD. $\square$

In the discussion following Definition 4.1 in [Kim 2021] it is stated that one may produce results for Kim-independence for the hyperimaginary extension M^{heq} of a first-order structure M parallel with those for first-order structures, provided that M^{heq} satisfies the existence axiom for forking (which, by the above theorem, is equivalent to the assumption that T satisfies this axiom). More generally, one can ask if our results on Kim-independence over models in thick NSOP_1 theories can be extended to arbitrary base sets assuming the existence axiom for forking:

Question 10.21. Suppose T is a thick positive NSOP_1 theory satisfying the existence axiom for forking. Can $\mathop{\smile\!\!\!\!\!\!|}^K$ be extended to an automorphism-invariant ternary relation between arbitrary small sets which satisfies the properties listed in Theorem 9.1?

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Henselianity in NIP $\mathbb{F}_p$-algebras

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We prove an assortment of results on (commutative and unital) NIP rings, especially $\mathbb{F}_p$-algebras. Let R be a NIP ring. Then every prime ideal or radical ideal of R is externally definable, and every localization $S^{-1}R$ is NIP. Suppose R is additionally an $\mathbb{F}_p$-algebra. Then R is a finite product of henselian local rings. Suppose in addition that R is integral. Then R is a henselian local domain, whose prime ideals are linearly ordered by inclusion. Suppose in addition that the residue field $R/\mathfrak{m}$ is infinite. Then the Artin–Schreier map $R \to R$ is surjective (generalizing the theorem of Kaplan, Scanlon, and Wagner for fields).

1. Introduction

The class of NIP theories has played a major role in contemporary model theory. See [Simon 2015] for an introduction to NIP. In recent years, much work has been done on the problem of classifying NIP fields and NIP rings. A conjectural classification of NIP fields has emerged through work of Anscombe, Halevi, Hasson, and Jahnke [Halevi et al. 2019; Anscombe and Jahnke 2019], and partial results towards this conjectural classification have been obtained by the author in the setting of finite dp-rank [Johnson 2015; 2020; 2021b].

NIP fields are closely connected to NIP valuation rings. Conjecturally:

- Every NIP valuation ring is henselian.
- Every infinite NIP field is elementarily equivalent to $\mathrm{Frac}(R)$ for some NIP nontrivial valuation ring R.

These conjectures form the basis for the proposed classification of NIP fields [Anscombe and Jahnke 2019], and are known to hold assuming finite dp-rank [Johnson 2020]. Additionally, the henselianity conjecture is known in positive characteristic: if R is a NIP valuation ring and $\mathrm{Frac}(R)$ has positive characteristic, then R is henselian [Johnson 2021a, Theorem 2.8].

More generally, one would like to understand (commutative) NIP rings, especially NIP integral domains. A first step in this direction is the recent work of d'Elbée and Halevi [2021] on dp-minimal integral domains. Among other things, they show

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that if R is a dp-minimal integral domain, then R is a local ring, the prime ideals of R are a chain, the localization of R at any nonmaximal prime is a valuation ring, and R is a valuation ring whenever its residue field is infinite.

In the present paper, we consider a NIP integral domain R such that $\mathrm{Frac}(R)$ has positive characteristic. By analogy with [d'Elbée and Halevi 2021], we show that R is a local ring whose primes ideals are linearly ordered by inclusion. Generalizing the earlier henselianity theorem for valuation rings, we show that R is a henselian local ring. These results may help to extend the work of d'Elbée and Halevi to "positive characteristic" NIP integral domains.

Main results. All rings are assumed to be commutative and unital. In Section 2 we consider a general NIP ring R. Our main results are the following:

- Any localization $S^{-1}R$ is interpretable in the Shelah expansion R^{Sh}, and is therefore NIP (Theorem 2.11).
- Any radical ideal in R is externally definable (Theorem 2.14).

In Section 3, we restrict to the case where R is an $\mathbb{F}_p$-algebra, and obtain significantly stronger results:

- R is a finite product of henselian local rings (Theorem 3.21).
- If R is an integral domain, then R is a henselian local domain (Theorem 3.22), and the prime ideals of R are linearly ordered by inclusion (Theorem 3.15).
- If R is a local integral domain with maximal ideal $\mathfrak{m}$ and $R/\mathfrak{m}$ is infinite, then the Artin–Schreier map $R \to R$ is surjective (Theorem 3.4).

The henselianity results generalize [Johnson 2021a, Theorem 2.8], which handled the case where R is a valuation ring. The surjectivity of the Artin–Schreier map generalizes a theorem of Kaplan, Scanlon, and Wagner [Kaplan et al. 2011, Theorem 4.4], which handled the case where R is a field.

2. General NIP rings

2A. ***Finite width.*** The *width* of a poset $(P, \leq)$ is the maximum size of an antichain in P. We write $\operatorname{Spec} R$ for the poset of prime ideals in R, ordered by inclusion. This is an abuse of notation, since we are forgetting the usual scheme and topology structure on $\operatorname{Spec} R$, and then adding the poset structure.

Fact 2.1. *Let R be a NIP ring. Then* $\operatorname{Spec} R$ *has finite width. Moreover, there is a uniform finite bound on the width of* $\operatorname{Spec} R'$ *for $R' \succeq R$.*

Fact 2.1 is proved by d'Elbée and Halevi [2021, Proposition 2.1, Remark 2.2], who attribute it to Pierre Simon.

Fact 2.1 has a number of useful corollaries, which we shall use in later sections. First of all, Dilworth's theorem gives the following corollary:

Corollary 2.2. *If R is a NIP ring, then* Spec R *is a finite union of chains.*

Another trivial corollary of Fact 2.1 is the following:

Corollary 2.3. *If R is a NIP ring, then R has finitely many maximal ideals and finitely many minimal prime ideals.*

Also, using Beth's implicit definability, we see the following:

Corollary 2.4. *If R is a NIP ring, then the maximal ideals of R are definable.*

For completeness, we give the proof. The proof uses the following form of Beth's theorem:

Fact 2.5. *Let M be an L_0-structure. Let L be a language extending L_0 and let T be an L-theory. Suppose there is a cardinal κ such that for any $M' \succeq M$ there are at most κ-many expansions of M' to a model of T. Then every such expansion is an expansion by definitions.*

Proof of Corollary 2.4. Let L_0 be the language of rings and L be $L_0 \cup \{P\}$, where P is a unary predicate symbol. Let T be the statement saying that P is a maximal ideal, i.e.,

$$\forall x, y : P(x) \wedge P(y) \to P(x+y),$$

$$P(0),$$

$$\forall x, y : P(x) \to P(x \cdot y),$$

$$\neg P(1),$$

$$\forall x : \neg P(x) \to \exists y : P(xy-1).$$

If $R' \succeq R$, then an expansion of R' to a model of T is the same thing as a maximal ideal of R'. The number of such maximal ideals is uniformly bounded by Fact 2.1, and so Fact 2.5 shows that each such maximal ideal is definable. $\square$

(Of course, there are other, more direct, algebraic proofs of Corollary 2.4.)

Recall that the Jacobson radical of a ring is the intersection of its maximal ideals.

Corollary 2.6. *Let R be a NIP integral domain. Then the Jacobson radical of R is nonzero.*

Proof. In a domain, the intersection of two nonzero ideals is nonzero. $\square$

Corollary 2.7. *Let R be a NIP integral domain that is not a field. Let $K =$ Frac(R). There is a nontrivial, nondiscrete Hausdorff field topology on K characterized by either of the following*:

- *The family of sets $\{aR : a \in K^\times\}$ is a neighborhood basis of* 0.
- *The set of nonzero ideals of R is a neighborhood basis of* 0.

Proof. Everything follows formally by [Prestel and Ziegler 1978, Example 1.2], except that we only get a *ring* topology. It remains to see that the map $x \mapsto 1/x$ is continuous. It suffices to consider continuity around $x = 1$. Let I be a nonzero ideal in R. We claim there is a nonzero ideal I' such that if $x \in 1 + I'$, then $1/x \in 1 + I$. Indeed, take $I' = I \cap J$, where J is the Jacobson radical. Suppose $x \in 1 + (I \cap J)$. Then $x - 1$ is in every maximal ideal, implying that x is in no maximal ideals, so $x \in R^\times$. Also, $x \in 1 + I$ implies that $1 - x \in I$, and then $x^{-1}(1 - x) \in I$, because x is a unit. But $x^{-1}(1 - x) = x^{-1} - 1$, and so $x^{-1} \in 1 + I$ as desired. □

Lemma 2.8. *If R and S are NIP rings and $R \equiv S$, then R and S have the same number of maximal ideals.*

Proof. It suffices to show that S has as many maximal ideals as R. By Corollaries 2.3 and 2.4 we can write the maximal ideals of R as $\phi_1(R, a_1), \dots, \phi_n(R, a_n)$ for some formulas ϕ_i and parameters a_i from R. Let $\psi(y_1, \dots, y_n)$ be the formula asserting

the sets $\phi_1(R, y_1), \dots, \phi_n(R, y_n)$ are pairwise distinct maximal ideals.

The formula ψ is satisfied by the tuple $(a_1, \dots, a_n)$ in R, so it is satisfied by some tuple in S, giving n distinct maximal ideals in S. □

2B. *Localizations.* If M is a structure, then M^{Sh} denotes the Shelah expansion of M. If M is NIP, then the definable sets in M^{Sh} are exactly the externally definable sets in M, and M^{Sh} is NIP [Simon 2015, Proposition 3.23, Corollary 3.24].

Say that a collection of sets $\mathcal{C}$ is "uniformly definable" in a structure M if $\mathcal{C} \subseteq \{X_a : a \in Y\}$ for some definable family of sets $\{X_a\}_{a \in Y}$.

Remark 2.9. Let M be a structure. Suppose $D = \bigcup_{i \in I} D_i$ is a directed union, and the D_i are uniformly definable in M. Then D is externally definable.

This is well known in certain circles, but here is the proof for completeness:

Proof. Take some $L(M)$-formula $\phi(x, y)$ such that $D_i = \phi(M, b_i)$ for some $b_i \in M^y$. Let $\Sigma(y)$ be the partial type

$$\{\phi(a, y) : a \in D\} \cup \{\neg\phi(a, y) : a \in M^x \setminus D\}.$$

Then $\Sigma(y)$ is finitely satisfiable, because for any $a_1, \dots, a_n \in D$ and $e_1, \dots, e_m \in M^x \setminus D$ we can find some i such that $D_i \supseteq \{a_1, \dots, a_n\}$, because the union is directed. Then $D_i \subseteq D$, so $D_i \cap \{e_1, \dots, e_m\} = \varnothing$. Thus b_i satisfies the relevant finite fragment of $\Sigma(y)$. By compactness there is a realization b of $\Sigma(y)$ in an elementary extension $N \succeq M$. Then $\phi(M, b) = D$, by definition of $\Sigma(y)$, so D is externally definable. □

Lemma 2.10. *Let R be a NIP ring. Let S be a multiplicative subset. Then there is an externally definable multiplicative subset $\bar{S}$ such that the localization $S^{-1}R$ is isomorphic (as an R-algebra) to $\bar{S}^{-1}R$.*

Proof. For any $x \in R$, let F_x denote the set of $y \in R$ such that $y \mid x$. Let $\bar{S} = \bigcup_{x \in S} F_x$. Note that if A is a ring and $f : R \to A$ is a homomorphism, then the following are equivalent:

- $f(s)$ is invertible for every $s \in S$.
- $f(x)$ is invertible for x, y, s with $xy = s$ and $s \in S$.
- $f(x)$ is invertible for x, s with $x \in F_s$ and $s \in S$.
- $f(x)$ is invertible for $x \in \bar{S}$.

Therefore $S^{-1}R$ and $\bar{S}^{-1}R$ represent the same functor, and are isomorphic.

It remains to see that $\bar{S}$ is externally definable. This follows by Remark 2.9 because the sets F_x are uniformly definable, and the union $\bigcup_{x \in S} F_x$ is a directed union. Indeed, if $x, y \in S$, then $xy \in S$ and $F_{xy} \supseteq F_x \cup F_y$. □

Theorem 2.11. *Let R be a NIP ring. Let S be a multiplicative subset. Then the localization $S^{-1}R$ and the homomorphism $R \to S^{-1}R$ are interpretable in R^{Sh}.*

Proof. By Lemma 2.10, we may replace S with an externally definable set $\bar{S}$, and then the result is clear. □

Corollary 2.12. *Let R be a NIP ring. Let S be a multiplicative subset. Then the localization $S^{-1}R$ is also NIP.*

Proof. The localization $S^{-1}R$ is interpretable in the NIP structure R^{Sh}. □

Corollary 2.12 generalizes part of [d'Elbée and Halevi 2021, Proposition 2.8(2)], dropping the assumptions that S is externally definable and R is integral.

Proposition 2.13. *Let R be a NIP ring. Let $\mathfrak{p}$ be a prime ideal in R. Then $\mathfrak{p}$ is externally definable.*

Proof. By Theorem 2.11, we can interpret $R \to R_{\mathfrak{p}}$ in R^{Sh}. The maximal ideal of $R_{\mathfrak{p}}$ is definable in $R_{\mathfrak{p}}$, as the set of nonunits. It pulls back to $\mathfrak{p}$ in R. Therefore $\mathfrak{p}$ is definable in R^{Sh}, hence externally definable in R. □

Proposition 2.13 generalizes a theorem of d'Elbée and Halevi, who proved that (certain) prime ideals in dp-minimal domains are externally definable [d'Elbée and Halevi 2021, Lemma 3.3].

Theorem 2.14. *Let R be a NIP ring. Let I be a radical ideal in R. Then I is externally definable.*

Proof. By Corollary 2.2, we can cover the set $\operatorname{Spec} R$ of prime ideals in R with finitely many chains $\mathcal{C}_1, \ldots, \mathcal{C}_n$. The ideal I is an intersection of prime ideals. Let $\mathfrak{p}_i$ be the intersection of the prime ideals $\mathfrak{p} \in \mathcal{C}_i$ with $\mathfrak{p} \supseteq I$. An intersection of a chain of prime ideals is prime, so $\mathfrak{p}_i$ is prime. Then I is a finite intersection $\bigcap_{i=1}^{n} \mathfrak{p}_i$. Each $\mathfrak{p}_i$ is externally definable by Proposition 2.13. □

Corollary 2.15. *Let R be a NIP ring. Let I be a radical ideal. The quotient R/I is NIP.*

Proof. The quotient R/I is interpretable in the NIP structure R^{Sh}. □

2C. ***Automatic connectedness.*** If G is a definable or type-definable group, then G^{00} is the smallest type-definable group of bounded index in G. In a NIP context, G^{00} always exists, and is type-definable over whatever parameters define G [Hrushovski et al. 2008, Proposition 6.1]

Proposition 2.16. *Let R be a NIP ring. Suppose that $R/\mathfrak{m}$ is infinite for every maximal ideal $\mathfrak{m}$ of R.*

(1) *If I is a definable ideal of R, then $I = I^{00}$.*

(2) *If R is a domain and $K = \mathrm{Frac}(R)$ and if I is a definable R-submodule of K, then $I = I^{00}$.*

In particular, in either case, I has no definable proper subgroups of finite index.

Proof. We may assume R is a monster model, i.e., κ-saturated for some big cardinal κ. "Small" will mean "cardinality less than κ", and "large" will mean "not small."

Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be the maximal ideals of R. By Corollary 2.3 there are only finitely many, and by Corollary 2.4 they are all definable. The quotients $R/\mathfrak{m}_i$ are infinite, hence large. Therefore every simple R-module is large. Every nontrivial R-module has a simple subquotient, so every nontrivial R-module is large.

Now suppose I is a definable ideal. If $a \in R$, then the map $I \to I$ sending x to ax must map I^{00} into I^{00}. Indeed, if we let $J = \{x \in I : ax \in I^{00}\}$, then J is a type-definable subgroup of I of bounded index, so $J \supseteq I^{00}$. Thus we see that for any $a \in R$, we have $aI^{00} \subseteq I^{00}$. In other words, I^{00} is an ideal. The quotient I/I^{00} is an R-module. By definition of G^{00}, the quotient I/I^{00} is small. We saw that nontrivial R-modules are large, so I/I^{00} must be trivial, implying $I = I^{00}$. This proves (1), and (2) is similar. □

3. NIP $\mathbb{F}_p$-algebras

3A. ***A variant of the Kaplan–Scanlon–Wagner theorem.*** In [Kaplan et al. 2011, Theorem 4.4], Kaplan, Scanlon, and Wagner show that if K is an infinite NIP field of characteristic $p > 0$, then the Artin–Schreier map $x \mapsto x^p - x$ is a surjection from K onto K. The same idea can be applied to certain local rings, as we will see in Theorem 3.4 below.

Before proving the theorem, we need some (well-known) lemmas on additive polynomials. Fix a field K of characteristic p. If $c \in K$, define

$$g_c(x) = x^p - c^{p-1}x.$$

The polynomial $g_c(x)$ defines an additive homomorphism from K to K. If V is a finite-dimensional $\mathbb{F}_p$-linear subspace of K (i.e., a finite subgroup of $(K, +)$), define

$$f_V(x) = \prod_{a \in V} (x - a). \tag{1}$$

We will see shortly that f_V is an additive homomorphism.

Lemma 3.1. *If $c \in K$ is nonzero, then $g_c(x) = f_{\mathbb{F}_p \cdot c}(x)$. In particular, $f_{\mathbb{F}_p \cdot c}(x)$ is an additive homomorphism.*

Proof. Note that $g_c(c) = 0$. Therefore, $\ker g_c$ contains the subgroup generated by c, which is $\mathbb{F}_p \cdot c$. Since g_c is monic of degree p, and $|\mathbb{F}_p \cdot c| = p$, we must have

$$g_c(x) = \prod_{a \in \mathbb{F}_p \cdot c} (x - a) = f_{\mathbb{F}_p \cdot c}(x). \qquad \square$$

Lemma 3.2. *Suppose $V_1 \subseteq V_2$ are finite-dimensional subspaces of K such that $\dim V_2 = \dim V_1 + 1$. Suppose f_{V_1} is an additive homomorphism on K. Then there is $c \in f_{V_1}(V_2)$ such that $f_{V_2} = g_c \circ f_{V_1}$, and in particular f_{V_2} is an additive homomorphism on K.*

Proof. Take $a \in V_2 \setminus V_1$ and let $c = f_{V_1}(a)$. Let $h = g_c \circ f_{V_1}$. Then h is an additive homomorphism on K, and it suffices to show that $h = f_{V_2}$. Note that if $x \in V_1$, then $h(x) = g_c(f_{V_1}(x)) = g_c(0) = 0$, since f_{V_1} vanishes on V_1. Additionally, $h(a) = g_c(f_{V_1}(a)) = g_c(c) = 0$. Thus the kernel of h contains V_1 as well as a. It therefore contains the group they generate, which is $V_1 + \mathbb{F}_p \cdot a = V_2$. If $d = \dim V_1$, then $|V_1| = p^d$ and $|V_2| = p^{d+1}$. The polynomial f_{V_1} is a monic polynomial of degree p^d, and g_c is a monic polynomial of degree p. Therefore the composition h is a monic polynomial of degree p^{d+1}. We have just seen that h vanishes on the set V_2 of size p^{d+1}, so $h(x)$ must be $\prod_{u \in V_2}(x - u) = f_{V_2}(x)$. $\square$

Lemma 3.3. *If V is a finite-dimensional subspace of K, then f_V is an additive homomorphism with kernel V.*

Proof. The fact that f_V is an additive homomorphism follows by induction on $\dim V$ using Lemma 3.2. The fact that $\ker f_V = V$ is immediate from the definition of f_V. $\square$

We now can prove our desired theorem on NIP local domains in positive characteristic:

Theorem 3.4. *Let $p > 0$ be a prime. Let R be a NIP $\mathbb{F}_p$-algebra with the following properties: R is a local ring, R is an integral domain with maximal ideal $\mathfrak{m}$, and the quotient field $k = R/\mathfrak{m}$ is infinite. Then $x \mapsto x^p - x$ is a surjection from R onto R.*

Proof. Let $K = \mathrm{Frac}(R)$. Note that if V is a finite-dimensional $\mathbb{F}_p$-subspace of R, then $f_V(x) \in R[x]$, and if $c \in R$, then $g_c(x) \in R[x]$.

Claim 3.5. *It suffices to find $c \in R^\times$ such that $g_c(x)$ is a surjection from R to R.*

Proof of claim. Note that $c^{-p} g_c(cx) = c^{-p}(c^p x^p - c^{p-1} cx) = x^p - x$. The maps $x \mapsto cx$ and $x \mapsto c^{-p}x$ are bijections on R, so if g_c is surjective then so is $g_1(x) = x^p - x$. □

For any $c \in R$, the polynomial $g_c(x)$ defines an additive map $R \to R$, whose image $g_c(R)$ is an additive subgroup of R. Let $\mathcal{G} = \{g_c(R) : c \in R\}$. By the Baldwin–Saxl theorem for NIP groups, there is some integer n such that if $G_1, \dots, G_n \in \mathcal{G}$, then there is some i such that

$$G_i \supseteq G_1 \cap \cdots \cap G_{i-1} \cap G_{i+1} \cap \cdots \cap G_n.$$

Fix such an $n \geq 2$.

The residue field k is infinite, and therefore we can find $\mathbb{F}_p$-linearly independent $\alpha_1, \dots, \alpha_n \in k$. Take $a_i \in R$ lifting $\alpha_i \in k$. Note $\alpha_i \neq 0$, so $a_i \notin \mathfrak{m}$, and thus $a_i \in R^\times$. Also note that the elements $\{a_1, \dots, a_{n-1}\}$ are $\mathbb{F}_p$-linearly independent in K.

Let $[n] = \{1, \dots, n\}$. If $S \subseteq [n]$ and $i \in [n]$, we write $S \cup i$ and $S \setminus i$ as abbreviations for $S \cup \{i\}$ and $S \setminus \{i\}$. Even worse, we sometimes abbreviate $\{i\}$ as i.

For $S \subseteq [n]$, let V_S be the $\mathbb{F}_p$-linear span of $\{a_i : i \in S\}$. Then V_S has dimension $|S|$. Let

$$f_S(x) := f_{V_S}(x) = \prod_{a \in V_S} (x - a).$$

This is a monic polynomial in $R[x]$. By Lemma 3.3 $f_S(x)$ induces an additive homomorphism $K \to K$, and therefore an additive homomorphism $R \to R$.

Note that $f_i(x) = f_{V_i}(x) = f_{\mathbb{F}_p \cdot a_i}(x) = g_{a_i}(x)$ by Lemma 3.1. By Claim 3.5, it suffices to show that f_i is a surjection from R to R, for at least one i.

If $S \subseteq [n]$ and $i \in [n] \setminus S$, then $V_{S \cup i}$ has dimension one more than V_S. By Lemma 3.2, there is some $c_{S,i} \in f_S(V_{S \cup i})$ such that $g_{c_{S,i}} \circ f_S = f_{S \cup i}$. Let $g_{S,i} := g_{c_{S,i}}$. Then

$$g_{S,i} \circ f_S = f_{S \cup i}.$$

Now $c_{S,i} \in f_S(V_{S \cup i})$, but $f_S(x) \in R[x]$ and $V_{S \cup i} \subseteq R$. Therefore $c_{S,i} \in R$, and $g_{S,i}(x) \in R[x]$.

Claim 3.6. *If $S \subseteq [n]$ and i, j are distinct elements of $[n] \setminus S$, then $c_{S,i}^{p-1} - c_{S,j}^{p-1} \notin \mathfrak{m}$.*

Proof of claim. Otherwise, the two polynomials $g_{S,i}(x)$ and $g_{S,j}(x)$ have the same reduction modulo $\mathfrak{m}$. From the identities $f_{S \cup i} = g_{S,i} \circ f_S$ and $f_{S \cup j} = g_{S,j} \circ f_S$, it follows that $f_{S \cup i} \equiv f_{S \cup j} \pmod{\mathfrak{m}}$. Let V'_S be the $\mathbb{F}_p$-linear span of $\{\alpha_i : i \in S\}$, or equivalently, the image of V_S under $R \to R/\mathfrak{m}$. By inspection, the reduction of f_S modulo $\mathfrak{m}$ is $\prod_{u \in V'_S}(x - u)$. Since $V'_{S \cup i} \neq V'_{S \cup j}$, it follows immediately that $f_{S \cup i}$ and $f_{S \cup j}$ cannot have the same reduction modulo $\mathfrak{m}$, a contradiction. □

Each of the groups $g_{[n]\setminus i,i}(R)$ is in the family $\mathcal{G}$. By choice of n, one of the factors in the intersection $\bigcap_{i=1}^n g_{[n]\setminus i,i}(R)$ is irrelevant. Without loss of generality, it is the first factor:

$$g_{[n]\setminus 1,1}(R) \supseteq \bigcap_{i=2}^{n} g_{[n]\setminus i,i}(R). \tag{2}$$

We claim that $f_1(x)$ defines a surjection from R to R. As $f_1(x) = g_{a_1}(x)$, this suffices, by Claim 3.5.

Take some $b_1 \in R$. It suffices to show that $b_1 \in f_1(R)$. Take some $b_\varnothing \in K^{\mathrm{alg}}$ such that $f_1(b_\varnothing) = b_1$. It suffices to show that $b_\varnothing \in R$. For $S \subseteq [n]$, define $b_S = f_S(b_\varnothing) \in K^{\mathrm{alg}}$. (When $S = \{1\}$ this recovers b_1, and when $S = \varnothing$ this recovers $b_\varnothing$, so the notation is consistent.) Note that

$$g_{S,i}(b_S) = g_{S,i}(f_S(b_\varnothing)) = f_{S\cup i}(b_\varnothing) = b_{S\cup i}. \tag{3}$$

Claim 3.7. *If* $1 \in S \subseteq [n]$, *then* $b_S \in R$.

Proof of claim. Take a minimal counterexample S. If $S = \{1\}$, then $b_S = b_1 \in R$. Otherwise, take $i \in S \setminus 1$ and let $S_0 = S \setminus i$. By choice of S, we have $b_{S_0} \in R$. Then $b_S = g_{S_0,i}(b_{S_0})$. But $g_{S_0,i}(x) \in R[x]$, so $b_S \in R$. ☐

In particular, $b_S \in R$ for $S = [n]$, as well as $S = [n] \setminus i$ for $i > 1$. Then

$$b_{[n]} = g_{[n]\setminus i,i}(b_{[n]\setminus i}) \in g_{[n]\setminus i,i}(R)$$

for $1 < i \leq n$. By (2), $b_{[n]} \in g_{[n]\setminus 1,1}(R)$. Take $v \in R$ such that $g_{[n]\setminus 1,1}(v) = b_{[n]}$. Then $g_{[n]\setminus 1,1}(v) = b_{[n]} = g_{[n]\setminus 1,1}(b_{[n]\setminus 1})$, and so

$$v - b_{[n]\setminus 1} \in \ker g_{[n]\setminus 1,1} = \mathbb{F}_p \cdot c_{[n]\setminus 1,1} \subseteq R.$$

Therefore $b_{[n]\setminus 1} \in R$. So we see that

$$b_{[n]\setminus i} \in R \quad \text{for all } 1 \leq i \leq n. \tag{4}$$

Claim 3.8. $b_\varnothing \in R$.

Proof of claim. Suppose otherwise. Take S maximal such that $b_S \notin R$. By Claim 3.7 and (4), S is neither $[n]$ nor $[n] \setminus i$ for $1 \leq i \leq n$. Therefore $[n] \setminus S$ contains at least two elements i, j. By choice of S, we have $b_{S\cup i} \in R$ and $b_{S\cup j} \in R$. By (3),

$$b_{S\cup i} = g_{S,i}(b_S) = b_S^p - c_{S,i}^{p-1} b_S \quad \text{and} \quad b_{S\cup j} = g_{S,j}(b_S) = b_S^p - c_{S,j}^{p-1} b_S.$$

Therefore

$$(c_{S,i}^{p-1} - c_{S,j}^{p-1}) b_S = b_{S\cup j} - b_{S\cup i} \in R.$$

By Claim 3.6, $c_{S,i}^{p-1} - c_{S,j}^{p-1} \in R \setminus \mathfrak{m} = R^\times$, and so $b_S \in R$, a contradiction. ☐

This completes the proof. We see that $b_\varnothing \in R$, and so $b_1 = f_1(b_\varnothing) \in f_1(R)$. As b_1 was an arbitrary element of R, it follows that f_1 gives a surjection from R to R. But $f_1(x) = g_{a_1}(x)$, and $a_1 \in R^\times$ (since its residue mod $\mathfrak{m}$ is the nonzero element α_1), and so we are done by Claim 3.5. □

3B. *Linearly ordering the primes.*

Lemma 3.9. *Let R be an $\mathbb{F}_p$-algebra that is integral and has exactly two maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Suppose that $R/\mathfrak{m}_1$ and $R/\mathfrak{m}_2$ are infinite. Then R isn't NIP.*

The proof uses an identical strategy to [Johnson 2021a, Lemma 2.6].

Proof. Suppose R is NIP. By Corollary 2.4, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are definable. Let $K = \mathrm{Frac}(R)$. Regard the localizations $R_{\mathfrak{m}_1}$ and $R_{\mathfrak{m}_2}$ as definable subrings of K. Note that $R_{\mathfrak{m}_1} \cap R_{\mathfrak{m}_2} = R$, by commutative algebra. (If $x \in K \setminus R$, then let $I = \{a \in R : ax \in R\}$; this is a proper ideal in R, so it is contained in some $\mathfrak{m}_i$, and then $I \subseteq \mathfrak{m}_i$ means precisely that $x \notin R_{\mathfrak{m}_i}$.)

Claim 3.10. *If $x \in R$, then the Artin–Schreier roots of x are in R.*

Proof of claim. The rings $R_{\mathfrak{m}_1}$ and $R_{\mathfrak{m}_2}$ satisfy the conditions of Theorem 3.4. (The residue field of $R_{\mathfrak{m}_i}$ is isomorphic to $R/\mathfrak{m}_i$, hence infinite.) Therefore, there are $y \in R_{\mathfrak{m}_1}$ and $z \in R_{\mathfrak{m}_2}$ such that $y^p - y = x = z^p - z$. Then $y - z$ is in the kernel of the Artin–Schreier map, which is $\mathbb{F}_p$, so $y \in z + \mathbb{F}_p \subseteq R_{\mathfrak{m}_2}$. As $y \in R_{\mathfrak{m}_1}$, this implies $y \in R_{\mathfrak{m}_1} \cap R_{\mathfrak{m}_2} = R$. Thus, at least one Artin–Schreier root (y) is in R. The other Artin–Schreier roots of x are the elements of $y + \mathbb{F}_p$, which are all in R. □

Let $J = \mathfrak{m}_1 \cap \mathfrak{m}_2$. This is the Jacobson radical of R. By Proposition 2.16, $J = J^{00}$, and there are no definable subgroups of finite index. Consider the sets

$$\Delta = \{(x, i, j) \in R \times \mathbb{F}_p \times \mathbb{F}_p : x - i \in \mathfrak{m}_1,\ x - j \in \mathfrak{m}_2\},$$
$$\Gamma = \{(x^p - x, i - j) : (x, i, j) \in \Delta\}.$$

Then $(\Delta, +)$ and $(\Gamma, +)$ are definable groups.

Claim 3.11. *Γ is the graph of a group homomorphism ψ from $(J, +)$ onto $(\mathbb{F}_p, +)$.*

Proof of claim. First, we show that $\Gamma \subseteq J \times \mathbb{F}_p$. Suppose that $(x, i, j) \in \Delta$. Then $x \equiv i \pmod{\mathfrak{m}_1}$, so $x^p - x \equiv i^p - i \equiv 0 \pmod{\mathfrak{m}_1}$, and $x^p - x \in \mathfrak{m}_1$. Similarly, $x^p - x \in \mathfrak{m}_2$, and therefore $x^p - x \in J$. Thus $(x^p - x, i - j) \in J \times \mathbb{F}_p$.

Next we show that Γ projects onto J. Take $y \in J$. By Claim 3.10 there is $x \in R$ with $x^p - x = y$. Then $x^p - x \equiv y \equiv 0 \pmod{\mathfrak{m}_1}$, so $x^p - x \equiv i \pmod{\mathfrak{m}_1}$ for some $i \in \mathbb{F}_p$. Similarly, $x^p - x \equiv j \pmod{\mathfrak{m}_2}$ for some $j \in \mathbb{F}_p$. Then $(x, i, j) \in \Delta$ and $(x^p - x, i - j) = (y, i - j) \in \Gamma$.

Next we show that the projection $\Gamma \to J$ is one-to-one. Otherwise, $\Gamma \to J$ has nontrivial kernel, so there is $(x, i, j) \in \Delta$ with $x^p - x = 0$ but $i - j \neq 0$. The fact that $x^p - x = 0$ implies $x \in \mathbb{F}_p$, and so $x \equiv i \pmod{\mathfrak{m}_1}$ implies $x = i$. Similarly, $x = j$. But then $i - j = 0$, a contradiction.

So now we see that $\Gamma \to J$ is one-to-one and onto, implying that Γ is the graph of some group homomorphism ψ from J to $\mathbb{F}_p$. It remains to show that ψ is onto. Equivalently, we must show that Γ projects onto $\mathbb{F}_p$. Let $i \in \mathbb{F}_p$ be given. By the Chinese remainder theorem, there is $x \in R$ such that $x \equiv i \pmod{\mathfrak{m}_1}$ and $x \equiv 0 \pmod{\mathfrak{m}_2}$. Then $(x, i, 0) \in \Delta$, so $(x^p - x, i - 0) \in \Gamma$. The element $(x^p - x, i)$ projects onto i. Equivalently, $\psi(x^p - x) = i$. □

Therefore there is a definable surjective group homomorphism $\psi : J \to \mathbb{F}_p$. The kernel $\ker \psi$ is a definable subgroup of J of index p. This contradicts Proposition 2.16. □

Lemma 3.12. *Let R be a NIP integral $\mathbb{F}_p$-algebra. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be prime ideals such that $R/\mathfrak{p}_1$ and $R/\mathfrak{p}_2$ are infinite. Then $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are comparable.*

Proof. Suppose otherwise. Let $S = R \setminus (\mathfrak{p}_1 \cup \mathfrak{p}_2)$. Then S is a multiplicative subset of R. Let $R' = S^{-1}R$. Then R' is NIP by Corollary 2.12. The ring R' has exactly two maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, where $\mathfrak{m}_i = \mathfrak{p}_i R'$. The map $R/\mathfrak{p}_i \to R'/\mathfrak{m}_i$ is injective, so $R'/\mathfrak{m}_i$ is infinite, for $i = 1, 2$. This contradicts Lemma 3.9. □

Lemma 3.13. *Let R be an $\mathbb{F}_p$-algebra that is integral and has exactly two maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Then R isn't NIP.*

Proof. Assume otherwise. Going to an elementary extension, we may assume that R is very saturated (Lemma 2.8). By the Chinese remainder theorem, there is some $a \in R$ such that $a \equiv 0 \pmod{\mathfrak{m}_1}$ but $a \equiv 1 \pmod{\mathfrak{m}_2}$.

Let $\Sigma(x)$ be the partial type saying that $x \in \mathfrak{m}_1$, $x \notin \mathfrak{m}_2$, and x does not divide a^n for any n.

Claim 3.14. *$\Sigma(x)$ is finitely satisfiable.*

Proof of claim. Let n be given. We claim there is an x such that $x \in \mathfrak{m}_1$, $x \notin \mathfrak{m}_2$, and x does not divide a^i for $i \leq n$. Take $x = a^{n+1}$. Then $x \equiv 0^{n+1} \equiv 0 \pmod{\mathfrak{m}_1}$, so $x \in \mathfrak{m}_1$. But $x \equiv 1^{n+1} \equiv 1 \pmod{\mathfrak{m}_2}$, so $x \notin \mathfrak{m}_2$. Finally, suppose $x = a^{n+1}$ divides a^i for some $i \leq n$. Then there is $u \in R$ with $ua^{n+1} = a^i$. Since R is a domain, we can cancel a factor of a^i from both sides, and see $ua^{n+1-i} = 1$. This implies that a is a unit, contradicting the fact that $a \in \mathfrak{m}_1$. □

By saturation, there is $a' \in R$ satisfying $\Sigma(x)$. The principal ideal (a') does not intersect the multiplicative set $S := a^{\mathbb{N}}$, by definition of $\Sigma(x)$. Let $\mathfrak{p}_1$ be maximal among ideals containing (a') and avoiding S. Then $\mathfrak{p}_1$ is a prime ideal. (In general, any ideal that is maximal among ideals avoiding a multiplicative set is prime.)

Now $\mathfrak{p}_1 \not\subseteq \mathfrak{m}_2$, because $a' \in \mathfrak{p}_1$ but $a' \notin \mathfrak{m}_2$. But $\mathfrak{p}_1$ must be contained in *some* maximal ideal, and so $\mathfrak{p}_1 \subseteq \mathfrak{m}_1$. The inclusion is strict, because $a \in \mathfrak{m}_1$ but $a \notin \mathfrak{p}_1$. Thus $\mathfrak{p}_1 \subsetneq \mathfrak{m}_1$ and $\mathfrak{p}_1 \not\subseteq \mathfrak{m}_2$. In particular, $\mathfrak{p}_1$ is not a maximal ideal.

Similarly, there is a nonmaximal prime ideal $\mathfrak{p}_2$ with $\mathfrak{p}_2 \subsetneq \mathfrak{m}_2$ and $\mathfrak{p}_2 \not\subseteq \mathfrak{m}_1$. Then $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are incomparable. Otherwise, say, $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subsetneq \mathfrak{m}_2$, and so $\mathfrak{p}_1 \subseteq \mathfrak{m}_2$, a contradiction. For $i = 1, 2$, the fact that $\mathfrak{p}_i$ is a nonmaximal prime ideal implies that $R/\mathfrak{p}_i$ is a nonfield integral domain, and therefore infinite. This contradicts Lemma 3.12. $\square$

Theorem 3.15. *Let R be a NIP integral $\mathbb{F}_p$-algebra. Then the prime ideals of R are linearly ordered by inclusion.*

Proof. The same proof as Lemma 3.12, using Lemma 3.13 instead of Lemma 3.9. $\square$

Corollary 3.16. *Let R be a NIP $\mathbb{F}_p$-algebra. Let $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{q}$ be prime ideals. If $\mathfrak{p}_i \supseteq \mathfrak{q}$ for $i = 1, 2$, then $\mathfrak{p}_1$ is comparable to $\mathfrak{p}_2$.*

Proof. Otherwise, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ induce incomparable primes in the NIP domain $R/\mathfrak{q}$. $\square$

3C. *Henselianity.*

Definition 3.17. A *forest* is a poset $(P, \leq)$ with the property that if $x \in P$, then the set $\{y \in P : y \geq x\}$ is linearly ordered.

Definition 3.18. A ring R is *good* if Spec R is a forest of finite width.

Lemma 3.19. (1) *If R is a NIP $\mathbb{F}_p$-algebra, then R is good.*

(2) *If R is good, then any quotient R/I is good.*

(3) *If R is good, then R is a finite product of local rings.*

Proof. (1) Fact 2.1 and Corollary 3.16.

(2) This is clear, since Spec R/I is a subposet of Spec R.

(3) We now break our usual convention, and regard Spec R as a scheme, or at least a topological space. By scheme theory, it suffices to write Spec R as a finite disjoint union of clopen sets U_i such that each U_i contains a unique closed point. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of R. There are finitely many because Spec R has finite width. Note that every prime ideal $\mathfrak{p} \in R$ satisfies $\mathfrak{p} \subseteq \mathfrak{m}_i$ for a unique i. (There is at least one i by Zorn's lemma, and at most one i because Spec R is a forest.) Let U_i be the set of primes below $\mathfrak{m}_i$. Then Spec R is a disjoint union of the U_i. It remains to show that each U_i is clopen. It suffices to show that each U_i is closed. Take $i = 1$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the minimal primes contained in $\mathfrak{m}_1$. (There are finitely many, because of finite width.) Let V_j be the set of primes containing $\mathfrak{p}_j$. Then V_j is a closed subset of Spec R — it is the closed subset cut out by the ideal $\mathfrak{p}_j$. Moreover, $V_j \subseteq U_1$, because Spec R is a forest. The sets $V_1, \ldots, V_m$ cover U_1, because every prime contains a minimal prime. Then U_1 is a finite union of closed sets $\bigcup_{i=1}^m V_i$, and so U_1 is closed. $\square$

Proposition 3.20. *Let R be a NIP local $\mathbb{F}_p$-algebra. Then R is a henselian local ring.*

Proof. By [Stacks 2005–, Lemma 04GG, condition (9)], it is sufficient to prove the following: any finite R-algebra is a product of local rings. Let S be a finite R-algebra. Let $a_1, \ldots, a_n$ be elements of S which generate S as an R-module. Each a_i is integral over R [Dummit and Foote 2004, Proposition 15.23], so there is a monic polynomial $P_i(x) \in R[x]$ such that $P_i(a) = 0$ in S. Then there is a surjective homomorphism

$$R[x_1, \ldots, x_n]/(P_1(x_1), \ldots, P_i(x_i)) \to S.$$

The ring on the left is interpretable in R — it is a finite-rank free R-module with basis the monomials $\prod_{i=1}^n x_i^{n_i}$ for $\bar{n} \in \prod_{i=1}^n \{0, 1, \ldots, \deg P_i - 1\}$. Therefore, the left-hand side is a NIP ring. By Lemma 3.19, it is good, S is good, and S is a finite product of local rings. □

Theorem 3.21. *Let R be a NIP $\mathbb{F}_p$-algebra. Then R is a finite product of henselian local rings.*

Proof. By Lemma 3.19, R is good, and R is a finite product of local rings. These local rings are easily seen to be interpretable in R, so they are also NIP. By Proposition 3.20, they are henselian local rings. □

Theorem 3.22. *Let R be a NIP, integral $\mathbb{F}_p$-algebra. Then R is a henselian local domain.*

Proof. R is a local ring by Theorem 3.15. So it is henselian by Proposition 3.20. □

Recall that a field K is *large* (also called *ample*) if every smooth irreducible K-curve with at least one K-point contains infinitely many K-points [Pop 2014]. By [Pop 2010, Theorem 1.1], if R is a henselian local domain that is not a field, then $\mathrm{Frac}(R)$ is large. Therefore we get the following corollary:

Corollary 3.23. *Let R be a NIP integral domain, and $K = \mathrm{Frac}(R)$. Suppose $R \neq K$ and K has positive characteristic. Then K is large.*

Large stable fields are classified [Johnson et al. 2020]. If we could extend this classification to large NIP fields, then Corollary 3.23 would tell us something very strong about NIP integral domains of positive characteristic.

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