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Keisler measures in the wild

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We investigate Keisler measures in arbitrary theories. Our initial focus is on Borel definability. We show that when working over countable parameter sets in countable theories, Borel definable measures are closed under Morley products and satisfy associativity. However, we also demonstrate failures of both properties over uncountable parameter sets. In particular, we show that the Morley product of Borel definable types need not be Borel definable (correcting an erroneous result from the literature). We then study various notions of generic stability for Keisler measures and generalize several results from the NIP setting to arbitrary theories. We also prove some positive results for the class of frequency interpretation measures in arbitrary theories, namely, that such measures are closed under convex combinations and commute with all Borel definable measures. Finally, we construct the first example of a complete type which is definable and finitely satisfiable in a small model, but not finitely approximated over any small model.

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Introduction

Finitely additive probability measures on definable sets were originally introduced by Keisler [1987] as a tool to study forking in NIP theories. Since then, Keisler measures have found extensive connections to various contexts in both pure and applied model theory. They played a pivotal role in resolving the Pillay conjectures on definably compact groups definable in o-minimal theories [Hrushovski et al. 2013] and, more generally, are a crucial tool in the study of definably amenable groups definable in NIP theories [Chernikov and Simon 2018; Onshuus and Pillay 2008]. Keisler measures are a vital component in the interplay between model theory and combinatorics (especially in connection to the regularity theorems) [Chernikov and Starchenko 2021; Conant et al. 2022]. Additionally, Keisler measures also arise naturally in continuous logic as types over models of the randomization [Ben Yaacov and Keisler 2009].

Despite this plethora of research, there is an obvious gap in the existing results: a clear structural understanding of Keisler measures only exists in NIP theories, and, with a few specialized exceptions, the deepest results concerning Keisler measures exist in that context. This paper lays the foundational groundwork for the study of Keisler measures outside the boundary of NIP. Our results demonstrate that the *general theory of Keisler measures* is fundamentally more complicated than previously thought. Broadly speaking, whereas Keisler measures in NIP theories can be sufficiently approximated by types, and so are tame, measures in arbitrary theories are far more sensitive to analytic and descriptive set-theoretic issues, and it is no longer possible to directly generalize proofs from types to measures. Indeed, we will develop several examples demonstrating novel and exotic behavior of arbitrary Keisler measures outside of NIP. However, we also prove positive results concerning the theory of "generically stable" Keisler measures, which further demonstrate that structural understanding is possible.

In arbitrary theories, invariant types can be "freely" amalgamated using the Morley product operation, sometimes also called the nonforking product. In NIP theories, this operation extends automatically to invariant Keisler measures, thanks to the result of Hrushovski and Pillay [2011] that any such measure in an NIP theory is *Borel definable*. While this need not hold outside of NIP, one can still define the Morley product of measures that are Borel definable. Thus the first main goal of this paper is to establish basic properties of Borel definable Keisler measures in arbitrary theories. We consider the two questions of whether Borel definability is preserved by Morley products, and whether the Morley product is associative for Borel definable measures. Despite the fundamental nature of these questions, the previous literature has been somewhat vague regarding the answers. A positive answer to the first question is stated without proof in [Hrushovski et al. 2013, Lemma 1.6]. Moreover, while associativity of the Morley product seems to be tacitly assumed in various places, it is only directly addressed in [Simon 2015] under the assumption of NIP (and, even in this case, a complete proof of associativity was given only recently; see Remark 2.14). The goal of Sections 2 and 3 is to clarify this situation. In Section 2, we show that if T is any countable theory, and $A \subset \mathcal{U}$ is countable, then the set of measures that are Borel definable over A is closed under Morley products, and associativity holds for such measures. On the other hand, we will see in Section 3 that both properties can fail without the extra countability assumptions (even in simple theories). In particular, this refutes the unproven claim in [Hrushovski et al. 2013].

The rest of the paper is devoted to developing various notions of "generic stability" for Keisler measures in arbitrary theories. We focus on three classes: measures that are *definable and finitely satisfiable* in a small model (or *dfs*), measures that are *finitely approximated* in a small model (or *fam*), and measures that are *frequency interpretation measures* with respect to a small model (or *fim*). Section 4 provides definitions and a review of basic facts about *fim*, *fam*, and *dfs* measures.

In NIP theories, the three classes of measures described above coincide, and a Keisler measure with these properties is called "generically stable". Outside of NIP theories, these properties are no longer equivalent, and thus one obtains three competing notions of generic stability for measures. An assessment of this competition was undertaken in [Conant and Gannon 2020], mostly focusing on types. The present article continues this work with a greater emphasis on measures. A recurring question is the extent to which fundamental results on generically stable Keisler measures in NIP theories can be generalized to arbitrary theories, and we will prove several results to this effect. These results help to clarify when NIP is playing a crucial role in a given result about measures, versus when a similar result can be obtained in general, perhaps after some appropriate modification of the working assumptions. In particular, a fundamental fact about NIP theories is that any Keisler measure can be locally approximated by types. This result is often used to replace measures by types in various arguments, and thus avoid the necessity of pure measure theory and integration techniques. On the other hand, our work will show that generalizations of certain results on NIP theories can indeed be obtained using more measure-theoretic proofs which, although possibly more complicated methodologically, are also shorter and in some cases more concise.

In Section 5, we focus on the question of commutativity for the Morley product of Borel definable Keisler measures. This is motivated by the result of Hrushovski, Pillay, and Simon [Hrushovski et al. 2013] that, in NIP theories, definable measures commute with finitely satisfiable measures and, moreover, *dfs* measures commute with arbitrary invariant measures. The goal of Section 5 is to obtain suitable generalizations of these results for arbitrary theories. We first show that in any theory, if μ is a definable measure, and ν is Borel definable and finitely satisfiable, then μ and ν commute provided that for any small model M, $\mu|_M$ has some definable global extension that commutes with ν (see Theorem 5.7). This recovers the corresponding fact from [Hrushovski et al. 2013] since, in NIP theories, any measure over a small model has a *smooth* global extension, and it is easy to show that smooth measures (in any theory) are definable and commute with all Borel definable measures. We also show later in the paper that, in Theorem 5.7, the extra assumption on restrictions of μ to small models (which is automatic in NIP theories) is necessary. In particular, we construct a theory with a *dfs* type and a definable measure that do not commute (see Proposition 7.14). Finally in Section 5, we show that in any theory, *fim* measures commute with Borel definable measures (see Theorem 5.16). In other words, the corresponding result for NIP theories from [Hrushovski et al. 2013] generalizes to arbitrary theories, provided one replaces *dfs* with *fim*.

In Section 6, we focus on further properties of *fim* measures. Evidence suggests that *fim* is the "right" notion of generic stability for measures in arbitrary theories. In particular, the notion of a generically stable *type* is well established in the literature, and Conant and Gannon [2020] showed that this notion coincides with *fim* when viewing types as $\{0, 1\}$-valued measures. In Theorem 6.2, we show that *fim* measures are closed under convex combinations. The analogous result for *dfs* and *fam* measures is quite easy to prove (see Proposition 4.11) and so, in light of [Hrushovski et al. 2013], Theorem 6.2 again generalizes known facts from the study of NIP theories. However, we will see that working directly with *fim* measures in general theories leads to significantly more complicated proofs. We finish Section 6 with a discussion of the still open question of whether *fim* measures are preserved by Morley products (an earlier draft of this article contained an erroneous proof of a positive answer).

In Section 7, we answer one of the main questions left open in [CG 2020], which is on the existence of a complete global measure that is *dfs* and not *fam* (an example involving a local type was given in [CG 2020]). We will first give a new local example of this phenomenon, which is built using subsets of the interval $[0, 1]$ of Lebesgue measure $\frac{1}{2}$. Then we develop this example into a more complicated theory with a complete *dfs* type that is not *fam*.

Section 8 focuses on examples of measures that are *fam* and not *fim*. We first show that a purported example from [Adler et al. 2014] of this phenomenon does not work. Then we revisit a different example from [CG 2020] in the theory of the generic K_s-free graph (for fixed $s \geq 3$). We develop further properties of this example, and correct an erroneous proof from [CG 2020]. Finally, we give a new example of a complete type that is *fam* and not *fim*, which is obtained by taking a certain reduct of the *dfs* and non-*fam* type from Section 7.

Corrigenda. For the sake of clarifying the literature, we summarize the incorrect results and proofs from previous work that are addressed in this article.

(1) We recall that the product of two Borel definable Keisler measures is Borel definable in the NIP setting. In [Hrushovski et al. 2013, Lemma 1.6], it is claimed, but not proved, that the Morley product of two Borel definable Keisler measures is Borel definable. We show here that this is not always true, even for Borel definable *types* (see Proposition 3.9).

(2) Example 1.7 of [Adler et al. 2014] describes a complete theory that is claimed to admit a global generically stable type p such that $p \otimes p$ is not generically stable. This claim is repeated in [CG 2020, Fact 5.4]. It turns out that p is not well defined, and we show here that this particular theory has no global nonalgebraic generically stable types (see Theorem 8.5). The question of whether Morley products preserve generic stability remains open. See the end of Section 6 for further discussion.

(3) Remark 4.2 of [CG 2020] makes an unjustified claim that *dfs*, *fam*, and *fim* measures are closed under localization at arbitrary Borel sets, which seems likely to be false. In Section 8, we supply correct proofs of the results in [CG 2020] that used this remark (see Proposition 8.2, Remark 8.3, and Theorem 8.10).

1. Basic definitions and notation

We start with some general notation that will be used throughout the paper. Let X be a set. Given a point $a \in X$, we let δ_a denote the Dirac measure on X concentrating at a. For $\bar{a} \in X^n$, we let $\mathrm{Av}(\bar{a})$ denote the "average" measure $(1/n)\sum_{i=1}^{n} \delta_{a_i}$. Given a (bounded) real-valued function f on X, define $\|f\|_\infty := \sup_{x \in X} |f(x)|$.

Given $r, s \in \mathbb{R}$ and $\varepsilon > 0$, we write $r \approx_\varepsilon s$ to mean that $|r - s| < \varepsilon$. Given an integer $n \geq 1$, let $[n] = \{1, \ldots, n\}$.

Now let T be a complete $\mathcal{L}$-theory with monster model $\mathcal{U}$. We work with formulas in the language $\mathcal{L}$ with parameters from $\mathcal{U}$. A formula $\varphi(x)$ is *over* $A \subseteq \mathcal{U}$ if all parameters in $\varphi(x)$ come from A. In this case we say $\varphi(x)$ is an $\mathcal{L}_A$*-formula*. An $\mathcal{L}$*-formula* is a formula without parameters. We will use x, y, z, etc., to denote tuples of variables, although at times we may also employ vector notation $\bar{x}$, $\bar{y}$, $\bar{z}$, etc., for clarity. As usual, we often partition the free variables in a formula $\varphi(x, y)$ into object variables x and parameter variables y.

Given $A \subseteq \mathcal{U}$, let $\mathrm{Def}_x(A)$ denote the Boolean algebra of $\mathcal{L}_A$-formulas with free variables x, up to equivalence modulo T expanded by constants for A. The corresponding Stone space of types is denoted by $S_x(A)$. Given an $\mathcal{L}_A$-formula $\varphi(x)$, we let $[\varphi(x)]$ denote the clopen set of types in $S_x(A)$ containing $\varphi(x)$.

We let $\mathfrak{M}_x(A)$ denote the space of Keisler measures (i.e., finitely additive probability measures) on $\mathrm{Def}_x(A)$. Recall that any $\mu \in \mathfrak{M}_x(A)$ determines a unique regular Borel probability measure $\tilde{\mu}$ on $S_x(A)$ such that if $\varphi(x)$ is an $\mathcal{L}_A$-formula then $\mu(\varphi(x)) = \tilde{\mu}([\varphi(x)])$ (and, furthermore, any regular Borel probability measure on $S_x(A)$ is of this form). See [Simon 2015, Section 7.1] for an explicit construction of $\tilde{\mu}$. By identifying μ and $\tilde{\mu}$, we can view μ as a regular Borel probability measure on $S_x(A)$. For further details on Borel measures and regularity, see Section A1. We will use the following special case of Fact A.4.

Fact 1.1. *Fix $A \subseteq \mathcal{U}$ and $\mu \in \mathfrak{M}_x(A)$.*

(a) *If $U \subseteq S_x(A)$ is open then*

$$\mu(U) = \sup\{\mu(\varphi(x)) : \varphi(x) \text{ is an } \mathcal{L}_A\text{-formula and } [\varphi(x)] \subseteq U\}.$$

(b) *If ν is a regular Borel probability measure on $S_x(A)$, and $\nu(\varphi(x)) = \mu(\varphi(x))$ for any $\mathcal{L}_A$-formula $\varphi(x)$, then $\mu = \nu$.*

Given $A \subseteq B \subseteq \mathcal{U}$ and a tuple x of variables, let $\rho^x_{B,A} : S_x(B) \to S_x(A)$ denote the restriction map. Note that $\rho^x_{B,A}$ is a continuous surjective map between compact Hausdorff spaces, and thus is a quotient map. Let ρ^x_A denote $\rho^x_{\mathcal{U},A}$. Given $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $A \subseteq \mathcal{U}$, we let $\mu|_A$ denote the restriction of μ to $\mathrm{Def}_x(A)$.

Remark 1.2. If $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $A \subseteq \mathcal{U}$ then, as a regular Borel measure on $S_x(A)$, $\mu|_A$ is the pushforward of μ to $S_x(A)$ along ρ^x_A. In other words, if $X \subseteq S_x(A)$ is Borel then $\mu|_A(X) = \mu((\rho^x_A)^{-1}(X))$. Indeed, by definition of $\mu|_A$, this holds when $X = [\varphi(x)]$ for some $\mathcal{L}_A$-formula $\varphi(x)$. Thus it holds for all Borel X by Fact 1.1(b), and since pushforwards preserve regularity in this context (see Fact A.3).

We write $A \subset \mathcal{U}$ to denote that A is a subset of $\mathcal{U}$ which is *small*, i.e., $\mathcal{U}$ is $|A|^+$-saturated and strongly $|A|^+$-homogeneous. A measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ is *invariant* if there is some $A \subset \mathcal{U}$ such that for any $\mathcal{L}$-formula $\varphi(x, y)$, if $b, b' \in \mathcal{U}^y$ have the same type over A, then $\mu(\varphi(x, b)) = \mu(\varphi(x, b'))$. In this case, we also say that μ is *invariant over A* or *A-invariant*.

Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is invariant over $A \subset \mathcal{U}$. Given an $\mathcal{L}_A$-formula $\varphi(x, y)$, define $F^{\varphi}_{\mu,A} : S_y(A) \to [0, 1]$ such that $F^{\varphi}_{\mu,A}(q) = \mu(\varphi(x, b))$ for some/any $b \models q$. Note that if $B \supseteq A$ then μ is invariant over B and, if $\varphi(x, y)$ is an $\mathcal{L}_A$-formula, then $F^{\varphi}_{\mu,B} = F^{\varphi}_{\mu,A} \circ \rho^y_{B,A}$.

A Keisler measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ is *Borel definable* if there is some $A \subset \mathcal{U}$ such that μ is A-invariant and $F^{\varphi}_{\mu,A}$ is a Borel map for any $\mathcal{L}$-formula $\varphi(x, y)$. In this case, we also say that μ is *Borel definable over A*. Note that if μ is Borel definable over A, then $F^{\varphi}_{\mu,A}$ is Borel for any $\mathcal{L}_A$-formula $\varphi(x, y)$ and, moreover, μ is Borel definable over any $B \supseteq A$ (see also [Gannon 2020, Proposition 2.22]).

Finally, we define the Morley product of Keisler measures. Given a Borel definable measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ and a measure $\nu \in \mathfrak{M}_y(\mathcal{U})$, we define a measure $\mu \otimes \nu$ in $\mathfrak{M}_{xy}(\mathcal{U})$ such that, given an $\mathcal{L}_{\mathcal{U}}$-formula $\varphi(x, y)$,

$$(\mu \otimes \nu)(\varphi(x, y)) = \int_{S_y(A)} F^{\varphi}_{\mu,A} \, d\nu|_A,$$

where $A \subset \mathcal{U}$ is any small set such that $\varphi(x, y)$ is over A and μ is Borel definable over A. One can show that this does not depend on the choice of A. The measure $\mu \otimes \nu$ is called the *Morley product* of μ and ν.

Remark 1.3. To help ease notation, we will write integrals $\int_{S_y(A)} f \, d\nu|_A$ simply as $\int_{S_y(A)} f \, d\nu$. In other words, the fact that we integrate with respect to $\nu|_A$ is implied by the domain of integration $S_y(A)$. When f is (or involves) a function of the form $F^{\varphi}_{\mu,A}$, we write $\int_{S_y(A)} F^{\varphi}_{\mu} \, d\nu$ instead of $\int_{S_y(A)} F^{\varphi}_{\mu,A} \, d\nu$.

Recall that $S_x(\mathcal{U})$ can be identified with a closed subset of $\mathfrak{M}_x(\mathcal{U})$ by viewing types as $\{0, 1\}$-valued measures. If $q \in S_y(\mathcal{U})$ is a type, then we have a well-defined Morley product $\mu \otimes q$ for any *invariant* $\mu \in \mathfrak{M}_x(\mathcal{U})$ since any function is integrable with respect to q as a Dirac measure. More explicitly, $(\mu \otimes q)(\varphi(x, y)) = \mu(\varphi(x, b))$, where μ is A-invariant, $\varphi(x, y)$ is over A, and $b \models q|_A$. If μ is a type $p \in S_x(\mathcal{U})$, then $p \otimes q$ is a type in $S_{xy}(\mathcal{U})$, and $\varphi(x, y) \in p \otimes q$ if and only if $\varphi(x, b) \in p$ (where b is as before). We recall the following easy exercise.

Fact 1.4. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$ are invariant. If μ is Borel definable, or if ν is a type, then $\mu \otimes \nu$ is invariant.*

2. Borel definability over countable sets

As explained in the introduction, one main goal of this paper is to settle the question of whether the Morley product of Keisler measures preserves Borel definability, and also to address associativity. In this section, we show that both properties hold when working over countable parameter sets in countable theories. We will approach this result from a general perspective that will lead to further facts about Borel definable measures, and also explain precisely how the situation turns complicated (and counterintuitive) over uncountable sets. This perspective will also lead to some useful conclusions for definable measures (see Section 2D).

2A. ***Fiber functions over Borel sets.*** Recall that if $\mu \in \mathfrak{M}_x(\mathcal{U})$ is Borel definable over $A \subset \mathcal{U}$, then it is Borel definable over any $B \supseteq A$. We now observe that Borel definability can also be dropped to smaller parameter sets, provided one still has invariance. The proof uses a result from [Holický and Spurný 2003], which can be viewed as a Borel variation on the universal property of quotient maps.

Theorem 2.1 [Holický and Spurný 2003]. *Suppose $\rho : X \to Y$ is a surjective continuous map between compact Hausdorff spaces. Then, for any $E \subseteq Y$, if $\rho^{-1}(E)$ is Borel then E is Borel. Therefore, if $f : Y \to Z$ is a map to a topological space Z, and $f \circ \rho$ is Borel, then f is Borel.*

Proof. The first claim is a special case of [Holický and Spurný 2003, Theorem 10]. The second claim follows from the first. Indeed, if $U \subseteq Z$ is open and $f \circ \rho$ is Borel, then $\rho^{-1}(f^{-1}(U))$ is a Borel set, and thus so is $f^{-1}(U)$. $\square$

Corollary 2.2. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is Borel definable, and invariant over $A \subset \mathcal{U}$. Then μ is Borel definable over A.*

Proof. This follows from Theorem 2.1 since if μ is Borel definable over $B \supseteq A$ then, for any $\mathcal{L}$-formula $\varphi(x, y)$, $F^{\varphi}_{\mu,B} = F^{\varphi}_{\mu,A} \circ \rho^{y}_{B,A}$. □

Remark 2.3. Despite the simplicity of the proof, Corollary 2.2 does not appear in previous literature, possibly due to the use of [Holický and Spurný 2003]. On the other hand, the analogue of this corollary for *definable* measures (which are discussed in Section 2D) is well known and follows from a similar proof. Indeed, if μ is definable over B and invariant over $A \subseteq B$, then $F^{\varphi}_{\mu,B}$ is continuous and thus $F^{\varphi}_{\mu,A}$ is continuous by the universal property of quotient maps. It follows that μ is definable over A.

Our next goal is to redefine $F^{\varphi}_{\mu,A}$ with an arbitrary Borel set $W(x, y) \subseteq S_{xy}(A)$ in place of $\varphi(x, y)$. The underlying idea is quite natural. We will "plug in" a parameter b for the y variables, and apply the measure μ. This perspective of treating Borel sets like formulas crops up in the literature, though often informally. We will see that while some techniques pass from formulas to Borel sets without any issues, there are certain places where things can go wrong. These subtleties will eventually lead to examples where Borel definable measures fail to be closed under Morley products, and where associativity of the Morley product fails. For this reason, we will proceed carefully with the next few definitions and basic observations, so as to ensure a solid foundation for the passage from formulas to Borel sets.

Definition 2.4. Given $A \subset \mathcal{U}$, we say that a set $W \subseteq S_x(\mathcal{U})$ is ρ^x_A*-invariant* if membership in W depends only on ρ^x_A, i.e., $W = (\rho^x_A)^{-1}(\rho^x_A(W))$.

Remark 2.5. If $W \subseteq S_x(\mathcal{U})$ is Borel and ρ^x_A-invariant for some $A \subset \mathcal{U}$, then $\rho^x_A(W)$ is a Borel set in $S_x(A)$ by Theorem 2.1.

Definition 2.6. Suppose $A \subset \mathcal{U}$ and $W \subseteq S_{xy}(A)$. Given $b \in \mathcal{U}^y$, we define

$$W(x, b) = \{p \in S_x(\mathcal{U}) : \mathrm{tp}(a, b/A) \in W \text{ for some/any } a \models p|_{Ab}\}.$$

Note that $W(x, b)$ is ρ^x_{Ab}-invariant.

Lemma 2.7. *Suppose $A \subset \mathcal{U}$ and $W \subseteq S_{xy}(A)$ is Borel.*

(a) *If $b \in \mathcal{U}^y$ then $W(x, b)$ is a Borel subset of $S_x(\mathcal{U})$.*

(b) *If $\mu \in \mathfrak{M}_x(\mathcal{U})$ is A-invariant, and $b, b' \in \mathcal{U}^y$ with $b \equiv_A b'$, then $\mu(W(x, b)) = \mu(W(x, b'))$.*

Proof. Both parts can be proved directly by induction on the $\mathbf{\Sigma}$-complexity of W, using only elementary steps. We will sketch alternative "high-level" arguments. For part (a), fix $b \in \mathcal{U}^y$ and let $X = \{q \in S_{xy}(A) : q(x, b) \text{ is consistent}\}$, which is closed in $S_{xy}(A)$. Set $\tau : X \to S_x(Ab)$ such that $\tau(q) = q(x, b)$. Then τ is surjective and continuous, and $\tau^{-1}(\tau(W \cap X)) = W \cap X$. So $\tau(W \cap X)$ is Borel by Theorem 2.1. Thus $W(x, b) = (\rho^x_{Ab})^{-1}(\tau(W \cap X))$ is Borel.

For part (b), suppose we have $b, b' \in \mathcal{U}^y$ and $\sigma \in \mathrm{Aut}(\mathcal{U}/A)$ such that $\sigma(b) = b'$. Then σ induces a homeomorphism of $S_x(\mathcal{U})$, which yields a regular Borel measure $\nu := \mu\sigma$ on $S_x(\mathcal{U})$. Since μ is A-invariant, it agrees with ν on clopen sets, and thus $\mu = \nu$ by Fact 1.1(b). So $\mu(W(x, b')) = \nu(W(x, b)) = \mu(W(x, b))$. $\square$

Definition 2.8. Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is invariant over $A \subset \mathcal{U}$, and $W \subseteq S_{xy}(A)$ is Borel. Define $F^W_{\mu,A} : S_y(A) \to [0, 1]$ such that $F^W_{\mu,A}(q) = \mu(W(x, b))$, where $b \models q$.

Note that $F^W_{\mu,A}$ is well defined by Lemma 2.7. We also note that if W is the *clopen* set determined by some $\mathcal{L}_A$-formula $\varphi(x, y)$, then $F^W_{\mu,A}$ coincides with $F^{\varphi}_{\mu,A}$.

2B. *Products and associativity.* In this subsection, we formulate some ad hoc conditions on Borel definable measures that allow one to prove preservation under Morley products and associativity. In the next subsection, we will see that these conditions hold over countable sets. We start with some motivation.

Consider a measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ that is Borel definable over some $A \subset \mathcal{U}$. Then maps of the form $F^W_{\mu,A}$ are Borel for any *clopen* set $W \subseteq S_{xy}(A)$. But we will eventually see that this is not enough to ensure $F^W_{\mu,A}$ is Borel for general Borel sets W. To obtain this, one needs to further assume that $F^U_{\mu,A}$ is Borel for any *open* $U \subseteq S_{xy}(A)$ (see Lemma 2.10). We will show that this assumption suffices to address preservation of Borel definability in Morley products. The issue of associativity, however, requires consideration of further subtleties. In particular, with μ as above, suppose that we have some fixed open set $U \subseteq S_{xy}(A)$ such that $F^U_{\mu,A}$ is Borel. Then, given some $\nu \in \mathfrak{M}_y(\mathcal{U})$, we have a well-defined integral $\int_{S_y(A)} F^U_\mu \, d\nu$. On the other hand, Fact 1.1(a) gives an explicit expression for $(\mu \otimes \nu)|_A(U)$ which, as we will see in later examples, need not be the same as the previous integral (note that if U is clopen then we do have such an equality by definition of the Morley product). Altogether, this discussion motivates the following definition.

Definition 2.9. Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is invariant over $A \subset \mathcal{U}$. We say μ is BD^+ over A if, for any y and any open $U \subseteq S_{xy}(A)$, the map $F^U_{\mu,A}$ is Borel. Moreover, we say μ is BD^{++} over A if it is BD^+ over A and, for any y, any open $U \subseteq S_{xy}(A)$, and any $\nu \in \mathfrak{M}_y(\mathcal{U})$, we have $(\mu \otimes \nu)|_A(U) = \int_{S_y(A)} F^U_\mu \, d\nu$.

Next we show that the defining properties of BD^+ and BD^{++} extend automatically from open sets to arbitrary Borel sets. For BD^+, this boils down to the fact that pointwise limits of Borel functions are Borel. For BD^{++} we will apply the dominated convergence theorem [Cohn 2013, Theorem 2.4.5].

Lemma 2.10. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is* BD$^+$ *over $A \subset \mathcal{U}$, and $W \subseteq S_{xy}(A)$ is Borel. Then $F^W_{\mu,A}$ is Borel. Moreover, if μ is* BD^{++} *over A then, for any $\nu \in \mathfrak{M}_y(\mathcal{U})$, we have $(\mu \otimes \nu)|_A(W) = \int_{S_y(A)} F^W_\mu \, d\nu$.*

Proof. We proceed by induction on the $\mathbf{\Sigma}$-complexity of W. The base case when W is open holds by assumption. So fix $1 < \alpha < \omega_1$ and assume the result for $\mathbf{\Sigma}^0_\beta$ subsets of $S_{xy}(A)$ for all $\beta < \alpha$. Suppose W is a $\mathbf{\Sigma}^0_\alpha$ set. Then, for $i < \omega$, we have $W_i \in \mathbf{\Sigma}^0_{\alpha_i}$ for some $\alpha_i < \alpha$, such that $W = \bigcup_{i<\omega} \neg W_i$. Since each $\mathbf{\Pi}^0_\beta$ class is closed under finite unions (see [Miller 1995, Theorem 2.1]), we may assume without loss of generality that $\neg W_i \subseteq \neg W_{i+1}$ for all $i < \omega$. Given $q \in S_y(A)$ and $b \models q$, we have

$$\begin{aligned} F^B_{\mu,A}(q) = \mu(W(x,b)) &= \lim_{i\to\infty} \mu\big((\neg W_i)(x,b)\big) \\ &= \lim_{i\to\infty}\big(1 - \mu(W_i(x,b))\big) = \lim_{i\to\infty}(1 - F^{W_i}_{\mu,A}(q)). \end{aligned}$$

By induction, $F^W_{\mu,A}$ is a pointwise limit of Borel functions, and thus is Borel. Moreover, if μ is BD^{++} over A, then, by induction and the dominated convergence theorem, we have

$$\begin{aligned} (\mu \otimes \nu)|_A(W) &= \lim_{i\to\infty} (\mu \otimes \nu)|_A(\neg W_i) \\ &= 1 - \lim_{i\to\infty} \int_{S_y(A)} F^{W_i}_\mu \, d\nu \\ &= \int_{S_y(A)} \lim_{i\to\infty} (1 - F^{W_i}_\mu)\, d\nu = \int_{S_y(A)} F^W_\mu \, d\nu. \qquad \square \end{aligned}$$

We can now prove the main result concerning BD$^+$ and BD^{++}.

Theorem 2.11. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is Borel definable over $A \subset \mathcal{U}$.*

(a) *If $\nu \in \mathfrak{M}_y(\mathcal{U})$ is* BD$^+$ *over A, then $\mu \otimes \nu$ is Borel definable over A.*
(b) *If $\nu \in \mathfrak{M}_y(\mathcal{U})$ is* BD^{++} *over A, then $((\mu \otimes \nu) \otimes \lambda)|_A = (\mu \otimes (\nu \otimes \lambda))|_A$ for any $\lambda \in \mathfrak{M}_z(\mathcal{U})$.*

Proof. Before proving the two statements, we will develop some preliminaries. Fix an $\mathcal{L}_A$-formula $\varphi(x, y, z)$. Then $F^\varphi_{\mu,A} : S_{yz}(A) \to [0, 1]$ is Borel, and so there is a sequence $(f_n)^\infty_{n=0}$ of simple Borel functions on $S_{yz}(A)$ converging pointwise to $F^\varphi_{\mu,A}$. (See Fact A.1; in fact, this convergence can be made uniform, but we will work with pointwise convergence in preparation for Remark 2.19.) For $n \geq 0$, write $f_n = \sum_{i=1}^{m_n} \alpha_{n,i} \mathbf{1}_{W_{n,i}}$, where $W_{n,i} \subseteq S_{yz}(A)$ is Borel and $\alpha_{n,i} \in [0, 1]$.

Given $c \in \mathcal{U}^z$, we set the following notation. Let $W^c_{n,i} := \rho^y_{Ac}(W_{n,i}(y, c))$, which is a Borel subset of $S_y(Ac)$ by Remark 2.5. Define the map $f^c_n = \sum_{i=1}^{m_n} \alpha_{n,i} \mathbf{1}_{W^c_{n,i}}$ on $S_y(Ac)$. Finally, let $\varphi_c(x, y)$ denote $\varphi(x, y, c)$.

Claim 1. *Fix $c \in \mathcal{U}^z$. Then $(f^c_n)^\infty_{n=0}$ converges pointwise to $F^{\varphi_c}_{\mu,Ac}$.*

Proof. Fix $q \in S_y(Ac)$, and let $s = \mathrm{tp}(b, c/A)$, where $b \models q$. Then, for any $i \leq m_n$, we have $s \in W_{n,i}$ if and only if $q \in W^c_{n,i}$. It follows that $f_n(s) = f^c_n(q)$ for any $n \geq 0$. Therefore

$$F^{\varphi_c}_{\mu,Ac}(q) = \mu(\varphi(x, b, c)) = F^{\varphi}_{\mu,A}(s) = \lim_{n\to\infty} f_n(s) = \lim_{n\to\infty} f^c_n(q). \qquad \dashv_{\text{claim}}$$

Now fix some A-invariant measure $\nu \in \mathfrak{M}_y(\mathcal{U})$. Given $n \geq 0$, define the function $h_n = \sum_{i=1}^{m_n} \alpha_{n,i} F^{W_{n,i}}_{\nu,A}$ on $S_z(A)$.

Claim 2. *$(h_n)_{n=0}^{\infty}$ converges pointwise to $F^{\varphi}_{\mu\otimes\nu,A}$.*

Proof. Fix $r \in S_z(A)$, and let $c \models r$. Then $\nu|_{Ac}(W^c_{n,i}) = F^{W_{n,i}}_{\nu,A}(r)$ for any $n \geq 0$ and $i \leq m_n$. So for any $n \geq 0$, we have

$$h_n(r) = \sum_{i=1}^{m_n} \alpha_{n,i} F^{W_{n,i}}_{\nu,A}(r) = \sum_{i=1}^{m_n} \alpha_{n,i}\, \nu|_{Ac}(W^c_{n,i}) = \int_{S_y(Ac)} f^c_n \, d\nu.$$

Therefore, by Claim 1 and the dominated convergence theorem, we have

$$F^{\varphi}_{\mu\otimes\nu,A}(r) = \int_{S_y(Ac)} F^{\varphi_c}_{\mu} \, d\nu = \lim_{n\to\infty} \int_{S_y(Ac)} f^c_n \, d\nu = \lim_{n\to\infty} h_n(r). \qquad \dashv_{\text{claim}}$$

We can now prove the theorem. For part (a), suppose $\nu \in \mathfrak{M}_y(\mathcal{U})$ is BD^+ over A. Then each function h_n above is Borel by Lemma 2.10. So $F^{\varphi}_{\mu\otimes\nu,A}$ is a pointwise limit of Borel functions by Claim 2, and thus is Borel. Since $\varphi(x, y, z)$ is an arbitrary $\mathcal{L}_A$-formula, we have that $\mu \otimes \nu$ is Borel definable over A.

Finally, for part (b), suppose $\nu \in \mathfrak{M}_y(\mathcal{U})$ is BD^{++} over A. Fix some $\lambda \in \mathfrak{M}_z(\mathcal{U})$. Then, for any $n \geq 0$ and $i \leq m_n$, we have $(\nu \otimes \lambda)|_A(W_{n,i}) = \int_{S_z(A)} F^{W_{n,i}}_{\nu} \, d\lambda$ by Lemma 2.10. Therefore

$$\begin{aligned}
&(\mu \otimes (\nu \otimes \lambda))(\varphi(x, y, z)) \\
&\qquad = \int_{S_{yz}(A)} F^{\varphi}_{\mu} \, d(\nu \otimes \lambda) = \lim_{n\to\infty} \int_{S_{yz}(A)} f_n \, d(\nu \otimes \lambda) \\
&\qquad = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_{n,i} (\nu \otimes \lambda)(W_{n,i}) = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_{n,i} \int_{S_z(A)} F^{W_{n,i}}_{\nu} \, d\lambda \\
&\qquad = \lim_{n\to\infty} \int_{S_z(A)} h_n \, d\lambda = \int_{S_z(A)} F^{\varphi}_{\mu\otimes\nu} \, d\lambda \\
&\qquad = ((\mu \otimes \nu) \otimes \lambda)(\varphi(x, y, z)).
\end{aligned}$$

Note that the second and sixth equalities again use dominated convergence. Since $\varphi(x, y, z)$ is an arbitrary $\mathcal{L}_A$-formula, we have $((\mu\otimes\nu)\otimes\lambda)|_A = (\mu\otimes(\nu\otimes\lambda))|_A$. $\square$

2C. *Countable sets.* Next we show that in a countable theory, Borel definability coincides with BD^{++} over countable parameter sets. This is another straightforward application of dominated convergence (similar to Lemma 2.10).

Lemma 2.12. *Assume T is countable, and suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is Borel definable over a countable set $A \subset \mathcal{U}$. Then μ is* BD^{++} *over A.*

Proof. Fix an open set $U \subseteq S_{xy}(A)$. Since T and A are countable, we can write $U = \bigcup_{n<\omega}[\varphi_n(x, y)]$, where each $\varphi_n(x, y)$ is an $\mathcal{L}_A$-formula and $\varphi_n(\mathcal{U}^{xy}) \subseteq \varphi_{n+1}(\mathcal{U}^{xy})$ for all $n < \omega$. Given $q \in S_y(A)$ and $b \models q$, we have

$$F^U_{\mu,A}(q) = \mu(U(x, b)) = \lim_{n\to\infty} \mu(\varphi_n(x, b)) = \lim_{n\to\infty} F^{\varphi_n}_{\mu,A}(q).$$

So $F^U_{\mu,A}$ is the pointwise limit of a countable sequence of Borel functions, and hence is Borel. Now fix another measure $\nu \in \mathfrak{M}_y(\mathcal{U})$. Then

$$\begin{aligned}(\mu \otimes \nu)(U) = \lim_{n\to\infty} (\mu \otimes \nu)(\varphi_n(x, y)) &= \lim_{n\to\infty} \int_{S_y(A)} F^{\varphi_n}_{\mu}\, d\nu \\ &= \int_{S_y(A)} \lim_{n\to\infty} F^{\varphi_n}_{\mu}\, d\nu = \int_{S_y(A)} F^U_{\mu}\, d\nu,\end{aligned}$$

where the third equality uses the dominated convergence theorem. □

Theorem 2.13. *Assume T is countable, and suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$ are Borel definable over a countable set $A \subset \mathcal{U}$. Then $\mu \otimes \nu$ is Borel definable over A and, for any $\lambda \in \mathfrak{M}_z(\mathcal{U})$, we have $(\mu \otimes \nu) \otimes \lambda = \mu \otimes (\nu \otimes \lambda)$.*

Proof. By Theorem 2.11(a) and Lemma 2.12, $\mu \otimes \nu$ is Borel definable over A. Note that μ and ν are Borel definable over any $B \supseteq A$. So ν is BD^{++} over any countable $B \supseteq A$ by Lemma 2.12. Therefore, for any $\lambda \in \mathfrak{M}_z(\mathcal{U})$, we have $((\mu \otimes \nu) \otimes \lambda)|_B = (\mu \otimes (\nu \otimes \lambda))|_B$ for any countable $B \supseteq A$ by Theorem 2.11(b). It follows that $(\mu \otimes \nu) \otimes \lambda = \mu \otimes (\nu \otimes \lambda)$. □

It is well known that the dominated convergence theorem does not hold for nets, and so the proof of Lemma 2.12 cannot be generalized to Borel definable measures over uncountable models. Indeed, in Section 3 we will give an example showing that Theorem 2.13 can fail without the countability assumptions.

Remark 2.14. Theorem 2.13 holds for NIP theories without the countability assumptions. In particular, Hrushovski and Pillay [2011] proved that if T is NIP then any invariant Keisler measure is Borel definable (see also [Simon 2015, Proposition 7.19]). Combined with Fact 1.4, it follows that Borel definability is preserved by Morley products in NIP theories. As for associativity of the Morley product in NIP theories, a proof sketch is given after Exercise 7.20 in [Simon 2015]. However, as pointed out recently by Krupiński, the argument tacitly uses assumptions along the lines of BD^{++} without justification. This motivated Conant and Gannon [2021] to write a different proof of associativity in NIP theories, which uses the existence of "smooth extensions". We will note a similar proof in Corollary 2.22.

2D. *Definable measures.* In this section we use the material developed above to prove some useful facts about *definable* measures. The definition of this notion is based on the following standard exercise (see also [Gannon 2020, Proposition 2.17]).

Fact 2.15. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is invariant over $A \subset \mathcal{U}$. Given an $\mathcal{L}$-formula $\varphi(x, y)$, the following are equivalent*:

(i) *$F^{\varphi}_{\mu,A}$ is continuous.*

(ii) *For any $\varepsilon > 0$, there are $\mathcal{L}_A$-formulas $\psi_1(y), \ldots, \psi_n(y)$ and real numbers $r_1, \ldots, r_n \in [0, 1]$ such that $\left\| F^{\varphi}_{\mu,A} - \sum_{i=1}^{n} r_i \mathbf{1}_{\psi_i} \right\|_\infty < \varepsilon$.*

(iii) *For any $\varepsilon > 0$, the set $\{b \in \mathcal{U}^y : \mu(\varphi(x, b)) \leq \varepsilon\}$ is type-definable over A.*

Definition 2.16. A measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ is *definable* if there is some $A \subset \mathcal{U}$ such that μ is A-invariant and, for any $\mathcal{L}$-formula $\varphi(x, y)$, the equivalent conditions of Fact 2.15 hold. In this case, we also say that μ is *definable over A*.

Note that condition (iii) of Fact 2.15 makes sense without assuming μ is invariant. Moreover, if (iii) holds for all $\mathcal{L}$-formulas $\varphi(x, y)$, then it follows that μ is A-invariant. Therefore, a measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ is definable over $A \subset \mathcal{U}$ if and only if, for any $\mathcal{L}$-formula $\varphi(x, y)$, condition (iii) holds.

In [CG 2020], it is shown that definable measures are closed under Morley products and satisfy associativity. Here we prove a more general associativity result when only one definable measure is involved. Note first that any definable measure is clearly Borel definable. The next lemma strengthens this fact.

Lemma 2.17. *If $\mu \in \mathfrak{M}_x(\mathcal{U})$ is definable over $A \subset \mathcal{U}$, then it is* BD^{++} *over A.*

Proof. Let $U \subseteq S_{xy}(A)$ be open. Similar to the proof of Lemma 2.12, we can write $U = \bigcup_{i \in I} [\varphi_i(x, y)]$ for some collection $\{\varphi_i(x, y) : i \in I\}$ of $\mathcal{L}_A$-formulas, where I is a directed partial order and for any $i, j \in I$, if $i \leq j$ then $\varphi_i(\mathcal{U}^{xy}) \subseteq \varphi_j(\mathcal{U}^{xy})$. Then $F^{U}_{\mu,A}$ is the pointwise limit of the *increasing* net $(F^{\varphi_i}_{\mu,A})_{i \in I}$. Moreover, given $r \in [0, 1]$, we claim that $(F^{U}_{\mu,A})^{-1}((r, 1]) = \bigcup_{i \in I} (F^{\varphi_i}_{\mu,A})^{-1}((r, 1])$. Indeed, $p \in (F^{U}_{\mu,A})^{-1}((r, 1])$ implies that $\mu(U(x, b)) \in (r, 1]$. By regularity, there exists some $\varphi_i(x, b)$ such that $\mu(\varphi_i(x, b)) \in (r, 1]$. The other direction is similar. Since each $F^{\varphi_i}_{\mu,A}$ is continuous, we now have that for any $r \in [0, 1]$, $(F^{U}_{\mu,A})^{-1}((r, 1])$ is open. Since sets of the form $(r, 1]$ generate the Borel σ-algebra on $[0, 1]$, it follows that $F^{U}_{\mu,A}$ is Borel (in fact, upper semicontinuous). Now, for any $\nu \in \mathfrak{M}_y(\mathcal{U})$, we have $\int_{S_y(A)} F^{U}_{\mu}\, d\nu = \lim_i \int_{S_y(A)} F^{\varphi_i}_{\mu}\, d\nu$ by the monotone convergence theorem for uniformly bounded increasing nets of continuous functions on compact Hausdorff spaces (see [Reed and Simon 1972, Theorem IV.15]). It follows that μ is BD^{++} over A, as in the proof of Lemma 2.12. $\square$

We can now prove the main associativity result for definable measures.

Theorem 2.18. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$ are Borel definable over some $A \subset \mathcal{U}$, and at least one of μ or ν is definable over A. Then $\mu \otimes \nu$ is Borel definable over A and, for any $\lambda \in \mathfrak{M}_z(\mathcal{U})$, we have $(\mu \otimes \nu) \otimes \lambda = \mu \otimes (\nu \otimes \lambda)$.*

Proof. First assume ν is definable over A. Then, by Lemma 2.17 and Theorem 2.11, $\mu \otimes \nu$ is Borel definable over A and $((\mu \otimes \nu) \otimes \lambda)|_A = (\mu \otimes (\nu \otimes \lambda))|_A$ for any $\lambda \in \mathfrak{M}_z(\mathcal{U})$. Moreover, note that if $B \supseteq A$ then μ is definable over B and ν is Borel definable over B. So this argument works over any $B \supseteq A$.

Now assume μ is definable over A. In this case, the proof is almost the same as that of Theorem 2.11 (applied to μ and ν), and so we just explain the necessary adjustments. In particular, since μ is definable, we can use Fact 2.15 to assume that the Borel sets $W_{n,i}$ in the proof of Theorem 2.11 are actually *clopen*. Therefore, one only needs Borel definability of ν to conclude that the maps $F_{\nu,A}^{W_{n,i}}$ and h_n are each Borel. This is all that is needed to conclude $\mu \otimes \nu$ is Borel definable over A. Finally, in the associativity argument, we do not need to assume ν is BD^{++} to know that $(\nu \otimes \lambda)|_A(W_{n,i}) = \int_{S_z(A)} F_\nu^{W_{n,i}}\, d\nu$. Indeed, because $W_{n,i}$ is clopen, this follows from the definition of the Morley product. So $((\mu \otimes \nu) \otimes \lambda)|_A = (\mu \otimes (\nu \otimes \lambda))|_A$ by the same steps. Once again, this argument works over any $B \supseteq A$. $\square$

Remark 2.19. Call an A-invariant measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ *Baire*-1 *definable over A* if for any $\mathcal{L}$-formula $\varphi(x, y)$, $F_{\mu,A}^{\varphi}$ is a function of Baire class 1, i.e., the pointwise limit of a sequence of continuous functions. Note that, as a property of measures, Baire-1 definability is stronger than Borel definability, but weaker than definability. We claim that if $\mu \in \mathfrak{M}_x(\mathcal{U})$ is Baire-1 definable over A, and $\nu \in \mathfrak{M}_y(\mathcal{U})$ is Borel definable over A, then $\mu \otimes \nu$ is Borel definable over A and, for any $\lambda \in \mathfrak{M}_z(\mathcal{U})$, we have $(\mu \otimes \nu) \otimes \lambda = \mu \otimes (\nu \otimes \lambda)$. Indeed, the proof of Theorem 2.11 only required pointwise limits, and so one can argue using the same adjustments as in Theorem 2.18 (together with the exercise that a Baire-1 function on a Stone space is a pointwise limit of finite linear combinations of indicator functions of clopen sets).

In light of Theorem 2.18, it is natural to ask if one gains any traction in proving associativity by assuming that the measure in the third position is definable. In Corollary 3.12, we will give an example of Borel definable types p and q, and a definable measure λ, such that $p \otimes q$ is Borel definable, but $(p \otimes q) \otimes \lambda \neq p \otimes (q \otimes \lambda)$. On the other hand, we do have the following result.

Corollary 2.20. *Fix $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$ such that μ, ν, and $\mu \otimes \nu$ are each Borel definable over some $A \subset \mathcal{U}$. Suppose $\lambda \in \mathfrak{M}_z(\mathcal{U})$ is such that $\lambda|_A$ has a definable global extension $\hat{\lambda} \in \mathfrak{M}_z(\mathcal{U})$ that commutes with μ, ν, and $\mu \otimes \nu$. Then $((\mu \otimes \nu) \otimes \lambda)|_A = (\mu \otimes (\nu \otimes \lambda))|_A$.*

Proof. Fix an $\mathcal{L}_A$-formula $\varphi(x, y, z)$. Then we have the following calculations (individual steps are justified afterward):

$$\begin{aligned}((\mu\otimes\nu)\otimes\lambda)(\varphi(x,y,z)) &= ((\mu\otimes\nu)\otimes\hat{\lambda})(\varphi(x,y,z))\\ &= (\hat{\lambda}\otimes(\mu\otimes\nu))(\varphi(x,y,z)) = ((\hat{\lambda}\otimes\mu)\otimes\nu)(\varphi(x,y,z))\\ &= ((\mu\otimes\hat{\lambda})\otimes\nu)(\varphi(x,y,z)) = (\mu\otimes(\hat{\lambda}\otimes\nu))(\varphi(x,y,z))\\ &= (\mu\otimes(\nu\otimes\hat{\lambda}))(\varphi(x,y,z)) = (\mu\otimes(\nu\otimes\lambda))(\varphi(x,y,z)).\end{aligned}$$

In the above calculations, the first and last equalities use $\lambda|_A = \hat{\lambda}|_A$, the third and fifth equalities use Theorem 2.18 and definability of $\hat{\lambda}$, and the remaining equalities use the commutativity assumptions on $\hat{\lambda}$. □

Remark 2.21. Note that in the previous result we do not need to assume that $\hat{\lambda}$ is definable *over* A. For example, given $p \in S_x(\mathcal{U})$ and $A \subset \mathcal{U}$, if $a \models p|_A$ and $\hat{p} = \mathrm{tp}(a/\mathcal{U})$, then $p|_A = \hat{p}|_A$, $\hat{p}$ is definable over $\{a\}$, and $\hat{p}$ commutes with any invariant measure. So if the measure λ in Corollary 2.20 is a type, then such a $\hat{\lambda}$ exists for any $A \subset \mathcal{U}$. (See Fact 3.1 below for a full account of associativity when the measure in the third position is a type.)

For a more interesting example of when Corollary 2.20 is applicable, one can turn to the class of NIP theories. Recall that if T is NIP then any invariant global Keisler measure is Borel definable by [Hrushovski and Pillay 2011]. Moreover, by the original work of Keisler [1987], any measure over a small model M of an NIP theory has a global extension that is *smooth*, i.e., it is the unique global extension of its restriction to some small model $N \succeq M$ (see also [Simon 2015, Proposition 7.9]). For example, if $p \in S_x(M)$ and $a \in \mathcal{U}^x$ realizes p, then $\mathrm{tp}(a/\mathcal{U})$ is a smooth global extension of p. It is not hard to show that smooth measures are definable and commute with all Borel definable measures (see [Hrushovski et al. 2013, Section 2]). So if T is NIP then any global measure λ satisfies the assumptions of Corollary 2.20 for any $A \subset \mathcal{U}$. Altogether, this reaffirms associativity of the Morley product for invariant measures in NIP theories (recall Remark 2.14).

Corollary 2.22. *If T is NIP then the Morley product of invariant measures is associative.*

3. Counterexamples in Borel definability

The goal of this section is to show that, over uncountable sets, the Morley product of two Borel definable measures need not be Borel definable and, moreover, that the Morley product can fail to be associative (even when all products involved are well defined and Borel definable). In fact, we will demonstrate this behavior in a relatively straightforward simple unstable (countable) theory. Before getting into this example, we first discuss some preliminaries.

3A. ***Strongly continuous measures.*** The purpose of this subsection is mainly to provide context for how Morley products can fail associativity. Let T be a complete $\mathcal{L}$-theory with monster model $\mathcal{U}$. It is well known (and easy to show) that the Morley product is associative with respect to invariant types (see [Simon 2015, Fact 2.20]), and so a failure of associativity must involve at least one "true" measure. Toward making this remark more precise, we state the following fact, which is left as an exercise (the mechanics of the proof are similar to that of Corollary 2.20).

Fact 3.1. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is Borel definable and $\nu \in \mathfrak{M}_y(\mathcal{U})$ is invariant. Then $(\mu \otimes \nu) \otimes r = \mu \otimes (\nu \otimes r)$ for any $r \in S_z(\mathcal{U})$.*

Next, we recall some terminology (from the theory of *charges* [Bhaskara Rao and Bhaskara Rao 1983]) that will be used to make the idea of a "true" measure more rigorous.

Definition 3.2. Given $A \subseteq \mathcal{U}$, a measure $\mu \in \mathfrak{M}_x(A)$ is *strongly continuous* if, for any $\varepsilon > 0$, there is a partition of $S_x(A)$ into finitely many clopen sets of measure less than ε.

Despite the use of the word "continuous" in the previous definition, we caution the reader that there is no general connection between strongly continuous measures and definable measures.

Remark 3.3. By compactness, a measure $\mu \in \mathfrak{M}_x(A)$ is strongly continuous if and only if $\mu(\{p\}) = 0$ for all $p \in S_x(A)$. Note also that if $\mu \in \mathfrak{M}_x(\mathcal{U})$ is strongly continuous, then there is some countable $A \subset \mathcal{U}$ such that $\mu|_A$ is strongly continuous.

Let $\mu \in \mathfrak{M}_x(\mathcal{U})$ be a Keisler measure. By the Sobczyk–Hammer decomposition theorem for finitely additive bounded charges (see [Bhaskara Rao and Bhaskara Rao 1983, Theorem 5.2.7]), one can write $\mu = \alpha_0\mu_0 + \sum_{n=1}^{\infty} \alpha_n p_n$, where each α_n is in [0, 1] with $\sum_{n=0}^{\infty} \alpha_n = 1$, each p_n is a type in $S_x(\mathcal{U})$, and either μ_0 is a strongly continuous measure in $\mathfrak{M}_x(\mathcal{U})$, or $\alpha_0 = 0$ and μ_0 is the identically zero measure (in this case, we call μ *atomic*). The next fact, which we leave as an exercise, follows from Fact 3.1 together with standard measure-theoretic computations.

Fact 3.4. *Fix $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $\lambda \in \mathfrak{M}_z(\mathcal{U})$. Let $\lambda = \alpha_0\lambda_0 + \sum_{n=1}^{\infty} \alpha_n p_n$ be the Sobczyk–Hammer decomposition of λ described above.*

(a) *Assume λ is atomic. If μ is Borel definable and ν is invariant, then $(\mu \otimes \nu) \otimes \lambda$ is well defined and equal to $\mu \otimes (\nu \otimes \lambda)$.*

(b) *Assume λ is not atomic. If μ, ν, and $\mu \otimes \nu$ are each Borel definable, then $(\mu \otimes \nu) \otimes \lambda = \mu \otimes (\nu \otimes \lambda)$ if and only if $(\mu \otimes \nu) \otimes \lambda_0 = \mu \otimes (\nu \otimes \lambda_0)$.*

In other words, this fact says that in any situation where the Morley product of Keisler measures fails associativity, the measure in the third coordinate cannot be atomic, and so the failure of associativity can be traced back to an underlying strongly continuous measure.

Finally, we recall a known result from the folklore characterizing the existence of strongly continuous Keisler measures.

Fact 3.5. *Given a complete theory T, the following are equivalent*:

(i) *T is totally transcendental* (*i.e.*, *every formula has ordinal Morley rank*).

(ii) *There is no strongly continuous measure in $\mathfrak{M}_x(\mathcal{U})$ for any x.*

(iii) *There is no strongly continuous measure in $\mathfrak{M}_x(\mathcal{U})$ for any tuple of variables x of length one.*

Proof. This follows from standard results on type spaces in totally transcendental theories, combined with various facts about strongly continuous measures (see [Bhaskara Rao and Bhaskara Rao 1983, Theorem 5.3.2, Lemma 5.3.8, Theorem 5.3.9]). See also [Chernikov and Gannon 2022, Fact 3.3], which discusses some details, including the relevance of [Keisler 1987, Lemma 1.7]. □

3B. *Relative measurability.* Later in this section, we will construct Borel definable global types p and q (in a specific theory) such that $p \otimes q$ is not Borel definable. In this case, one might wonder if $p \otimes q$ still admits Morley products with restricted classes of measures with nice behavior (e.g., if r is a type then $(p \otimes q) \otimes r$ is well defined since $p \otimes q$ is still invariant). However, our construction will show that $p \otimes q$ can be arbitrarily bad. This will be made precise using the following notions.

Definition 3.6. Let X be a topological space. Given a Borel measure μ on X, a subset of X is called *μ-measurable* if it is measurable with respect to the completion of μ. A subset Z of X is called a *Bernstein set* if both Z and $X \setminus Z$ nontrivially intersect every uncountable closed subset of X.

Recall that, in the above context, a subset of X is μ-measurable if and only if it is of the form $B \cup E$, where B is Borel and E is contained in a μ-null Borel set. It is also a standard fact that any Polish space contains a Bernstein set (see, e.g., Theorem 4 in [Just and Weese 1996, Chapter 11]).

Lemma 3.7. *Suppose $\rho : X \to Y$ is a surjective continuous map between compact Hausdorff spaces, and μ is a regular Borel measure on X. Let ν be the pushforward of μ along ρ, and assume that any singleton in Y is ν-null. Then, for any Bernstein set $Z \subseteq Y$, the set $\rho^{-1}(Z)$ is not μ-measurable.*

Proof. Suppose $\rho^{-1}(Z)$ is μ-measurable. Since $Y \setminus Z$ is also a Bernstein set, we may assume without loss of generality that $\mu(\rho^{-1}(Z)) > 0$. By regularity, there is a closed set $C \subseteq \rho^{-1}(Z)$ such that $\mu(C) > 0$. Since $\rho(C)$ is closed, and contained in Z, it must be countable. So $\nu(\rho(C)) = 0$ by assumption on ν and countable additivity. But then $\mu(C) \leq \mu\big(\rho^{-1}(\rho(C))\big) = \nu(\rho(C)) = 0$, which is a contradiction. □

Corollary 3.8. *Let T be a complete $\mathcal{L}$-theory with monster model $\mathcal{U}$. Fix $A \subseteq \mathcal{U}$ and suppose $\mu \in \mathfrak{M}_x(A)$ is strongly continuous. Then there is a subset of $S_x(A)$ that is not μ-measurable.*

Proof. Choose a countable set $A_0 \subseteq A$, and a countable sublanguage $\mathcal{L}_0 \subseteq \mathcal{L}$, such that if ν is the restriction of $\mu|_{A_0}$ to $(\mathcal{L}_0)_{A_0}$-formulas, then ν is still strongly continuous. Let $X = S_x^{\mathcal{L}}(A)$ and $Y = S_x^{\mathcal{L}_0}(A_0)$, and define $\rho : X \to Y$ to be the composition of ρ^x_{A,A_0} with restriction to $\mathcal{L}_0$. Then ν is the pushforward of μ along ρ (as in Remark 1.2). Not that any singleton in Y is ν-null by strong continuity of μ. Since Y is Polish, there is a Bernstein set $Z \subseteq Y$. Altogether, $\rho^{-1}(Z) \subseteq S_x(A)$ is not μ-measurable by Lemma 3.7. □

3C. ***The random ternary relation.*** Given any finite relational language $\mathcal{L}$, the class of finite $\mathcal{L}$-structures is a Fraïssé class, and the complete theory of the corresponding Fraïssé limit is $\aleph_0$-categorical and has quantifier elimination (this follows from [Hodges 1993, Theorem 7.4.1]). It is well known, and not hard to prove, that any theory obtained this way is supersimple of SU-rank 1, and is also unstable if and only if $\mathcal{L}$ contains a relation of arity at least 2.

In this subsection, we work with the theory T_R obtained in the above fashion, where $\mathcal{L}$ consists of a single *ternary* relation $R(x, y, z)$. Throughout this section, $\mathcal{U}$ is a monster model of T_R. We will first show that in T_R, Borel definability of measures is not preserved by Morley products. So this refutes the unproven claim in [Hrushovski et al. 2013, Lemma 1.6]. In fact, we show that the product of Borel definable *types* is not necessarily Borel definable.

Proposition 3.9. *There are Borel definable $p, q \in S_1(\mathcal{U})$ such that $p \otimes q$ is not Borel definable.*

Proof. Fix an infinite set $B \subset \mathcal{U}$ and an arbitrary set $Z \subseteq S_1(B)$ such that $\kappa := |Z| \geq |B|$. We will construct types $p, q \in S_1(\mathcal{U})$ satisfying the following properties:

(i) p and q are Borel definable over some $A \supseteq B$ of cardinality κ.

(ii) For any $c \in \mathcal{U}$, $R(x, y, c) \in p \otimes q$ if and only if $\text{tp}(c/B) \in Z$.

In particular, setting $Z^* = (\rho^z_{A,B})^{-1}(Z)$, we have $F^{R(x,y;z)}_{p\otimes q,A} = \mathbf{1}_{Z^*}$ by (ii). So if Z is not Borel then $p \otimes q$ is not Borel definable by Theorem 2.1 and Corollary 2.2. Note also that $S_1(B)$ has a topological basis of size $|B|$, and thus has at most $2^{|B|}$ Borel subsets. On the other hand, $S_1(B)$ has $2^{2^{|B|}}$ subsets (since it has size $2^{|B|}$). So there are non-Borel choices for the set Z above.

Fix $A \supseteq B$ of cardinality κ. Given an A-invariant type $p \in S_{\bar{x}}(\mathcal{U})$ and a formula $\varphi(\bar{x}; \bar{y})$, we define $dp(\varphi) = \{s \in S_{\bar{y}}(A) : \varphi(\bar{x}; \bar{b}) \in p \text{ for } \bar{b} \models s\}$ (in particular, we have $F^{\varphi}_{p,A} = \mathbf{1}_{dp(\varphi)}$). Let x, y, z be tuples of variables of length one, and let $R_1(x; y, z)$ and $R_2(y; x, z)$ be partitions of $R(x, y, z)$. Enumerate $A = \{a_i : i < \kappa\}$ and $Z = \{r_i : i < \kappa\}$. We construct the desired types p and q.

First, define $p \in S_x(\mathcal{U})$ so that the positive instances of R in p (which actually involve x) are precisely those of the form $R(x, b, c)$, where $b, c \in \mathcal{U}$ are such that $R(a_i, b, c)$ holds for some $i < \kappa$. Note that p is A-invariant. To prove Borel definability of p, it suffices by quantifier elimination to focus on atomic formulas; and by definition of p, we only need to consider $R_1(x; y, z)$. By construction, $dp(R_1) = \bigcup_{i<\kappa}[R(m_i, y, z)]$, which is open.

Now define $q \in S_y(\mathcal{U})$ so that the positive instances of R in q (which actually involve y) are precisely those of the form $R(a, y, c)$, where $a, c \in \mathcal{U}$ are such that $a = a_i$ and $c \models r_i$ for some $i < \kappa$. Then q is A-invariant by construction. We claim that q is Borel definable. By quantifier elimination, and the definition of q, it suffices to consider $R_2(y; x, z)$. Given $i < \kappa$, set $K_i = \{s \in S_{xz}(A) : s_z|_B = r_i\}$, and note that K_i is closed. From the definition of q, we have that $dq(R_2) = \bigcup_{i<\kappa} K_i \cap [x = a_i]$, which is an infinite union of closed sets. However, if we set $K = \bigcap_{i<\kappa} K_i \cup [x \neq a_i]$ and $U = \bigcup_{i<\kappa}[x = a_i]$, then K is closed, U is open, and $dq(R_2) = K \cap U$.

We have now built p and q satisfying (i). It remains to show that $p\otimes q$ satisfies (ii). It is easy to check that any positive instance of R in $p \otimes q$ involving the variables x and y must have the form $R(x, y, c)$ for some $c \in \mathcal{U}$. So fix $c \in \mathcal{U}$. Using the definition of p, we have $R(x, y, c) \in p\otimes q$ if and only if there are $i < \kappa$ and $b \models q|_{Ac}$ such that $R(a_i, b, c)$. Using the definition of q, we conclude that $R(x, y, c) \in p \otimes q$ if and only if $c \models r_i$ for some $i < \kappa$. Altogether, we have property (ii). $\square$

We now use the previous construction to produce Borel definable types $p, q \in S_1(\mathcal{U})$, and a measure $\lambda \in \mathfrak{M}_1(\mathcal{U})$, such that the Morley product of $p \otimes q$ with λ is not well defined. First, using Fact 3.5, we may fix a strongly continuous measure $\lambda \in \mathfrak{M}_1(\mathcal{U})$ (one can even choose λ to be *definable* via Lemma 3.11 below). Let $B \subset \mathcal{U}$ be an infinite set such that $\lambda|_B$ is strongly continuous. By Corollary 3.8, there is a set $Z \subseteq S_1(B)$ that is not $\lambda|_B$-measurable. Now choose $p, q \in S_1(\mathcal{U})$ as in the proof of Proposition 3.9; in particular, $R(x, y, c) \in p \otimes q$ if and only if $\mathrm{tp}(c/B) \in Z$. Then $(p \otimes q) \otimes \lambda$ is not well defined since $F^R_{p\otimes q,B} = \mathbf{1}_Z$. Note that $p \otimes (q \otimes \lambda)$ is well defined, however, and so this also produces a rather cheap failure of associativity.

Next we will demonstrate a more substantial failure of associativity in which all Morley products involved are well defined. Intuitively speaking, the construction uses something like the "first-year probability theory paradox" that the measure of $[0, 1]$ is 1, yet the measure of $\{x\}$ for any $x \in [0, 1]$ is 0.

Proposition 3.10. *Suppose $\lambda \in \mathfrak{M}_1(\mathcal{U})$ is a strongly continuous measure. Then there are types $p, q \in S_1(\mathcal{U})$ such that p, q, and $p \otimes q$ are Borel definable, the measures $q\otimes\lambda$, $(p\otimes q)\otimes\lambda$, and $p\otimes(q\otimes\lambda)$ are well defined, but $((p\otimes q)\otimes\lambda)(R(x, y, z)) = 1$ and $(p \otimes (q \otimes \lambda))(R(x, y, z)) = 0$.*

Proof. We use similar notation as in the proof of Proposition 3.9. Using Remark 3.3, we may choose infinite $B \subseteq A \subset \mathcal{U}$ such that $\kappa := |A| = 2^{|B|}$ and $\lambda|_A(\{r\}) = 0$ for all $r \in S_1(A)$. Enumerate $A = \{a_i : i < \kappa\}$ and $S_1(B) = \{r_i : i < \kappa\}$.

Define $p \in S_x(\mathcal{U})$ so that the positive instances of R in p are precisely those of the form $R(x, b, c)$, where $b, c \in \mathcal{U}$ are such that $R(a_i, b, c)$ holds for some $i < \kappa$. Define $q \in S_y(\mathcal{U})$ so that the positive instances of R in q are precisely those of the form $R(a, y, c)$, where $a, c \in \mathcal{U}$ are such that $a = a_i$ and $c \models r_i$ for some $i < \kappa$. In other words, p and q are exactly as in the proof of Proposition 3.9, if one chooses $Z = S_z(B)$ in the definition of q. So p, q, and $p \otimes q$ are Borel definable over A. (Note that, in the general construction from the proof of Proposition 3.9, $p \otimes q$ is Borel definable if and only if Z is Borel.)

Note that $q \otimes \lambda$, $(p \otimes q) \otimes \lambda$, and $p \otimes (q \otimes \lambda)$ are well defined since the leftmost term in each product is Borel definable. Let $\eta_1 = (p \otimes q) \otimes \lambda$ and $\eta_2 = p \otimes (q \otimes \lambda)$. Then $\eta_1(R(x, y, z)) = 1$ since $F^{R(x,y;z)}_{p\otimes q,A}$ takes the constant value 1 on $S_z(A)$. It remains to show that $\eta_2(R(x, y, z)) = 0$.

By definition, $\eta_2(R(x, y, z)) = (q \otimes \lambda)|_A(U)$, where $U := dp(R(x; y, z)) = \bigcup_{i<\kappa}[R(a_i, y, z)]$. So U is an open set in $S_{yz}(A)$. Moreover, if $i < \kappa$ then

$$(q \otimes \lambda)(R(a_i, y, z)) = \lambda|_A\big(dq(R(a_i, y; z))\big) = \lambda|_A(\{r_i\}) = 0.$$

So by compactness, Fact 1.1(a), and finite additivity of $(q \otimes \lambda)|_A$, we have

$$\eta_2(R(x, y, z)) = (q \otimes \lambda)|_A(U) \leq \sup_{I\in[\kappa]^{<\omega}} \sum_{i\in I}(q \otimes \lambda)|_A(R(a_i, y, z)) = 0. \qquad \square$$

In fact, we can strengthen the previous result using the existence of a *definable* strongly continuous measure in T_R, namely, the "coin-flipping" measure, which independently assigns R-neighborhoods measure $\frac{1}{2}$. Similar measures on the random graph are studied by Albert [1994].

Lemma 3.11. *There is an $\varnothing$-definable strongly continuous measure in $\mathfrak{M}_1(\mathcal{U})$.*

Proof. Let λ be the unique measure in $\mathfrak{M}_1(\mathcal{U})$ satisfying the property that if $\theta_1(x), \dots, \theta_n(x)$ are pairwise distinct (positive) instances of R in one free variable, and $\psi_i(x)$ is either $\theta_i(x)$ or $\neg\theta_i(x)$, then

$$\lambda(\psi_1(x) \wedge \cdots \wedge \psi_n(x)) = \frac{1}{2^n}.$$

The justification that such a measure exists is given in Section A2.

To see that λ is strongly continuous, fix $n > 0$ and distinct $a_1, \dots, a_n \in \mathcal{U}$. Suppose $\theta(x) = \bigwedge_{i=1}^n \theta_i(x)$, where $\theta_i(x)$ is either $R(x, a_i, a_i)$ or $\neg R(x, a_i, a_i)$. Then $\lambda(\theta(x)) = 1/2^n$. Since the collection of all such $\theta(x)$ forms a finite partition of $\mathcal{U}^x$, we conclude that λ is strongly continuous.

Finally, to see that λ is $\varnothing$-definable, fix a formula $\varphi(x; y_1, \dots, y_n)$. By quantifier-elimination, we may assume φ is a conjunction of atomic and negated atomic formulas. We may also assume without loss of generality that φ contains $y_i \neq y_j$

for all $i \leq j$. Note that $\lambda(X) = 0$ for any finite $X \subseteq \mathcal{U}$. Altogether, every consistent instance of φ has the same measure. Therefore for any formula $\theta(x; y_1, \ldots, y_n) := \varphi(x; y_1, \ldots, y_n) \bigwedge_{1 \leq i < j \leq n}^{n} y_i \neq y_j$, the map $F^{\theta}_{\lambda,\varnothing} : S_{\bar{y}}(\varnothing) \to [0, 1]$ is well defined and constant, and in particular, continuous. Hence λ is $\varnothing$-definable. □

The two previous results together yield a strong failure of associativity for the Morley product in T_R, which also provides a counterpoint to Theorem 2.18.

Corollary 3.12. *There are $p, q \in S_1(\mathcal{U})$ and $\lambda \in S_1(\mathcal{U})$ such that p, q, and $p \otimes q$ are Borel definable, and λ is $\varnothing$-definable, but $((p \otimes q) \otimes \lambda)(R(x, y, z)) = 1$ and $(p \otimes (q \otimes \lambda))(R(x, y, z)) = 0$.*

Note that in the previous result, $q \otimes \lambda$ and $(p \otimes q) \otimes \lambda$ are also Borel definable by Theorem 2.18. One can further show that if λ is the specific measure from Lemma 3.11, then $p \otimes (q \otimes \lambda)$ is Borel definable. But this involves a somewhat technical case analysis so we omit the details.

4. Fim fam flim flam

We now change our overall focus from Borel definability to stronger notions motivated by the study of model-theoretic tameness. In this section, we review several properties which, in the setting of NIP theories, characterize a canonical notion of "generic stability" for invariant Keisler measures. These properties are referred to using the descriptors *fim*, *fam*, and *dfs*, which stand for *frequency interpretation measure*, *finitely approximated measure*, and *definable and finitely satisfiable*, respectively. See Definition 4.1 below for full details. Much of the motivation for studying these notions comes from the fundamental result, due to Hrushovski, Pillay, and Simon [Hrushovski et al. 2013], that if T is NIP then *fim*, *fam*, and *dfs* are equivalent. More precisely, we have the following implications:

$$\textit{fim} \implies \textit{fam} \implies \textit{dfs} \overset{\text{NIP}}{\implies} \textit{fim}.$$

The first implication is clear from the definitions (given below), the second is a standard exercise (e.g., [Gannon 2020, Proposition 2.30]; see also Proposition 4.3 below), and the third is [Hrushovski et al. 2013, Theorem 3.2]. The purpose of this section is to rapidly review the parade of definitions and basic facts about *fim*, *fam*, and *dfs* that we will need for later results.

Let T be a complete theory with monster model $\mathcal{U}$. Given measures $\mu, \nu \in \mathfrak{M}_x(\mathcal{U})$, and some $\mathcal{L}_{\mathcal{U}}$-formula $\varphi(x, y)$, we write $\mu \approx^{\varphi}_{\varepsilon} \nu$ to denote that $\mu(\varphi(x, b)) \approx_{\varepsilon} \nu(\varphi(x, b))$ for all $b \in \mathcal{U}^y$. Note that if μ and ν are invariant over $A \subset \mathcal{U}$ and $\varphi(x, y)$ is an $\mathcal{L}_A$-formula, then $\mu \approx^{\varphi}_{\varepsilon} \nu$ if and only if $\|F^{\varphi}_{\mu,A} - F^{\varphi}_{\nu,A}\|_{\infty} < \varepsilon$.

Given a Borel definable measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ and some $n \geq 1$, we define $\mu^{(n)} \in \mathfrak{M}_{x_1 \ldots x_n}(\mathcal{U})$ by setting $\mu^{(1)} = \mu_{x_1}$ and $\mu^{(n+1)} = \mu_{x_{n+1}} \otimes \mu^{(n)}_{x_1 \ldots x_n}$. Note that even if $\mu^{(n)}$ is not Borel definable for some n, the product involved in the definition of $\mu^{(n+1)}$ is still well defined. Also, if μ is a type $p \in S_x(\mathcal{U})$, then one only needs invariance in order to define $p^{(n)}$.

We now recall the definitions of the properties mentioned above. For various reasons, these notions are more effective when formulated over small models, rather than arbitrary parameter sets. Thus we will now shift our focus to small models.

Definition 4.1. Fix $\mu \in \mathfrak{M}_x(\mathcal{U})$.

(1) μ is *finitely satisfiable in $M \prec \mathcal{U}$* if for any $\mathcal{L}_{\mathcal{U}}$-formula $\varphi(x)$, if $\mu(\varphi(x)) > 0$, then $\varphi(x)$ is realized in M.

(2) μ is *dfs* if there is some $M \prec \mathcal{U}$ such that μ is definable over M and finitely satisfiable in M. In this case, we also say that μ is *dfs over M*.

(3) μ is *fam* ("finitely approximated measure") if there is some $M \prec \mathcal{U}$ such that, for any $\mathcal{L}$-formula $\varphi(x, y)$ and any $\varepsilon > 0$, there is $\bar{a} \in (M^x)^n$ such that $\mu \approx^{\varphi}_{\varepsilon} \mathrm{Av}(\bar{a})$. In this case, we also say that M is *fam over M*.

(4) μ is *fim* ("frequency interpretation measure") if there is some $M \prec \mathcal{U}$ such that, for any $\mathcal{L}$-formula $\varphi(x, y)$, there is a sequence $(\theta_n(x_1, \dots, x_n))_{n=1}^{\infty}$ of consistent $\mathcal{L}_M$-formulas satisfying the following properties:
   (a) For any $\varepsilon > 0$ there is some $n(\varepsilon) \geq 1$ such that, if $n \geq n(\varepsilon)$ and $\bar{a} \models \theta_n$, then $\mu \approx^{\varphi}_{\varepsilon} \mathrm{Av}(\bar{a})$.
   (b) $\lim_{n\to\infty} \mu^{(n)}(\theta_n) = 1$.

   In this case, we also say that M is *fim over M*.

The definition of *fim* implicitly assumes that the Morley products $\mu^{(n)}$ are well defined. This is justified by the fact that condition (i) ensures μ is *fam* (since we work over small models), and thus definable. We also note the following easy facts.

Remark 4.2. Fix $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $M \prec \mathcal{U}$.

(1) If μ is finitely satisfiable in M then it is invariant over M.

(2) If $\bar{a} \in (M^x)^n$ then $\mathrm{Av}(\bar{a}) \in \mathfrak{M}_x(\mathcal{U})$ is *fim* over M.

Next we take the opportunity to provide a novel characterization of *dfs*, which is formulated using *fam*-like behavior. In particular, we show that *dfs* is equivalent to being "piecewise" *fam*. This result also provides further evidence that *dfs* is a natural notion in its own right, rather than just an arbitrary combination of two separate notions.

Proposition 4.3. *Fix $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $M \prec \mathcal{U}$. Then μ is dfs over M if and only if for any $\mathcal{L}$-formula $\varphi(x, y)$ and any $\varepsilon > 0$, there are tuples $\bar{a}_1 \in (M^x)^{n_1}, \dots, \bar{a}_k \in (M^x)^{n_k}$ and $\mathcal{L}_M$-formulas $\psi_1(y), \dots, \psi_k(y)$ partitioning M^y such that for any $b \in \mathcal{U}^y$, if $\mathcal{U} \models \psi_i(b)$ then $\mu(\varphi(x, b)) \approx_{\varepsilon} \mathrm{Av}(\bar{a}_i)(\varphi(x, b))$.*

Proof. Suppose first that μ satisfies the latter condition. Then μ is clearly invariant over M and, for any $\mathcal{L}$-formula $\varphi(x,y)$, $F^{\varphi}_{\mu,M}$ is a uniform limit of continuous functions on $S_y(M)$ (specifically piecewise constant functions with clopen pieces). Therefore $F^{\varphi}_{\mu,M}$ is continuous for any $\varphi(x,y)$, whence μ is definable over M. Finite satisfiability of μ in M is straightforward and left to the reader (the argument is nearly identical to showing that *fam* over M implies finitely satisfiable in M).

Conversely, suppose μ is *dfs* over M. Fix an $\mathcal{L}$-formula $\varphi(x,y)$. Then $F^{\varphi}_{\mu,M}$ is continuous. Let $\mathcal{F}$ be the set of functions $F^{\varphi}_{\mathrm{Av}(\bar{a}),M}$ for $\bar{a}\in(M^x)^{<\omega}$. In particular, $\mathcal{F}$ is a set of continuous functions from $S_y(M)$ to $[0,1]$. To ease notation, let $g=F^{\varphi}_{\mu,M}$ and $X=S_y(M)$.

Claim. *For any $q\in X$ and any $\varepsilon>0$, there is some $f\in\mathcal{F}$ such that $|f(q)-g(q)|<\varepsilon$.*

Proof. Fix $q\in X$ and $b\models q$. There are three cases: either $\mu(\varphi(x,b))=0$, $\mu(\varphi(x,b))=1$, or $\mu(\varphi(x,b))\in(0,1)$.

In the first case, we can find $a\in M^x$ such that $\neg\varphi(a,b)$ holds. In the second case, we can find $a\in M^x$ such that $\varphi(a,b)$ holds. Therefore, in either case, if $f=F^{\varphi}_{\mathrm{Av}(a),M}=\mathbf{1}_{\varphi(a,y)}$, then $f\in\mathcal{F}$ and $f(q)=g(q)$.

In the third case, we can find $a_+,a_-\in M^x$ such that $\varphi(a_+,b)$ and $\neg\varphi(a_-,b)$ both hold. Fix $\varepsilon>0$, and choose a rational $r=m/n\in[0,1]$ such that $|r-g(q)|<\varepsilon$. Let $\bar{a}=(a_1,\dots,a_n)$ with $a_i=a_+$ for $1\leq i\leq m$ and $a_i=a_-$ for $m<i\leq n$. If $f=F^{\varphi}_{\mathrm{Av}(\bar{a}),M}$ then $f\in\mathcal{F}$ and $f(q)=r$, hence $|f(q)-g(q)|<\varepsilon$. $\dashv_{\text{claim}}$

Fix $\varepsilon>0$. We will now find a clopen partition $A_1,\dots,A_k$ of X, along with functions $f_1,\dots,f_k\in\mathcal{F}$, such that for any $q\in X$ and $1\leq i\leq k$, if $q\in A_i$ then $|f_i(q)-g(q)|<\varepsilon$. By choice of $\mathcal{F}$ and g, this will finish the proof.

For each $q\in X$, we apply the claim to find $f_q\in\mathcal{F}$ such that $|f_q(x)-g(q)|<\frac{1}{3}\varepsilon$. Since g and f_q are continuous, we can also find a clopen neighborhood B_q of q such that for any $p\in B_q$, $|f_q(p)-f_q(q)|<\frac{1}{3}\varepsilon$ and $|g(p)-g(q)|<\frac{1}{3}\varepsilon$. In particular, this implies that for any $p\in B_q$,

$$|f_q(p)-g(p)|\leq|f_q(p)-f_q(q)|+|f_q(q)-g(q)|+|g(q)-g(p)|<\varepsilon.$$

By compactness, we can find a finite sequence $q_1,\dots,q_k$ such that $B_{q_1},\dots,B_{q_k}$ covers X. Setting $f_i=f_{q_i}$ and $A_i=B_{q_i}\setminus\bigcup_{j<i}B_{q_j}$, we have the required functions and partition (after possibly decreasing k and discarding any empty A_i). $\square$

Note that $\mu\in\mathfrak{M}_x(\mathcal{U})$ is *fam* over $M\prec\mathcal{U}$ if and only if it satisfies the conditions of the previous proposition with $k=1$.

Next we review basic properties about approximations.

Definition 4.4. Given a measure $\mu\in\mathfrak{M}_x(\mathcal{U})$, an $\mathcal{L}_{\mathcal{U}}$-formula $\varphi(x,y)$, an integer $n\geq 1$, and some $C\subseteq[0,1]$, let $\bar{x}=(x_1,\dots,x_n)$ and define

$$\mathrm{Av}^n_C(\mu,\varphi)=\left\{\bar{a}\in\mathcal{U}^{\bar{x}}:\left|\mu(\varphi(x,b))-\mathrm{Av}(\bar{a})(\varphi(x,b))\right|\in C\text{ for all }b\in\mathcal{U}^y\right\}.$$

Note that a measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ is *fam* over $M \prec \mathcal{U}$ if and only if, for any $\mathcal{L}$-formula $\varphi(x, y)$ and $\varepsilon > 0$, there is some $n \geq 1$ such that $\mathrm{Av}^n_{<\varepsilon}(\mu, \varphi) \cap (M^x)^n \neq \varnothing$.

Lemma 4.5. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is Borel definable and $\varphi(x, y)$ is an $\mathcal{L}_{\mathcal{U}}$-formula. Then, for any $\bar{a} \in \mathrm{Av}^n_{<\varepsilon}(\mu, \varphi)$ and any $\nu \in \mathfrak{M}_y(\mathcal{U})$, we have*

$$(\mu \otimes \nu)(\varphi(x, y)) \approx_\varepsilon (\mathrm{Av}(\bar{a}) \otimes \nu)(\varphi(x, y)) = \frac{1}{n} \sum_{n=1}^{n} \nu(\varphi(a_i, y)).$$

Proof. This is a straightforward calculation (integrate over $S_y(M)$, where $M \prec \mathcal{U}$ is such that $\varphi(x, y)$ is over M, μ is Borel definable over M, and $\bar{a} \in M^{\bar{x}}$). □

Next we work toward a characterization of *fim* (Proposition 4.8 below), which will be useful in several later results. Recall that a set (in $\mathcal{U}$) is *co-type-definable* if its complement is type-definable.

Lemma 4.6. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is definable over $A \subset \mathcal{U}$. Then, for any $\mathcal{L}_A$-formula $\varphi(x, y)$ and any $n \geq 1$ and $\varepsilon > 0$, $\mathrm{Av}^n_{\leq\varepsilon}(\mu, \varphi)$ is type-definable over A, and $\mathrm{Av}^n_{<\varepsilon}(\mu, \varphi)$ is co-type-definable over A.*

Proof. Fix $\varphi(x, y)$, $n \geq 1$, and $\varepsilon > 0$. For any closed set $C \subseteq [0, 1]$, we have a well-defined partial y-type "$\mu(\varphi(x, y)) \in C$" over A. Let $q_C(\bar{x}, y)$ be the type defined by

$$\bigwedge_{i=0}^{n} \left(\left(\mathrm{Av}(\bar{x})(\varphi(x, y)) = \frac{i}{n} \right) \to \left(\mu(\varphi(x, y)) \in C + \frac{i}{n} \right) \vee \left(\mu(\varphi(x, y)) \in \frac{i}{n} - C \right) \right).$$

Then $(\bar{a}, b) \models q_C$ if and only if $\left| \mu(\varphi(x, b)) - \mathrm{Av}(\bar{a})(\varphi(x, b)) \right| \in C$. Now $\mathrm{Av}^n_{\leq\varepsilon}(\mu, \varphi)$ is defined by $\forall y\, q_{[0,\varepsilon]}(\bar{x}, y)$, and $\neg\, \mathrm{Av}^n_{<\varepsilon}(\mu, \varphi)$ is defined by $\exists y\, q_{[\varepsilon,1]}(\bar{x}, y)$, both of which are types over A. □

Definition 4.7. Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\varphi(x, y)$ is an $\mathcal{L}$-formula. Given $\varepsilon > 0$, we say that a sequence $(\chi_n(x_1, \ldots, x_n))_{n=1}^{\infty}$ of $\mathcal{L}_{\mathcal{U}}$-formulas is a *$(\varphi, \varepsilon)$-approximation sequence for* μ if, for all $n \geq 1$, we have $\mathrm{Av}^n_{\leq\varepsilon/2}(\mu, \varphi) \subseteq \chi_n(\mathcal{U}^{\bar{x}}) \subseteq \mathrm{Av}^n_{<\varepsilon}(\mu, \varphi)$. We also say that such a sequence is *over $A \subset \mathcal{U}$* if each χ_n is an $\mathcal{L}_A$-formula.

Note that, by Lemma 4.6, if $\mu \in \mathfrak{M}_x(\mathcal{U})$ is definable over $A \subset \mathcal{U}$, then for any $\mathcal{L}$-formula $\varphi(x, y)$ and $\varepsilon > 0$, there is a (φ, ε)-approximation sequence for μ over A (but the formulas in the sequence may be unsatisfiable).

Proposition 4.8. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is definable over $M \prec \mathcal{U}$. Then the following are equivalent*:

(i) *μ is fim over M.*

(ii) *For any $\mathcal{L}$-formula $\varphi(x, y)$ and $\varepsilon > 0$, there is a (φ, ε)-approximation sequence $(\chi_n)_{n=0}^{\infty}$ for μ over M such that* $\lim_{n\to\infty} \mu^{(n)}(\chi_n) = 1$.

(iii) *For any $\mathcal{L}$-formula $\varphi(x, y)$ and $\varepsilon > 0$, if $(\chi_n)_{n=0}^{\infty}$ is a (φ, ε)-approximation sequence for μ over M, then* $\lim_{n\to\infty} \mu^{(n)}(\chi_n) = 1$.

Proof. (i) $\Rightarrow$ (iii) Assume μ is *fim* over M. Fix an $\mathcal{L}$-formula $\varphi(x, y)$. Since μ is *fim*, there are formulas $(\theta_n(x_1, \dots, x_n))_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \mu^{(n)}(\theta_n) = 1$ and, for all $\varepsilon > 0$, we have $\theta_n(\mathcal{U}^{\bar{x}}) \subseteq \mathrm{Av}^n_{\leq\varepsilon/2}(\mu, \varphi)$ for sufficiently large n. Now fix $\varepsilon > 0$, and let $(\chi_n)_{n=1}^{\infty}$ be a (φ, ε)-approximation sequence for μ over M. Then $\theta_n(\mathcal{U}^{\bar{x}}) \subseteq \mathrm{Av}^n_{\leq\varepsilon/2}(\mu, \varphi) \subseteq \chi_n(\mathcal{U}^{\bar{x}})$ for sufficiently large n, and so $\lim_{n\to\infty} \mu^{(n)}(\chi_n) = 1$.

(iii) $\Rightarrow$ (ii) This is trivial since approximation sequences for μ exist by Lemma 4.6.

(ii) $\Rightarrow$ (i) Assume μ satisfies (ii), and fix an $\mathcal{L}$-formula $\varphi(x, y)$. For all $i \geq 1$, we have an $\mathcal{L}_M$-formula $\chi_{n_i}(x_1, \dots, x_{n_i})$ such that $\chi_{n_i}(\mathcal{U}^{x_1 \dots x_{n_i}}) \subseteq \mathrm{Av}^{n_i}_{<1/i}(\mu, \varphi)$ and $\mu^{(n_i)}(\chi_{n_i}) \geq 1 - 1/i$. This suffices to prove that μ is *fim* over M (similar to the proof of [CG 2020, Proposition 3.2]). $\square$

We now summarize the situation concerning the analogue of Corollary 2.2 for various properties of measures.

Proposition 4.9. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is fim (resp., fam, finitely satisfiable in some small model, definable, or Borel definable), and also invariant over $M \prec \mathcal{U}$. Then μ is fim over M (resp., fam over M, finitely satisfiable in M, definable over M, or Borel definable over M).*

Proof. Corollary 2.2 provides the Borel definable case, and the definable case is similar (see Remark 2.3). The finitely satisfiable case is a straightforward modification of the proof for types, e.g., as in [Simon 2015, Lemma 2.18]. See [Gannon 2020, Proposition 2.18] for details. It remains to consider *fim* and *fam*.

Suppose μ is *fam*. Fix an $\mathcal{L}$-formula $\varphi(x, y)$ and $\varepsilon > 0$. We want to find $n \geq 1$ such that $\mathrm{Av}^n_{<\varepsilon}(\mu, \varphi) \cap M^{\bar{x}} \neq \varnothing$. By assumption, there is some $n \geq 1$ and $\bar{a}^*$ such that $\bar{a}^* \in \mathrm{Av}^n_{\leq\varepsilon/2}(\mu, \varphi) \cap \mathcal{U}^{\bar{x}}$. Since μ is definable and M-invariant, it is definable over M. So $\mathrm{Av}^n_{\leq\varepsilon/2}(\mu, \varphi)$ is type-definable over M, and contained in $\mathrm{Av}^n_{<\varepsilon}(\mu, \varphi)$, which is co-type-definable over M. Therefore we may find an $\mathcal{L}_M$-formula $\chi(x_1, \dots, x_n)$ such that $\mathrm{Av}^n_{\leq\varepsilon/2}(\mu, \varphi) \subseteq \chi(\mathcal{U}^{\bar{x}}) \subseteq \mathrm{Av}^n_{<\varepsilon}(\mu, \varphi)$. Then $\mathcal{U} \models \chi(\bar{a}^*)$ and so, since $M \prec \mathcal{U}$, there is $\bar{a} \in (M^x)^n$ such that $M \models \chi(\bar{a})$. So $\bar{a} \in \mathrm{Av}^n_{<\varepsilon}(\mu, \varphi) \cap M^{\bar{x}}$.

Finally, suppose μ is *fim*. Fix an $\mathcal{L}$-formula $\varphi(x, y)$ and $\varepsilon > 0$. Then there is a $(\varphi, \varepsilon/2)$-approximation sequence $(\chi_n^*)_{n=0}^{\infty}$ for μ such that $\lim_{n\to\infty} \mu^{(n)}(\chi_n^*) = 1$. As before, μ is definable over M. Let $(\chi_n)_{n=0}^{\infty}$ be (φ, ε)-approximation sequence for μ over M. Then $\chi_n^*(\mathcal{U}^{\bar{x}}) \subseteq \chi_n(\mathcal{U}^{\bar{x}})$ for all $n \geq 1$, which implies $\lim_{n\to\infty} \mu^{(n)}(\chi_n) = 1$. So μ is *fim* over M by Proposition 4.8. $\square$

Next we summarize what is known about the preservation of various properties with respect to Morley products.

Proposition 4.10. *Fix $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $M \prec \mathcal{U}$.*

(a) *If μ and ν are fam (resp., definable) over M, then $\mu\otimes\nu$ is fam (resp., definable) over M.*

(b) *μ and ν are finitely satisfiable in M, and μ is Borel definable or ν is a type, then $\mu\otimes\nu$ is finitely satisfiable in M.*

Proof. (a) See Propositions 2.10 and 2.6 of [CG 2020].

(b) See [Hrushovski et al. 2013, Lemma 1.6]. The authors there do not explicitly assume μ is Borel definable (which is needed for $\mu\otimes\nu$ to be well defined). Instead, they assume T is NIP so that this becomes automatic. See also [Gannon 2020, Proposition 2.25]. □

In contrast to the previous result, we have already seen that Borel definable measures are not necessarily closed under Morley products. In [CG 2020] it is claimed that this is also the case for *fim* measures, due to an example from [Adler et al. 2014]. However, that example turns out not to work (see Section 8A) and it remains an open question whether *fim* measures are closed under Morley products (see the end of Section 6 for further discussion).

Finally, in preparation for the main result in Section 6, we make some easy observations about convex combinations.

Proposition 4.11. *Fix $M\prec\mathcal{U}$, and suppose $\mu,\nu\in\mathfrak{M}_x(\mathcal{U})$ are fam over M (resp., finitely satisfiable in M, definable over M, or Borel definable over M). Then, for any $r\in[0,1]$, $\lambda:=r\mu+(1-r)\nu$ is fam over M (resp, finitely satisfiable in M, definable over M, or Borel definable over M).*

Proof. Fix a formula $\varphi(x,y)$. Note that, in any case, μ is invariant over M. If μ and ν are definable (resp., Borel definable), then $F^{\varphi}_{\mu,M}$ and $F^{\varphi}_{\nu,M}$ are continuous (resp., Borel), and so $F^{\varphi}_{\lambda,M}=rF^{\varphi}_{\mu,M}+(1-r)F^{\varphi}_{\nu,M}$ is continuous (resp., Borel), which implies λ is definable (resp., Borel definable) with respect $\varphi(x,y)$. Also, if μ and ν are both finitely satisfiable in M, then it is clear that λ is too.

Finally, suppose μ and ν are *fam* over M. Fix $\varepsilon>0$. Then there are $\bar{a},\bar{b}\in(M^x)^{<\omega}$ such that $\mu\approx^{\varphi}_{\varepsilon}\operatorname{Av}(\bar{a})$ and $\nu\approx^{\varphi}_{\varepsilon}\operatorname{Av}(\bar{b})$. Let $\eta=r\operatorname{Av}(\bar{a})+(1-r)\operatorname{Av}(\bar{b})$. Then $\lambda\approx^{\varphi}_{\varepsilon}\eta$, and it is easy to find some $\bar{c}\in(M^x)^{<\omega}$ such that $\eta\approx^{\varphi}_{\varepsilon}\operatorname{Av}(\bar{c})$. So $\lambda\approx^{\varphi}_{2\varepsilon}\operatorname{Av}(\bar{c})$. This shows λ is *fam* over M. □

Once again, *fim* measures are missing from the previous result. We will show in Theorem 6.2 that *fim* measures are also closed under convex combinations.

5. Commuting measures

Let T be a complete $\mathcal{L}$-theory with monster model $\mathcal{U}$. In this section, we investigate pairs of measures that commute. Let us start with some results from the literature. The first is an easy exercise.

Proposition 5.1. *If $p \in S_x(\mathcal{U})$ is definable and $\nu \in \mathfrak{M}_y(\mathcal{U})$ is finitely satisfiable in some small model, then $p \otimes \nu = \nu \otimes p$.*

Proof. The argument is similar to that of [Hrushovski and Pillay 2011, Lemma 3.4] (which assumes ν is a type). Fix an $\mathcal{L}_{\mathcal{U}}$-formula $\varphi(x, y)$ and let $M \prec \mathcal{U}$ be such that $\varphi(x, y)$ is over M, p is definable over M, and ν is finitely satisfiable in M. Choose an $\mathcal{L}_M$-formula $\psi(y)$ such that $\varphi(x, b) \in p$ if and only if $\mathcal{U} \models \psi(b)$. Let $a \models p|_M$. Then we have $\varphi(a, M^y) = \psi(M^y)$, and so $\nu(\varphi(a, y) \triangle \psi(y)) = 0$ since ν is finitely satisfiable in M. Therefore $(p \otimes \nu)(\varphi(x, y)) = \nu(\psi(y)) = \nu(\varphi(a, y)) = (\nu \otimes p)(\varphi(x, y))$. □

An obvious question is whether the previous result holds when p is replaced by a definable global measure (and ν is also Borel definable so that both products are well defined). We will show later on that this is not the case (see Example 5.9). However, Hrushovski, Pillay, and Simon [Hrushovski et al. 2013] proved the following generalization and elaboration of Proposition 5.1 in the setting of NIP theories.

Theorem 5.2 [Hrushovski et al. 2013]. *Assume T is NIP.*

(a) *If $\mu \in \mathfrak{M}_x(\mathcal{U})$ is definable, and $\nu \in \mathfrak{M}_y(\mathcal{U})$ is finitely satisfiable in some small model, then $\mu \otimes \nu = \nu \otimes \mu$.*

(b) *If $\mu \in \mathfrak{M}_x(\mathcal{U})$ is dfs, then $\mu \otimes \nu = \nu \otimes \mu$ for any invariant $\nu \in \mathfrak{M}_y(\mathcal{U})$.*

(c) *If $\mu \in \mathfrak{M}_x(\mathcal{U})$ is invariant, then it is dfs if and only if $\mu_x \otimes \mu_{x'} = \mu_{x'} \otimes \mu_x$.*

In this section, we pursue results along the lines of adapting Theorem 5.2 to arbitrary theories. First, we briefly note that outside of NIP, self-commuting measures (in the sense of Theorem 5.2(c)) need not have any special properties.

Example 5.3. Let T be the theory of the random graph. Then any invariant global type in a *one* free variable commutes with itself. On the other hand, for any $M \prec \mathcal{U}$ and $Z \subseteq S_1(M)$, there is a unique nonalgebraic M-invariant type $p_Z \in S_1(\mathcal{U})$ such that $E(x, b) \in p_Z$ if and only if $\operatorname{tp}(b/M) \in Z$. So p_Z is Borel definable (resp., definable) if and only if Z is Borel (resp., clopen). Note also that the "generic" definable types $p_\varnothing$ and $p_{S_1(M)}$ are not finitely satisfiable in M. In fact, T has no nontrivial *dfs* global measures (see [CG 2020, Theorem 4.9]).

The first goal this section is a suitable generalization of Theorem 5.2(a) for arbitrary theories. The original proof of this theorem relied on a fundamental property of measures in the NIP setting, namely, that any Keisler measure can be *locally uniformly approximated* by averaging on a finite collection of types in the support of the given measure. Using this, the authors of [Hrushovski et al. 2013] were able to reduce the problem of whether a finitely satisfiable *measure* commutes with a definable measure to the question of whether a finitely satisfiable *type* commutes with a definable measure (note the duality to Proposition 5.1 in this statement).

That being said, the proof of this "easier" problem remained nontrivial and still required the use of NIP, along with the weak law of large numbers. Unfortunately there are two major obstacles one finds when trying to directly adapt this proof of Theorem 5.2(a) to the general setting. First, Keisler measures in the wild do not admit approximations by types as discussed above. Secondly, and more importantly, the statement in total generality is false. Indeed, Proposition 7.14 gives an example of a *dfs* type and a definable measure that do not commute. Therefore the dual version of Proposition 5.1 alluded to above fails outside of NIP.

Fortunately however, one can give a simpler proof of Theorem 5.2(a) in the NIP context by treating smooth extensions of measures as analogous to realizations of types, along with some elementary topology (see [Gannon 2022, Proposition 3.6]). By embracing this ideology, and widening the focus to commuting extensions of measures, we will recover a "deviant" generalization of Theorem 5.2(a), which applies to general theories and has a purely topological proof. This generalization is given in Theorem 5.7 below. We start with some topological lemmas.

Recall that $\mathfrak{M}_x(\mathcal{U})$ is a compact Hausdorff space under the subspace topology induced from $[0,1]^{\mathrm{Def}_x(\mathcal{U})}$.

Lemma 5.4. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is definable. Then, for any $\mathcal{L}_{\mathcal{U}}$-formula $\varphi(x,y)$, the map $\nu \mapsto (\mu \otimes \nu)(\varphi(x,y))$ is continuous from $\mathfrak{M}_y(\mathcal{U})$ to* $[0,1]$.

Proof. This involves similar calculations as in the proofs of [Chernikov and Gannon 2022, Proposition 6.3] and [CG 2020, Proposition 2.6]. Fix an $\mathcal{L}_{\mathcal{U}}$-formula $\varphi(x,y)$, and fix $A \subset \mathcal{U}$ such that $\varphi(x,y)$ is over A and μ is definable over A. Fix $\varepsilon > 0$. By Fact 2.15, there are $\mathcal{L}_A$-formulas $\psi_1(y), \dots, \psi_n(y)$ and $r_1, \dots, r_n \in [0,1]$ such that $\left\| F^{\varphi}_{\mu,A} - \sum_{i=1}^{n} r_i \mathbf{1}_{\psi_i(y)} \right\|_\infty < \varepsilon$. Hence, for arbitrary $\nu \in \mathfrak{M}_y(\mathcal{U})$, we have

$$(\mu \otimes \nu)(\varphi(x,y)) = \int_{S_y(A)} F^{\varphi}_{\mu}\, d\nu \approx_\varepsilon \int_{S_y(A)} \sum_{i=1}^{n} \mathbf{1}_{\psi_i(y)}\, d\nu = \sum_{i=1}^{n} r_i \nu(\psi_i(y)).$$

By definition of the topology on $\mathfrak{M}_y(\mathcal{U})$, the map $\nu \mapsto \nu(\psi_i(y))$ is continuous. So the map $\nu \mapsto \sum_{i=1}^{n} r_i \nu(\psi_i(y))$ is also continuous since it is a linear combination of continuous functions. Since ν was arbitrary we have

$$\sup_{\nu \in \mathfrak{M}_y(\mathcal{U})} \left| (\mu \otimes \nu)(\varphi(x,y)) - \sum_{i=1}^{n} r_i \nu(\psi_i(y)) \right| < \varepsilon.$$

Therefore $\nu \mapsto (\mu \otimes \nu)(\varphi(x,y))$ is a uniform limit of continuous functions, and hence is continuous. □

In the subsequent results, we will consider pairs of global measures in the same variable sort, which have the same restriction to some small model. Thus we take a moment to point out various subtleties that arise. In particular, suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$

is Borel definable over $A \subset \mathcal{U}$, and ν, $\hat{\nu} \in \mathfrak{M}_y(\mathcal{U})$ are such that $\nu|_A = \hat{\nu}|_A$. Then we trivially have $(\mu \otimes \nu)|_A = (\mu \otimes \hat{\nu})|_A$. But note that this can fail if μ is only Borel definable over some larger $B \supseteq A$, since in this case the Morley products with μ must be computed with respect to $\nu|_B$ and $\hat{\nu}|_B$ (even when applied to $\mathcal{L}_A$-formulas). It is also important to point out that if we instead have μ, $\hat{\mu} \in \mathfrak{M}_x(\mathcal{U})$ Borel definable over A, with $\mu|_A = \hat{\mu}|_A$, then one *cannot* necessarily conclude $(\mu \otimes \nu)|_A = (\hat{\mu} \otimes \nu)|_A$ for a given $\nu \in \mathfrak{M}_y(\mathcal{U})$.

Definition 5.5. Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is Borel definable over $A \subset \mathcal{U}$. Then a Borel definable measure $\nu \in \mathfrak{M}_y(\mathcal{U})$ *A-commutes* with μ if $(\mu \otimes \nu)|_A = (\nu \otimes \mu)|_A$. Define $C^{\mu}_y(A)$ to be the set of measures in $\mathfrak{M}_y(\mathcal{U})$ that are Borel definable over A and A-commute with μ.

As we will see below, Theorem 5.2(a) can be viewed as a question of when $C^{\mu}_y(A)$ contains certain limit points. The next lemma describes a technical scenario in which this can happen.

Lemma 5.6. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is definable over A, and $\nu \in \mathfrak{M}_y(\mathcal{U})$ is a Borel definable measure, which is the limit of a net $(\nu_i)_{i \in I}$ from $C^{\mu}_y(A)$. If $\mu|_A$ has a definable global extension $\hat{\mu} \in \mathfrak{M}_x(\mathcal{U})$, which A-commutes with ν and ν_i for all $i \in I$, then $\nu \in C^{\mu}_y(A)$.*

Proof. We first note that ν is A-invariant, and thus Borel definable over A by Corollary 2.2. Now fix an $\mathcal{L}_A$-formula $\varphi(x, y)$. Then we have the following calculations (individual steps are justified afterward):

$$\begin{aligned}(\mu \otimes \nu)(\varphi(x, y)) &= \lim_{i \in I}(\mu \otimes \nu_i)(\varphi(x, y)) = \lim_{i \in I}(\nu_i \otimes \mu)(\varphi(x, y)) \\ &= \lim_{i \in I}(\nu_i \otimes \hat{\mu})(\varphi(x, y)) = \lim_{i \in I}(\hat{\mu} \otimes \nu_i)(\varphi(x, y)) \\ &= (\hat{\mu} \otimes \nu)(\varphi(x, y)) = (\nu \otimes \hat{\mu})(\varphi(x, y)) \\ &= (\nu \otimes \mu)(\varphi(x, y)).\end{aligned}$$

The first and fifth equalities above use Lemma 5.4; the second equality uses the assumption that ν_i is in $C^{\mu}_y(A)$; the third and seventh equalities use $\mu|_A = \hat{\mu}|_A$; and the fourth and sixth equalities use the commutativity assumptions on $\hat{\mu}$. $\square$

Note that in the statement of Lemma 5.6, we do not need to assume that $\hat{\mu}$ is definable *over A*. For example, if μ is a type then such a $\hat{\mu}$ exists as in Remark 2.21. So we see that if $p \in S_x(\mathcal{U})$ is definable over $A \subset \mathcal{U}$, then the set of A-invariant measures in $\mathfrak{M}_y(\mathcal{U})$ that A-commute with p is closed (as usual, when working with types, Borel definability can be weakened to invariance). However, when μ is a measure, the existence of $\hat{\mu}$ as in Lemma 5.6 is a nontrivial assumption.

We can now prove a generalization of Theorem 5.2(a) for arbitrary theories.

Theorem 5.7. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is definable over $M \prec \mathcal{U}$, and $\nu \in \mathfrak{M}_y(\mathcal{U})$ is Borel definable and finitely satisfiable in M. If $\mu|_M$ has a definable global extension that M-commutes with ν, then $(\mu \otimes \nu)|_M = (\nu \otimes \mu)|_M$.*

Proof. Let X denote the convex hull of $\{\delta_a : a \in M^y\}$ in $\mathfrak{M}_y(\mathcal{U})$. Then it is not hard to show that ν is in the closure of X (see also [Chernikov and Gannon 2022, Proposition 2.11]). Moreover, by an easy calculation, a measure in X commutes with *every* invariant measure. Thus the hypotheses of Lemma 5.6 are satisfied, and so we have $\nu \in C^{\mu}_y(M)$. □

Remark 5.8. Theorem 5.2(a) is a consequence of Theorem 5.7 together with the fundamental results on NIP theories discussed before Corollary 2.22. Indeed, suppose T is NIP, $\mu \in \mathfrak{M}_x(\mathcal{U})$ is definable, and ν is finitely satisfiable in some $M \prec \mathcal{U}$. Without loss of generality, μ is definable over M. Moreover, ν is M-invariant and hence Borel definable over M. Finally, $\mu|_M$ has a global extension that is smooth, and thus is definable and commutes with ν. Since all of this works over any $N \succeq M$, we have $\mu \otimes \nu = \nu \otimes \mu$ by Theorem 5.7.

In light of Proposition 5.1, it is natural to ask whether the assumption on $\mu|_M$ in Theorem 5.7 is necessary. A counterexample, which we only mention now, will be given later in the paper.

Example 5.9. There is a complete theory T, a definable measure $\mu \in \mathfrak{M}_x(\mathcal{U})$, and a *dfs* type $q \in S_y(\mathcal{U})$, such that $\mu \otimes q \neq q \otimes \mu$. See Section 7D for details.

On the other hand, the following question remains open.

Question 5.10. Do any two *dfs* global measures commute? (Note that for types this is a special case of Proposition 5.1.)

The next goal of this section is to show that *fim* measures commute with Borel definable measures. In other words, Theorem 5.2(b) generalizes to arbitrary theories, provided that *dfs* is replaced by *fim* (which is equivalent in NIP). This result also generalizes the easier fact that *smooth* measures commute with Borel definable measures [Hrushovski et al. 2013]. In analogy to the comparison between Proposition 5.1 and Theorem 5.7, we will also see that *fim types* commute with *invariant* measures. However, in this case the overall structure of the proof for types is not that much different than for measures. So to avoid repetitive arguments, we will use the relative notion of measurability from Section 3B.

Definition 5.11. Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is invariant over $A \subset \mathcal{U}$, and $\nu \in \mathfrak{M}_y(\mathcal{U})$. Then μ is *ν-measurable over A* if, for any $\mathcal{L}_A$-formula $\varphi(x, y)$, the map $F^{\varphi}_{\mu,A}$ is $\nu|_A$-measurable, i.e., $(F^{\varphi}_{\mu,A})^{-1}(U)$ is $\nu|_A$-measurable for any open $U \subseteq [0, 1]$.

Let us note the two examples of interest.

Example 5.12. Fix $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $A \subset \mathcal{U}$.

(1) If μ is Borel definable over A then it is ν-measurable over A for any $\nu \in \mathfrak{M}_y(\mathcal{U})$.

(2) If μ is invariant over A then it is q-measurable over A for any $q \in S_y(\mathcal{U})$.

Suppose that $\mu \in \mathfrak{M}_x(\mathcal{U})$ is invariant over $A \subset \mathcal{U}$, and $\nu \in \mathfrak{M}_y(\mathcal{U})$, and μ is ν-measurable over A. Given an $\mathcal{L}_A$-formula $\varphi(x, y)$, we set $(\mu \otimes_A \nu)(\varphi(x, y)) = \int_{S_y(A)} F_\mu^\varphi \, d\nu$. This yields a well-defined Keisler measure $\mu \otimes_A \nu$ in $\mathfrak{M}_{xy}(A)$. Note that if μ is either Borel definable over A, or ν is a type in $S_y(\mathcal{U})$, then $\mu \otimes_A \nu = (\mu \otimes \nu)|_A$. However, unlike the situation with Borel definability, it is possible for a measure μ to be ν-measurable over some $A \subset \mathcal{U}$, but not ν-measurable over any proper $B \supset A$ (see Proposition A.8 for an example). So we may not have a well-defined global product $\mu \otimes \nu$.

Next, we recall the weak law of large numbers (a special case of Chebyshev's inequality). Our formulation of this result follows [Simon 2015, Proposition B.4], except we have sharpened the bound slightly (in a way that is evident from how Chebyshev is applied).

Fact 5.13. *Suppose $(\Omega, \mathcal{B}, \mu)$ is a probability space, and fix $X \in \mathcal{B}$ and $\varepsilon > 0$. Given $n \geq 1$, let μ^n denote the usual product measure on Ω^n, and let*

$$X_{n,\varepsilon} = \{\bar{a} \in \Omega^n : \mu(X) \approx_\varepsilon \operatorname{Av}(\bar{a})(X)\}.$$

Then $X_{n,\varepsilon}$ is μ^n-measurable, and $\mu^n(X_{n,\varepsilon}) \geq 1 - \mu(X)(1 - \mu(X))/(\varepsilon^2 n)$.

The next lemma uses Fact 5.13 to highlight the leverage obtained when working with *fim* measures. This distinction is further discussed after Theorem 5.16.

Lemma 5.14. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is fim over $M \prec \mathcal{U}$. Fix an $\mathcal{L}$-formula $\varphi(x, y)$ and $\mu|_M$-measurable sets $X_1, \ldots, X_n \subseteq S_x(M)$. Then, for any $\varepsilon > 0$, there is an integer $k \geq 1$ and a sequence $(a_1, \ldots, a_k) \in (\mathcal{U}^x)^k$ such that $\mu \approx_\varepsilon^\varphi \operatorname{Av}(\bar{a})$ and $\mu|_M(X_i) \approx_\varepsilon \operatorname{Av}(\bar{a})|_M(X_i)$ for all $1 \leq i \leq n$.*

Proof. The argument is similar to various parts of Section 3 in [Hrushovski et al. 2013] (see, e.g., Lemma 3.6 there). Fix $\varepsilon > 0$. Choose $\mathcal{L}_M$-formulas $\theta_k(x_1, \ldots, x_k)$ such that $\lim_{k\to\infty} \mu^{(k)}(\theta_k) = 1$ and, for k sufficiently large, if $\theta_k(\bar{a})$ holds then $\mu \approx_\varepsilon^\varphi \operatorname{Av}(\bar{a})$. For $1 \leq i \leq n$ and $k \geq 1$, define

$$\begin{aligned} X_{i,k} &= \{(p_1, \ldots, p_k) \in S_x(M)^k : \mu|_M(X_i) \approx_\varepsilon \operatorname{Av}(\bar{p})(X_i)\}, \\ Y_{i,k} &= \{p \in S_{x_1,\ldots,x_k}(M) : (p|_{x_1}, \ldots, p|_{x_k}) \in X_{i,k}\}. \end{aligned}$$

Then each set $Y_{i,k}$ is $\mu^{(k)}|_M$-measurable and $\mu^{(k)}|_M(Y_{i,k}) = (\mu|_M)^k(X_{i,k})$. So we have that $\lim_{k\to\infty} \mu^{(k)}(Y_{i,k}) = 1$ by Fact 5.13. Choose k large enough so that $\mu^{(k)}(\theta_k), \mu^{(k)}(Y_{1,k}), \ldots, \mu^{(k)}(Y_{n,k})$ are each strictly greater than $n/(n+1)$. Then there is some $p \in [\theta_k] \cap Y_{1,k} \cap \cdots \cap Y_{n,k}$. Let $\bar{a} \in \mathcal{U}^k$ realize p. Then $\bar{a}$ satisfies the desired conditions. $\square$

We now prove a proposition that provides the heart of the result that *fim* measures commute with Borel definable measures.

Proposition 5.15. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is fim over $M \prec \mathcal{U}$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$ is μ-measurable over M. Then $(\mu \otimes \nu)|_M = \nu \otimes_M \mu$.*

Proof. Fix an $\mathcal{L}_M$-formula $\varphi(x, y)$ and some $\varepsilon > 0$. Let $\varphi^*(y, x)$ denote the same formula $\varphi(x, y)$, but with the roles of object and parameter variables exchanged. Because $F^{\varphi^*}_{\nu,M}$ is bounded and $\mu|_M$-measurable, it can be approximated uniformly by simple $\mu|_M$-measurable functions (see Fact A.1). So there are $\mu|_M$-measurable sets $X_1, \ldots, X_n \subseteq S_x(M)$, and $r_1, \ldots, r_n \in [0, 1]$ such that $\| F^{\varphi^*}_{\nu,M} - \sum_{i=1}^n r_i \mathbf{1}_{X_i} \|_\infty < \varepsilon$. By Lemma 5.14, there is some $\bar{a} \in (\mathcal{U}^x)^k$ such that $\mu \approx^\varphi_\varepsilon \mathrm{Av}(\bar{a})$ and $\mu(X_i)|_M \approx_{\varepsilon/n} \mathrm{Av}(\bar{a})|_M(X_i)$ for all $1 \leq i \leq n$. Let $p_j = \mathrm{tp}(a_j/M)$. Then

$$
\begin{aligned}
(\mu \otimes \nu)(\varphi(x, y)) &\approx_\varepsilon (\mathrm{Av}(\bar{a}) \otimes \nu)(\varphi(x, y)) = \frac{1}{k}\sum_{j=1}^k \nu(\varphi(a_j, y)) = \frac{1}{k}\sum_{j=1}^k F^{\varphi^*}_{\nu,M}(p_j) \\
&\approx_\varepsilon \frac{1}{k}\sum_{j=1}^k \sum_{i=1}^n r_i \mathbf{1}_{X_i}(p_j) \\
&= \frac{1}{k}\sum_{i=1}^n \sum_{j=1}^k r_i \delta_{a_j}|_M(X_i) = \sum_{i=1}^n r_i \,\mathrm{Av}(\bar{a})|_M(X_i) \\
&\approx_\varepsilon \sum_{i=1}^n r_i \mu|_M(X_i) = \int_{S_x(M)} \sum_{i=1}^n r_i \mathbf{1}_{X_i}\, d\mu \\
&\approx_\varepsilon \int_{S_x(M)} F^{\varphi^*}_\nu \, d\mu = (\nu \otimes_M \mu)(\varphi(x, y)).
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, we have the desired result. □

Theorem 5.16. (a) *If $\mu \in \mathfrak{M}_x(\mathcal{U})$ is fim and $\nu \in \mathfrak{M}_y(\mathcal{U})$ is Borel definable, then $\mu \otimes \nu = \nu \otimes \mu$.*

(b) *If $p \in S_x(\mathcal{U})$ is fim and $\nu \in \mathfrak{M}_y(\mathcal{U})$ is invariant, then $p \otimes \nu = \nu \otimes p$.*

Proof. In light of Example 5.12, this follows immediately from Proposition 5.15. □

A natural question is whether one can get away with using *fam* measures in place of *fim* measures in Theorem 5.16. However, note that in the proof of Proposition 5.15, we need to approximate μ simultaneously on instances of the formula $\varphi(x, y)$ *as well as* finitely many Borel sets. A priori, the assumption of *fam* is not sufficient to obtain this level of approximation. That being said, it is perhaps worth emphasizing that this kind of approximation for a measure μ is all that is needed to prove that μ commutes with any Borel definable measure. Thus it may be worth pursuing the question of whether this is strictly weaker than the *fim* assumption.

On the other hand, if in Proposition 5.15 we restrict to the case where ν is definable, then we only need to approximate μ on instances of $\varphi(x, y)$ and finitely many *clopen* sets (i.e., formulas). In this case, such an approximation is possible when μ is only *fam*. These observations yield the next result, which was first shown by the second author using different methods (see [Gannon 2022, Corollary 3.8]).

Proposition 5.17. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is fam and $\nu \in \mathfrak{M}_y(\mathcal{U})$ is definable. Then $\mu \otimes \nu = \nu \otimes \mu$.*

Proof. Following the proof of Proposition 5.15, we can use definability of ν (and Fact 2.15) to ensure each set X_i is clopen (say given by the formula $\psi_i(x)$). Then, since μ is *fam*, we may choose an (ε/n)-approximation $(a_1, \ldots, a_k)$ for the finite set of *formulas* $\{\varphi(x, y), \psi_1(x), \ldots, \psi_n(x)\}$. The rest of the calculations are the same. $\square$

In [CG 2020], it is shown that a type $p \in S_x(\mathcal{U})$ is *fim* over $M \prec \mathcal{U}$ if and only if it is *generically stable over M*, i.e., p is M-invariant and there does not exist an $\mathcal{L}$-formula $\varphi(x, y)$, a sequence $(b_i)_{i<\omega}$ from $\mathcal{U}^y$, and a Morley sequence $(a_i)_{i<\omega}$ in $p|_M$, such that $\mathcal{U} \models \varphi(a_i, b_j)$ if and only if $i \leq j$. So we have shown that generically stable types commute with invariant measures.

Remark 5.18. There is a more direct proof that generically stable types commute with invariant *types*. For the sake of completeness, we include the argument (which is similar to the proof that in NIP theories, *dfs* types commute with invariant types [Simon 2015, Proposition 2.33]). Suppose $p \in S_x(\mathcal{U})$ is generically stable, and $q \in S_y(\mathcal{U})$ is invariant. Fix $\varphi(x, y) \in q \otimes p$. Let $M \prec \mathcal{U}$ be such that $\varphi(x, y)$ is over M, p is generically stable over M, and q is M-invariant. Let $(a_i)_{i<\omega}$ be a Morley sequence in p over M, and fix $b \models q|_{Ma_{<\omega}}$. Then, for all $i < \omega$, we have $(a_i, b) \models (q \otimes p)|_M$ and so $\varphi(a_i, b)$ holds. By generic stability, p coincides with the average type of $(a_i)_{i<\omega}$ (see [CG 2020, Section 3]), and so $\varphi(x, b) \in p$. Thus $\varphi(x, y) \in p \otimes q$ since $b \models q|_M$.

It is natural to ask at this point whether commuting with all Borel definable measures characterizes *fim*. This turns out to not be the case.

Example 5.19. There is a complete theory T and a *fam* (but not *fim*) global type that commutes with every invariant measure. See Section 8B for details.

On the other hand, we know from Example 5.9 that *dfs* is not sufficient to ensure commuting with Borel definable measures. So this leaves the following questions.

Question 5.20. Let T be a complete theory, and fix $\mu \in \mathfrak{M}_x(\mathcal{U})$.

(1) Suppose μ is *fam*. Does μ commute with every Borel definable measure?

(2) Suppose μ commutes with every Borel definable measure. Is μ *fam*?

6. Closure properties of *fim* measures

In this section, we focus specifically on frequency interpretation measures, which seem to provide the "right" generalization of the notion of generically stable types (recall the discussion before Remark 5.18) to the setting of Keisler measures. Our goal is to investigate preservation of *fim* under natural operations on Keisler measures. We first consider convex combinations.

One fundamental difference between the space of types and the space of Keisler measures is that the latter admits a convex structure. More explicitly, given any two Keisler measures μ and ν in $\mathfrak{M}_x(\mathcal{U})$ and any real number r in the interval $[0, 1]$, one can construct the measure $r\mu + (1-r)\nu \in \mathfrak{M}_x(\mathcal{U})$. Thus the question arises as to which collections of measures are preserved under this construction, or which subsets of the space $\mathfrak{M}_x(\mathcal{U})$ are *convex*. While it is easily observed that the classes of *dfs* and *fam* measures form convex sets (see Proposition 4.11), this property does not obviously hold for the class of *fim* measures. In this section, we demonstrate that the class of *fim* measures is also convex. This result provides some fundamental geometric information about the space of *fim* measures, and also provides a process for making new *fim* measures from old ones. For example, the average of finitely many *fim* measures is still *fim*. In fact, showing this even in the case of *fim* types is nontrivial.

The proof that *fim* is preserved under convex combinations will require the following tail bound for a binomial distribution. Note that, for any real number r and integer $n \geq 1$, we have $\sum_{X \subseteq [n]} r^{|X|}(1-r)^{n-|X|} = 1$. Given $n \geq 1$, $\varepsilon > 0$, and $r \in [0, 1]$, let $\mathcal{P}_{r,\varepsilon}(n)$ denote the collection of subsets $X \subseteq [n]$ such that $|X|/n \approx_\varepsilon r$.

Fact 6.1. *If $r \in [0, 1]$ and $\varepsilon > 0$ then*

$$\sum_{X \in \mathcal{P}_{r,\varepsilon}(n)} r^{|X|}(1-r)^{n-|X|} \geq 1 - \frac{r(1-r)}{\varepsilon^2 n}.$$

Proof. This follows from Chebyshev's inequality applied to the binomial distribution $B(n, r)$ $\big($the sum above is precisely $P\big(|B(n, r) - rn| < \varepsilon n\big)\big)$. Equivalently, apply Fact 5.13 with $\Omega = \{0, 1\}$, $\mathcal{B} = \mathcal{P}(\Omega)$, $X = \{1\}$, and $\mu(X) = r$. □

Theorem 6.2. *Suppose $\mu, \nu \in \mathfrak{M}_x(\mathcal{U})$ are fim over $M \prec \mathcal{U}$, and fix $r \in [0, 1]$. Then $\lambda = r\mu + (1-r)\nu$ is fim over M.*

Proof. Note that μ and ν are definable over M, and thus so is λ by Proposition 4.11. Fix a formula $\varphi(x, y)$. For each $\varepsilon > 0$, let $(\chi^\mu_{n,\varepsilon})^\infty_{n=0}$, $(\chi^\nu_{n,\varepsilon})^\infty_{n=0}$, and $(\chi^\lambda_{n,\varepsilon})^\infty_{n=0}$ be (φ, ε)-approximation sequences over M for μ, ν, and λ, respectively. By Proposition 4.8, we have that for every $\varepsilon > 0$, $\lim_{n\to\infty} \mu^{(n)}(\chi^\mu_{n,\varepsilon}) = 1$, and likewise for ν. We need to show $\lim_{n\to\infty} \lambda^{(n)}(\chi^\lambda_{n,\varepsilon}) = 1$ for all $\varepsilon > 0$. So fix some $\varepsilon > 0$. Without loss of generality, assume $\varepsilon < \min\big\{\frac{1}{2}r, \frac{1}{2}(1-r)\big\}$. What we will end up showing is that for any $\delta > 0$, there is an integer $n(\delta) \geq 1$ such that if $n \geq n(\delta)$

then $\lambda^{(n)}(\chi^{\lambda}_{n,6\varepsilon}) > (1-\delta)^3$. Given this, we can conclude $\lim_{n\to\infty}\lambda^{(n)}(\chi^{\lambda}_{n,6\varepsilon}) = 1$. Since $\varepsilon > 0$ is arbitrarily small, this suffices to yield the desired result.

Given $n \geq 1$ and $X \subseteq [n]$, let

$$\lambda_{n,X} = \bigotimes_{i=1}^{n} \begin{Bmatrix} \mu, & i \in X \\ \nu, & i \notin X \end{Bmatrix}.$$

Note that $\lambda_{n,X}$ is well defined by associativity for definable measures. By linearity of the Morley product, we have that for any $n \geq 1$,

$$\lambda^{(n)} = \sum_{X\subseteq[n]} r^{|X|}(1-r)^{n-|X|}\lambda_{n,X}.$$

Now fix some $\delta > 0$. Choose $n_* \geq 1$ so that if $n \geq \frac{1}{2}rn_*$ then $\mu^{(n)}(\chi^{\mu}_{n,\varepsilon}) > 1-\delta$, and if $n \geq \frac{1}{2}(1-r)n_*$ then $\nu^{(n)}(\chi^{\nu}_{n,\varepsilon}) > 1-\delta$. Since $\varepsilon < \min\left\{\frac{1}{2}r, \frac{1}{2}(1-r)\right\}$, for any $n \geq n_*$ and any $m \leq n$, if $m/n \approx_\varepsilon r$ then $m \geq \frac{1}{2}rn_*$ and $n-m \geq \frac{1}{2}(1-r)n_*$.

Suppose that $n \geq n_*$ and $X \in \mathcal{P}_{r,\varepsilon}(n)$. We will show $\lambda_{n,X}(\chi^{\lambda}_{n,6\varepsilon}) > (1-\delta)^2$. Let $m = |X|$. By construction, we have $m \geq \frac{1}{2}rn_*$ and $n-m \geq \frac{1}{2}(1-r)n_*$. Enumerate $X = \{i_1,\ldots,i_m\}$ and $[n]\setminus X = \{j_1,\ldots,j_{n-m}\}$. Consider the formula $\Phi(x_1,\ldots,x_n) := \chi^{\mu}_{m,\varepsilon}(x_{i_1},\ldots,x_{i_m}) \wedge \chi^{\nu}_{n-m,\varepsilon}(x_{j_1},\ldots,x_{j_{n-m}})$. Then we have

$$\lambda_{n,X}(\Phi) = \mu^{(m)}(\chi^{\mu}_{m,\varepsilon})\cdot\nu^{(n-m)}(\chi^{\nu}_{n-m,\varepsilon}) > (1-\delta)^2.$$

Furthermore, for any $\bar{a} \models \Phi$ and $b \in \mathcal{U}^y$, we have

$$\begin{aligned}\lambda(\varphi(x,b)) &= r\mu(\varphi(x,b)) + (1-r)\nu(\varphi(x,b))\\ &\approx_{2\varepsilon} \frac{m}{n}\mu(\varphi(x,b)) + \frac{n-m}{n}\nu(\varphi(x,b))\\ &\approx_{\varepsilon} \frac{m}{n}\operatorname{Av}(a_{i_1},\ldots,a_{i_m})(\varphi(x,b)) + \frac{n-m}{n}\operatorname{Av}(a_{j_1},\ldots,a_{j_{n-m}})(\varphi(x,b))\\ &= \operatorname{Av}(a_1,\ldots,a_n)(\varphi(x,b)).\end{aligned}$$

Thus $\Phi(\mathcal{U}^{\bar{x}}) \subseteq \operatorname{Av}^n_{<3\varepsilon}(\lambda,\varphi) \subseteq \chi^{\lambda}_{n,6\varepsilon}(\mathcal{U}^{\bar{x}})$, and so $\lambda_{n,X}(\chi^{\lambda}_{n,6\varepsilon}) \geq \lambda_{n,X}(\Phi) > (1-\delta)^2$.

Finally, fix $n \geq n(\delta) := \max\{n_*, r(1-r)/(\varepsilon^2\delta)\}$. We show $\lambda^{(n)}(\chi^{\lambda}_{n,6\varepsilon}) > (1-\delta)^3$. Indeed, we have just shown that $\lambda_{n,X}(\chi^{\lambda}_{n,6\varepsilon}) > (1-\delta)^2$ for all $X \in \mathcal{P}_{r,\varepsilon}(n)$. So

$$\begin{aligned}\lambda^{(n)}(\chi^{\lambda}_{n,6\varepsilon}) &= \sum_{X\subseteq[n]} r^{|X|}(1-r)^{n-|X|}\lambda_{n,X}(\chi^{\lambda}_{n,2\varepsilon})\\ &\geq \sum_{X\in\mathcal{P}_{r,\varepsilon}(n)} r^{|X|}(1-r)^{n-|X|}\lambda_{n,X}(\chi^{\lambda}_{n,2\varepsilon})\\ &> (1-\delta)^2\Bigg(\sum_{X\in\mathcal{P}_{r,\varepsilon}(n)} r^{|X|}(1-r)^{n-|X|}\Bigg) > (1-\delta)^3,\end{aligned}$$

where the final inequality uses Fact 6.1 and choice of n. □

Finally, we discuss the question of whether *fim* measures are closed under the Morley product. In [CG 2020], a negative answer to this question was claimed by the first two authors, due to an example from [Adler et al. 2014] of a generically stable type p such that $p \otimes p$ is not generically stable. However, a gap in the proof was later noted by the third author. In fact, we will show in Section 8A that the ambient theory defined in [Adler et al. 2014] admits no nontrivial *dfs* measures. In an earlier draft of this paper, we further claimed that *fim* measures are indeed closed under Morley products, but an error in our proof was found by Silvain Rideau and Paul Wang. Thus the question remains open, and so we close this section with some remarks on the underlying subtleties.

Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$ are *fim* over some $M \prec \mathcal{U}$. To analyze the question of *fim* for $\mu \otimes \nu$, we consider a formula $\varphi(x, y, z)$ which, without loss of generality, is over M. Let $(\theta_n(x_1, \dots, x_n))_{n=0}^{\infty}$ and $(\chi_n(y_1, \dots, y_n))_{n=0}^{\infty}$ be sequences of $\mathcal{L}_M$-formulas obtained by applying the definition of *fim* to μ and ν with respect to the relevant bipartitions of $\varphi(x, y, z)$. If we let $\psi_n(x_1, y_1, \dots, x_n, y_n)$ denote the $\mathcal{L}_M$-formula $\theta_n(x_1, \dots, x_n) \wedge \chi_n(y_1, \dots, y_n)$, then it is not difficult to show that $\lim_{n\to\infty}(\mu \otimes \nu)^{(n)}(\psi_n) = 1$ and, for any $\varepsilon > 0$, if n is sufficiently large then $\mu \otimes \nu \approx^{\varphi}_{\varepsilon} \mathrm{Av}(a_i, b_j)_{i,j\leq n}$ for any $(\bar{a}, \bar{b}) \models \psi_n$. However, this does not (a priori) imply the same conclusion for $\mathrm{Av}(a_i, b_i)_{i\leq n}$, which would be needed to conclude $\mu \otimes \nu$ is *fim* using this argument. Indeed, despite obtaining arbitrarily good approximations along the full array $(a_i, b_j)_{i,j\leq n}$, there could be very different behavior on the "diagonal". This suggests that perhaps the original intuition from [Adler et al. 2014] is correct after all, and counterexamples may exist in sufficiently complicated theories. Such issues will be considered in future work, along with other questions about *fim* measures and generic stability.

7. Examples: *dfs* and not *fam*

One of the main questions left open in [CG 2020] was on the existence of a global measure that is *dfs* but not *fam*. What was done in [CG 2020] was a local version of this phenomenon. Specifically, it was shown that for any $s > r \geq 3$, if $T_{r,s}$ is the theory of the generic K^r_s-free r-uniform hypergraph (where K^r_s is the complete r-uniform hypergraph on s vertices), then there is a formula $\varphi(x, y)$ and a φ-type in $S_{\varphi}(\mathcal{U})$ that is *dfs* and not *fam* with respect to $\varphi(x, y)$. However, it is also proved in [CG 2020] that this type cannot be extended to a *dfs* global type (or measure).

The goal of this section is to construct complete theory with a *dfs* global type that is not *fam*. Our theory, denoted by $T^{\infty}_{1/2}$, will be much more complicated than $T_{r,s}$ (although the proof of *dfs* and not *fam* will be easier in some ways). Therefore, we will first construct a less complicated theory $T_{1/2}$ with a complete *quantifier-free* type q_0 that is *dfs* and not *fam*. We will then note some problems that arise when investigating quantifier elimination for $T_{1/2}$ which, in particular, suggest that finding a complete *dfs* extension of q_0 is likely to be difficult, if not impossible. This will motivate the construction of $T^{\infty}_{1/2}$, a complicated variation of $T_{1/2}$ which admits a global complete type q that is *dfs* and not *fam*. We will also construct a definable measure μ in $T^{\infty}_{1/2}$ that does not commute with q (as promised in Example 5.9).

7A. ***Sets that are half full.*** In this section we define $T_{1/2}$. This theory seems to represent the paradigm for the combinatorial separation of *dfs* from *fam*. We will work with the interval $[0, 1)$. Given $n \geq 1$, let $\mathcal{I}_n = \big\{[(i-1)/n, i/n) : 1 \leq i \leq n\big\}$.

Let $\mathcal{L} = \{P, Q, \sqsubseteq\}$, where P and Q are unary sorts and $\sqsubseteq$ is a binary relation on $P \times Q$. Define an $\mathcal{L}$-structure $M_{1/2}$ such that

- $P(M_{1/2})$ is the interval $[0, 1)$,
- $Q(M_{1/2})$ is the set of subsets of $[0, 1)$ obtained as the union of exactly n distinct intervals in $\mathcal{I}_{2n}$ for some $n \geq 1$, and
- $\sqsubseteq^{M_{1/2}}$ is the membership relation.

Note that any set in $Q(M_{1/2})$ has Lebesgue measure $\frac{1}{2}$.

Define $T_{1/2} = \mathrm{Th}(M_{1/2})$. Let $\mathcal{U} \succ M_{1/2}$ be a monster model. Define

$$q_0(y) = \{a \sqsubseteq y : a \in P(\mathcal{U})\}.$$

Note that $q_0 \models y \neq b$ for all $b \in Q(\mathcal{U})$. So q_0 determines a unique complete quantifier-free type, which is $\varnothing$-definable with respect to quantifier-free formulas.

Proposition 7.1. *q_0 is finitely satisfiable in $M_{1/2}$, but not finitely approximated in $M_{1/2}$ with respect to $x \sqsubseteq y$.*

Proof. We first show q_0 is finitely satisfiable in $M_{1/2}$. Fix $a_1, \ldots, a_n \in P(\mathcal{U})$. We need to find some $b \in Q(M_{1/2})$ such that $a_i \sqsubseteq b$ holds for all $1 \leq i \leq n$. Let $b_1, \ldots, b_k \in Q(M_{1/2})$ enumerate the sets obtained as the union of exactly n intervals in $\mathcal{I}_{2n}$ $\big($and thus $k = \binom{2n}{n}\big)$. Then $M_{1/2}$ satisfies the $\mathcal{L}_{M_{1/2}}$-sentence saying that for any n elements from P, there is some b_t containing them.

Now we show that q_0 is not finitely approximated in $M_{1/2}$ with respect to $x \sqsubseteq y$. Fix a tuple $\bar{b} \in Q(M_{1/2})^n$. We find $a \in [0, 1)$ such that $|\{1 \leq i \leq n : a \sqsubseteq b_i\}| \leq \frac{1}{2}n$. Define $f : [0, 1) \to \{0, 1, \ldots, n\}$ such that $f(a) = |\{1 \leq i \leq n : a \sqsubseteq b_i\}|$, i.e., $f = \sum_{i=1}^{n} \mathbf{1}_{b_i}$. Then f is integrable and we have

$$\int_0^1 f\,dx = \int_0^1 \sum_{i=1}^{n} \mathbf{1}_{b_i}\,dx = \sum_{i=1}^{n} \int_0^1 \mathbf{1}_{b_i}\,dx = \frac{1}{2}n.$$

So there is some $a \in [0, 1)$ such that $f(a) \leq \frac{1}{2}n$, as desired. $\square$

Altogether, q_0 is a complete quantifier-free global type, which is both $\varnothing$-definable and finitely satisfiable in $M_{1/2}$. By $\varnothing$-definability, we conclude that q_0 is finitely satisfiable in any small model, but not finitely approximated in any small model. However, the theory $T_{1/2}$ does not have quantifier elimination. For example, we can define relations on Q of the form $f(\bar{x}) = g(\bar{y})$, where f and g are terms in the language of Boolean algebras. A natural route to a theory with reasonable quantifier elimination might involve replacing $Q(M_{1/2})$ by a suitable Boolean algebra. But, as we show in the next section, this would cause problems for finding nontrivial *dfs* types. Therefore we will abandon $T_{1/2}$, and replace it with a more complicated theory that is able to sidestep the obstacles created by Boolean algebras.

7B. *Interlude on dfs types in Boolean algebras.*

Proposition 7.2. *Let T be the complete theory of a Boolean algebra, and suppose $p \in S_1(\mathcal{U})$ is dfs over $M \prec \mathcal{U}$. Then p is realized in M.*

Proof. We use the symbols $\sqcup$, $\sqcap$, ${}^{\mathrm{c}}$, $\top$, $\bot$, and $\sqsubseteq$ to denote the join, meet, complement, top element, bottom element, and induced partial order, respectively. Choose $\mathcal{L}_M$-formulas $\varphi(y)$ and $\psi(y)$ such that, given $c \in \mathcal{U}$, we have $p \models x \sqsubseteq c$ if and only if $\varphi(c)$, and $p \models c \sqsubseteq x$ if and only if $\psi(c)$. By [CG 2020, Proposition 2.9], we may assume $\varphi(y)$ is a Boolean combination of formulas of the form $a \sqsubseteq y$ for $a \in M$, and $\psi(y)$ is a Boolean combination of formulas of the form $y \sqsubseteq a$ for $a \in M$. So we can choose $a_1, \dots, a_k, b_1, \dots, b_\ell \in M$ and finite sets $X_1, \dots, X_k, Y_1, \dots, Y_k \subseteq M$ such that

$$\varphi(y) = \bigvee_{i=1}^{k} \Big((a_i \sqsubseteq y) \wedge \bigwedge_{m \in X_i} \neg(m \sqsubseteq y) \Big),$$

$$\psi(y) = \bigvee_{i=1}^{\ell} \Big((y \sqsubseteq b_i) \wedge \bigwedge_{n \in Y_i} \neg(y \sqsubseteq n) \Big).$$

Note that both $\varphi(y)$ and $\psi(y)$ are consistent since we have $\varphi(\top)$ and $\psi(\bot)$. So we can discard any inconsistent disjuncts in either formula. It then follows that $\varphi(a_i)$ holds for all $1 \le i \le k$, and $\psi(b_i)$ holds for all $1 \le i \le \ell$. So $p \models x \sqsubseteq a_i$ for all $i \le k$, and $p \models b_i \sqsubseteq a$ for all $i \le \ell$. Thus, if $a = a_1 \sqcap \cdots \sqcap a_k$ and $b = b_1 \sqcup \cdots \sqcup b_\ell$, then we have $p \models b \sqsubseteq x \sqsubseteq a$.

Since p is consistent, we know that $b \sqsubseteq a$. Moreover, for any $c \in \mathcal{U}$, if $b \sqsubset c \sqsubset a$ then $\neg\varphi(c) \wedge \neg\psi(c)$ holds, and so $p \models \neg(x \sqsubseteq c \vee c \sqsubseteq x)$.

Let $B = \{m \in M : b \sqsubseteq m \sqsubseteq a\}$. We view B as a Boolean algebra, with bottom element b and top element a. We will show that $|B| \le 2$, from which it follows that $M \models \neg\exists x(b \sqsubset x \sqsubset a)$, and so p must be realized by a or by b. Toward a contradiction, suppose $|B| > 2$.

Let P be an ultrafilter over B. Then the type $\{b \sqsubset x \sqsubseteq m : m \in P\}$ is finitely satisfiable, and thus realized by some $c \in \mathcal{U}$. Note that $b \sqsubset c \sqsubset a$. Let $d = b \sqcup (a \sqcap c^{\mathrm{c}})$ (i.e., d is the complement of c in $B(\mathcal{U})$). Then $b \sqsubset d \sqsubset a$ as well. Therefore

$p \models (b \sqsubseteq x \sqsubseteq a) \wedge \neg(c \sqsubseteq x) \wedge \neg(x \sqsubseteq d)$. So we can find some $m \in M$ realizing this formula. Then $m \in B$ since $b \sqsubseteq m \sqsubseteq a$. Since $\neg(c \sqsubseteq m)$, we must have $m \notin P$. So $b \sqcup (a \sqcap m^{\mathrm{c}}) \in P$, and therefore $c \sqsubseteq b \sqcup (a \sqcap m^{\mathrm{c}})$. It follows that $m \sqsubseteq d$, which contradicts the choice of m. □

After relativizing the previous argument, we have the following conclusion.

Corollary 7.3. *Let T be a complete theory, and suppose $p \in S_x(\mathcal{U})$ is a global type, which is dfs over some $M \prec \mathcal{U}$. Given an $\mathcal{L}_M$-formula $\varphi(x) \in p$, if there is an M-definable Boolean algebra on $\varphi(M^x)$, then p is realized in $\varphi(M^x)$.*

Remark 7.4. We briefly note that one can construct *measures* in Boolean algebras that are nontrivial in the sense of Definition 8.1. (This was posed as a question in a preliminary version of the paper.) In particular, let ABA be the theory of atomless Boolean algebras, and fix $\mathcal{U} \models \mathsf{ABA}$. We construct a nontrivial *dfs* (in fact, smooth) measure $\sigma \in \mathfrak{M}_1(\mathcal{U})$.

Say that an $\mathcal{L}_{\mathcal{U}}$-formula $\varphi(x)$ (in a single free variable) is *convex* if whenever $\varphi(a)$ holds, $\varphi(c)$ holds, and $a \sqsubseteq b \sqsubseteq c$, then $\varphi(b)$ holds. It is not too hard to show, using quantifier elimination for ABA, that every formula in a single free variable is equivalent to a (finite) disjunction of convex formulas.

Let $\mathcal{H}$ be the Boolean algebra of subsets of $[0, 1)$ generated by half-open intervals of the form $[a, b)$ with $0 \le a < b \le 1$. Note that $\mathcal{H}$ is an atomless Boolean algebra. We will regard $\mathcal{H}$ as an elementary substructure of $\mathcal{U}$. Given an $\mathcal{L}_{\mathcal{U}}$-formula $\varphi(x)$, we define $X(\varphi) = \{r \in [0, 1] : \mathcal{U} \models \varphi((0, r])\}$.

For any convex $\mathcal{L}_{\mathcal{U}}$-formula $\varphi(x)$, it is easy to see that $X(\varphi)$ is an interval. Thus, for any $\mathcal{L}_{\mathcal{U}}$-formula $\psi(x)$, we have that $X(\psi)$ is a finite union of intervals and is thus Lebesgue measurable. Define a measure σ by setting $\sigma(\psi(x))$ equal to the Lebesgue measure of $X(\psi)$. Then σ is clearly not a trivial measure. Moreover, σ is the unique global extension of $\sigma{\restriction}\mathcal{H}$ (this follows using an argument analogous to the proof that Lebesgue measure restricted to the interval $[0, 1]$ is smooth in DLO).

7C. *Amalgamated Boolean algebras.* In this section, we construct a theory, denoted by $T^{\infty}_{1/2}$, which admits a complete *dfs* global type that is not *fam*. This theory will be obtained from $T_{1/2}$ by making various modifications, which are guided by two opposing forces. On the one hand, quantifier elimination is made easier by working with a richer structure in the Q sort, such as a Boolean algebra. So we start with the Boolean algebra $\mathcal{H}$ from Remark 7.4, i.e., the subalgebra of $\mathcal{P}([0, 1))$ generated by half-open intervals $[a, b)$. On the other hand, Corollary 7.3 tells us that using $\mathcal{H}$ in the Q sort will destroy any chance of obtaining a nontrivial *dfs* type. In order to fix this issue, we will replace $\mathcal{H}$ by a certain algebraic lattice, denoted by $\mathcal{Q}$, which will be obtained by taking infinitely many disjoint copies of $\mathcal{H}$ and identifying all of the top elements and all of the bottom elements. By doing this, we will be able to construct a complete global type, which is built from the same local instance of *dfs* and not *fam* found in $T_{1/2}$, but is also able to avoid concentrating on any "standard" Boolean algebra defined in $\mathcal{U}$, and thus maintain *dfs* globally.

We now work toward the definition of $T^{\infty}_{1/2}$, starting with a precise description of $\mathcal{Q}$ (which, in fact, will be a lattice with a complement operator). First, let $\mathcal{H}$ be as defined above, and set $\mathcal{H}_0 = \mathcal{H} \setminus \{\varnothing, [0, 1)\}$. Set $\mathcal{Q} = (\mathbb{N} \times \mathcal{H}_0) \cup \{\perp, \top\}$, where $\perp$ and $\top$ are new symbols. Define the structure $(\mathcal{Q}, \sqcap, \sqcup, {}^{\mathsf{c}}, \perp, \top)$ as follows.

* Given $m, n \in \mathbb{N}$ and $X, Y \in \mathcal{H}_0$,

$$(m, X) \sqcap (n, Y) = \begin{cases} (m, X \cap Y) & \text{if } m = n, \\ \perp & \text{if } m \neq n, \end{cases}$$
$$(m, X) \sqcup (n, Y) = \begin{cases} (m, X \cup Y) & \text{if } m = n, \\ \top & \text{if } m \neq n, \end{cases}$$
$$(m, X)^{\mathsf{c}} = (m, [0, 1) \setminus X).$$

* Given $b \in \mathcal{Q}$,

$$\perp \sqcap b = b \sqcap \perp = \perp, \quad \perp \sqcup b = b \sqcup \perp = b,$$
$$\top \sqcap b = b \sqcap \top = b, \quad \top \sqcup b = b \sqcup \top = \top.$$

* $\perp^{\mathsf{c}} = \top$ and $\top^{\mathsf{c}} = \perp$.

In fact, $\mathcal{Q}$ is an orthocomplemented lattice; however, general familiarity with such structures will not be necessary, and the reader will only need the definition above. We let $\mathcal{Q}_0$ denote $\mathcal{Q} \setminus \{\perp, \top\} = \mathbb{N} \times \mathcal{H}_0$.

The next goal is to describe the set that will play the same role held by $[0, 1)$ in the P sort of $T_{1/2}$. Since $\mathcal{Q}$ is not a Boolean algebra of subsets of some ground set, we first need to define an analogous notion of "membership" in elements of $\mathcal{Q}$.

Definition 7.5. Define a binary relation $\sqsubseteq$ on $[0, 1)^{\mathbb{N}} \times \mathcal{Q}$ such that $a \sqsubseteq b$ if and only if either $b = \top$ or $b = (n, X) \in \mathcal{Q}_0$ and $a(n) \in X$.

In order to obtain quantifier elimination for $T^{\infty}_{1/2}$, we will not use $[0, 1)^{\mathbb{N}}$ in the P sort, but rather a countable subset S satisfying certain properties (described in Lemma 7.8 below). We first need some terminology.

Definition 7.6. A *cube* C is a nonempty subset of $[0, 1)^{\mathbb{N}}$ that is of the form $C = \prod_{n \in \mathbb{N}} [a_n, b_n)$ with $a_n = 0$ and $b_n = 1$ for all but finitely many $n \in \mathbb{N}$. If, moreover, a_n and b_n are rational for all n, then C is a *rational cube*.

The following facts, which will use later on, are easy to verify.

Fact 7.7. *Let B be the Boolean algebra on $[0, 1)^{\mathbb{N}}$ generated by sets of the form $\{a \in [0, 1)^{\mathbb{N}} : a \sqsubseteq b\}$ for all $b \in \mathcal{Q}$.*

(a) *B is the same as the Boolean algebra generated by cubes.*

(b) *Every element of B can be written as a disjoint union of finitely many cubes.*

(c) *Suppose* $f : [0, 1)^{\mathbb{N}} \to \mathbb{R}$ *is a finite linear combination of indicator functions of sets in B. Then f can be written as* $\sum_{i<k} r_i \mathbf{1}_{C_i}$, *where* $\{C_i\}_{i<k}$ *is a partition of* $[0, 1)^{\mathbb{N}}$ *into disjoint cubes and each r_i is in the image of f.*

Lemma 7.8. *There is a countably infinite set* $S \subseteq [0, 1)^{\mathbb{N}}$ *such that*

(i) *S nontrivially intersects any cube in* $[0, 1)^{\mathbb{N}}$, *and*

(ii) *the map* $(a, n) \to a(n)$ *from* $S \times \mathbb{N}$ *to* $[0, 1]$ *is injective.*

Proof. Let $\{C_\ell\}_{\ell\in\mathbb{N}}$ be an enumeration of all rational cubes. We will build the set $S = \{a_\ell\}_{\ell\in\mathbb{N}}$ in stages. At stage ℓ, pick $a_\ell \in [0, 1)^{\mathbb{N}}$ so that

(1) $a_\ell \in C_\ell$,

(2) a_ℓ is an injective map whose range is disjoint from those of a_i for each $i < \ell$.

Since at each stage we have only used countably many elements of $[0, 1)$, it is clear that we can always make such a choice of a_ℓ. Now, since any cube in $[0, 1)^{\mathbb{N}}$ contains a rational cube, we have condition (i) of the lemma by part (1) of the construction. Condition (ii) of the lemma follows from part (2). □

We now have all of the ingredients necessary to define $T^{\infty}_{1/2}$. In order to mimic the behavior in Proposition 7.1, we need to be able to pick out sets in $\mathcal{Q}$ of "measure" $\frac{1}{2}$. With an eye toward quantifier elimination, we will add a third sort R for the ordered group of reals, and a unary function from Q to R for the appropriate measure. Altogether, we define $\mathcal{L} = \{P, Q, R, \sqsubseteq, \sim, \ell, \sqcap, \sqcup, {}^{\mathsf{c}}, \bot, \top, +, <, 0, 1\}$, where

* P, Q, and R are unary sorts,
* $\sqsubseteq$ is a binary relation on $P \times Q$,
* $\sim$ is a binary relation on Q,
* ℓ is a unary function from Q to R,
* $\{\sqcap, \sqcup, {}^{\mathsf{c}}, \bot, \top\}$ is the language of "lattices with complements" on Q, and
* $\{+, <, 0, 1\}$ is the language of ordered abelian groups on R, with an additional constant symbol 1.

Now let λ denote the Lebesgue measure on $[0, 1)$. We define an $\mathcal{L}$-structure $M^{\infty}_{1/2}$ via the following interpretation of $\mathcal{L}$:

* $P(M^{\infty}_{1/2})$ is a fixed set $S \subseteq [0, 1)^{\mathbb{N}}$ as in Lemma 7.8.
* $(Q(M^{\infty}_{1/2}), \sqcap, \sqcup, {}^{\mathsf{c}}, \bot, \top)$ is $(\mathcal{Q}, \sqcap, \sqcup, {}^{\mathsf{c}}, \bot, \top)$.
* $(R(M^{\infty}_{1/2}), +, <, 0, 1)$ is $(\mathbb{R}, +, <, 0, 1)$.
* $\sqsubseteq$ is as described in Definition 7.5 (but restricted to $P(M^{\infty}_{1/2}) \times \mathcal{Q}$).

* If $b, c \in Q(M^{\infty}_{1/2})$, then $b \sim c$ holds if and only if b and c are both in $\mathcal{Q}_0$ and have the same first coordinates.
* If $b \in Q(M^{\infty}_{1/2})$, then

$$\ell(b) = \begin{cases} \lambda(X) & \text{if } b = (n, X) \in \mathbb{N} \times \mathcal{H}_0, \\ 0 & \text{if } b = \bot, \\ 1 & \text{if } b = \top. \end{cases}$$

Finally, we define $T^{\infty}_{1/2} = \mathrm{Th}(M^{\infty}_{1/2})$.

Let $\mathcal{U} \succ M^{\infty}_{1/2}$ be a monster model. For the rest of this section, we let y denote a variable (of length one) in the Q sort. Define

$$q_1(y) = \{a \sqsubseteq y : a \in P(\mathcal{U})\} \cup \left\{\ell(y) = \tfrac{1}{2}\right\} \cup \{y \not\sim b : b \in Q(\mathcal{U})\}.$$

We now observe that q_1 is a partial global type exhibiting the same behavior found in Proposition 7.1.

Proposition 7.9. *q_1 is finitely satisfiable in $M^{\infty}_{1/2}$, but not finitely approximated in $M^{\infty}_{1/2}$ with respect to the formula $\psi(y, x) := (x \sqsubseteq y) \wedge \left(\ell(y) = \frac{1}{2}\right)$.*

Proof. We first show q_1 is finitely satisfiable in $M^{\infty}_{1/2}$. Fix $a_1, \ldots, a_n \in P(\mathcal{U})$ and $b_1, \ldots, b_m \in Q(\mathcal{U})$. We need to find some $c \in \mathcal{Q}$ such that $\ell(c) = \frac{1}{2}$, $a_i \sqsubseteq c$ holds for all $1 \leq i \leq n$, and $c \not\sim b_i$ for all $1 \leq i \leq m$. Fix $s \in \mathbb{N}$ such that, for all $1 \leq i \leq m$, $b_i \not\sim (s, X)$ for any $X \in \mathcal{H}_0$. Let $X_1, \ldots, X_k \in \mathcal{H}_0$ enumerate the sets obtained as the union of exactly n intervals in $\mathcal{I}_{2n}$, and consider $c_t = (s, X_t) \in \mathcal{Q}$ for $t \leq k$. Then by construction there exists some $t_* \leq k$ such that $\mathcal{U} \models a_i \sqsubseteq c_{t_*}$ for all $1 \leq i \leq n$, as desired.

Now we show that q_1 is not finitely approximated in $M^{\infty}_{1/2}$ with respect to $\psi(y, x)$. Fix a tuple $\bar{b} \in \mathcal{Q}^n$. We find $a \in P(M^{\infty}_{1/2})$ such that $|\{1 \leq i \leq n : \psi(b_i, a)\}| \leq \frac{1}{2}n$. Without loss of generality, we may assume $\ell(b_i) = \frac{1}{2}$ for all $1 \leq i \leq n$. Let $\lambda^{\mathbb{N}}$ denote the measure on $[0, 1)^{\mathbb{N}}$ obtained from the product of λ on $[0, 1)$. Define $f : [0, 1)^{\mathbb{N}} \to \{0, 1, \ldots, n\}$ such that $f(a) = |\{1 \leq i \leq n : a \sqsubseteq b_i\}|$. Given $1 \leq i \leq n$, set $B_i = \{a \in [0, 1)^{\mathbb{N}} : a \sqsubseteq b_i\}$. Then each B_i is $\lambda^{\mathbb{N}}$-measurable, with $\lambda^{\mathbb{N}}(B_i) = \ell(b_i) = \frac{1}{2}$, and $f = \sum_{i=1}^{n} \mathbf{1}_{B_i}$. Therefore f is $\lambda^{\mathbb{N}}$-integrable and

$$\int_{[0,1)^{\mathbb{N}}} f \, d\lambda^{\mathbb{N}} = \int_{[0,1)^{\mathbb{N}}} \sum_{i=1}^{n} \mathbf{1}_{B_i} \, d\lambda^{\mathbb{N}} = \sum_{i=1}^{n} \int_{[0,1)^{\mathbb{N}}} \mathbf{1}_{B_i} \, d\lambda^{\mathbb{N}} = \frac{1}{2}n.$$

By Fact 7.7, $f = \sum_{j<k} r_j \mathbf{1}_{C_j}$, where $\{C_j\}_{j<k}$ is a partition of $[0, 1)^{\mathbb{N}}$ into disjoint cubes and each r_j is in $\{0, 1, \ldots, n\}$. So it must be the case that $r_j \leq \frac{1}{2}n$ for some $j < k$. By our choice of $P(M^{\infty}_{1/2})$, there must be some $a \in P(M^{\infty}_{1/2}) \cap C_j$, which therefore has $f(a) \leq \frac{1}{2}n$, as desired. □

We now turn to the main goal, which is to show that q_1 extends to a complete global type in $S_Q(\mathcal{U})$ that is *dfs* and not *fam* (here $S_Q(\mathcal{U})$ denotes the space of complete global types concentrating on the Q sort). In fact, we will show that q_1 determines a unique complete type in $S_Q(\mathcal{U})$ and that this type has the desired properties. The first step is quantifier elimination.

Theorem 7.10. *$T^{\infty}_{1/2}$ has quantifier elimination.*

The proof of the previous theorem is rather involved and so, to avoid stalling the primary exposition, we have cordoned off the details in Section A3. So let us now continue toward the main goal.

Lemma 7.11. *For any $\mathcal{L}$-formula $\varphi(y,\bar{z})$, there is an $\mathcal{L}$-formula $\psi(\bar{z})$ such that, for any $\bar{b}\in\mathcal{U}^{\bar{z}}$, if $\mathcal{U}\models\psi(\bar{b})$ then $q_1\models\varphi(y,\bar{b})$, and if $\mathcal{U}\models\neg\psi(\bar{b})$ then $q_1\models\neg\varphi(y,\bar{b})$.*

Proof. Let $\mathcal{L}_Q$ and $\mathcal{L}_R$ denote the restrictions of $\mathcal{L}$ to the Q and R sorts, respectively. We first claim that, for any $\mathcal{L}_Q$-term $t(y,\bar{z})$, there is a term $s(y,\bar{z})$ of the form y, y^{c}, or $u(\bar{z})$ such that $q\models t(y,\bar{b})=s(y,\bar{b})$ for all $\bar{b}\in\mathcal{U}^{\bar{z}}$. Indeed, this can be proved by induction on terms. The main point is that, for any $b\in\mathcal{U}$, $q_1\models y\not\sim b$, and so $q_1\models(y\sqcap b=\bot)\wedge(y\sqcup b=\top)$.

Now we prove the lemma. It suffices to assume φ is atomic. We consider cases.

Suppose $\varphi(y,\bar{z},x)$ is $x\sqsubseteq t(y,\bar{z})$, where $t(y,\bar{z})$ is an $\mathcal{L}_Q$-term. Let s be as in the initial claim. If s is y, let $\psi(\bar{z},x)$ be $x=x$. If s is y^{c}, let $\psi(\bar{z},x)$ be $x\neq x$. If s is $u(\bar{z})$, let $\psi(\bar{z},x)$ be $x\sqsubseteq u(\bar{z})$.

Suppose $\varphi(y,\bar{z})$ is $t_1(y,\bar{z})\asymp t_2(y,\bar{z})$, where t_1,t_2 are $\mathcal{L}_Q$-terms and $\asymp$ is $=$ or $\sim$. Let s_1 and s_2 be as in the initial claim. If $s_1,s_2\in\{y,y^{\mathsf{c}}\}$, then let $\psi(\bar{z})$ be $\bar{z}=\bar{z}$ if s_1 and s_2 are the same or $\asymp$ is $\sim$, and let $\psi(\bar{z})$ be $\bar{z}\neq\bar{z}$ otherwise. If s_1 and s_2 are $u_1(\bar{z})$ and $u_2(\bar{z})$, let $\psi(\bar{z})$ be $u_1(\bar{z})\asymp u_2(\bar{z})$. Otherwise, let $\psi(\bar{z})$ be $\bar{z}\neq\bar{z}$.

Finally, suppose $\varphi(y,\bar{z},\bar{w})$ is $f\big(\ell(t_1(y,\bar{z})),\dots,\ell(t_n(y,\bar{z})),\bar{w}\big)\asymp 0$, where $\asymp$ is $=$ or $<$, each t_i is an $\mathcal{L}_Q$-term, and $f(v_1,\dots,v_n,\bar{w})$ is an $\mathcal{L}_R$-term. Let $s_1,\dots,s_n$ be as in the initial claim. Recall that $q\models\ell(y)=\ell(y^{\mathsf{c}})=\frac{1}{2}$. Let $v_i(\bar{z})$ be either $\frac{1}{2}$, if s_i is y or y^{c}, or $\ell(u_i(\bar{z}))$ if s_i is $u_i(\bar{z})$. Then we may take $\psi(\bar{z},\bar{w})$ to be $f(v_1(\bar{z}),\dots,v_n(\bar{z}),\bar{w})\asymp 0$. □

Corollary 7.12. *There is a unique complete type $q\in S_Q(\mathcal{U})$ extending q_1. Moreover, q is $\varnothing$-definable and finitely satisfiable in any small model, but is not fam.*

Proof. By Lemma 7.11, there is a unique type $q\in S_Q(\mathcal{U})$ extending q_1, and q is $\varnothing$-definable. By Proposition 7.9, q is finitely satisfiable in $M^{\infty}_{1/2}$ (in particular, for any $\varphi(x)\in q$ there is some $\psi(x)\in q_1$ such that $\psi(x)$ implies $\varphi(x)$), but not *fam* over $M^{\infty}_{1/2}$. Since q is $\varnothing$-invariant, the same is true over any small model. □

7D. *Failure to commute.* Let $\mathcal{U}\models T^{\infty}_{1/2}$. In this section, we construct a definable measure $\mu\in\mathfrak{M}_P(\mathcal{U})$ that does not commute with the *dfs* type $q\in S_Q(\mathcal{U})$ from the last section. This justifies the claim made in Example 5.9. Throughout this section, x and y denote tuples of variables of length one in the P and Q sorts, respectively.

Recall that in the standard model $M^{\infty}_{1/2}$ of $T^{\infty}_{1/2}$, the set $P(M^{\infty}_{1/2})$ is a subset of $[0,1)^{\mathbb{N}}$, which is equipped with the product Lebesgue measure $\lambda^{\mathbb{N}}$. Suppose $\varphi(x)$ is an $\mathcal{L}_{M^{\infty}_{1/2}}$-formula in the P sort. By quantifier elimination, the subset of $P(M^{\infty}_{1/2})$ defined by $\varphi(x)$ differs by finitely many elements from a set of the form $P(M^{\infty}_{1/2}) \cap \bigcup_{i<n} C_i$, with each C_i a cube. Furthermore, the set $\bigcup_{i<n} C_i$ is uniquely determined by $\varphi(x)$. The map from $M^{\infty}_{1/2}$-definable subsets of $P(M^{\infty}_{1/2})$ to subsets from $[0,1)^{\mathbb{N}}$ defined by taking each $\varphi(x)$ to the corresponding $\bigcup_{i<n} C_i$ is a Boolean algebra homomorphism. Let λ^* be the pullback of $\lambda^{\mathbb{N}}$ along this homomorphism. Then λ^* is automatically a finitely additive probability measure on the $M^{\infty}_{1/2}$-definable subsets of $P(M^{\infty}_{1/2})$, so it is an element of $\mathfrak{M}_P(M^{\infty}_{1/2})$.

Lemma 7.13. *There is a unique $\varnothing$-definable measure $\mu \in \mathfrak{M}_P(\mathcal{U})$ extending λ^*.*

Proof. Let $M = M^{\infty}_{1/2}$. The proof amounts to showing that $\lambda^* \in \mathfrak{M}_P(M)$ is "$\varnothing$-definable". This notion of definability for Keisler measures over small models is similar to that for global measures. See Section A4 for details. We will apply Remark A.22 and Theorem A.24. Fix an $\mathcal{L}$-formula $\Phi(x;\bar{w},\bar{y},\bar{z})$ which is a conjunction of atomic and negated atomic formulas, where $\bar{w}$ is of sort P, $\bar{y}$ is of sort Q, and $\bar{z}$ is of sort R. Define the map $f^{\Phi}_{\lambda^*} : (\bar{b},\bar{c},\bar{d}) \mapsto \lambda^*(\Phi(x;\bar{b},\bar{c},\bar{d}))$ from $M^{\bar{w}\bar{y}\bar{z}}$ to $[0,1]$. We want to show that $f^{\Phi}_{\lambda^*}$ has an $\varnothing$-invariant continuous extension to $S_{\bar{w}\bar{y}\bar{z}}(M)$ (see also Definition A.18 and surrounding discussion).

Write $\Phi(x;\bar{w},\bar{y},\bar{z})$ as $\varphi(x,\bar{y}) \wedge \theta(x,\bar{w}) \wedge \chi(\bar{w},\bar{y},\bar{z})$, where

* $\chi(\bar{w},\bar{y},\bar{z})$ is some $\mathcal{L}$-formula not mentioning x,
* $\theta(x,\bar{w}) := \bigwedge_{i=1}^{m}(x =_i w_i)$, where $=_i$ is $=$ or $\neq$, and
* $\varphi(x,\bar{y}) := \bigwedge_{i=1}^{n}(x \sqsubseteq_i t_i(\bar{y}))$, where $\sqsubseteq_i$ is $\sqsubseteq$ or $\not\sqsubseteq$, and each $t_i(\bar{x})$ is an $\mathcal{L}_Q$-term.

If some $=_i$ is $=$ then $f^{\Phi}_{\lambda^*}$ is identically 0, in which case our task is trivial. So we may assume that each $=_i$ is $\neq$. In this case, we have $f^{\Phi}_{\lambda^*} = (f^{\varphi}_{\lambda^*} \circ \rho)\mathbf{1}_{\chi}$, where $\rho : M^{\bar{w}\bar{y}\bar{z}} \to M^{\bar{y}}$ is the subtuple map. So it suffices to show that $f^{\varphi}_{\lambda^*}$ has an $\varnothing$-invariant continuous extension to $S_{\bar{y}}(M)$.

Let $\sim^*$ denote the equivalence relation on Q obtained from $\sim$ by making $\top$ and $\bot$ singleton classes, i.e., $y \sim^* y'$ if and only if $(y \sim y') \vee (y = y' = \top) \vee (y = y' = \bot)$. Note that $\sim^*$ is $\varnothing$-definable. Let Σ denote the set of partitions of $[n]$. Given $\sigma \in \Sigma$, let $\theta_\sigma(\bar{y})$ be the conjunction of $t_i(\bar{y}) \sim^* t_j(\bar{y})$ for all σ-related $i, j \in [n]$, and $t_i(\bar{y}) \not\sim^* t_j(\bar{y})$ for all σ-unrelated $i, j \in [n]$. Let $\varphi_\sigma(x,\bar{y})$ be the $\mathcal{L}$-formula $\varphi_\sigma(x,\bar{y}) \wedge \theta_\sigma(\bar{y})$. Then $f^{\varphi}_{\lambda^*} = \sum_{\sigma\in\Sigma} f^{\varphi_\sigma}_{\lambda^*}$. So it suffices to show that each $f^{\varphi_\sigma}_{\lambda^*}$ has an $\varnothing$-invariant continuous extension to $S_{\bar{y}}(M)$.

Now fix $\sigma \in \Sigma$, and let $\sigma = \{\sigma_1,\ldots,\sigma_k\}$. For $1 \le j \le k$, define the $\mathcal{L}_Q$-term

$$u_j(\bar{y}) := \prod_{i\in\sigma_j} \begin{Bmatrix} t_i(\bar{y}), & \sqsubseteq_i \text{ is } \sqsubseteq \\ t_i(\bar{y})^{\mathsf{c}}, & \sqsubseteq_i \text{ is } \not\sqsubseteq \end{Bmatrix},$$

and let $\zeta_j(x,\bar{y})$ be the $\mathcal{L}$-formula $x \sqsubseteq u_j(\bar{y})$. Then, for any $\bar{b} \in M^{\bar{y}}$, we have

$$f^{\varphi_\sigma}_{\lambda^*}(\bar{b}) = \lambda^*(\varphi(x,\bar{b})) = \prod_{j=1}^{k} \ell(u_j(\bar{b})) = \prod_{j=1}^{k} f^{\zeta_j}_{\lambda^*}(\bar{b})$$

(note that here we are suppressing compositions with "subtuple" functions, as in the first reduction from Φ to φ). So it suffices to show that each $f^{\zeta_j}_{\lambda^*}$ has an $\varnothing$-invariant continuous extension to $S_{\bar{y}}(M)$. For this, we apply Fact A.19. In particular, fix $1 \leq j \leq k$ and $\varepsilon < \delta$ in $[0,1]$. Let $\psi_j(\bar{y})$ be the $\mathcal{L}$-formula $\ell(u_j(\bar{y})) < \alpha$, where $\alpha \in (\varepsilon, \delta)$ is rational. Then

$$\{\bar{b} \in M^{\bar{y}} : f^{\zeta_j}_{\lambda^*}(\bar{b}) \leq \varepsilon\} \subseteq \psi_j(M) \subseteq \{\bar{b} \in M^{\bar{y}} : f^{\zeta_j}_{\lambda^*}(\bar{b}) < \delta\},$$

as desired. □

Now, since p and μ are both definable, we have the Morley products $\mu \otimes q$ and $q \otimes \mu$, which are also both definable.

Proposition 7.14. *$(\mu \otimes q)(x \sqsubseteq y) = \frac{1}{2}$ and $(q \otimes \mu)(x \sqsubseteq y) = 1$.*

Proof. Note that $F^{\sqsubseteq}_{q,M}$ is the constant function 1, and so $(q \otimes \mu)(x \sqsubseteq y) = 1$. On the other hand, we have $(\mu \otimes q)(x \sqsubseteq y) = \mu(x \sqsubseteq b)$, where $b \models q|_{M^{\infty}_{1/2}}$. So $\ell(b) = \frac{1}{2}$. For any $c \in \mathcal{Q}$, if $\ell(c) = \frac{1}{2}$ then $\mu(x \sqsubseteq c) = \lambda^*(x \sqsubseteq c) = \frac{1}{2}$. By Remark A.23, this also holds for all $c \in Q(\mathcal{U})$, and thus $\mu(x \sqsubseteq b) = \frac{1}{2}$. □

With Question 5.10 in mind, we point out that μ is not finitely satisfiable in $M^{\infty}_{1/2}$ (and hence not in any small model). Indeed, if $b \models q|_{M^{\infty}_{1/2}}$, then, by the previous proof, $\mu(x \not\sqsubseteq b) = \frac{1}{2}$, but $x \not\sqsubseteq b$ has no solution in $M^{\infty}_{1/2}$ by definition of q. In fact, any *dfs* measure in $\mathfrak{M}_P(\mathcal{U})$ must be a sum of countably many weighted Dirac measures at points in $P(\mathcal{U})$. This assertion follows from an argument nearly identical to the proof of [CG 2020, Theorem 4.9], together with the following genericity property in $M^{\infty}_{1/2}$: for any finite disjoint sets $A, B \subset P(M^{\infty}_{1/2})$, there is some $c \in Q(M^{\infty}_{1/2})$ such that $a \sqsubseteq c$ for all $a \in A$ and $b \not\sqsubseteq c$ for all $b \in B$ (indeed, by construction of $P(M^{\infty}_{1/2})$, such an element c can be found in $\{n\} \times \mathcal{H}_0$ for any $n \in \mathbb{N}$).

Note that Lemma 7.13, Proposition 7.14, and Proposition 5.17 provide another demonstration that the type q is not *fam*. By Theorem 5.7, we also have the following conclusion for λ^*.

Corollary 7.15. *No definable extension of λ^* in $\mathfrak{M}_P(\mathcal{U})$ commutes with q. In particular, λ^* has no smooth (or even fim) global extensions in $\mathfrak{M}_P(\mathcal{U})$.*

Remark 7.16. Furthermore, the restriction of λ^* to the language $\mathcal{L}_{PQ}$ is an example of a measure with no definable extension (and is the first such example that we are aware of). Let λ^*_{PQ} be this restriction. Assume that λ^*_{PQ} has some definable extension ν over some $N \succeq M_{PQ}$. Then we must have some $\mathcal{L}_N$-formula $\psi(y)$ such that for any $b \in P^N$, if $\nu(x \sqsubseteq b) < \frac{1}{3}$ then $\psi(b)$ holds, and if $\nu(x \sqsubseteq b) > \frac{2}{3}$

then $\psi(b)$ fails to hold. This implies that for any $b \in P^{M_{PQ}}$, if $\lambda^*_{PQ}(x \sqsubseteq b) < \frac{1}{3}$ then $\psi(b)$ holds, and if $\nu(x \sqsubseteq b) > \frac{2}{3}$ then $\psi(b)$ fails to hold.

Find a $\sim$-class C in M_{PQ} such that no parameter of ψ is contained in C. Let A be the set of P-parameters in ψ. For each $a \in A$, we can find $b_a \in C$ such that $a \sqsubseteq b_a$ and $\lambda^*_{PQ}(x \sqsubseteq b_a) < \frac{1}{4}(1/|A|)$. By construction, we have that $\lambda^*_{PQ}\big(x \sqsubseteq \bigsqcup_{a\in A} b_a\big) < \frac{1}{4}$. Thus, we can find c and d in C with $\lambda^*_{PQ}(x \sqsubseteq c) < \frac{1}{3}$ and $\lambda^*_{PQ}(x \sqsubseteq d) > \frac{2}{3}$ such that c and d are disjoint from every b_a.

By quantifier elimination (Proposition A.15), c and d have the same type over A, hence $\mathcal{U} \models \psi(c) \leftrightarrow \psi(d)$. But $\lambda^*_{PQ}(x \sqsubseteq c) < \frac{1}{3}$ and $\lambda^*_{PQ}(x \sqsubseteq d) > \frac{2}{3}$, which is a contradiction. Therefore λ^*_{PQ} has no definable extensions.

Remark 7.17. We showed above that q does not admit *fam* approximations for $\psi(y, x) := (x \sqsubseteq y) \wedge \big(\ell(y) = \frac{1}{2}\big)$ within error $\frac{1}{2}$ (i.e., $\mathrm{Av}^n_{<1/2}(q, \psi) = \varnothing$ for all n). Given a fixed rational $\varepsilon \in (0, 1)$, let q_ε be the result of replacing $\frac{1}{2}$ with ε in the definition of q_1. Then similar arguments show that q_ε determines a complete *dfs* type, with no *fam* approximations for $(x \sqsubseteq y) \wedge (\ell(y) = \varepsilon)$ within error $1 - \varepsilon$. In other words, one can construct arbitrarily terrible failures of *fam* approximations for a *dfs* type. This is in contrast to the result for $T_{r,s}$ from [CG 2020], mentioned at the beginning of this section, which produced a *dfs* local φ-type with no *fam* approximations for φ within error $(r-1)!/(r-1)^{r-1}$.

The previous modification also results in a more severe failure of symmetry in that we obtain $(\mu \otimes q_\varepsilon)(x \sqsubseteq y) = \varepsilon$ and $(q_\varepsilon \otimes \mu)(x \sqsubseteq y) = 1$.

Finally, we reiterate that $T^\infty_{1/2}$ is much more complicated than $T_{r,s}$, both in its construction and with respect to its classification in neostability. In particular, $T_{r,s}$ is supersimple, while $T^\infty_{1/2}$ interprets the theory of atomless Boolean algebras and thus has TP_2 and SOP.

Question 7.18. Is there a simple theory T and a global type $p \in S_x(\mathcal{U})$ such that p is *dfs* and not *fam*?

In general, it would be interesting to find less complicated examples of complete theories separating *dfs* and *fam*, even at the level of measures.

8. Examples: *fam* and not *fim*

In this section, we discuss examples of Keisler measures that are *fam* and not *fim*, starting with a reexamination of some examples given in [CG 2020]. We first note in Section 8A that one of those examples relied on an erroneous claim from [Adler et al. 2014], and we show that in fact the ambient theory in this example has no nontrivial *dfs* measures. So this reduces the previously known examples of theories with *fam* and non-*fim* measures to essentially one family, namely, the generic K_s-free graphs for some fixed $s \geq 3$. In Section 8B, we will develop more features of this example, and then give a correct proof of a certain result from [CG 2020]. Finally, we will show that in the reduct of $T^{\infty}_{1/2}$ obtained by forgetting the measure ℓ, the corresponding reduct of the *dfs* and non-*fam* type in Section 7C becomes a new example of a *fam* and non-*fim* complete type.

8A. ***Parametrized equivalence relations.*** Let T^*_{feq2} denote the model completion of the theory of parametrized equivalence relations in which each equivalence class has size 2. In [Adler et al. 2014, Example 1.7], it is claimed that this theory admits a generically stable (and thus *fim*) type $p \in S_1(\mathcal{U})$ such that $p^{(2)}$ is not generically stable, and this was elaborated on in [CG 2020, Section 5.1]. However, it turns out that the type p suggested in [Adler et al. 2014] is not well defined (see Remark 8.6 for details). Here we show that, in fact, there are no nontrivial *dfs* measures in T^*_{feq2}. We first recall some definitions. Let T be a complete $\mathcal{L}$-theory with monster model $\mathcal{U}$.

Definition 8.1. A measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ is *trivial* if it can be written as $\sum_{n=0}^{\infty} r_n \delta_{a_n}$, where $a_n \in \mathcal{U}^x$ and $r_n \in [0, 1]$, with $\sum_{n=0}^{\infty} r_n = 1$. We say that T is *dfs-trivial* if every *dfs* measure is trivial.

Note that a type $p \in S_x(\mathcal{U})$ is trivial if and only if it is realized in $\mathcal{U}^x$. It is clear that any trivial measure is *fim*. The following result was implicitly claimed in [CG 2020], but the proof used an unjustified assumption involving localization of measures, namely, [CG 2020, Remark 4.2]. In reality, the argument only requires a very weak version of this remark, which is easily proved. So we clarify the details.

Proposition 8.2. *A theory T is dfs-trivial if and only if for every x of length one and every dfs measure $\mu \in \mathfrak{M}_x(\mathcal{U})$, there is some $b \in \mathcal{U}$ such that $\mu(x = b) > 0$.*

Proof. As noted in the proof of [CG 2020, Proposition 4.3], the forward direction is trivial. For the reverse implication, assume that for every x of length one and every *dfs* measure $\mu \in \mathfrak{M}_x(\mathcal{U})$, there is some $b \in \mathcal{U}$ such that $\mu(x = b) > 0$. To show that T is *dfs*-trivial, it suffices by [CG 2020, Proposition 4.5], to show that for any x of length one, every *dfs* measure in $\mathfrak{M}_x(\mathcal{U})$ is trivial. (The cited result from [CG 2020] works in a one-sorted setting for simplicity; however the proof is by induction on the length of a tuple of variables, which need not be all in the same sort for the argument to work.) For this, it suffices by the proof of [CG 2020, Proposition 4.3], to show that for an arbitrary variable tuple $\bar{x}$, *dfs* measures in $\mathfrak{M}_{\bar{x}}(\mathcal{U})$ are closed under the following special case of localization. In particular, suppose $\mu \in \mathfrak{M}_{\bar{x}}(\mathcal{U})$ is *dfs*, and let $X = S_{\bar{x}}(\mathcal{U}) \setminus S$, where S is a fixed countable set $S \subseteq S_{\bar{x}}(\mathcal{U})$ of *realized* types, with $\mu(S) < 1$. Let μ_0 be the localization μ_0 of μ at X, i.e., $\mu_0(\varphi(\bar{x})) := \mu(\varphi(\bar{x}) \cap X)/\mu(X)$. Then we claim that μ_0 is *dfs*.

Fix $M \prec \mathcal{U}$ such that μ is *dfs* over M, and any type in S is realized in M. We show that μ_0 is *dfs* over M. We may assume that $\mu(S) > 0$, since otherwise

$\mu_0 = \mu$. It is clear that μ_0 is finitely satisfiable in M. Consider the trivial measure $\mu_1 = (1/\mu(S)) \sum_{\bar{a}\in S} \mu(\bar{x} = \bar{a})\delta_{\bar{a}}$, which is definable over M. Set $r := \mu(X)$, and notice that $\mu = r\mu_0 + (1-r)\mu_1$. So for any $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, we have $F^{\varphi}_{\mu_0,M} = (1/r)(F^{\varphi}_{\mu,M} - (1-r)F^{\varphi}_{\mu_1,M})$, and thus $F^{\varphi}_{\mu_0,M}$ is continuous since it can be written as a linear combination of continuous functions. Therefore μ_0 is definable over M. $\square$

Remark 8.3. For the sake of clarifying the literature, we note that Proposition 8.2 (and its proof) can be used in place of Remark 4.2 and Proposition 4.3 in [CG 2020] to recover the proofs of Theorems 4.8, 4.9, and 5.10(a) in [CG 2020]. The only other use of Remark 4.2 in [CG 2020] is in the proof of Theorem 5.10(b), which we address in the next subsection (see Theorem 8.10 and preceding discussion).

Next, we describe a way to lift a *dfs* measure to an imaginary sort. Suppose $E(x, y)$ is a definable equivalence relation on $\mathcal{U}$. We extend E to tuples from $\mathcal{U}^n$ in the obvious way. We view $\mathcal{U}/E$ as a structure in a relational language $\mathcal{L}_0$ such that for any E-invariant formula $\varphi(x_1, \dots, x_n)$, we have an n-ary relation symbol R_φ interpreted as $\varphi(\mathcal{U}^n)/E$. Note that any quantifier-free $\mathcal{L}_0$-formula is equivalent to R_φ for some equivariant φ.

Now suppose we have a *dfs* measure μ in $\mathfrak{M}_1(\mathcal{U})$. Given an E-invariant formula $\varphi(x; y_1, \dots, y_n)$ and $b_1, \dots, b_n \in \mathcal{U}/E$, define $\mu_0(R_\varphi(x; b_1, \dots, b_n)) = \mu(\varphi(x; b_1^*, \dots, b_n^*))$, where b_i^* is a representative of b_i in $\mathcal{U}$.

Proposition 8.4. *μ_0 is a dfs measure on quantifier-free $\mathcal{L}_0(\mathcal{U})$-formulas.*

Proof. First note that $\mu_0(R_\varphi(x; b_1, \dots, b_n))$ does not depend on the choice of representatives by E-invariance, and so μ_0 is well defined. From there one easily shows that μ_0 is a finitely additive probability measure on quantifier-free $\mathcal{L}_0(\mathcal{U})$-formulas.

Fix $M \prec \mathcal{U}$ such that μ is *dfs* over M. We show that μ_0 is *dfs* over M/E. First, fix some $R_\varphi(x; y_1, \dots, y_n)$ and $b_1, \dots, b_n \in \mathcal{U}/E$, with $\mu_0(R_\varphi(x; b_1, \dots, b_n)) > 0$. Then there is $a \in M$ such that $\mathcal{U} \models \varphi(a; b_1^*, \dots, b_n^*)$. Therefore $[a]_E \in M/E$ and $\mathcal{U}/E \models R_\varphi([a]_E; b_1, \dots, b_n)$.

Finally, fix an E-invariant formula $\varphi(x; y_1, \dots, y_n)$ and some $\varepsilon > 0$. Define

$$X = \{\bar{b} \in (\mathcal{U}/E)^n : \mu_0(R_\varphi(x; \bar{b})) \le \varepsilon\}.$$

Then $X = Y/E$, where $Y = \{\bar{b}^* \in \mathcal{U}^n : \mu(\varphi(x; \bar{b}^*)) \le \varepsilon\}$. Since μ is *dfs* over M, there is a small collection $\{\psi_i(\bar{y}; \bar{x}) : i \in I\}$ of Boolean combinations of $\varphi^*(\bar{y}; x_i)$, and tuples $\bar{a}_i^*$ from M, such that $Y = \bigcap_{i\in I} \psi_i(\mathcal{U}; \bar{a}_i^*)$. Let $\bar{a}_i = [\bar{a}_i^*]_E$. Then we have $X = \bigcap_{i\in I} R_{\psi_i}(\mathcal{U}/E; \bar{a}_i)$. Therefore μ_0 is definable over M. $\square$

We now return to T^*_{feq2}. This theory is in a two-sorted language $\mathcal{L}$ with sorts O and P, and a ternary relation $E_z(x, y)$ on $O_x \times O_y \times P_z$. Then T^*_{feq2} is the model completion of the $\mathcal{L}$-theory asserting that for any z, $E_z(x, y)$ is an equivalence relation in which all classes have size 2. We have quantifier elimination after expanding by the function $f : P \times O \to O$ such that, for any $z \in P$, the induced function $f_z : O \to O$ swaps the two elements in any E_z-class.

Theorem 8.5. *T^*_{feq2} is dfs-trivial.*

Proof. Suppose not. By Proposition 8.2, there is some *dfs* measure $\mu \in \mathfrak{M}_1(\mathcal{U})$ such that $\mu(x = b) = 0$ for all $b \in \mathcal{U}$. Fix a parameter $e \in P(\mathcal{U})$. Let $E(x, y)$ be the equivalence relation on $\mathcal{U}$ which coincides with $E_e(x, y)$ on $O(\mathcal{U})$ and equality on $P(\mathcal{U})$. Let μ_0 be the (quantifier-free) *dfs* measure induced on $\mathcal{U}/E$ as above. Then $\mu_0(x = b) = 0$ for any $b \in \mathcal{U}/E$. Indeed, if $b = [b^*]_E$ for some $b^* \in \mathcal{U}$ then, since $[b^*]_E$ is finite, we have $\mu_0(x = b) = \mu(E(x, b^*)) = 0$.

Finally, we show that the theory T_{rbg} of the random bipartite graph is a (strong) reduct of $\mathcal{U}/E$ using $\mathcal{L}_0$-formulas. Given this, we will obtain a contradiction to *dfs*-triviality of T_{rbg} (see [CG 2020, Theorem 4.9]). We work with T_{rbg} in the language of bipartite graphs with unary predicates U and V, and a binary relation R on $U \times V$. We interpret $U = O(\mathcal{U})/E$ and $V = P(\mathcal{U})\backslash\{e\}$ (note that both $O(x)$ and $P(x) \wedge x \neq e$ are E-invariant). We then interpret $R(x, y)$ on $U \times V$ as $R_\varphi(x, y) \wedge y \neq e$, where $\varphi(x, y)$ is the formula $f_e(x) = f_y(x)$. To check that $\varphi(x, y)$ is E-invariant, note that if $b \in \mathcal{P}(\mathcal{U})$ and $a, a' \in O(\mathcal{U})$ are distinct and E-equivalent, then

$$f_e(a) = f_b(a) \iff a' = f_b(a) \iff a = f_b(a') \iff f_e(a') = f_b(a').$$

Let us now verify that $(U, V; R) \models T_{\text{rbg}}$. First, fix finite disjoint sets $X, Y \subseteq U$. We want to find some $b \in V$ such that $R(x, b)$ holds for all $x \in X$ and $\neg R(y, b)$ holds for all $y \in Y$. Without loss of generality, assume $Y = \{[a_i] : i < n\}$, where n is even. Set $Z = \bigcup_{x \in X \cup Y} x \subseteq O(\mathcal{U})$. Then we have a well-defined partition $\mathcal{P} = X \cup \big\{\{a_i, a_j\} : \{i, j\} \in \mathcal{S}\big\} \cup \big\{\{f_e(a_{2i}), f_e(a_{2i+1})\} : i < \frac{1}{2}n\big\}$ of Z into two-element sets. So there is $b \in \mathcal{P}(\mathcal{U}) \setminus \{e\}$ such that E_b partitions Z according to $\mathcal{P}$. Then b satisfies the desired properties.

Now suppose $X, Y \subseteq \mathcal{P}(\mathcal{U})$ are finite and disjoint. We want to find some $a \in O(\mathcal{U})$ such that $R([a], x)$ holds for all $x \in X$ and $\neg R([a], y)$ holds for all $y \in Y$. Since $X \cup \{e\}$ is still disjoint from Y, there is $a \in O(\mathcal{U})$ such that $f_x(a) = f_y(a)$ for all $x, y \in X \cup \{e\}$, and $f_y(a) \neq f_e(a)$ for all $y \in Y$. Then a satisfies the desired properties. $\square$

Remark 8.6. In [CG 2020, Remark 5.2], it is claimed that any unary definable subset of the object sort O of T^*_{feq2} is finite or cofinite. This was used to justify the claim in [Adler et al. 2014, Example 1.7] that T^*_{feq2} admits a global generically stable type p such that $p^{(2)}$ is not generically stable. While p is not explicitly defined in [Adler et al. 2014], it is implied to be the unique nonalgebraic global type in O, the existence of which is equivalent to the remark from [CG 2020] described above. But this remark is false, e.g., consider instances of the formula $\varphi(x; y, z)$ given by $f_y(x) = f_z(x)$.

8B. *Henson graphs.* Fix $s \geq 3$ and let T_s denote the theory of the generic K_s-free graph, in the language with a binary relation symbol E. Let $\mathcal{U} \models T_s$ be a monster model. By quantifier elimination, we have a unique nonrealized type $p_E \in S_1(\mathcal{U})$ containing $\neg E(x, b)$ for all $b \in \mathcal{U}$. In [CG 2020], it is proved that p_E is *fam*, but not *fim*. The proof of *fam* was a combinatorial argument relying on growth rates of certain Ramsey numbers. On the other hand, the failure of *fim* for p_E is easy to see (modulo the equivalence of *fim* and generic stability for types), since one can clearly witness the order property for $E(x, y)$ using Morley sequences in p_E.

The first goal of this section is to show that p_E commutes with any invariant measure in T_s (as promised in Example 5.19). First, we state a well-known Borel–Cantelli-type result on finitely additive probability measures.

Fact 8.7. *For any $\varepsilon > 0$ and $k \geq 1$, there is some $\delta > 0$ and $n \geq 1$ such that the following holds. Let B be a Boolean algebra, and μ a finitely additive probability measure on B. Suppose $x_1, \ldots, x_n \in B$ and $\mu(x_i) \geq \varepsilon$ for all $i \in [n]$. Then there is a k-element set $I \subseteq [n]$ such that $\mu\big(\prod_{i \in I} x_i\big) \geq \delta$.*

A standard consequence of the previous fact is that if $\mu \in \mathfrak{M}_x(\mathcal{U})$ is an M-invariant measure (in any theory) and $\varphi(x)$ is an $\mathcal{L}_{\mathcal{U}}$-formula that forks over M, then $\mu(\varphi(x)) = 0$.

Lemma 8.8. *Suppose $\mu \in \mathfrak{M}_1(\mathcal{U})$ is invariant over $M \prec \mathcal{U}$. Then $\mu(E(x, a)) = 0$ for some/any $a \models p_E|_M$.*

Proof. Consider the formula $\varphi(x, y_1, \ldots, y_{s-1}) := \bigwedge_{i=1}^{s-1} E(x, y_i)$. Then, for any pairwise distinct $a_1, \ldots, a_{s-1} \models p_E|_M$, the formula $\varphi(x, \bar{a})$ forks over M by [Conant 2017, Corollary 4.8], and thus $\mu(\varphi(x, \bar{a})) = 0$. Now let $\varepsilon = \mu(E(x, a))$, where a is some/any realization of $p_E|_M$. Toward a contradiction, suppose $\varepsilon > 0$. Let $n \geq 1$ and $\delta > 0$ be as in Fact 8.7, with $k = s-1$. Choose pairwise distinct $a_1, \ldots, a_n \models p_E|_M$. Then there is an $(s-1)$-element set $I \subseteq [n]$ such that $\mu(\varphi(x, \bar{a}_I)) \geq \delta > 0$, which is a contradiction. $\square$

Corollary 8.9. *Suppose $\mu \in \mathfrak{M}_n(\mathcal{U})$ is invariant. Then $p_E \otimes \mu = \mu \otimes p_E$.*

Proof. Let $\nu_1 = p_E \otimes \mu$ and $\nu_2 = \mu \otimes p_E$. It suffices to show that ν_1 and ν_2 agree on formulas that are conjunctions of atomics and negated atomics. So fix such an $\mathcal{L}_{\mathcal{U}}$-formula $\varphi(x, \bar{y})$, where $|\bar{y}| = n$. Without loss of generality, $\varphi(x, \bar{y})$ is of the form

$$\bigwedge_{i=1}^{n} E^{\varepsilon_i}(x, y_i) \wedge \bigwedge_{i=1}^{n} (x =_i y_i) \wedge \psi(x) \wedge \theta(\bar{y}),$$

where $\varepsilon_i \in \{0, 1\}$, E^1 is E, E^0 is $\neg E$, $=_i$ is either $=$ or $\neq$, and $\psi(x) \wedge \theta(\bar{y})$ is some $\mathcal{L}_{\mathcal{U}}$-formula. Fix $M \prec \mathcal{U}$ such that $\varphi(x, \bar{y})$ is over M and μ is invariant over M.

Given $q \in S_{\bar{y}}(M)$, we have

$$F^{\varphi}_{p_E}(q) = \begin{cases} 0 & \text{if some } \varepsilon_i = 1, \text{ some } =_i \text{ is } =, \ \theta(\bar{y}) \notin q, \text{ or } \psi(x) \notin p, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore

$$\nu_1(\varphi(x, y)) = \begin{cases} 0 & \text{if some } \varepsilon_i = 1, \text{ some } =_i \text{ is } =, \text{ or } \psi(x) \notin p, \\ \mu(\theta(\bar{y})) & \text{otherwise.} \end{cases}$$

On the other hand, $\nu_2(\varphi(x, y)) = \mu(\varphi(a, \bar{y}))$, where $a \models p_E|_M$. If some $\varepsilon_i = 1$ then $\mu(\varphi(a, \bar{y})) = 0$ by Lemma 8.8. If some $=_i$ is $=$, then $\mu(\varphi(a, \bar{y})) = 0$ since $a \notin M$ and μ is M-invariant. If $\psi(x) \notin p$, then clearly $\mu(\varphi(a, \bar{y})) = 0$. So we may assume all ε_i are 0, all $=_i$ are $\neq$, and $\psi(x) \in p$. Since $\mu(\neg E(a, y_i)) = \mu(a \neq y_i) = 1$ for all i, we have $\mu(\varphi(a, \bar{y})) = \mu(\theta(\bar{y}))$. So $\nu_1(\varphi(x, \bar{y})) = \nu_2(\varphi(x, \bar{y}))$. □

In [CG 2020, Theorem 5.10], the first two authors made two more assertions about Keisler measures in T. First, it was claimed that a measure $\mu \in \mathfrak{M}_1(\mathcal{U})$ is *dfs* if and only if it is *fam*, and in this case μ is a convex combination of p_E and a trivial measure. Second, it is claimed that every *fim* measure in $\mathfrak{M}_1(\mathcal{U})$ is trivial. As indicated by Remark 8.3, both statements relied on an erroneous remark concerning localization of measures, and the first statement is easily recovered using the corrected proof of Proposition 8.2. On the other hand, the second result is more complicated, and so we take the opportunity here to provide a correct proof.

Theorem 8.10. *Any fim measure in $\mathfrak{M}_1(\mathcal{U})$ is trivial.*

Proof. Suppose $\mu \in \mathfrak{M}_1(\mathcal{U})$ is *fim*. By [CG 2020, Theorem 5.10(a)], we can write $\mu = rp_E + (1 - r)\nu$ for some $r \in [0, 1]$ and trivial $\nu \in \mathfrak{M}_1(\mathcal{U})$. Toward a contradiction, suppose $r > 0$. Set $\varepsilon = r/2$ and fix $n \geq r(1-r)/(\varepsilon^2(1-\varepsilon))$. Since μ is *fim*, there is a formula $\theta(x_1, \dots, x_n)$ such that $\mu^{(n)}(\theta(\bar{x})) \geq 1 - \varepsilon$ and if $\bar{a} \models \theta$ then $\mu \approx^E_\varepsilon \operatorname{Av}(\bar{a})$.

Fix $M \prec \mathcal{U}$ such that $\theta(\bar{x})$ is over M and ν is a weighted sum of Dirac measures at points in M. By quantifier elimination, we may write

$$\theta(\bar{x}) = \bigvee_{t=1}^{k} \psi_t(\bar{x}),$$

where each $\psi_t(\bar{x})$ is a consistent conjunction of atomic and negated atomics. Given $1 \leq t \leq k$, call a set $X \subseteq [n]$ *t-good* if $\psi_t(\bar{x})$ does not prove a formula of the $x_i = b$ for some $i \in X$ and $b \in M$, or of the form $E(x_i, x_j)$ for some $i, j \in X$.

Suppose first that, for some $1 \leq t \leq k$, we have a t-good set $X \subseteq [n]$ of size at least εn. Then we can find a realization $\bar{a} \models \theta(\bar{x})$ such that $a_i \notin M$ for all $i \in X$, and $\neg E(a_i, a_j)$ for all $i, j \in X$. We may then choose $b \in \mathcal{U}$ such that $E(a_i, b)$ holds for all $i \in X$ and $\neg E(b, m)$ holds for all $m \in M$. Note that $\mu(E(x, b)) = 0$. On the other hand, $\operatorname{Av}(\bar{a})(E(x, b)) \geq |X|/n \geq \varepsilon$, which contradicts the choice of $\theta(\bar{x})$.

So now we can assume that for all $1 \leq t \leq k$, any t-good set $X \subseteq [n]$ has size strictly less than εn. Given $X \subseteq [n]$, set

$$r_X = r^{|X|}(1-r)^{n-|X|} \quad \text{and} \quad \mu_X = \bigotimes_{i=1}^{n} \begin{Bmatrix} p_E, & i \in X \\ \nu, & i \notin X \end{Bmatrix}.$$

By Fact 6.1 and choice of n, we have

$$\begin{aligned} \mu^{(n)}(\theta(\bar{x})) = \sum_{X \subseteq [n]} r_X \mu_X(\theta(\bar{x})) &\leq \sum_{X \notin \mathcal{P}_{r,\varepsilon}(n)} r_X + \sum_{X \in \mathcal{P}_{r,\varepsilon}(n)} r_X \mu_X(\theta(\bar{x})) \\ &< 1 - \varepsilon + \sum_{X \in \mathcal{P}_{r,\varepsilon}(n)} r_X \mu_X(\theta(\bar{x})). \end{aligned}$$

Now fix $X \in \mathcal{P}_{r,\varepsilon}(n)$. Then $|X| \geq rn/2 = \varepsilon n$, and so X is not t-good for any $1 \leq t \leq k$. Fix $1 \leq t \leq k$. Then, for any t, either $\psi_t(\bar{x})$ contains a conjunct $x_i = b$ for some $i \in X$ and $b \in M$, or a conjunct $E(x_i, x_j)$ for some $i, j \in X$. In the first case we have $\mu_X(\psi_t(\bar{x})) \leq \mu_X(x_i = b) = p_E(x_i = b) = 0$; and in the second case we have $\mu_X(\psi_t(\bar{x})) \leq \mu_X(E(x_i, x_j)) = (p_E \otimes p_E)(E(x_i, x_j)) = 0$. Altogether, we have $\mu_X(\psi_t(\bar{x})) = 0$ for all $1 \leq t \leq k$, and so $\mu_X(\theta(\bar{x})) = 0$. By the inequalities above, it follows that $\mu^n(\theta(\bar{x})) < 1 - \varepsilon$, which contradicts the choice of $\theta(\bar{x})$. $\square$

8C. ***A new example of a fam and non-fim complete type.*** In this section, we show that a certain reduct of the *dfs* and non-*fam* type built in Section 7C is *fam* and non-*fim*. First, we prove a technical lemma regarding *fam* types in the presence of quantifier elimination.

Lemma 8.11. *Assume T has quantifier elimination, and fix $p \in S_x(\mathcal{U})$. Suppose there exists a sequence of tuples $(\bar{c}_n)_{n\in\omega}$ such that for every atomic formula $\theta(x, \bar{y})$ and every $\varepsilon > 0$ there exists $N(\varepsilon, \theta)$ so that for all $n > N(\varepsilon, \theta)$, $p \approx^{\theta}_{\varepsilon} \mathrm{Av}(\bar{c}_n)$. Then p is finitely approximated over any small model containing $(\bar{c}_n)_{n\in\omega}$.*

Proof. We first note that for any formula $\psi(x, \bar{y})$ and any tuple $\bar{a}$ of points in $\mathcal{U}^x$, we have $p \approx^{\psi}_{\varepsilon} \mathrm{Av}(\bar{a})$ if and only if $p \approx^{\neg\psi}_{\varepsilon} \mathrm{Av}(\bar{a})$. Let $\gamma(x, \bar{y}) = \bigwedge_{j\in J} \theta_j(x, \bar{y})$, where for each j, the formula $\theta_j(x, \bar{y})$ is either an atomic formula or the negation of an atomic formula. Fix $\varepsilon > 0$. For each $\theta_j(x, \bar{y})$, choose $N_j = N(\varepsilon/|J|, \theta_j)$ as in the statement of the lemma and fix $n > \max\{N_j : j \in J\}$. First, assume that $\gamma(x, \bar{b}) \in p$. For each $j \in J$, $\theta_j(x, \bar{b}) \in p$ and so

$$\begin{aligned} \mathrm{Av}(\bar{c}_n)(\gamma(x, \bar{b})) &= 1 - \mathrm{Av}(\bar{c}_n)\Big(\bigvee_{j\in J} \neg\theta_j(x, \bar{b})\Big) \\ &\geq 1 - \sum_{j\in J} \mathrm{Av}(\bar{c}_n)(\neg\theta_j(x, \bar{b})) \geq 1 - \sum_{j\in J} \frac{\varepsilon}{|J|} = 1 - \varepsilon. \end{aligned}$$

On the other hand, if $\neg\gamma(x, \bar{b}) \in p$, then there exists some fixed $j \in J$ such that $\theta_j(x, \bar{b}) \notin p$. Since $p \approx^{\theta_j}_{\varepsilon/|J|} \mathrm{Av}(\bar{c}_n)$, we have $\mathrm{Av}(\bar{c}_n)(\gamma(x, \bar{b})) \leq \mathrm{Av}(\bar{c}_n)(\theta_j(x, \bar{b})) \leq \varepsilon$.

Now assume that $\rho(x, \bar{y}) = \bigvee_{i \in I} \gamma_i(x, \bar{y})$, where each $\gamma_i(x, \bar{y})$ is as before (i.e., a conjunction of atomic and negated atomic formulas). By the previous paragraph, we can choose $m \in \omega$ so that for any $i \in I$, then $p \approx^{\gamma_i}_{\varepsilon/|I|} \mathrm{Av}(\bar{c}_m)$. First, assume that $\rho(x, \bar{b}) \in p$. Then there exists some fixed $i \in I$ so that $\gamma_i(x, b) \in p$. So $\mathrm{Av}(\bar{c}_m)(\rho(x, \bar{b})) \geq \mathrm{Av}(\bar{c}_m)(\gamma_i(x, \bar{b})) \geq 1 - \varepsilon$. Finally, assume that $\neg\rho(x, \bar{b}) \in p$. So for each $i \in I$, we have that $\neg\gamma_i(x, \bar{b}) \in p$. Then we have the computation

$$\mathrm{Av}(\bar{c}_m)(\rho(x, \bar{b})) \leq \sum_{i \in I} \mathrm{Av}(\bar{c}_m)(\gamma_i(x, \bar{b})) < |I|\left(\frac{\varepsilon}{|I|}\right) = \varepsilon. \qquad \square$$

Let $\mathcal{L}_{PQ}$ be the reduct of the language described in Section 7C to just the P and Q sort. Let $q_{PQ}(y)$ be the reduct of the type from Corollary 7.12 to the language $\mathcal{L}_{PQ}$. By Proposition A.15, the reduct $T_{PQ} := T^{\infty}_{1/2}|\mathcal{L}_{PQ}$ has quantifier elimination. From this it is not hard to show that $q_{PQ}(y)$ is axiomatized by the formulas

* $a \sqsubseteq y$ for every $a \in P(\mathcal{U})$,
* $y \neq \top$, and
* $y \not\sim b$ for every $b \in Q(\mathcal{U})$.

By quantifier elimination for T_{PQ}, and essentially the same arguments as in Section 7C, q_{PQ} determines a unique $\varnothing$-definable complete type, which is finitely satisfiable in any small model. However, we will now show that by dropping the measure sort, q_{PQ} in fact becomes *fam*, but is still not *fim*.

Proposition 8.12. *q_{PQ} is fam and not fim.*

Proof. We first show q_{PQ} is *fam*. Fix $n < \omega$. For each $i, j < n$, we let $d_{i,j} = \big(i, [j/n, (j+1)/n)\big) \in Q(M^{\infty}_{1/2})$ and define the tuple $\bar{c}_n = (d^{c}_{i,j})_{i,j<n}$. We show that $(\bar{c}_n)_{n<\omega}$ satisfies the conditions of Lemma 8.11 with respect to the type q_{PQ}. By quantifier elimination (Proposition A.15), we can then conclude that q_{PQ} is *fam*.

First, note that for any $a \in P(\mathcal{U})$, we clearly have $\mathrm{Av}(\bar{c}_n)(a \sqsubseteq y) = (n-1)/n$. So the conditions of Lemma 8.11 are satisfied for the atomic formula $x \sqsubseteq y$. Now consider an atomic formula of the form $t(\bar{x}, y) \asymp s(\bar{x}, y)$, where $\asymp$ is either $=$ or $\sim$ and t and s are terms in the Q sort. For any tuple, $\bar{b}$ of elements of $Q(\mathcal{U})$ it is not too hard to see that if d is an element of $Q(\mathcal{U}) \setminus \{\top, \bot\}$ that is not $\sim$-equivalent to any b_i, then $t(\bar{b}, d) \asymp s(\bar{b}, d)$ holds if and only if $t(\bar{b}, y) \asymp s(\bar{b}, y) \in q_{PQ}(y)$. Note that for any tuple $\bar{b}$ of elements of $Q(\mathcal{U})$ at most $|\bar{x}| = |\bar{b}|$ of the elements of $\bar{c}_n$ can be $\sim$-equivalent to some b_i. This implies that we always have that $\mathrm{Av}(\bar{c}_n)(t(\bar{b}, y) \asymp s(\bar{b}, y)) \geq (n - |\bar{x}|)/n$ if $q_{PQ}(y)$ satisfies $t(\bar{b}, y) \asymp s(\bar{b}, y)$ and $\mathrm{Av}(\bar{c}_n)(t(\bar{b}, y) \asymp s(\bar{b}, y)) \leq |\bar{x}|/n$ if $q_{PQ}(y)$ does not satisfy $t(\bar{b}, y) \asymp s(\bar{b}, y)$.

Finally, we show q_{PQ} is not *fim*. Recall that for types, *fim* is equivalent to generically stable (see [CG 2020, Section 3]). Let $(b_i)_{i<\omega}$ be a Morley sequence in $q_{PQ}(y)$ over some set of parameters. Then $b_i \not\sim b_j$ for all $i \neq j$. It follows (using compactness) that for any $I \subseteq \omega$, there is an $a_I \in P(\mathcal{U})$ such that for any $i < \omega$, $a_I \sqsubseteq b_i$ if and only if $i \in I$. Thus $q_{PQ}(y)$ is not generically stable and so not *fim*. $\square$

9. Concluding remarks

A recurring theme in the previous work is that, outside of NIP theories, the study of Keisler measures is much more complicated and requires confrontation with a greater amount of pure measure theory. We have also seen that much of the aberrant behavior involving Morley products and Borel definable measures can be found in a very straightforward simple unstable theory, namely, the random ternary relation (see Proposition A.8 for another example of bad behavior in this theory). This suggests that a coherent study of Keisler measures in simple theories may need to focus on very different questions, as compared to NIP theories. On the other hand, since our counterexamples were all built using a generic *ternary* relation, perhaps it is possible to recover some good behavior in the setting of 2-dependent theories (see [Chernikov et al. 2019] for the definition of k-dependence).

Question 9.1. Is the product of two Borel definable Keisler measures in a 2-dependent theory again Borel definable? If so, does associativity hold for Borel definable measures in 2-dependent theories?

Despite the bad news for Borel definability, our results on notions of "generic stability" for measures corroborate the philosophy of [CG 2020] that interesting results exist outside of NIP. For example, we have shown further evidence that *fim* measures are a sufficiently well-behaved class in general theories. Moreover, results such as the weak law of large numbers continue to be effective tools for studying *fim* measures outside of NIP. However, these developments are somewhat dampened by the fact that, while we have now found interesting and exotic *dfs* and *fam* measures in independent theories, there is a concerning dearth of examples of nontrivial *fim* measures. As for *dfs* and *fam*, our work in Section 5 shows that some nice behavior can be recovered, and several interesting open questions remain. In particular, our results further highlight the power of Keisler's original result on the existence of smooth extensions in NIP theories, and we have demonstrated that this phenomenon remains powerful without a global NIP assumption. More specifically, we have shown that several results about measures in NIP theories generalize to measures in arbitrary theories, as along as one assumes that the measures in question admit extensions with various properties exhibited by smooth measures.

Appendix

A1. ***Borel measures.*** Let (X, Σ) be a *measure space*, i.e., X is a set and Σ is a σ-algebra of subsets of X. A function $f : X \to [0, \infty)$ is Σ*-measurable* if $f^{-1}(U) \in \Sigma$

for all open $U \subseteq [0, \infty)$. A Σ-measurable function on X is Σ-*simple* if its image is finite.

Fact A.1. *If $f : X \to [0, \infty)$ is Σ-measurable then there is a sequence $(f_n)_{n=1}^{\infty}$ of Σ-simple functions converging pointwise to f. Moreover, $(f_n)_{n=1}^{\infty}$ converges uniformly to f on any subset of X for which f is bounded.*

Remark A.2. If $f : X \to [0, 1]$ is Σ-measurable, then in the previous fact one can take $f_n = \sum_{i=0}^{n-1}(i/n)\mathbf{1}_{B_i}$, where $B_i = f^{-1}\big((i/n, (i+1)/n]\big)$ for $0 \leq i \leq n-1$. Indeed, for any $n \geq 1$, we have $\|f - f_n\|_\infty \leq 1/n$.

Now assume X is a compact Hausdorff space and Σ is the σ-algebra of Borel subsets of X. Let μ be a *Borel measure* on X, i.e., a countably additive function $\mu : \Sigma \to [0, 1]$ such that $\mu(\varnothing) = 0$. We call μ a *Borel probability measure* if, moreover, $\mu(X) = 1$. Also, μ is called *regular* if, for any Borel set $B \subseteq X$,

$$\sup\{\mu(C) : C \subseteq B,\ C \text{ is closed}\} = \mu(B) = \inf\{\mu(U) : B \subseteq U,\ U \text{ is open}\}.$$

Given a continuous surjective map $\rho : X \to Y$, with Y compact Hausdorff, and a regular Borel measure μ on X, the *pushforward of μ along ρ* is the Borel measure ν on Y defined by $\nu(B) = \mu(\rho^{-1}(B))$ for any Borel $B \subseteq Y$.

Fact A.3. *Suppose $\rho : X \to Y$ is a continuous surjective function between compact Hausdorff spaces, and μ is a regular Borel probability measure on X. Then the pushforward of μ along ρ is regular.*

Finally, we note some facts about totally disconnected compact Hausdorff spaces.

Fact A.4. *Suppose X is a totally disconnected compact Hausdorff space, and let μ be a regular Borel probability measure on X.*

(a) *If $U \subseteq X$ is open, then $\mu(U) = \sup\{\mu(K) : K \subseteq U,\ K$ is clopen$\}$.*

(b) *If ν is a regular Borel probability measure on X, and $\nu(K) = \mu(K)$ for all clopen $K \subseteq X$, then $\mu = \nu$.*

Proof. Part (a) is straightforward. Part (b) follows from part (a) and regularity. □

A2. ***Measures on independent sets.*** The primary goal of this section is to construct the measure defined in the proof of Lemma 3.11. We take a somewhat broader approach of independent interest. Let B be a Boolean algebra, with join, meet, complement, top element, bottom element, and induced partial order denoted by $\sqcup$, $\sqcap$, ${}^{\mathsf{c}}$, $\top$, $\bot$, and $\sqsubseteq$, respectively. Given $X \subseteq B$, let $X^{\mathsf{c}} = \{x^{\mathsf{c}} : x \in X\}$, and assuming X is finite, let $\bigsqcap X = \bigsqcap_{x \in X} x$.

Definition A.5. A subset $F \subseteq B$ is *independent* if $\bigsqcap X \sqcap \bigsqcap Y^{\mathsf{c}} \neq \bot$ for any finite disjoint $X, Y \subseteq F$.

The following lemma is certainly well known, but we were unable to find a suitable reference.

Lemma A.6. *Suppose $F \subseteq B$ is independent, and let $f : F \to [0, 1]$ be a function. Then there is a finitely additive probability measure μ on B such that, for any finite disjoint $X, Y \subseteq F$,*

$$\mu\Big(\bigsqcap X \sqcap \bigsqcap Y^{\mathrm{c}}\Big) = \prod_{x\in X} f(x) \cdot \prod_{x\in Y}(1 - f(x)).$$

Proof. We first observe that we can reduce to the case that F is finite and generates B. Indeed, given a finite subset $E \subseteq F$, let B_E be the subalgebra generated by E. Suppose that for all finite $E \subseteq F$, we have a finitely additive probability measure μ_E on B_E satisfying the desired conclusion for all finite disjoint $X, Y \subseteq E$. We can extend each μ_E arbitrarily to some finitely additive probability measure μ^*_E on B (e.g., by [Łoś and Marczewski 1949]; see also [Starchenko 2017, Theorem 3.7]). Then $(\mu^*_E)_E$ is a net in the compact space of all finitely additive probability measures on B, and thus has a subnet converging to some measure μ with the desired properties.

So now assume F is finite and generates B. Let $n = |F|$. Since F is independent, B has 2^n atoms, which are precisely the elements of the form $a_X := \bigsqcap X \sqcap \bigsqcap(F\setminus X)^{\mathrm{c}}$ for $X \subseteq F$. A direct calculation show that the unique measure μ satisfying $\mu(a_X) = \prod_{x\in X} f(x) \cdot \prod_{x\in F\setminus X}(1 - f(x))$ has the desired properties.

Thus we can view B as the event space of the experiment of flipping n independent coins (identified with the elements of F). If we assign $x \in F$ the probability $f(x)$ of landing heads, then the resulting probability function is the desired finitely additive measure on B. □

Corollary A.7. *Let T be a complete theory with monster model $\mathcal{U}$, and suppose $\mathcal{F} \subseteq \mathrm{Def}_x(\mathcal{U})$ is independent. Then, for any $f : \mathcal{F} \to [0, 1]$, there is some $\mu \in \mathfrak{M}_x(\mathcal{U})$ such that, for any finite disjoint $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{F}$,*

$$\mu\Big(\bigcap_{A\in\mathcal{X}} A \cap \bigcap_{B\in\mathcal{Y}} \neg B\Big) = \prod_{A\in\mathcal{X}} f(A) \cdot \prod_{B\in\mathcal{Y}}(1 - f(B)).$$

Proof. This follows directly from Lemma A.6. □

We now return to the theory T_R of the random ternary relation R, defined in Section 3C. Let $\mathcal{U} \models T_R$ be a monster model. Let $\mathcal{F}$ be the collection of (positive) instances of R in one free variable. Then $\mathcal{F}$ is independent by the extension axioms for T_R. So we can apply Corollary A.7 with the constant $\frac{1}{2}$ function to obtain a measure $\lambda \in \mathfrak{M}_1(\mathcal{U})$ such that if $\theta_1(x), \ldots, \theta_n(x)$ are pairwise distinct (positive) instances of R in one free variable, and $\psi_i(x)$ is either $\theta_i(x)$ or $\neg\theta_i(x)$, then

$$\lambda(\psi_1(x) \wedge \cdots \wedge \psi_n(x)) = \frac{1}{2^n}.$$

This finishes the construction of the measure defined in the proof of Lemma 3.11.

We can use a similar construction to justify a claim made after Example 5.12. Fix any countably infinite set $A \subset \mathcal{U}$. Define $\nu \in \mathfrak{M}_1(\mathcal{U})$ in the same way as λ above, except start by insisting that any instance of R involving only parameters from A has ν-measure 0, and all other instances of R have ν-measure $\frac{1}{2}$. Now view ν as a measure in $\mathfrak{M}_y(\mathcal{U})$. Fix a Bernstein set $Z \subseteq S_{yz}(A)$, and define $p \in S_x(\mathcal{U})$ such that the positive instances of R in p are precisely those of the form $R(x, b, c)$, where $\operatorname{tp}(b, c/A) \in Z$. Note that p and ν are A-invariant.

Proposition A.8. *The type p is ν-measurable over A, but not ν-measurable over any proper extension $B \supset A$.*

Proof. Note that $\nu|_A$ coincides with the unique type in $S_y(A)$ that contains the negation of any instance of R involving y and parameters from A. Therefore p is ν-measurable over A since any $f : S_y(A) \to [0, 1]$ is $\nu|_A$-measurable.

Fix a proper extension $B \supset A$, and fix some $c \in B \setminus A$. Let $D = dp(R(x, y, c)) := \{q \in S_y(B) : R(x, b, c) \in p$ for some $b \models q\}$. We claim that D is not $\nu|_B$-measurable, and thus p is not ν-measurable over B. First, since $c \notin A$, and A is infinite, it follows that $\nu|_{Ac}$ is strongly continuous (as in the proof of Lemma 3.11). Let $\rho = \rho^y_{B,Ac}$ and let $f : S_y(Ac) \to S_{yz}(A)$ be the natural inclusion map. Then $\rho(D) = f^{-1}(Z)$, and $D = \rho^{-1}(\rho(D))$ (by A-invariance of p). So, by Lemma 3.7, it suffices to show that $f^{-1}(Z)$ is a Bernstein set in $S_y(Ac)$. To see this, note that $X := f(S_y(Ac))$ is a closed set in $S_{yz}(A)$, whence $Z \cap X$ is a Bernstein set in X, and so $f^{-1}(Z)$ is a Bernstein set in $S_y(Ac)$ as well. □

A3. *Quantifier elimination for $T^{\infty}_{1/2}$*. In this section, we prove that theory $T^{\infty}_{1/2}$ defined in Section 7C has quantifier elimination (this was stated in Theorem 7.10).

Recall that $\mathcal{H}$ is the Boolean algebra on $[0, 1)$ generated by sets of the form $[a, b)$ with $0 \leq a < b \leq 1$. Note that every element of $\mathcal{H}$ can be expressed as a (possibly empty) finite union of half-open intervals of this form, and therefore, in particular, $\mathcal{H}$ contains no singletons and is an atomless Boolean algebra. Recall also that λ denotes the Lebesgue measure on $[0, 1)$.

We start with the following easy fact.

Fact A.9. *For any $X \in \mathcal{H}$, any finite $A \subset X$, and any real number $r \in (0, \lambda(X))$, there is $Y \in \mathcal{H}$ such that $A \subset Y \subset X$ and $\lambda(Y) = r$.*

Lemma A.10. *For any finite set $B \subset Q(M^{\infty}_{1/2})$, the substructure of $Q(M^{\infty}_{1/2})$ generated by B is exhausted by elements of the form $\bigsqcup C$, where C is some set of elements of the form $\bigsqcap D$ for some $D \subseteq B \cup \{b^{\mathsf{c}} : b \in B\}$ (where $\sqcup\varnothing = \perp$ and $\sqcap\varnothing = \top$).*

In particular, for any set of variables $\bar{x}$ of sort Q, there is a fixed finite list of $\mathcal{L}_Q$-terms $\{t_i(\bar{x})\}_{i<n}$ such that for any $\bar{b} \in Q(M^{\infty}_{1/2})$, $\{t_i(\bar{b})\}_{i<n}$ exhausts the $\mathcal{L}_Q$-substructure of $Q(M^{\infty}_{1/2})$ generated by B.

Proof. The first statement is easy to check, so for the second statement we may take $\{t_i(\bar{x})\}_{i<n}$ to be an enumeration of all disjunctive normal form formal Boolean combinations of the variables $\bar{x}$. $\square$

Let $\mathcal{L}_{PQ}$ be the sublanguage of $\mathcal{L}$ obtained by removing the sort R and all associated symbols, and let T_{PQ} denote the reduct of $T^{\infty}_{1/2}$ to $\mathcal{L}_{PQ}$.

Corollary A.11. *For any $M \models T^{\infty}_{1/2}$, $Q(M)$ is locally finite with respect to $\mathcal{L}_{PQ}$ (i.e., every finite subset of $Q(M)$ generates a finite $\mathcal{L}_{PQ}$-substructure).*

Let M be a model of $T^{\infty}_{1/2}$. Then $\sqcap$ and $\sqcup$ are lattice operations on $Q(M)$. We will denote the induced partial order by $\sqsubseteq$. Furthermore, $\sim$ is an equivalence relation on $Q_0(M) := Q(M) \setminus \{\bot, \top\}$. Given $b \in Q_0(M)$, we set $[b]_\sim = \{c \in Q_0(M) : b \sim c\}$ and $[b]^*_\sim := [b]_\sim \cup \{\bot, \top\}$. We refer to $[b]_\sim$ as the *$\sim$-class* of b. Note that $([b]^*_\sim, \sqcap, \sqcup, {}^{\mathsf{c}}, \bot, \top)$ is an atomless Boolean algebra for any $b \in Q_0(M)$. We also emphasize that all of this notation depends implicitly on the ambient model M.

Definition A.12. Given a finite substructure $B \subset Q(M)$, an element $b \in B$ is *minimal* if it is not $\bot$ and is minimal with regards to the partial order $\sqsubseteq$.

Lemma A.13. *For any $M \models T^{\infty}_{1/2}$, any finite $\mathcal{L}_Q$-substructure $B \subset Q(M)$, and any $a \in P(M)$, there is, for each $\sim$-class C with representatives in B, a unique minimal $b \in B \cap C$ such that $a \sqsubset\!\!\!\!- b$.*

Proof. This is clearly a first-order property that holds in $M^{\infty}_{1/2}$. $\square$

We will need the following lemma and proposition for quantifier elimination of $T^{\infty}_{1/2}$.

Lemma A.14. *Let M be a model of T_{PQ}. Let $Y \subset P(M)$ be some finite set. For any $\sim$-class C of M, and any sequence $c_0, \dots, c_{n-1} \in C \cup \{\top\}$ with $c_i \sqcap c_j = \bot$ for each $i < j < n$ and with $\bigsqcup_{i<n} c_i = \top$, there exists a family $\{d_a\}_{a \in Y}$ of elements of C such that*

- *for any distinct $a, a' \in Y$, $d_a \sqcap d_{a'} = \bot$,*
- *for each $a \in Y$, $d_a \sqsubset c_i$ for some $i < n$,*
- *for each $i < n$, $\bigsqcup\{d_a : d_a \sqsubseteq c_i\} \sqsubset c_i$, and*
- *for each $a \in Y$, $a \sqsubset\!\!\!\!- d_a$.*

Proof. First note that for each fixed n, the stated property is equivalent to some sentence in $\mathcal{L}_{PQ}$, so it is sufficient to show that the property holds in M.

So fix a finite set $Y \subset P(M)$, a $\sim$-class C in M, and a finite sequence $c_0, \dots, c_{n-1}$ in C with $c_i \sqcap c_j = \bot$ for each $i < j < n$ and with $\bigsqcup_{i<n} c_i = \top$. Let $k \in \mathbb{N}$ be such that C is $\{k\} \times \mathcal{H}_0$.

For each $i < n$, let $Y_i = \{a \in Y : a \sqsubset\!\!\!\!- c_i\}$. Fix some $i < n$. Call an element of $[0, 1]$ a *special point of c_i* if it either is $a(k)$ for some in $a \in Y_i$ or is in one of the righthand endpoints of one of the constituent intervals in the subset of $[0, 1)$

determined by c_i (where if $c_i = \top$ then this is $[0, 1)$). Note that by our choice of $P(M)$, we have $a(k) \neq a'(k)$ for any distinct $a, a' \in Y_i$. So each element of Y_i corresponds to a unique special point of c_i.

For each $a \in Y_i$, let $d_a = \big(k, \big[a(k), \frac{1}{2}(a(k)+r)\big)\big)$, where r is the smallest special point of c_i strictly greater than $a(k)$. Since no element of Y_i yields a righthand endpoint interval in c_i, we clearly have that $d_a \sqsubset c_i$. We also clearly have that $a \sqsubseteq d_a$, that for any distinct $a, a' \in Y_i$, $d_a \sqcap d_{a'} = \bot$, and that $\bigsqcup\{d_a : d_a \sqsubseteq c_i\} = \bigsqcup_{a \in Y_i} d_a \sqsubset c_i$.

Since $d_a \sqsubset c_i$ for each $a \sqsubseteq c_i$, we have that for any distinct $a, a' \in Y$, $d_a \sqcap d_{a'} = \bot$, and so the family $\{d_a\}_{a \in Y}$ fulfills the requirements of the lemma. $\square$

Proposition A.15. *The theory T_{PQ} is ω-categorical and has quantifier elimination.*

Proof. Let M_0 and M_1 be two countable models of T_{PQ}. Let (Y_0, C_0) and (Y_1, C_1) be isomorphic finite substructures of M_0 and M_1, respectively, and let $f : (Y_0, C_0) \to (Y_1, C_1)$ be a fixed isomorphism. All we need to show is that

- $*$ for any $a_0 \in P(M_0)$, there exists an $a_1 \in P(M_1)$ such that f extends to an isomorphism from $(Y_0 a_0, C_0)$ to $(Y_1 a_1, C_1)$ sending a_0 to a_1, and
- $*$ for any $b_0 \in Q(M_0)$, there exists a $b_1 \in Q(M_1)$ such that f extends to an isomorphism from $(Y_0, \langle C_0 b_0 \rangle)$ to $(Y_1, \langle C_1 b_1 \rangle)$ sending b_0 to b_1.

Note that the same follows with 0 and 1 switched by symmetry. Also note that since we are only assuming that f is an isomorphism, rather than an elementary map, this is sufficient to show quantifier elimination.

Suppose we have (Y_0, C_0), (Y_1, C_1), f, and $a_0 \in P(M_0)$. Assume $a_0 \notin Y_0$. By Lemma A.13, for each $\sim$-class D in C_0, there is a unique minimal $b_D \in D \subset C_0$ such that $a_0 \sqsubseteq b_D$. The elements $f(b_D)$ (for each $\sim$-class D that intersects C_0) are pairwise $\sim$-inequivalent. $T^{\infty}_{1/2}$ says that for any finite set X of pairwise $\sim$-inequivalent elements of $\mathcal{Q}_0$, there are infinitely many elements of P which are $\sqsubseteq$-in each element of X. Therefore we may find $a_1 \in P(M_1)$ that is $\sqsubseteq$-in each $f(b_D)$ and that is not in Y_1, and we get that the obvious extension of f from $(Y_0 a_0, C_0)$ to $(Y_1 a_1, C_1)$ is an isomorphism.

Suppose we have (Y_0, C_0), (Y_1, C_1), f, and $b_0 \in Q(M_0)$. Assume that $b_0 \notin C_0$. There are two cases: either b_0 is $\sim$-equivalent to some element of C_0 or it is not. If b_0 is not $\sim$-equivalent to an element of C_0, then we may choose $b_1 \in \mathcal{Q}_0$ so that b_1 is not $\sim$-equivalent to any element of C_1 and, for $a \in Y_0$, we have $a \sqsubseteq b_0$ if and only if $f(a) \sqsubseteq b_1$. Then f extends to an isomorphism from $(Y_0, \langle C_0 b_0 \rangle)$ to $(Y_1, \langle C_1 b_1 \rangle)$ sending b_0 to b_1. Now suppose b_0 is $\sim$-equivalent to some element of C_0. Let E be the set of minimal elements of $[b_0]_\sim \cap C_0$. For each $e \in E$ and $a \in Y_0$ with $a \sqsubseteq e$, one of the following cases holds:

- $*$ $b_0 \sqcap e = \bot$ and $a \not\sqsubseteq b_0 \sqcap e$,
- $*$ $\bot \sqsubset b_0 \sqcap e \sqsubset e$ and $a \not\sqsubseteq b_0 \sqcap e$,

* $\bot \sqsubset b_0 \sqcap e \sqsubset e$ and $a \sqsubseteq b_0 \sqcap e$, or
* $b_0 \sqcap e = e$ and $a \sqsubseteq b_0 \sqcap e$.

Let $E = \{c_0, \ldots, c_{n-1}\}$. The list $\{f(c_0), \ldots, f(c_{n-1})\}$ satisfies the requirements of Lemma A.14, so we can apply with Y_1 to get a family $\{d_x\}_{x \in Y_1}$ satisfying that

* for any distinct $x, x' \in Y_1,\ d_x \sqcap d_{x'} = \bot$,
* for each $x \in Y_1,\ d_x \sqsubset f(c_i)$ for some $i < n$,
* for each $i < n,\ \bigsqcup\{d_x : d_x \sqsubseteq f(c_i)\} \sqsubset f(c_i)$, and
* for each $x \in Y_1,\ x \sqsubseteq d_x$.

Given the third condition, we can also clearly find, for each $i < n$, an element h_i satisfying that $\bot \sqsubset h_i \sqsubset f(c_i) \sqcap \big(\bigsqcup\{d_x : d_x \sqsubseteq f(c_i)\}\big)^{\mathsf{c}}$. In particular, each h_i satisfies $x \not\sqsubseteq h_i$ for all $x \in Y_1$.

Now let

$$b_1 = \bigsqcup_{i<n} \bigsqcup_{\substack{a \in Y_0, \\ a \sqsubseteq c_i}} \begin{cases} \bot & \text{if } b_0 \sqcap c_i = \bot, \\ h_i & \text{if } \bot \sqsubset b_0 \sqcap c_i \sqsubset c_i \text{ and } a \not\sqsubseteq b_0 \sqcap c_i, \\ d_{f(a)} & \text{if } \bot \sqsubset b_0 \sqcap c_i \sqsubset c_i \text{ and } a \sqsubseteq b_0 \sqcap c_i, \\ f(c_i) & \text{if } b_0 \sqcap c_i = c_i. \end{cases}$$

By construction, we now have that for any $e \in E$ and $a \in Y_0$ with $a \sqsubseteq e$,

* $b_1 \sqcap f(e) = \bot$ and $f(a) \not\sqsubseteq b_1$ if and only if $b_0 \sqcap e = \bot$ and $a \not\sqsubseteq b_0$,
* $\bot \sqsubset b_1 \sqcap f(e) \sqsubset f(e)$ and $f(a) \not\sqsubseteq b_1$ if and only if $\bot \sqsubset b_0 \sqcap e \sqsubset e$ and $a \not\sqsubseteq b_0$,
* $\bot \sqsubset b_1 \sqcap f(e) \sqsubset f(e)$ and $f(a) \sqsubseteq b_1$ if and only if $\bot \sqsubset b_0 \sqcap e \sqsubset e$ and $a \sqsubseteq b_0$,
* $b_1 \sqcap f(e) = f(e)$ and $f(a) \sqsubseteq b_1$ if and only if $b_0 \sqcap e = e$ and $a \sqsubseteq b_0$.

These conditions are enough to imply that f extends (uniquely) to an isomorphism from $(Y_0, \langle C_0 b_0 \rangle)$ to $(Y_1, \langle C_1 b_1 \rangle)$ sending b_0 to b_1. □

Fact A.16. *The reduct of $T^{\infty}_{1/2}$ to the sort R is the theory of ordered divisible abelian groups with a nonzero constant. This theory has quantifier elimination.*

We can now prove Theorem 7.10 ($T^{\infty}_{1/2}$ has quantifier elimination).

Proof of Theorem 7.10. For any finite tuples of variables $\bar{x}$ of sort P and $\bar{y}$ of sort Q and any type $p(\bar{x}, \bar{y}) \in S_{\bar{x}\bar{y}}(T_{PQ})$, let $\tau_p(\bar{x}, \bar{y})$ be some fixed quantifier-free formula isolating p (note that τ_p exists by Proposition A.15).

Quantifier elimination can be established by showing that for any tuples of variables $\bar{x}$ of sort P, $\bar{y}$ of sort Q, and $\bar{z}$ of sort R and for any formula of the form

$$\psi(\bar{x}, \bar{y}, \bar{z}) := \tau_p(\bar{x}, \bar{y}) \wedge \varphi\big(\ell(t_0(\bar{y})), \ell(t_1(\bar{y})), \ldots, \ell(t_{k-1}(\bar{y})), \bar{z}\big),$$

each of $\exists x_0 \psi$, $\exists y_0 \psi$, and $\exists z_0 \psi$ is equivalent to a quantifier-free formula, where

* $p \in S_{\bar{x}\bar{y}}(T_{PQ})$ is some type (which, by Proposition A.15, we may view as a complete quantifier-free type in $\mathcal{L}_{PQ}$),
* $\varphi(\bar{w}, \bar{z})$ is a quantifier-free atomic $\mathcal{L}_R$-formula, and
* $t_j(\bar{y})$ is an $\mathcal{L}_Q$-term for each $j < k$.

(To see that this is sufficient, note that every quantifier-free formula is logically equivalent to a disjunction of formulas of the same form as $\psi(\bar{x}, \bar{y}, \bar{z})$.)

Let $\bar{\ell}$ be shorthand for the tuple $\big(\ell(t_0(\bar{y})), \ell(t_1(\bar{y})), \dots, \ell(t_{k-1}(\bar{y}))\big)$. (So we will write φ as $\varphi(\bar{\ell}, \bar{z})$.) Let $\bar{x}_*$ be $\bar{x}$ without x_0, and let $\bar{y}_*$ and $\bar{z}_*$ be defined similarly.

Eliminating P quantifiers. Consider $\exists x_0 \psi(\bar{x}, \bar{y}, \bar{z})$. Because φ does not actually contain x_0, $\exists x_0 \psi(\bar{x}, \bar{y}, \bar{z})$ is logically equivalent to $\varphi(\bar{\ell}, \bar{z}) \wedge \exists x_0 \tau_p(\bar{x}, \bar{y})$. By quantifier elimination for T_{PQ}, this is equivalent to $\varphi(\bar{\ell}, \bar{z}) \wedge \tau_{p|\bar{x}_*\bar{y}}(\bar{x}_*, \bar{y})$, which is quantifier-free.

Reducing Q quantifiers to R quantifiers. The type $p(\bar{x}, \bar{y})$ fully determines the $\mathcal{L}_{PQ}$-isomorphism type of the substructure of PQ generated by $\bar{x}\bar{y}$. Let E be a set of terms $s(\bar{y}_*)$ corresponding to the minimal elements of the $\mathcal{L}_Q$-substructure of Q generated by $\bar{y}_*$. (These terms exist for any given type p by Lemma A.10.) Without loss of generality, assume that y_0 is not $\top$ or $\bot$ (modulo p). Let $E_\sim$ be the set of terms in E that are $\sim$-equivalent to y_0 (modulo p) and let $E_{\not\sim}$ be the set of those that are not $\sim$-equivalent to y_0 (modulo p).

Claim. *In ψ, we may assume that each $t_i(\bar{y})$ is either $s(\bar{y}_*)$ for some $s \in E_{\not\sim}$ or $s(\bar{y}_*) \sqcap y_0$ or $s(\bar{y}_*) \sqcap y_0^{\mathsf{c}}$ for some $s \in E_\sim$.*

Proof. For every term of the form $\ell(t_j(\bar{y}))$, either

* there is some $E_0 \subseteq E_{\not\sim}$ such that $\ell(t_j(\bar{y})) = \sum_{s \in E_0} \ell(s(\bar{y}_*))$, or
* there are some $E_1, E_2 \subseteq E_\sim$ such that

$$\ell(t_j(\bar{y})) = \sum_{s \in E_1} \ell(s(\bar{y}_*) \sqcap y_0) + \sum_{s \in E_2} \ell(s(\bar{y}_*) \sqcap y_0^c)$$

(modulo p). By substituting these expressions into φ, we get the claim. $\dashv_{\text{claim}}$

In light of the claim, we will split $\bar{\ell}$ into three subtuples $\bar{\ell}^{\not\sim}$, $\bar{\ell}^{\sim}_{\sqcap}$, and $\bar{\ell}^{\sim}_{\mathsf{c}}$, where

* $\bar{\ell}^{\not\sim}$ is a list of all terms of the form $\ell(s(\bar{y}_*))$ for $s \in E_{\not\sim}$,
* $\bar{\ell}^{\sim}_{\sqcap}$ is a list of all terms of the form $\ell(s(\bar{y}_*) \sqcap y_0)$ for $s \in E_\sim$, and
* $\bar{\ell}^{\sim}_{\mathsf{c}}$ is a list of all terms of the form $\ell(s(\bar{y}_*) \sqcap y_0^{\mathsf{c}})$ for $s \in E_\sim$ (in the same order).

So now we will think of φ as $\varphi(\bar{\ell}^{\not\sim}, \bar{\ell}^{\sim}_{\sqcap}, \bar{\ell}^{\sim}_{\mathsf{c}}, \bar{z})$. It will also be useful to have the notation $\bar{\ell}^{\sim}$ for a list of all terms of the form $\ell(s(\bar{y}_*))$ for $s \in E_\sim$ (also in the same order). Note that $\bar{\ell}^{\not\sim}$ and $\bar{\ell}^{\sim}$ do not contain the variable y_0.

The core idea for reducing the quantifier $\exists y_0$ to some quantifiers in the R sort is that once one fixes $\bar{a}$ in P and $\bar{b}$ in Q satisfying $p|_{\bar{x}\bar{y}_*}$ as well as some $\bar{c}$ in R, the existence of some d such that $\psi(\bar{a}, d\bar{b}, \bar{c})$ holds depends *only* on the existence of some values $\{m_s\}_{s \in E_\sim}$ for $\ell(s(\bar{b}) \sqcap d)$ which are consistent with the existing measures of elements of E as well as whatever requirements are imposed by the formula φ. In order to be consistent with the existing measures of elements of E it is necessary and sufficient that for each $s \in E_\sim$,

* if p requires that $s(\bar{y}_*) \sqcap y_0 = \bot$, then $m_s = 0$,
* if p requires that $\bot \sqsubset s(\bar{y}_*) \sqcap y_0 \sqsubset s(\bar{y}_*)$, then $0 < m_s < \ell(s(\bar{y}_*))$, and
* if p requires that $s(\bar{y}_*) \sqcap y_0 = s(\bar{y}_*)$, then $m_s = \ell(s(\bar{y}_*))$.

Let $\bar{m}$ and $\bar{u}$ be two new tuples of R-variables in the same order as $\bar{\ell}^\sim$ (and so also in the same order as $\bar{\ell}^\sim_\sqcap$ and $\bar{\ell}^\sim_\mathsf{c}$). Rather than writing literal numerical indices for $\bar{m}$ and $\bar{u}$, we will write expressions such as m_s to mean the variable in $\bar{m}$ in the same position as $\ell(s(\bar{y}_*))$ in $\bar{\ell}^\sim$. We need a formula $\eta(\bar{m}, \bar{u})$ expressing these compatibility requirements. So, to accomplish this, let

$$\eta(\bar{m}, \bar{u}) := \bigwedge_{s \in E_\sim} \begin{cases} m_s = 0, & p(\bar{x}, \bar{y}) \models s(\bar{y}_*) \sqcap y_0 = \bot, \\ 0 < m_s < u_s, & p(\bar{x}, \bar{y}) \models \bot \sqsubset s(\bar{y}_*) \sqcap y_0 \sqsubset s(\bar{y}_*), \\ m_s = u_s, & p(\bar{x}, \bar{y}) \models s(\bar{y}_*) \sqcap y_0 = s(\bar{y}_*). \end{cases}$$

It's easy to see that the compatibility condition for $\bar{m}$ is equivalent to $\eta(\bar{m}, \bar{\ell}^\sim)$.

Now we can reduce the $\exists y_0$ quantifier to $\exists \bar{m}$. Consider the formula

$$\chi(\bar{x}, \bar{y}_*, \bar{z}) := \tau_{p|\bar{x}\bar{y}_*}(\bar{x}, \bar{y}_*) \wedge \exists \bar{m}\big[\eta(\bar{m}, \bar{\ell}^\sim) \wedge \varphi(\bar{\ell}^{\not\sim}, \bar{m}, \bar{\ell}^\sim - \bar{m}, \bar{z})\big],$$

where $\bar{\ell}^\sim - \bar{m}$ is the tuple whose elements are $\ell(s(\bar{y}_*)) - m_s$ for $s \in E_\sim$.

Claim. *$\chi(\bar{x}, \bar{y}_*, \bar{z})$ is logically equivalent to $\exists y_0 \psi(\bar{x}, \bar{y}, \bar{z})$.*

Proof. Fix $\bar{a}$ in P, $\bar{b}$ in Q, and $\bar{c}$ in R.

($\Leftarrow$) Suppose that there exists some d in Q such that $\psi(\bar{a}, d\bar{b}, \bar{c})$ holds. Clearly we have that $\bar{a}\bar{b} \models p|_{\bar{x}\bar{y}_*}$, so $\tau_{p|\bar{x}\bar{y}_*}(\bar{a}, \bar{b})$ holds. By setting m_s equal to $\ell(s(\bar{b}) \sqcap d)$ for each $s \in E_\sim$, we get that the second part of $\chi(\bar{a}, \bar{b}, \bar{c})$ holds. (Noting that $\ell(s(\bar{b}) \sqcap d^\mathsf{c}) = \ell(s(\bar{b})) - \ell(s(\bar{b}) \sqcap d)$.) Therefore $\chi(\bar{a}, \bar{b}, \bar{c})$ holds.

($\Rightarrow$) Suppose that $\chi(\bar{a}, \bar{b}, \bar{c})$ holds. Let $\bar{e}$ be the tuple of elements of R witnessing the $\exists \bar{m}$ quantifier. For each $s \in E_\sim$, choose f_s such that

* if p requires that $s(\bar{y}_*) \sqcap y_0 = \bot$, then $f_s = \bot$,
* if p requires that $\bot \sqsubset s(\bar{y}_*) \sqcap y_0 \sqsubset s(\bar{y}_*)$, then f_s is some element satisfying $\bot \sqsubset f_s \sqsubset s(\bar{b})$, $\ell(f_s) = e_s$, and $a_i \sqsubseteq f_s$ if and only if $p \models x_i \sqsubseteq y_0$, and
* if p requires that $s(\bar{y}_*) \sqcap y_0 = s(\bar{y}_*)$, then $f_s = s(\bar{b})$.

This is always possible for each $s \in E_{\sim}$ since $\eta(\bar{e}, \bar{\ell}^{\sim}(\bar{b}))$ holds and since the theory $T^{\infty}_{1/2}$ entails that, for any $y \in Q$, any disjoint finite sets $P_{\sqsubseteq}$ and $P_{\not\sqsubseteq}$ of elements of P, and any $m \in (0, \ell(z))$, there exists a $z \in Q$ with $\bot \sqsubset z \sqsubset y$ and $\ell(z) = m$ such that $g \sqsubseteq z$ for every $g \in P_{\sqsubseteq}$ and $g \not\sqsubseteq z$ for every $g \in P_{\not\sqsubseteq}$. (Note that this is a family of first-order statements that hold in $M^{\infty}_{1/2}$.) Finally, let

$$d = \bigsqcup_{s \in E_{\sim}} f_s.$$

By quantifier elimination for T_{PQ}, we have that $\bar{a}d\bar{b} \models p$ (where $\bar{a}$ is assigned to $\bar{x}$ and $d\bar{b}$ to $\bar{y}$), so $\tau_p(\bar{a}, d\bar{b})$ holds. Since $\varphi\big(\bar{\ell}^{\not\sim}(\bar{b}), \bar{e}, \bar{\ell}^{\sim}(\bar{b}) - \bar{e}, \bar{c}\big)$ holds and since $\bar{e} = \bar{\ell}^{\sim}_{\sqcap}(\bar{b}, d)$, we have that $\psi(\bar{a}, d\bar{b}, \bar{c})$ holds, whence $\exists y_0 \psi(\bar{a}, \bar{b}, \bar{c})$ holds. $\dashv_{\text{claim}}$

Therefore, once we can show that we can eliminate quantifiers of sort R, we will have shown that we can eliminate quantifiers of sort Q.

Eliminating R quantifiers. The formula $\exists z_0 \psi(\bar{x}, \bar{y}, \bar{z})$ is logically equivalent to $\tau_p(\bar{x}, \bar{y}) \wedge \exists z_0 \varphi(\bar{\ell}, \bar{z})$. By quantifier elimination for T_R, $\exists z_0 \varphi(\bar{v}, \bar{z})$ (which is an $\mathcal{L}_R$-formula) is logically equivalent to some $\mathcal{L}_R$-formula $\theta(\bar{v}, \bar{z}_*)$. Therefore we have that $\exists z_0 \psi(\bar{x}, \bar{y}, \bar{z})$ is logically equivalent to $\tau_p(\bar{x}, \bar{y}) \wedge \theta(\bar{\ell}, \bar{z}_*)$, which is a quantifier-free formula.

Altogether, because we can reduce Q-quantifiers to R-quantifiers and eliminate P- and R-quantifiers, we can eliminate quantifiers in general, and $T^{\infty}_{1/2}$ admits quantifier elimination. □

Remark A.17. As an aside, quantifier elimination for $T^{\infty}_{1/2}$ implies that the R sort is stably embedded and that any types $p(x)$ in the P sort and $r(y)$ in the R sort are weakly orthogonal (i.e., $p(x) \cup r(y)$ axiomatizes a complete type).

A4. *Heirs of definable measures.* The purpose of this section is to discuss definability for Keisler measures over small models, and prove that such definable measures have definable global extensions. This material is known in the folklore, especially from the perspective of continuous logic (see Remark A.25). See also [Hrushovski et al. 2008, Remark 2.7] and [Starchenko 2017, Remark 3.20]. However, since a complete account does not appear in the literature, we take the opportunity in this appendix to provide complete definitions and some details of various proofs.

Let T be a complete theory with monster model $\mathcal{U}$. Throughout this section, we fix a model $M \preceq \mathcal{U}$ and an arbitrary parameter set $A \subseteq M$.

Definition A.18. Given $\mu \in \mathfrak{M}_x(\mathcal{U})$ and an $\mathcal{L}$-formula $\varphi(x, y)$, define the map $f^{\varphi}_{\mu} : M^y \to [0, 1]$ such that $f^{\varphi}_{\mu}(b) = \mu(\varphi(x, b))$.

We view M^x as a dense subset of $S_x(M)$ by identifying $a \in M^x$ with the isolated type $\text{tp}(a/M)$. A function $f : S_x(M) \to [0, 1]$ is called *A-invariant* if $f(p) = f(q)$ whenever $p|_A = q|_A$.

Fact A.19. *Given a measure $\mu \in \mathfrak{M}_x(M)$ and an $\mathcal{L}$-formula $\varphi(x, y)$, the following are equivalent*:

(i) f^{φ}_{μ} *extends to an A-invariant continuous function from $S_y(M)$ to* $[0, 1]$.

(ii) *For any $\varepsilon > 0$, there are $\mathcal{L}_A$-formulas $\psi_1(y), \dots, \psi_n(y)$ such that*

- $*$ $\{\psi_i(y) : 1 \leq i \leq n\}$ *is a partition of M^y, and*
- $*$ *for any $1 \leq i \leq n$, if $b_1, b_2 \in \psi_i(M)$ then $|f^{\varphi}_{\mu}(b_1) - f^{\varphi}_{\mu}(b_2)| < \varepsilon$.*

(iii) *For any $\varepsilon < \delta$ in $[0, 1]$, there is an $\mathcal{L}_A$-formula $\psi(y)$ such that*

$$\{b \in M^y : f^{\varphi}_{\mu}(b) \leq \varepsilon\} \subseteq \psi(M) \subseteq \{b \in M^y : f^{\varphi}_{\mu}(b) < \delta\}.$$

Proof. This is a standard exercise in topology, which is similar to Fact 2.15 and could be phrased entirely for functions on arbitrary Stone spaces. The direction requiring the most work is (ii) $\Rightarrow$ (i). So we note that this task can be simplified using *Taĭmanov's theorem*, which is a classical result that characterizes when a function on a dense subset of a space X can be extended to a continuous function on X. See [Blair 1976] for details. $\square$

Definition A.20. A measure $\mu \in \mathfrak{M}_x(M)$ is *A-definable* if, for any $\mathcal{L}$-formula $\varphi(x, y)$, the map f^{φ}_{μ} satisfies the equivalent properties in Fact A.19.

The previous definition appears also in [Starchenko 2017, Definition 3.19] (using characterization (ii) of Fact A.19). Note that this definition does not conflict with the formulation of definability for global measures. In particular, if we take M to be the monster $\mathcal{U}$, then Fact A.19 aligns with Fact 2.15.

Remark A.21. Let f be a map from M^x to a compact Hausdorff space X. Then f is called *A-definable* if for any closed $C \subseteq X$ and open $U \subseteq X$, with $C \subseteq U$, there is some A-definable set $D \subseteq M^x$ such that $f^{-1}(C) \subseteq D \subseteq f^{-1}(U)$. In particular, condition (ii) of Fact A.19 is equivalent to A-definability of f^{φ}_{μ}.

Remark A.22. Suppose T has quantifier elimination in the language $\mathcal{L}$. Then $\mu \in \mathfrak{M}_x(M)$ is A-definable if and only if the equivalent properties of Fact A.19 hold for any $\mathcal{L}$-formula $\varphi(x, y)$, which is a conjunction of atomic and negated atomic $\mathcal{L}$-formulas. Indeed, if every formula $\varphi(x, y)$ of the described form satisfies condition (i) of Fact A.19, then so does every quantifier-free $\mathcal{L}$-formula by inclusion–exclusion.

The main result of this section says that definable global "heirs" of definable measures exist and are unique. The uniqueness aspect is a consequence of the following observation, which also makes explicit the analogy to heirs of types.

Remark A.23. Fix $\mu \in \mathfrak{M}_x(M)$, and suppose that $\hat{\mu} \in \mathfrak{M}_x(\mathcal{U})$ is an A-definable extension of μ. Then, for any $\mathcal{L}$-formula $\varphi(x, y)$, $F^{\varphi}_{\hat{\mu},M}$ is an A-invariant continuous extension of f^{φ}_{μ}. Therefore μ is A-definable. Moreover, for any $\mathcal{L}$-formula $\varphi(x, y)$

and any open set $U \subseteq [0,1]$, if $b \in \mathcal{U}^y$ and $\hat{\mu}(\varphi(x,b)) \in U$, then there is some $\psi(y) \in \mathrm{tp}(b/A)$ such that $\hat{\mu}(\varphi(x,c)) \in U$ for any $c \in \psi(\mathcal{U})$ (so, in particular, $\mu(\varphi(x,c)) \in U$ for some $c \in M^y$). It follows that $\hat{\mu}$ is the unique A-definable global extension of μ.

Theorem A.24. *Suppose that $\mu \in \mathfrak{M}_x(M)$ is A-definable. Then μ has a unique A-definable extension $\hat{\mu} \in \mathfrak{M}_x(\mathcal{U})$.*

Proof. By Remark A.23, it suffices to just show that $\hat{\mu}$ exists. Given an $\mathcal{L}$-formula $\varphi(x,y)$ and $b \in M^y$, define

$$\hat{\mu}(\varphi(x,b)) = \hat{f}^{\varphi}_{\mu}(\mathrm{tp}(b/M)),$$

where $\hat{f}^{\varphi}_{\mu}$ is the continuous A-invariant extension of f^{φ}_{μ} to $S_y(M)$. Assuming $\hat{\mu}$ is a well-defined Keisler measure, it follows that $\hat{\mu}$ is an A-invariant global extension of μ, and $F^{\varphi}_{\hat{\mu},M} = \hat{f}^{\varphi}_{\mu}$ for any $\mathcal{L}$-formula $\varphi(x,y)$. In particular, $\hat{\mu}$ is A-definable.

To show that $\hat{\mu}$ is well defined, we need to verify that any inconsistent $\mathcal{L}_{\mathcal{U}}$-formula has measure 0, and that finite additivity holds. So first fix an $\mathcal{L}$-formula $\varphi(x,y)$ and $b \in M^y$ such that $\varphi(x,b)$ is inconsistent. Then $\mathrm{tp}(b/M)$ is in the clopen set $C := [\forall x \neg\varphi(x,y)] \subseteq S_y(M)$. Since f^{φ}_{μ} is identically 0 on $C \cap M^y$, which is dense in C, we have $\hat{f}^{\varphi}_{\mu}(\mathrm{tp}(b/M)) = 0$, as desired. Next, to verify finite additivity, fix $\mathcal{L}$-formulas $\varphi(x,y)$ and $\psi(x,z)$, and let $\theta(x;y,z)$ and $\chi(x;y,z)$ denote $\varphi(x,y) \vee \psi(x,z)$ and $\varphi(x,y) \wedge \psi(x,z)$, respectively. We need to show that if $bc \in \mathcal{U}^{yz}$ then

$$\hat{f}^{\theta}_{\mu}(\mathrm{tp}(bc/M)) = \hat{f}^{\varphi}_{\mu}(\mathrm{tp}(b/M)) + \hat{f}^{\psi}_{\mu}(\mathrm{tp}(c/M)) - \hat{f}^{\chi}_{\mu}(\mathrm{tp}(bc/M)).$$

Note that the previous equation holds for any $bc \in M^{yz}$ since μ is a Keisler measure and the $\hat{f}_{\mu}$-maps extend the f_{μ}-maps. So fix some $bc \in \mathcal{U}^{yz}$, and let $(b_i c_i)_{i \in I}$ be a net of points in M^{yz} such that $\lim_{i \in I} b_i c_i = \mathrm{tp}(bc/M)$ (recall that we identify points from M with realized types over M). Then we have the following computation:

$$\begin{aligned}
\hat{f}^{\theta}_{\mu}(\mathrm{tp}(bc/M)) &= \lim_{i \in I} \hat{f}^{\theta}_{\mu}(b_i c_i) = \lim_{i \in I}\big(\hat{f}^{\varphi}_{\mu}(b_i) + \hat{f}^{\psi}_{\mu}(c_i) - \hat{f}^{\chi}_{\mu}(b_i c_i)\big) \\
&= \lim_{i \in I} \hat{f}^{\varphi}_{\mu}(b_i) + \lim_{i \in I} \hat{f}^{\psi}_{\mu}(c_i) - \lim_{i \in I} \hat{f}^{\chi}_{\mu}(b_i c_i) \\
&= \hat{f}^{\varphi}_{\mu}(\mathrm{tp}(b/M)) + \hat{f}^{\psi}_{\mu}(\mathrm{tp}(c/M)) - \hat{f}^{\chi}_{\mu}(\mathrm{tp}(bc/M)). \qquad \square
\end{aligned}$$

Remark A.25. The proof of Theorem A.24 can be understood in an abstract way with continuous logic. For the case of types, one can show that a definable type p over a model M has a canonical definable global extension by arguing that the theory of M "knows" that the defining schema of p gives a complete type. This is essentially the same as the argument we have presented here. A continuous real valued function on a type space is the same thing as a definable predicate in the sense of continuous logic (or a formula if one has a broad enough notion of formula). Given a definable measure μ over a model M, the theory of M "knows" that the defining schema of μ gives a measure, and so it follows that the same schema gives a global measure.

Finally, we take a brief moment to note the existence of "coheirs" for measures over small models. This is not used in any part of the paper, but it is thematically relevant to the aims of this part of the appendix.

Proposition A.26. *For any $\mu \in \mathfrak{M}_x(M)$ there is some $\hat{\mu} \in \mathfrak{M}_x(\mathcal{U})$ such that $\hat{\mu}|_M = \mu$ and $\hat{\mu}$ is finitely satisfiable in M.*

Proof. Define $\mathfrak{M}_x(\mathcal{U}, M) := \{\mu \in \mathfrak{M}_x(\mathcal{U}) : \mu \text{ is finitely satisfiable in } M\}$. Let ρ be the restriction of $\rho^x_M : \mathfrak{M}_x(\mathcal{U}) \to \mathfrak{M}_x(M)$ to $\mathfrak{M}_x(\mathcal{U}, M)$. We want to show that ρ is surjective. Note that $\mathfrak{M}_x(\mathcal{U}, M)$ is closed in $\mathfrak{M}_x(\mathcal{U})$, and so ρ is a continuous map between compact Hausdorff spaces. Thus $\mathrm{im}(\rho)$ is closed in $\mathfrak{M}_x(M)$. Moreover, $\mathrm{im}(\rho)$ is a dense subset of $\mathfrak{M}_x(M)$ (consider the image of $\{\mathrm{Av}(\bar{a}) : \bar{a} \in (M^x)^{<\omega}\}$). Therefore ρ is surjective. □

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Quasi groupes de Frobenius dimensionnels

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Nous nous intéressons à une classe de groupes, les quasi groupes de Frobenius (avec involutions), dont la structure interne généralise celle des groupes classiques $\mathrm{GA}_1(\mathbb{C})$, $\mathrm{PGL}_2(\mathbb{C})$ et $\mathrm{SO}_3(\mathbb{R})$: un sous-groupe et ses conjugués, d'indice fini dans leur normalisateur et d'intersection mutuelle triviale, recouvrent "génériquement" le groupe ambiant. Dans la perspective de la théorie des modèles, nous travaillons avec l'hypothèse de l'existence d'une bonne notion de dimension sur les ensembles définissables (il faut distinguer le cas o-minimal et le cas rangé). Nous accordons une attention particulière au cas rangé. En étudiant la géométrie d'incidence induite par les involutions, nous esquissons une classification des quasi groupes de Frobenius et nous déterminons ainsi sous quelles conditions des groupes classiques peuvent être identifiés dans un cadre dimensionnel.

We are interested in a class of groups, quasi Frobenius groups (with involutions), whose internal structure generalizes that of the classical groups $\mathrm{GA}_1(\mathbb{C})$, $\mathrm{PGL}_2(\mathbb{C})$ and $\mathrm{SO}_3(\mathbb{R})$: a subgroup and its conjugates, of finite index in their normalizer and trivial mutual intersection, cover "generically" the ambient group. From the perspective of model theory, we work with the hypothesis of the existence of a good notion of dimension on definable sets (we must distinguish between the o-minimal case and the ranked case). We pay special attention to the ranked case. By studying the geometry of incidence induced by involutions, we sketch a classification of quasi Frobenius groups and thus determine under which conditions classical groups can be identified in a dimensional framework.

1. Introduction

Les groupes algébriques sur un corps algébriquement clos sont des exemples importants de groupes rangés (aussi appelés groupes de rang de Morley fini) ; leurs ensembles définissables peuvent être munis d'une dimension finie aux propriétés remarquables. À cet égard, il est possible d'établir une analogie avec les groupes de Lie réels et plus généralement avec les groupes définissables dans les structures o-minimales. Ces derniers peuvent être également décrits du point de vue d'une fonction de dimension qui présente de nombreux points communs avec le rang

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de Morley (elle est additive et définissable). Dans les deux cas, nous disposons d'une notion appropriée de généricité qui s'applique aux parties définissables de dimension maximale. Dans la perspective de la théorie des modèles, il semble donc possible de décrire dans un cadre commun — celui des groupes *dimensionnels*, dont la définition est donnée à la section 2 — les groupes de Lie réels et les groupes algébriques sur un corps algébriquement clos. Comme références générales pour les groupes rangés, nous renvoyons à [Borovik et Nesin 1994; Altınel et al. 2008] ; en ce qui concerne les groupes définissables dans une structure o-minimale, on pourra consulter [Otero 2008].

On peut illustrer ce principe méthodologique en considérant les groupes $\mathrm{SO}_3(\mathbb{R})$ et $\mathrm{PGL}_2(\mathbb{C})$. Ce sont des groupes dimensionnels qui présentent une structure interne très proche, se reflétant dans la géométrie d'incidence induite par les involutions. Comme le font remarquer A. Deloro et J. Wiscons [2020] et avant eux A. Nesin, les involutions forment un plan projectif dans $\mathrm{SO}_3(\mathbb{R})$; les axiomes sont satisfaits seulement génériquement dans $\mathrm{PGL}_2(\mathbb{C})$. Il s'agira de déterminer dans quelle mesure cette géométrie des involutions permet de caractériser ces groupes classiques parmi les groupes dimensionnels.

Pour établir cette théorie commune, nous sommes conduits à introduire les définitions suivantes :

Définition 1.0.1. Un groupe G contenant un sous-groupe propre C est un quasi groupe de Frobenius si C est d'indice fini dans son normalisateur et TI, i.e., $C^g \cap C = \{1\}$ pour tout $g \notin N_G(C)$. On dit que C est un quasi complément de Frobenius.

Définition 1.0.2. Soit $C < G$ un quasi groupe de Frobenius. On note n l'indice de C dans $N_G(C)$. On dit que G est

— de degré pair si n est un entier pair,

— de degré impair si n est un entier impair.

Si le degré est égal à 1, on parle plutôt de groupe de Frobenius.

Dans la suite de l'article, sauf mention explicite du contraire, le terme de « degré » ne désignera pas le degré de Morley.

Nous cherchons à classifier les quasi groupes de Frobenius dimensionnels. Nous insisterons en particulier sur les configurations de degré pair dont l'étude forme le cœur de l'article. En général, nous supposerons que les groupes en question sont $U_2^\perp$: il n'y a pas de 2-groupe abélien élémentaire infini. On dira qu'un groupe contenant des involutions et $U_2^\perp$ est de *type impair*. Du point de vue des quasi groupes de Frobenius de degré pair, nous pouvons formuler les deux conjectures suivantes :

Conjecture 1.0.3. *(1)* [Deloro et Wiscons 2020]. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable et connexe. Supposons* *que G est de degré pair et de type impair. Alors G est isomorphe à* $\mathrm{PGL}_2(K)$ *pour un corps algébriquement clos K de caractéristique différente de* 2.

(2) Soit $C < G$ un quasi groupe de Frobenius définissable dans une structure o-minimale, connexe, tel que C est définissable et connexe. Supposons que G contient des involutions. Si $\bigcup_G C^g$ contient toutes les translations, c'est-à-dire tous les produits de deux involutions, alors G est isomorphe à $\mathrm{SO}_3(R)$ *pour un corps réel clos R.*

L'étude de ce type de configurations trouve son origine dans [Nesin 1989; Nesin et al. 1991]. Les groupes $\mathrm{PGL}_2(K)$ et $\mathrm{SO}_3(R)$ y sont identifiés parmi les groupes dimensionnels de petite dimension (inférieure ou égale à trois).

Nous évoquerons également les groupes de Frobenius mais nous restreignons notre attention au cas rangé. La conjecture suivante guidera notre étude :

Conjecture 1.0.4 [Borovik et Nesin 1994]. *Un groupe de Frobenius rangé $C < G$ est scindé, i.e., s'écrit $U \rtimes C$ pour un sous-groupe définissable U.*

Cette conjecture est vérifiée pour les groupes de Frobenius algébriques et finis. Dans le cas rangé, les travaux de Nesin constituent une première élucidation de la structure des groupes de Frobenius. Remarquons que $\mathrm{GA}_1(\mathbb{C}) \simeq \mathbb{C}^+ \rtimes \mathbb{C}^\times$, le groupe des transformations affines complexes, est un exemple de groupe de Frobenius rangé connexe. Si les involutions induisent également une géométrie d'incidence, cette dernière se distingue assez nettement de la géométrie observée précédemment. Pour plus de détails, nous renvoyons à [Clausen et Tent 2021]. En ce qui concerne les structures o-minimales, la classification des groupes de Frobenius et en particulier des groupes strictement 2-transitifs est clarifiée par [Macpherson et al. 2000; Tent 2000]. En effet, dans ce dernier article, il est établi que les groupes strictement 2-transitifs définissables dans une structure o-minimale sont isomorphes aux groupes des transformations affines d'un corps gauche (interprétable) ; ce sont en particulier des groupes de Frobenius scindés.

Dans cet article, nous introduisons d'abord le cadre approprié pour définir la notion de groupe dimensionnel et nous montrons sous quelles conditions il est possible d'exploiter la structure des quasi groupes de Frobenius pour interpréter $\mathrm{SO}_3(K)$, pour un corps K interprétable (section 2).

Nous étudions ensuite plus précisément le cas o-minimal et nous démontrons le théorème suivant (section 3) :

Théorème 1.0.5. *Soit $C < G$ un quasi groupe de Frobenius définissable dans une structure o-minimale, connexe, tel que C est définissable et connexe. Supposons que G contient des involutions. On suppose que $\bigcup_G C^g$ contient tous les* 2*-éléments et toutes les translations (produits de deux involutions). Alors :*

(1) G est un groupe semi-simple, c'est-à-dire qu'il ne contient pas de sous-groupe abélien normal infini.

(2) *Si on suppose de plus C nilpotent, alors $G/Z(G) \simeq \mathrm{SO}_3(R)$ pour un corps réel clos R interprétable.*

Nous développons ensuite notre analyse dans le contexte rangé (section 4). Nous formulons d'abord quelques résultats généraux sur les quasi groupes de Frobenius (structure induite, structure de la 2-torsion, résolubilité et étude du groupe de Weyl). Une attention particulière est ensuite portée à l'étude des translations. Nous montrons sous l'hypothèse de la générosité des sous-groupes de Borel (l'ensemble des éléments formé par un sous-groupe définissable connexe résoluble maximal et ses conjugués est générique dans le groupe ambiant) les deux théorèmes suivants :

Théorème 1.0.6. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est de degré pair et de type impair. Si les sous-groupes de Borel sont généreux, alors $G \simeq \mathrm{PGL}_2(K)$ pour un corps K (rangé) algébriquement clos de caractéristique différente de 2.*

Théorème 1.0.7. *Soit $C < G$ un groupe de Frobenius rangé connexe de type impair avec C résoluble et G non résoluble. Si les sous-groupes de Borel sont généreux, alors C n'est pas nilpotent.*

À cet égard, soulignons le parallélisme entre notre approche et l'étude des configurations de type « CiBo » ; voir [Cherlin et Jaligot 2004; Deloro 2007; Deloro et Jaligot 2016]. Les théorèmes précédents constituent un analogue de certains résultats de [Cherlin et Jaligot 2004; Deloro et Jaligot 2016] où une hypothèse de minimalité (« minimal simple connexe » ou « $N_\circ^\circ$ ») a été remplacée par l'hypothèse « quasi groupe de Frobenius ». Les techniques employées dans ces travaux jouent un rôle prépondérant dans notre classification des quasi groupes de Frobenius rangés.

2. Groupes dimensionnels et théorème de Bachmann

Étant donné une structure M, on dira qu'un groupe G définissable dans M est *dimensionnel* si on peut munir les ensembles définissables de G et de ses puissances cartésiennes d'une dimension finie dim satisfaisant les axiomes suivants ; voir [Baro et al. 2012; 2014] :

(A_1) Si f est une fonction définissable entre deux sous-ensembles définissables A et B, alors $\{b \in B : \dim(f^{-1}(b)) = n\}$ est définissable.

(A_2) Si f est une fonction définissable surjective entre deux sous-ensembles définissables A et B, telle que les fibres sont de dimension constante égale à m, alors $\dim(A) = \dim(B) + m$.

(A_3) A est fini si et seulement si $\dim(A) = 0$.

(A_4) $\dim(A \cup B) = \max\{\dim(A), \dim(B)\}$.

Les groupes de rang de Morley fini, les groupes définissables dans une structure o-minimale, dans le corps des nombres p-adiques, dans un corps P-minimal sont des exemples de groupes dimensionnels.

On parlera de groupe *dimensionnel*$^+$ si de plus le groupe satisfait la condition de chaîne descendante pour les sous-groupes définissables et si la dimension peut être étendue aux sous-ensembles interprétables du groupe. En particulier, on peut attacher une dimension aux sections définissables (les quotients d'un sous-groupe définissable par un autre sous-groupe définissable).

Les groupes définissables dans une structure o-minimale [Edmundo 2003, Theorem 7.2] et les groupes rangés (par définition) sont des exemples de groupes dimensionnels$^+$, mais ce n'est plus le cas des groupes définissables dans le corps des nombres p-adiques, par exemple.

Un groupe dimensionnel$^+$ G admet une *composante connexe* G°, soit un plus petit sous-groupe définissable d'indice fini : il suffit de considérer l'intersection de tous les sous-groupes définissables d'indice fini. On dira qu'il est *connexe* si $G^\circ = G$. De plus, chaque sous-ensemble $X \subseteq G$ est contenu dans un plus petit sous-groupe définissable, appelé *enveloppe définissable* et noté $d(X)$. Remarquons que, comme dans le cas rangé, certaines propriétés sont préservées lorsque l'on passe à la clôture définissable d'un sous-groupe (non définissable) H [Baro et al. 2012, Remark 3.3 et Fact 3.4] :

— Soit $K \subseteq G$. Si H est K-invariant, alors $d(H)$ est K-invariant.

— Si H est nilpotent de classe n, alors $d(H)$ est nilpotent de classe n.

La condition de chaîne descendante pour les sous-groupes définissables rend également possible le relèvement de la torsion [Baro et al. 2012, Fact 3.9] :

Soit H un sous-groupe définissable normal tel que xH est un p-élément dans G/H. Alors il existe un p-élément dans xH.

L'existence d'une dimension finie permet de caractériser facilement les parties qui sont, en un certain sens, « larges ». On dira qu'une partie Y est *générique* si elle contient une partie définissable X telle que $\dim(X) = \dim(G)$. On dira que X est *large* si $\dim(G \setminus X) < \dim(G)$. De plus, étant donné un sous-groupe définissable H, l'équation $\dim(G) = \dim(G/H) + \dim(H)$ montre qu'un sous-groupe définissable H est d'indice fini dans G si et seulement si H est générique dans G.

Passons maintenant à l'étude des quasi groupes de Frobenius dimensionnels$^+$. La notion de quasi groupe de Frobenius a été définie à la définition 1.0.1. Soit $C < G$ un quasi groupe de Frobenius dimensionnel$^+$ connexe tel que C est définissable et connexe. On remarque que $N_G(C)^\circ = C$ et que $C_G(g)^\circ \leq C$, pour tout $1 \neq g \in C$: $C_G(g) \leq N_G(C)$ par la propriété TI et $C_G(g)^\circ \leq N_G(C)^\circ = C$.

Notation. Soit $C < G$ un quasi groupe de Frobenius. Pour $x \in \bigcup_G C^g$, on note C_x le conjugué de C contenant x.

Lemme 2.0.1. *Soit $C < G$ un quasi groupe de Frobenius dimensionnel$^+$ connexe tel que C est définissable et connexe. Alors $\bigcup_G C^g$ est générique dans G.*

Démonstration. En raisonnant comme dans [Jaligot 2006], l'additivité et la définissabilité de la dimension permettent de conclure. □

Avant de poursuivre l'analyse des quasi groupes de Frobenius dimensionnels$^+$ connexes, il nous faut souligner une différence essentielle entre le contexte rangé et le contexte o-minimal. Pour un groupe connexe rangé G, la notion de largeur et celle de généricité coïncident alors que ce n'est pas forcément le cas pour un groupe connexe définissable dans une structure o-minimale. En particulier, deux parties génériques ne s'intersectent pas forcément non trivialement.

De plus, l'action des automorphismes définissables involutifs peut être caractérisée de façon commode dans le contexte rangé, mais pas forcément dans le cas o-minimal : si un automorphisme involutif définissable d'un groupe rangé connexe a un nombre fini de points fixes, alors l'action se fait par inversion et le groupe est abélien. Or, la compréhension de l'action des involutions se révèle indispensable à notre étude des quasi groupes de Frobenius dimensionnels$^+$ connexes. Nous sommes donc conduits à proposer un nouvel axiome pour identifier une sous-classe des groupes dimensionnels$^+$:

(A_5) Pour tout sous-groupe définissable connexe H et tout automorphisme involutif définissable i de H, si $C_H(i)$ est fini, alors H est abélien inversé par i.

Nous ne savons pas si un tel axiome (vérifié par les groupes rangés) reste valable pour une sous-classe naturelle de groupes définissables dans une structure o-minimale ; par exemple, pour les groupes définissablement simples et/ou définissablement compacts. Dans la suite, nous indiquerons à quel moment (A_5) semble requis.

Remarque 2.0.2. Soient C un sous-groupe abélien définissable connexe et i un automorphisme involutif définissable ; alors (A_5) n'est plus nécessaire pour montrer que l'action est par inversion. En effet, posons $X = \{c^i c^{-1} : c \in C\}$ (c'est un ensemble définissable inversé par i). Les fibres de l'application définissable $\phi : C \to X$ définie par $\phi(c) = c^i c^{-1}$ ont même dimension que $C_C(i)$, qui est un groupe fini. Par conséquent, l'additivité de la dimension nous donne $\dim(X) = \dim(C)$. Le sous-groupe formé par les éléments inversés par i est générique et donc il est égal à C par connexité.

Nous allons poursuivre l'étude des translations menée dans [Deloro et Wiscons 2020] et nous utiliserons les méthodes de l'école de Bachmann pour interpréter $SO_3(K)$ dans un cadre dimensionnel.

Jusqu'à la fin de la section, soit $C < G$ un quasi groupe de Frobenius dimensionnel$^+$ connexe tel que C est définissable et connexe. Supposons que G est de type impair et que $\bigcup_G C^g$ contient tous les 2-éléments. Dans le cas rangé, l'hypothèse $U_2^\perp$ suffit ; voir lemme 4.2.4. On suppose également que toutes les translations (les produits de deux involutions) sont contenues dans $\bigcup_G C^g$.

Lemme 2.0.3. *Le sous-groupe C est d'indice pair dans son normalisateur. De plus, il existe une involution $1 \neq k \in N_G(C) \setminus C$.*

Démonstration. Soit i une involution contenue dans C et $j \neq i$ une autre involution contenue dans un conjugué C_j distinct de C. Par hypothèse, la translation $x = ij$ est contenue dans un conjugué C_x de C. On a donc $C_G(x)^\circ \leq C_x$; de plus, i et j appartiennent à $N_G(C_G(x)^\circ) \leq N_G(C_x)$. Si i et j étaient contenues dans C_x, alors on aurait d'une part $C_x \cap C \neq \{1\}$ et d'autre part $C_x \cap C_j \neq \{1\}$ et finalement $C = C_x = C_j$: contradiction. Par conséquent, i et j appartiennent à $N_G(C_x) \setminus C_x$. □

On dit qu'un groupe est *semi-simple* s'il n'a pas de sous-groupe abélien normal infini. Soit $C < G$ un quasi groupe de Frobenius dimensionnel$^+$ connexe tel que C est définissable et connexe. Alors G a un centre fini. En effet, $Z(G) \leq N_G(C)$, donc $Z(G)^\circ \leq C$. En prenant des conjugués, $Z(G)^\circ = \{1\}$.

Lemme 2.0.4. *Le groupe G est semi-simple.*

Démonstration. Soit A un sous-groupe abélien normal infini. Quitte à considérer la composante connexe de son enveloppe définissable, on peut supposer que A est définissable et connexe. Si $A \cap C \neq \{1\}$, soit $1 \neq x \in A \cap C$, alors $A \leq C_G(x)^\circ \leq C$ et donc $C = G$: contradiction. On a donc $A \cap \bigcup_G C^g = \{1\}$. Le groupe A est en particulier 2-divisible car il ne contient pas d'involutions. Soit i une involution ; comme on peut supposer que $i \in C$, $C_A(i)^\circ \leq A \cap C = \{1\}$, le sous-groupe $C_A(i)$ est donc fini et il existe un élément $a \in A$ tel que $a^i a^{-1} = b \in A$ est non trivial et inversé par i. Mais alors $1 \neq b^2 \in A$ et $b^2 = bib^{-1}i = i^{b^{-1}}i \in \bigcup_{g \in G} C^g$: contradiction. Par conséquent, il n'y a pas de sous-groupe abélien normal infini. □

On suppose désormais que G satisfait également (A_5).

Lemme 2.0.5. *Soit k une involution telle que $k \in N_G(C) \setminus C$. Alors $C_C(k)$ est fini. De plus, C est abélien 2-divisible inversé par k (et donc $C \subseteq I \cdot I$).*

Démonstration. Supposons que $C_C(k)$ est infini, on a donc $C_C(k)^\circ \leq C_k \cap C$ et finalement $C_k = C$: contradiction. Par (A_5), l'action de k est donc bien par inversion et C est abélien. C'est un groupe 2-divisible, car il contient un nombre fini d'involutions : l'élévation au carré définit un homomorphisme de groupe définissable de noyau fini. Puisque le groupe est connexe, c'est un homomorphisme surjectif. □

Remarque 2.0.6. Nous n'utilisons plus l'axiome (A_5) dans la suite.

Lemme 2.0.7. *Le sous-groupe (abélien) C contient une unique involution et toutes les involutions de $N_G(C) \setminus C$ appartiennent au même translaté de C.*

Démonstration. Supposons par l'absurde que $i \neq j$ sont deux involutions distinctes appartenant à C. Soit $\alpha \in N_G(C) \setminus C$ une involution. Puisque l'involution α agit par inversion, elle commute avec l'involution $x = ij$; mais i appartient à $N_G(C_\alpha) \setminus C_\alpha$ et inverse donc C_α. Il en est de même pour j, et donc $x \in C_G(C_\alpha)$. D'après le lemme 2.0.5, $x \in C_\alpha \cap C = \{1\}$: contradiction. Ceci montre l'unicité.

Soient $j, k \in N_G(C) \setminus C$; puisque l'action est par inversion, $jk \in C_G(C)$. Mais alors $C \leq C_G(jk)^\circ \leq C_{jk}$ et donc $C_{jk} = C$ et $jk \in C$. Finalement, $k \in jC$ et donc toutes les involutions de $N_G(C) \setminus C$ sont égales modulo C. Réciproquement, le translaté jC est composé uniquement d'involutions. □

En procédant comme dans la démonstration du théorème 3.31 de [Poizat 1987], on peut munir l'ensemble des involutions d'une structure de plan projectif : à chaque involution i (point), on associe la ligne $D_i = \{k \in I : ki = ik\} \setminus \{i\} = \{k \in I : k \in N_G(C_i) \setminus C_i\} = kC_i$ pour chaque $k \in N_G(C_i) \setminus C_i$.

Dans ce cas, par deux points distincts $i \neq j$ passe une unique ligne à savoir D_k où k est l'unique involution de C_{ij}. En effet, i et j inversent $C_{ij} = C_k$ mais n'appartiennent pas à C_{ij}. De plus, si $i, j \in D_\ell \neq D_k$, alors $i, j \in N_G(C_\ell) \setminus C_\ell$ et donc $x = ij \in C_\ell$, finalement $C_\ell = C_x = C_k$: contradiction.

De plus, deux lignes D_i et D_j s'intersectent en un seul point k, l'unique involution contenue dans C_{ij}. En effet, pour cette involution, on a bien $k \in N_G(C_i) \setminus C_i$ et $k \in N_G(C_j) \setminus C_j$, car $k \in C_G(i) \cap C_G(j)$. Soit $\ell \neq k \in D_j \cap D_i$, alors $\ell k \in C_i \cap C_j$, et $C_i = C_j$: contradiction. En particulier, pour deux involutions distinctes $i \neq j$, il existe une unique involution k distincte de i et de j qui commute avec les deux.

Finalement, on remarque que trois involutions distinctes i, j, k sont colinéaires si et seulement si ijk est une involution : ijk est une involution (si et seulement si ij est inversé par k et $ij \neq k$) si et seulement si $k, i, j \in D(\ell)$ où ℓ est l'involution contenue dans C_{ij}. On peut dès lors appliquer une forme du théorème de Bachmann :

Fait 2.0.8 [Borovik et Nesin 1994, Fact 8.15; Schröder 1982]. *Soit G un groupe contenant des involutions. Si on peut munir l'ensemble des involutions I d'une structure de plan projectif tel que trois involutions i, j et k sont colinéaires si et seulement si ijk est une involution, alors le sous-groupe $\langle I \rangle = I \cdot I$ est isomorphe à $\mathrm{SO}_3(K, f)$ pour un corps K interprétable dans G et une forme quadratique anisotrope f définie sur K^3.*

Il suffit de remarquer que $\langle I \rangle = I \cdot I = \bigcup_G C^g$ est un sous-groupe définissable de même dimension que G (lemme 2.0.1) et donc lui est égal par connexité. Notamment, $G \simeq \mathrm{SO}_3(K, f)$.

Dans un univers rangé, un corps interprétable K est algébriquement clos [Macintyre 1971] et toute forme quadratique définie sur un K-espace vectoriel de dimension supérieure ou égale à deux est alors forcément isotrope. Par conséquent, comme corollaire, on obtient le théorème A de [Deloro et Wiscons 2020] :

Théorème 2.0.9. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable et connexe. Supposons que G est de type impair. Alors $\bigcup_G C^g$ ne contient pas toutes les translations.*

3. Le cas o-minimal

Cette section est consacrée à la démonstration du théorème 1.0.5 :

Théorème 1.0.5. *Soit $C < G$ un quasi groupe de Frobenius définissable dans une structure o-minimale, connexe, tel que C est définissable et connexe. Supposons que G contient des involutions. On suppose que $\bigcup_G C^g$ contient tous les 2-éléments et toutes les translations (produits de deux involutions). Alors :*

(1) G est un groupe semi-simple, c'est-à-dire qu'il ne contient pas de sous-groupe abélien normal infini.

(2) Si on suppose de plus C nilpotent, alors $G/Z(G) \simeq \mathrm{SO}_3(R)$ pour un corps réel clos R interprétable.

Notons que nous affaiblissons l'hypothèse de commutativité de la remarque 2.0.2 en supposant que le quasi complément est seulement nilpotent.

Nous renvoyons à [Peterzil et al. 2000] pour les théorèmes concernant la structure des groupes semi-simples définissables dans une structure o-minimale. Nous mobilisons également la théorie des sous-groupes de Carter et de Cartan développée dans [Baro et al. 2014]. Un *sous-groupe de Cartan Q* est un sous-groupe nilpotent maximal tel que pour tout sous-groupe $X \leq Q$ normal d'indice fini, X est un sous-groupe d'indice fini dans $N_G(X)$. D'après [Baro et al. 2014], on sait que les sous-groupes de Cartan d'un groupe définissable dans une structure o-minimale sont définissables. De plus, les *sous-groupes de Carter*, i.e., les sous-groupes définissables connexes nilpotents d'indice fini dans leur normalisateur correspondent aux composantes connexes des sous-groupes de Cartan. Il est possible de caractériser précisément la structure des sous-groupes de Cartan des groupes définissablement simples définissables dans une structure o-minimale (et plus généralement des sous-groupes définissables semi-simples).

Fait 3.0.1 [Baro et al. 2014, Theorem 58]. *Soit G un groupe définissablement simple définissable dans une structure o-minimale. Alors les sous-groupes de Cartan sont abéliens et ont tous même dimension.*

Soit $C < G$ un quasi groupe de Frobenius définissable dans une structure o-minimale, connexe, tel que C est définissable et connexe. Supposons que G contient des involutions et que $\bigcup_G C^g$ contient les 2-éléments et les translations. Nous savons déjà que G est semi-simple (lemme 2.0.4). Nous supposons de plus que C est nilpotent (c'est donc un sous-groupe de Carter).

Lemme 3.0.2. *Le groupe $G/Z(G)$ est un quasi groupe de Frobenius avec comme quasi complément de Frobenius $CZ(G)/Z(G)$.*

Démonstration. On note que $Z(G)$ est fini. Supposons $\alpha \in \overline{C} \cap \overline{C^g} \setminus \{\bar{1}\}$, relevé par a. Alors $a = c_1 z_1 = c_2^g z_2$ avec $c_1, c_2 \in C \setminus \{1\}$, et $z_1, z_2 \in Z(G)$. On a $C_G(a)^\circ = C_G(c_1)^\circ \leq C$ et donc $C = C^g$; ceci montre que $\overline{C}$ est TI. Soit maintenant $\nu \in N_{\overline{G}}(\overline{C})$, relevé par $n \in G$. Alors $C^n \leq C \cdot Z(G)$, donc par connexité $C^n \leq C$ et $n \in N_G(C)$. Le groupe $N_{\overline{G}}(\overline{C})/\overline{C}$ est donc fini. □

Nous pouvons passer à la démonstration du théorème 1.0.5. D'après [Peterzil et al. 2000, Theorem 4.1], le groupe $G/Z(G)$ est isomorphe à un produit fini de groupes simples définissables, les H_i. Comme $G/Z(G)$ a des involutions, on peut supposer que H_1 contient des involutions. Soit $i \in H_1 \cap \overline{C}_i$, alors $H_2 \leq C_{\overline{G}}(i)^\circ \leq \overline{C}_i$: contradiction. Le groupe $G/Z(G)$ est donc définissablement simple. D'après le fait 3.0.1, ses sous-groupes de Carter sont abéliens (et de même dimension). Or, $\overline{C}$ est un sous-groupe de Carter ; c'est donc un groupe abélien. On peut désormais appliquer les résultats de la section précédente et par conséquent, $G/Z(G) \simeq \mathrm{SO}_3(K, f)$.

Montrons désormais que le corps K interprétable est réel clos et que la forme quadratique anisotrope correspond à un produit scalaire. Nous avons besoin du fait suivant :

Fait 3.0.3 [Pillay 1988, Theorem 3.9]. *Un corps commutatif infini K définissable dans une structure o-minimale est soit réel clos* (*et* $\dim(K) = 1$), *soit algébriquement clos.*

Puisque la forme quadratique est anisotrope, le corps K est réel clos. Finalement, le théorème d'inertie de Sylvester nous permet de conclure que $G/Z(G) \simeq \mathrm{SO}_3(K)$ où K est réel clos.

4. Le cas rangé

4.1. *Préliminaires généraux.* Nous rappelons désormais plusieurs faits plus ou moins spécialisés sur les groupes rangés qui seront utilisés dans cet article. On pourra consulter [Borovik et Nesin 1994] pour une caractérisation axiomatique de la dimension dans ce contexte. Lorsqu'on parlera de groupe algébrique, on entend groupe des K-points dans un corps algébriquement clos K. Comme pour les structures o-minimales, nous dirons qu'une partie définissable X d'un groupe rangé G est générique si $\mathrm{RM}(X) = \mathrm{RM}(G)$.

Générosité. Rappelons que dans un groupe connexe rangé, deux parties génériques s'intersectent toujours non trivialement. On dira qu'un sous-groupe définissable H est *généreux* si $\bigcup_{g \in G} H^g$ est générique. On dira également qu'un sous-groupe définissable H est *presque autonormalisant* si H est d'indice fini dans son normalisateur. Enfin, un sous-groupe définissable H est *génériquement disjoint de ses* *conjugués* dans G si $H \setminus \bigcup_{g \in G \setminus N(H)} (H \cap H^g)$ est générique dans H. On rappelle plusieurs résultats concernant la générosité et la généricité ; voir [Jaligot 2006].

Fait 4.1.1. *(1)* [Altınel et al. 2008, Chapter IV, Lemma 1.2]. *Soient G un groupe rangé connexe et H un sous-groupe définissable presque autonormalisant et génériquement disjoint de ses conjugués. Alors H est généreux dans G.*

(2) [Jaligot 2006, Lemma 2.4]. *Si $A \le B \le C$ sont trois sous-groupes définissables avec B connexe, alors la générosité est transitive.*

Indécomposables. Nous supposerons connus les raisonnements par indécomposable [Borovik et Nesin 1994, §5.4] ; nous ferons également usage du résultat de Zilber concernant l'interprétation d'un corps à partir d'une action définissable sur un groupe abélien [Borovik et Nesin 1994, Theorem 9.1].

Théorie de Sylow. Les groupes rangés peuvent être analysés via leurs éléments de torsion. Il existe une théorie des p-sous-groupes de Sylow (p-sous-groupes maximaux) mais elle n'est pleinement développée que pour $p = 2$ ou dans le contexte des groupes (connexes) résolubles rangés. D'après [Borovik et Nesin 1994, §6.4], les 2-Sylow sont conjugués et leurs composantes connexes s'écrivent comme produit central d'un 2-groupe abélien divisible et d'un groupe d'exposant fini définissable connexe.

Définition 4.1.2. Soit G un groupe rangé et soit $S^\circ = T * B$ la composante connexe d'un 2-sous-groupe de Sylow, où T est abélien divisible et B d'exposant fini.

(1) Si $S^\circ = \{1\}$, alors G est de type dégénéré.

(2) Si $T \neq \{1\}$ et $B \neq \{1\}$, alors G est de type mixte.

(3) Si $T \neq \{1\}$ et $B = \{1\}$, alors G est de type impair.

(4) Si $T = \{1\}$ et $B \neq \{1\}$, alors G est de type pair.

Dans cette perspective, un *p-groupe unipotent* est un p-groupe nilpotent définissable connexe d'exposant fini. On dira qu'un groupe rangé est $U_p^\perp$ s'il ne contient pas de p-groupe unipotent. Un *p-tore* est a contrario un p-groupe abélien divisible isomorphe à une somme finie de p-groupes de Prüfer. Le p-rang de Prüfer d'un groupe rangé désigne alors le p-rang d'un p-tore maximal. Le cas $p = 2$ revêt une importance particulière en raison du résultat de conjugaison évoqué précédemment. De plus, bien qu'on ne dispose pas d'un analogue du théorème de Feit–Thompson qui éliminerait le cas dégénéré, les involutions, comme dans le cas des groupes simples finis, sont souvent au cœur de l'analyse. Nous renvoyons à [Borovik et Nesin 1994, §10] pour plus de détails sur les propriétés des involutions dans notre contexte. Nous supposerons également connues les propriétés des sous-groupes définissables *fortement inclus* ; voir [Borovik et Nesin 1994, Theorem 10.19].

Dans le cas d'un groupe connexe, la structure de la p-torsion satisfait la propriété additionnelle suivante :

Fait 4.1.3 [Borovik et al. 2007]. *Soit G un groupe rangé connexe dont les p-sous-groupes de Sylow sont finis. Alors G ne contient pas de p-éléments.*

On cite une autre conséquence de l'analyse de la torsion dans les groupes connexes :

Fait 4.1.4 [Borovik et al. 2007; Altınel et al. 2008, Chapter IV, Corollary 4.18]. *Soit G un groupe rangé connexe. Alors $C_G(g)^\circ$ est infini pour tout $g \in G$.*

Groupes résolubles. L'étude d'un groupe rangé passe en général par une compréhension de ses *sous-groupes de Borel*, i.e., de ses sous-groupes définissables connexes résolubles maximaux. Plus généralement, nous allons nous intéresser aux propriétés des sous-groupes définissables (connexes) résolubles.

Fait 4.1.5 [Borovik et Nesin 1994, Corollary 9.9]. *Soit G un groupe rangé connexe résoluble. Alors G' est un sous-groupe définissable connexe nilpotent.*

Notions de tore. En exploitant l'analogie avec les groupes algébriques affines, on peut introduire pour les groupes rangés une notion de tore (éléments semi-simples) et de radical unipotent. Idéalement un groupe connexe rangé résoluble devrait pouvoir s'écrire comme un produit semi-direct d'un radical unipotent et d'un tore.

On caractérise maintenant la semi-simplicité dans le contexte des groupes rangés. Nous pourrons également mieux comprendre la nature des éléments génériques. Un sous-groupe abélien divisible est un *tore décent* si c'est la clôture définissable de sa torsion ; c'est un *bon tore* si cette propriété est héréditaire pour les sous-groupes définissables infinis. Le groupe multiplicatif d'un corps de caractéristique positive est un exemple typique de bon tore comme le prouve le fait suivant qui se base sur des travaux de F. O. Wagner [2001] :

Fait 4.1.6 [Altınel et al. 2008, Chapter I, Proposition 4.20]. *Soit K un corps rangé de caractéristique positive. Alors $K^\times$ est un bon tore.*

Remarquons également que le résultat suivant de Wagner permet de conjecturer la non-existence des *mauvais corps* en caractéristique positive :

Fait 4.1.7 [Wagner 2003]. *S'il existe une infinité de p-nombres premiers de Mersenne, c'est-à-dire de nombres premiers de la forme $(p^n - 1)/(p - 1)$, alors il n'existe pas de mauvais corps de caractéristique positive $p > 0$.*

Grâce aux travaux de G. Cherlin et T. Altınel puis d'O. Frécon notamment, on sait que le comportement des tores décents présente des similitudes avec celui des tores algébriques.

Fait 4.1.8. *(1)* [Cherlin 2005]. *Soit G un groupe rangé. Alors les tores décents maximaux sont conjugués.*

(2) [Frécon 2006, Lemma 3.1]. *Soient G un groupe rangé et N un sous-groupe normal définissable. Soit T un tore décent maximal. Alors TN/N est un tore décent maximal de G/N et tous les tores décents maximaux ont cette forme.*

(3) [Altınel et Burdges 2008, Theorem 1]. *Soit T un tore décent d'un groupe rangé connexe G. Alors $C_G(T)$ est connexe, généreux dans G, et aussi $N_G(C_G(T))^\circ = C_G(T)$.*

On cite également un fait qui décrit la p-torsion lorsque le groupe est $U_p^\perp$.

Fait 4.1.9 [Burdges et Cherlin 2009, Theorem 3]. *Soit G un groupe rangé connexe, et soit g un p-élément tel que $C_G(g)^\circ$ est $U_p^\perp$. Alors g appartient à un p-tore.*

Il s'agit d'un principe de *toralité*.

4.2. *Résultats généraux.* Nous commençons à élaborer une théorie générale des quasi groupes de Frobenius rangés connexes. Un certain nombre de résultats valables pour les groupes de Frobenius rangés (étudiés dans [Borovik et Nesin 1994]) se généralisent à ce cadre plus général. Remarquons que nous faisons usage de manière cruciale d'arguments impliquant l'étude de la généricité et de la générosité dans les groupes connexes (ce qui les rend difficilement généralisables au cas o-minimal).

Structure induite. Dans un quasi groupe de Frobenius rangé connexe, l'intersection d'un sous-groupe définissable connexe avec un quasi complément de Frobenius définissable connexe induit une structure de quasi groupe de Frobenius connexe.

Lemme 4.2.1 (à comparer avec [Borovik et Nesin 1994, Lemma 11.10]). *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable et connexe. Soit $H < G$ un sous-groupe définissable connexe. Si $1 < (H \cap C) < H$ alors $(H \cap C) < H$ est encore un quasi groupe de Frobenius. De plus, $H \cap C$ est infini et connexe. Finalement, pour tout conjugué C_1 de C tel que $(H \cap C_1) \neq \{1\}$, il existe $1 \neq h \in H$, tel que $H \cap C_1 = (H \cap C)^h$.*

Démonstration. Montrons que $H \cap C$ est TI dans H. Supposons que $(H \cap C)^h \cap (H \cap C) \neq \{1\}$ pour un $h \in H$; on a donc $C^h \cap C \neq \{1\}$ et par conséquent $C^h = C$. Le même raisonnement montre que $N_H(H \cap C) \leq N_G(C)$ et donc $N_H(H \cap C)^\circ \leq N_G(C)^\circ = C$, i.e., $N_H(H \cap C)^\circ \leq H \cap C$. Le groupe $H \cap C$ est donc TI dans H et d'indice fini dans son normalisateur dans H. On remarque que pour $1 \neq c \in H \cap C$, $C_H(c) \leq N_H(H \cap C)$ et donc $C_H(c)^\circ \leq N_H(H \cap C)^\circ$. Or, d'après le fait 4.1.4, $C_H(c)$ est infini, donc $H \cap C$ est infini.

Si $H \cap C$ n'était pas connexe alors $(H \cap C)^\circ$ et $(H \cap C) \setminus (H \cap C)^\circ = X$ seraient généreux. En effet, $(H \cap C)^\circ$ et X sont invariants sous l'action de $N_H(H \cap C)$; de plus, si X^h ou $(H \cap C)^{\circ^h}$ intersecte $(H \cap C)$ non trivialement alors $(H \cap C) \cap (H \cap C)^h$ est non trivial et donc $h \in N_H(H \cap C)$. Par conséquent, les ensembles disjoints $\bigcup_H X^h$ et $\bigcup_H (H \cap C)^{\circ^h}$ sont génériques, ce qui contredit la connexité de H. Soit

$(H \cap C_1) \neq \{1\}$. Par ce qui précède, c'est un sous-groupe généreux dans H, tout comme $(H \cap C)$. Par connexité de H, il existe $1 \neq h \in H$ tel que $(H \cap C) \cap (H \cap C_1)^h \neq \{1\}$. Par conséquent, $C_1^h = C$ et finalement $(H \cap C) = (H \cap C_1)^h$. $\square$

Remarque 4.2.2. À notre connaissance, le quasi complément de Frobenius pourrait ne pas être définissable dans la pure structure de groupe, contrairement à ce qui se passe dans les groupes de Frobenius rangés [Borovik et Nesin 1994, Lemma 11.19]. Cependant, en raisonnant comme dans la démonstration précédente, on obtient le résultat suivant : si $C < G$ est un quasi groupe de Frobenius rangé connexe, avec C définissable, alors C est infini connexe. Dans la suite, nous préciserons seulement que le quasi complément de Frobenius est définissable puisque la connexité en découle.

Un critère de scission. On démontre un critère de scission pour les quasi groupes de Frobenius rangés connexes avec complément définissable.

Proposition 4.2.3 (à comparer avec le lemme 11.21 de [Borovik et Nesin 1994]). *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons qu'il existe des involutions et qu'elles sont toutes contenues dans $\bigcup_G C^g$. On suppose qu'il existe un groupe A définissable connexe normal infini d'intersection triviale avec C. Alors il existe un groupe définissable connexe abélien A_1 tel que $G = A_1 \rtimes C$.*

Démonstration. Soit $i \in C$ une involution, alors i agit par inversion sur A ; sinon, $C_A(i)^\circ \leq C$: contradiction. Le groupe A est donc abélien uniquement 2-divisible ; si A contenait une involution j, alors on aurait $A \leq C_j$: contradiction. On considère maintenant $C_G(A)^\circ$. On a bien $C_G(A)^\circ \cap C_i = \{1\}$; sinon, soit $1 \neq x \in C_G(A)^\circ \cap C_i$, alors $A \leq C_G(x)^\circ \leq C_i$: contradiction. L'involution i agit aussi par inversion sur $C_G(A)^\circ$, qui est donc abélien. On a de plus $C_G(C_G(A)^\circ)^\circ = C_G(A)^\circ$, donc quitte à remplacer A par $C_G(A)^\circ$, on peut supposer que $C_G(A)^\circ = A$.

Soit $g \in G$, alors $[i, g] \leq C_G(A)$. En effet, i et i^g agissent par inversion et $[i, g] = ii^g$. Par conséquent, $[i, G] \leq C_G(A)^\circ = A$. Soit $g \in G$, on considère $a = [i, g]$; puisque A est 2-divisible, il existe $a' \in A$ tel que $a = a'^2 = [i, a']$. Par conséquent, on a

$$[i, ga'^{-1}] = ia'g^{-1}iga'^{-1} = a'^i[i, g]a'^{-1} = a'^{-1}a'^2a'^{-1} = 1.$$

On a donc $G = AC_G(i) = C_G(i)A$, mais $C_G(i) \cap A = \{1\}$ (sinon, A contiendrait un élément a tel que $a^i = a = a^{-1}$, i.e., une involution : contradiction). Ainsi, $G = A \rtimes C_G(i)$. On déduit que $C_G(i) = C_G(i)^\circ = C$ et donc $G = A \rtimes C$. $\square$

2-*torsion.* Ce sont les quasi groupes de Frobenius contenant des involutions qui nous intéressent principalement. Nous montrons un certain nombre de résultats qui clarifient la situation de la 2-torsion dans ce contexte.

Lemme 4.2.4 [Deloro et Wiscons 2020, Proposition 1]. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est de type impair. Alors les 2-éléments appartiennent à des tores décents et sont contenus dans $\bigcup_G C^g$.*

Pour les groupes de Frobenius connexes, nous avons la situation suivante :

Fait 4.2.5 [Borovik et Nesin 1994, Lemma 11.20]. *Soit $C < G$ un groupe de Frobenius rangé connexe avec involutions. Alors les involutions de G sont conjuguées et C contient au plus une involution.*

Proposition 4.2.6. *Soit $C < G$ un groupe de Frobenius rangé connexe avec involutions. Alors G est $U_2^\perp$ si et seulement si C contient des involutions.*

Démonstration. Supposons que G contient un 2-groupe abélien élémentaire infini A. Supposons par l'absurde que C contient des involutions. Puisque par le fait 4.2.5 les involutions forment une classe de conjugaison, on a $I \subseteq \bigcup_G C^g$. En particulier, il existe $1 \neq x \in A \cap C$ (quitte à conjuguer) et $A \leq C_G(x) \leq C$. Mais C contient au plus une involution par le fait 4.2.5 : contradiction. Donc C ne contient pas d'involutions.

Réciproquement, supposons que C ne contient pas d'involutions. Alors les 2-sous-groupes de Sylow° de G sont d'exposant fini, et en particulier contiennent des 2-groupes abéliens élémentaires infinis. En effet, soit S un 2-sous-groupe de Sylow° et supposons qu'il contient un 2-tore ; on peut l'étendre en un tore décent maximal T. Mais $C_G(T)$ est généreux et il existe donc $1 \neq y \in C_G(T) \cap C^g$. Par conséquent, $T \leq C_G(y) \leq C^g$: contradiction. □

Corollaire 4.2.7. *Soit G un groupe de Frobenius rangé connexe. Alors G ne peut pas être de type mixte.*

Remarque 4.2.8. T. Clausen et K. Tent [2021, Proposition 6.2] aboutissent à un résultat similaire.

Nous rappelons maintenant quelques faits concernant les groupes de type impair dont le rang de Prüfer est égal à 1.

Fait 4.2.9 [Deloro et Jaligot 2010, Proposition 27]. *Soit G un groupe rangé connexe de type impair, de rang de Prüfer égal à 1. Alors il y a trois possibilités pour le type d'isomorphisme des 2-sous-groupes de Sylow (S° est un 2-tore de rang 1) :*

(1) $S = S^\circ \rtimes \langle i \rangle$ *pour i une involution agissant sur S° par inversion (type* $\mathrm{PGL}_2(\mathbb{C})$*).*

(2) $S = S^\circ \cdot \langle w \rangle$ *pour w un élément d'ordre 4 agissant sur S° par inversion. On a de plus que w^2 est l'involution de S° (type* $\mathrm{SL}_2(\mathbb{C})$*).*

(3) $S = S^\circ$.

Fait 4.2.10 [Deloro 2009]. *Soit G un groupe rangé et $U_p^\perp$, et soit $x \in G$ un p-élément tel que $x^{p^n} \in Z(G)$. Alors l'exposant du groupe $C_G(x)/C_G(x)^\circ$ divise p^n.*

La proposition suivante caractérise complètement la structure de la 2-torsion des quasi groupes de Frobenius rangés connexes de degré pair et de type impair :

Fait 4.2.11 [Deloro et Wiscons 2020, Proposition 1]. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est de degré pair et de type impair. Alors G a les mêmes 2-sous-groupes de Sylow que* $\mathrm{PGL}_2(\mathbb{C})$. *De plus, on a $N_G(C) = C \rtimes \langle k \rangle$ avec k une involution agissant par inversion. En particulier, C est un groupe abélien.*

On remarque que dans ce cas $N_G(C) \setminus C$ est composé d'involutions agissant par inversion. En effet, $ckck = cc^{-1} = 1$; soit j une telle involution, on a $C^j = C$ et l'action est par inversion car $C_j \neq C$.

On peut déterminer de la même manière la structure des 2-sous-groupes de Sylow pour les groupes de Frobenius rangés connexes de type impair. En effet, le rang de Prüfer est égal à 1 (fait 4.2.5).

Proposition 4.2.12. *Soit $C < G$ un groupe de Frobenius rangé connexe de type impair. Alors les 2-sous-groupes de Sylow de G sont soit connexes, soit du type* $\mathrm{SL}_2(\mathbb{C})$

Démonstration. Soit S un 2-sous-groupe de Sylow et S° sa composante connexe ; on peut l'étendre à un tore décent maximal T. Il existe donc un conjugué C^g tel que $T \leq C^g$. On note i l'involution contenue dans C^g. Soit $j \neq i$ une involution et soit C_j le conjugué de C qui la contient. Si j agissait par inversion sur S° alors j commuterait avec i. Par conséquent, $1 \neq i \in C_j$: contradiction. Donc les 2-sous-groupes de Sylow ne sont pas du type $\mathrm{PGL}_2(\mathbb{C})$, et on conclut par le fait 4.2.9. □

Pour les quasi groupes de Frobenius rangés connexes de degré impair supérieur à 1 et de type impair, nous n'avons pas de contrôle sur le rang de Prüfer. Cependant, les involutions forment une classe de conjugaison.

Fait 4.2.13 [Deloro et Wiscons 2020, Proposition 1]. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est de type impair et de degré impair. Alors $N_G(C)$ est un sous-groupe fortement inclus. En particulier, les involutions forment une classe de conjugaison dans G.*

Remarque 4.2.14. Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est de degré pair et de type impair. Alors les involutions sont conjuguées. En effet, le rang de Prüfer est égal à 1.

4.3. ***Le cas résoluble.*** Avant d'étudier le cas résoluble, rappelons un certain nombre de résultats utiles sur les sous-groupes de Carter, i.e., les sous-groupes définissables connexes nilpotents presque autonormalisants.

Fait 4.3.1. *(1)* [Frécon et Jaligot 2008, Theorem 3.11]. *Soit G un groupe rangé. Alors G contient un sous-groupe de Carter.*

(2) [Jaligot 2006, Theorem 3.1]. *Soit G un groupe rangé connexe. Alors les sous-groupes de Carter généreux sont conjugués.*

(3) [Wagner 1994, Theorem 29]. *Soit G un groupe rangé connexe résoluble. Alors ses sous-groupes de Carter sont conjugués.*

(4) [Frécon 2009, Corollary 2.10]. *Soit G un groupe rangé. Alors tout tore décent de G est contenu dans un sous-groupe de Carter.*

(5) [Frécon 2000, théorèmes 1.1 et 1.2]. *Soit G un groupe rangé connexe résoluble. Alors tout sous-groupe H contenant un sous-groupe de Carter est définissable, connexe et autonormalisant.*

(6) [Frécon et Jaligot 2008, Theorem 3.11]. *Soit G un groupe rangé résoluble. Alors ses sous-groupes de Carter sont généreux et génériquement disjoints de leurs conjugués.*

(7) [Altınel et al. 2008, Chapter I, Corollary 8.30]. *Soit G un groupe rangé connexe résoluble. Alors $G = G'Q$ où Q est un sous-groupe de Carter.*

Lemme 4.3.2 (à comparer avec [Borovik et Nesin 1994, Lemma 11.32]). *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est résoluble. Alors G n'est pas nilpotent et $G = G' \rtimes C$ avec C abélien. De plus, on peut interpréter un corps K tel que $C \hookrightarrow K^\times$.*

Démonstration. Soit H un sous-groupe définissable connexe nilpotent normal. Alors $H \cap C = \{1\}$. En effet, soit $1 \neq c \in H \cap C$, alors $Z(H)^\circ$ (infini, car H est infini connexe nilpotent) commute avec c et donc $Z(H)^\circ \leq C$. Mais alors $H \leq C_G(Z(H)^\circ)^\circ \leq C$ et donc $C = G$: contradiction. Le groupe G est en particulier non nilpotent. Le groupe G' est un sous-groupe définissable connexe nilpotent normal et donc $G' \cap C = \{1\}$. Par conséquent, C est abélien et c'est un sous-groupe de Carter ; il est en particulier autonormalisant par le fait 4.3.1. D'après le fait 4.3.1, on a donc $G = G' \rtimes C$.

Soit $B \leq G'$ un sous-groupe définissable connexe abélien C-minimal. On a bien $C_C(B) = \{1\}$. Soit $1 \neq c \in C_C(B)$, alors $B \leq C_G(c)^\circ \leq C$: contradiction. On peut donc appliquer [Borovik et Nesin 1994, Theorem 9.1] pour conclure. □

Corollaire 4.3.3. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est résoluble. Alors G est un groupe de Frobenius.*

Lemme 4.3.4. *Soit $G = U \rtimes C$ un groupe rangé où U est définissable connexe nilpotent et chaque $c \neq 1$ agit sur U en fixant au plus un nombre fini de points. Alors $U \setminus \{1\} = G \setminus \bigcup_G C^g$.*

Démonstration. Par l'exercice 10 (a) à la page 98 de [Borovik et Nesin 1994], on a $U = [U, c]$ pour $c \in C$. Soit $g = uc \in G \setminus U$ avec $u \in U$ et $c \in C$. Soit $u_1 \in U$ tel que $u = [u_1^{-1}, c^{-1}]$, alors $g^{u_1} \in C$. □

Récapitulons maintenant les résultats obtenus dans le cas résoluble.

Théorème 4.3.5. *Soit $C < G$ est un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est résoluble, alors* :

(1) $C < G$ *est un groupe de Frobenius.*

(2) C *est abélien.*

(3) G *n'est pas nilpotent.*

(4) $G' = \left(G \setminus \left\{\bigcup_G C^g\right\}\right) \cup \{1\}$.

(5) $G = G' \rtimes C$.

(6) Il existe un corps définissable K *tel que* $C \hookrightarrow K^\times$ *et* $\mathrm{RM}(C) \leq \mathrm{RM}(G')$.

4.4. *Le groupe de Weyl.* Cette section est une digression concernant la notion de groupe de Weyl au sein des quasi groupes de Frobenius rangés connexes. Une façon d'analyser cette classe de groupes consiste à s'intéresser à $N_G(C)/C$, une forme de *groupe de Weyl*. En effet, pour un groupe algébrique réductif, le groupe de Weyl désigne $N_G(T)/C_G(T)$ où T est un tore (algébrique) maximal. Or, pour les groupes de Frobenius algébriques connexes, le complément de Frobenius est précisément un tore maximal ; on renvoie à [Hertzig 1961, Proposition 1]. De même dans $\mathrm{PGL}_2(K)$ qui est l'exemple paradigmatique de quasi groupe de Frobenius algébrique de degré pair, le quasi complément de Frobenius est un tore (algébrique) maximal.

En général dans le contexte rangé, plusieurs notions de groupes de Weyl peuvent être envisagées. Si on suppose que le quasi complément de Frobenius est résoluble, alors elles coïncident toutes pour les quasi groupes de Frobenius rangés connexes de type impair. Nos résultats s'inspirent largement de [Altınel et al. 2013] qui étudie les groupes de Weyl des groupes simples minimaux connexes rangés, i.e., des groupes simples rangés tels que les sous-groupes définissables connexes propres sont résolubles.

Il nous faut d'abord décrire la structure des sous-groupes de Carter dans les quasi groupes de Frobenius rangés connexes.

Lemme 4.4.1. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable et résoluble. Alors C contient un sous-groupe de Carter de G généreux dans G. De plus, un sous-groupe de Carter de G est soit généreux contenu dans un conjugué de C, soit il n'est pas généreux, intersecte trivialement C et ses conjugués, et ne contient pas de torsion divisible.*

Démonstration. D'après le fait 4.3.1, le groupe C contient un sous-groupe de Carter Q. Or, d'après ce même fait, les sous-groupes de Carter d'un groupe connexe résoluble sont autonormalisants, conjugués et généreux. Puisque C est généreux et connexe, par le fait 4.1.1, le groupe Q est un sous-groupe de Carter de G généreux dans G, car $N_G(Q)^\circ = N_C(Q)^\circ = Q$. En particulier, tous les sous-groupes de Carter de C sont des sous-groupes de Carter généreux de G.

Si Q_1 est un sous-groupe de Carter généreux de G, alors $Q_1 \cap C$ (quitte à conjuguer C) est non trivial. Puisque $Z(Q_1)^\circ \neq \{1\}$, on a $Q_1 \leq C$. Inversement, si Q_1 est un sous-groupe de Carter qui intersecte C non trivialement, alors $Q_1 \leq C$ (condition du normalisateur) et c'est un sous-groupe de Carter généreux. Si Q_1 intersecte trivialement C et ses conjugués, il ne contient pas de torsion divisible. □

Proposition 4.4.2 (à comparer avec [Altınel et al. 2013, Proposition 3.2]). *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est de type impair. Alors on a $N_G(C) = CN_G(T)$ où T est un tore décent maximal de C. Si de plus C est résoluble alors $W(G) = N_G(T)/C_G(T) \simeq N_G(Q)/Q \simeq N_G(C)/C$ où Q est un sous-groupe de Carter généreux de G.*

Démonstration. Soit T un tore décent maximal de C. D'après le fait 4.1.8, les tores décents maximaux de C sont conjugués dans C. On applique un argument de Frattini au groupe $N_G(C)$: soit $n \in N_G(C)$, on a $T^n \leq C$ et par conjugaison des tores décents maximaux, il existe $c \in C$ tel que $c^{-1}n \in N_G(T)$; par conséquent $N_G(C) = CN_{N_G(C)}(T)$. Mais $N_G(T) \leq N_G(C)$; par conséquent, $N_G(C) = CN_G(T)$.

Supposons maintenant que C est résoluble. Soit Q un sous-groupe de Carter généreux de G. Quitte à conjuguer, on peut supposer que $T \leq Q \leq C$. On rappelle que les sous-groupes de Carter de C sont conjugués et autonormalisants. Un argument de Frattini appliqué à $N_G(C)$ nous donne $N_G(C) = CN_G(Q)$, car $N_{N_G(C)}(Q) = N_G(Q)$. Par conséquent, $N_G(C)/C \simeq N_G(Q)/(C \cap N_G(Q)) = N_G(Q)/Q$. De plus, $Q \leq C_G(T) \leq C$; donc $C_G(T)$ est définissable connexe résoluble. Il suit que Q est autonormalisant dans $C_G(T)$. Cette fois-ci un argument de Frattini appliqué à $N_G(T)$ nous donne $N_G(T) = C_G(T)N_G(Q)$, car $N_G(Q) \leq N_G(T)$ puisque T est caractéristique dans Q. Puisque Q est autonormalisant dans $C_G(T)$, on a $N_G(T)/C_G(T) \simeq N_G(Q)/Q$. Finalement, $W(G) \simeq N_G(Q)/Q \simeq N_G(C)/C$. □

On obtient de façon similaire la proposition suivante (pour plus de détails, nous renvoyons à [Zamour 2022, chapitre 4]) :

Proposition 4.4.3 (à comparer avec [Altınel et al. 2013, Corollary 3.4]). *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable et résoluble. Alors pour un p-tore maximal non trivial S, on a $N_G(S)/C_G(S) \simeq W(G)$.*

Corollaire 4.4.4. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable et résoluble. Supposons que G est de degré impair supérieur à* 1 *et de type impair. Alors le* 2*-rang de Prüfer est supérieur ou égal à* 2*.*

Démonstration. Supposons que le sous-groupe C contient un 2-tore maximal T_2 de rang de Prüfer égal à 1. D'après la proposition 4.4.3, $N_G(T_2)/C_G(T_2) \simeq N_G(C)/C \neq \{1\}$. Or, d'après le fait 29 de [Deloro et Jaligot 2010], $\mathrm{Aut}(T_2) \simeq Z_2^\times$ et le seul automorphisme d'ordre fini non trivial est l'inversion. Le groupe $N_G(C)/C$ serait d'ordre 2 : contradiction. $\square$

4.5. *À la recherche d'un sous-groupe de Borel fortement standard.* L'étude des involutions et des translations constitue un pan essentiel de l'analyse des groupes de type impair. Rappelons que A. Deloro et J. Wiscons montrent un résultat très important sur la situation des translations dans notre contexte (théorème 2.0.9) :

Théorème 2.0.9. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable et connexe. Supposons que G est de type impair. Alors $\bigcup_{g \in G} C^g$ ne contient pas toutes les translations.*

Remarque 4.5.1. Si $C < G$ est un groupe de Frobenius connexe de type impair, alors aucune translation n'est contenue dans $\bigcup_{g \in G} C^g$. Soit $x = ij \in C^g$, alors $i, j \in N_G(C^g) = C^g$, et donc $i = j$, car le complément de Frobenius contient une unique involution.

Sur la base de ce théorème, nous allons montrer l'existence d'un sous-groupe de Borel non nilpotent qui intersecte C (quitte à conjuguer) dans une partie infinie, mais qui n'est pas contenu dans C. Idéalement, on conjecture qu'un tel sous-groupe de Borel devrait contenir tout C.

Définition 4.5.2. Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Soit B un sous-groupe de Borel. Alors :
— B est faiblement standard si $1 < (B \cap C) < B$.
— B est standard si $1 < (B \cap C) < B$ et $B \cap C$ contient des involutions.
— B est fortement standard si $C < B$.

Autrement dit, nous allons montrer que G contient un sous-groupe de Borel faiblement standard. D'après le théorème 4.3.5, un sous-groupe de Borel B faiblement standard n'est pas nilpotent, c'est un groupe de Frobenius tel que $B = B' \rtimes (B \cap C)$ avec $B \cap C \hookrightarrow K^\times$ pour un corps K définissable. Si B est standard, le sous-groupe $(B \cap C)$ contient des involutions qui inversent donc le groupe abélien B'. Finalement, un sous-groupe de Borel fortement standard B s'écrit sous la forme $B = B' \rtimes C$. Remarquons que dans certaines configurations de type CiBo le quasi complément C est précisément un sous-groupe de Borel. L'existence d'un sous-groupe de Borel fortement standard est donc une hypothèse très forte qui permettrait d'éliminer des configurations pathologiques.

On utilise le terme « standard », car dans le cas d'un quasi groupe de Frobenius rangé connexe de degré pair, nous cherchons à identifier un sous-groupe de Borel dont la structure est similaire à celle des sous-groupes de Borel de $\mathrm{PGL}_2(K)$ à savoir isomorphes au groupe strictement 2-transitif $K^+ \rtimes K^\times$. Remarquons que dans $\mathrm{PGL}_2(\mathbb{C})$ un sous-groupe de Borel est généreux mais il n'est pas génériquement disjoint de ses conjugués. En effet, soient $B \neq B^g$ deux sous-groupes de Borel ; ils s'intersectent dans un tore maximal T qui est autocentralisant dans G (groupe réductif). Or, $C_B(T) = T$ est généreux dans B ; par conséquent $\mathrm{RM}\big(B \setminus \bigcup_{G/N_G(B)} B^g\big) < \mathrm{RM}(B)$. Pour les groupes de Frobenius connexes, nous cherchons plutôt à montrer qu'ils sont résolubles.

Tout d'abord, voici un lemme inspiré de la démonstration du théorème B de [Deloro et Wiscons 2020].

Lemme 4.5.3. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est de type impair. Soient i, j deux involutions. Si C est $\langle i, j\rangle$-invariant, alors $ij \in \bigcup_G C^g$.*

Démonstration. Si la configuration est de degré impair, alors $i, j \in C$. Reste le cas de degré pair. Alors $i, j \in N_G(C)$. Si les deux involutions sont dans C, $1 = ij \in C$ (fait 4.2.11). Si une seule est dans C, alors elle est centralisée par l'autre : ij est une involution donc elle appartient à $\bigcup_G C^g$. Enfin, si $i, j \in N_G(C) \setminus C$, alors $ij \in C_G(C)$ et donc ou bien $ij \in C$, ou bien ij est une involution contenue dans un conjugué de C. □

Nous pouvons maintenant énoncer le théorème :

Théorème 4.5.4. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est de type impair. Alors il existe un sous-groupe A définissable connexe abélien inversé par (au moins) deux involutions $i \neq j$, tel que $A \cap \bigcup_G C^g = \{1\}$, et maximal pour ces propriétés. De plus, on a les propriétés suivantes :*

(1) $C_G(A)^\circ = A$.

(2) A est TI.

(3) Soit $N_G(A) = N$. Alors A n'est pas presque autonormalisant et $N^\circ = A \rtimes (N^\circ \cap C)$.

Démonstration. Par le théorème 2.0.9, on trouve, pour deux involutions $i \neq j$, une translation $x = ij$ qui n'est pas contenue dans $\bigcup_G C^g$. On considère $A = C_G(x)^\circ$, qui est infini par le fait 4.1.4. Remarquons qu'il s'agit d'un sous-groupe $\langle i, j\rangle$-invariant tel que $C_A(i)^\circ = C_G(i, j)^\circ = C_A(j)^\circ$. Supposons que le groupe $\langle i, j\rangle$-invariant $C_A(i)^\circ$ est infini. On a donc $C_A(i)^\circ \leq C_G(i)^\circ \leq C_i$. Puisque C_i est TI, il est aussi $\langle i, j\rangle$-invariant et donc $x \in \bigcup_G C^g$ (lemme 4.5.3) : contradiction. Par conséquent, le groupe $C_A(i) = C_A(j)$ est fini et donc A est un groupe définissable connexe abélien inversé par i et j. De plus, $A \cap \bigcup_G C^g = \{1\}$; le groupe A est en particulier uniquement 2-divisible.

Il existe donc un sous-groupe définissable connexe abélien inversé par au moins deux involutions et d'intersection triviale avec les conjugués de C. On prend A maximal pour ces propriétés et on pose $i \neq j$ deux involutions l'inversant. Montrons tout d'abord $C_G(A)^\circ = A$. On considère $C_G(A)^\circ$; c'est un groupe $\langle i, j\rangle$-invariant, inversé par i, j. De plus, $C_G(A)^\circ \cap \bigcup_G C^g = \{1\}$. Par maximalité de A et puisque A est abélien, on a $C_G(A)^\circ = A$. Montrons maintenant que le groupe A est TI. Supposons par l'absurde qu'il existe un conjugué $A' \neq A$ et $1 \neq x' \in A' \cap A$. L'élément x' est une translation inversée par i. De plus, $A, A' \leq C_G(x')^\circ = A_1$. En raisonnant comme précédemment, on obtient que le groupe A_1 est abélien inversé par i et j. Ainsi, $A_1 = C_G(A)^\circ = A = A'$: contradiction. Donc A est TI.

Le sous-groupe A n'est pas presque autonormalisant : si $N_G(A)^\circ = A$, alors le groupe A est généreux et intersecte donc non trivialement un conjugué de C : contradiction. On pose $N = N_G(A)$. Remarquons que $i, j \in N$. Si les groupes $C_N(i)$ et $C_N(j)$ étaient finis alors i, j inverseraient le groupe abélien N°. En particulier, $N^\circ = C_G(A)^\circ = A$: contradiction. Par conséquent, $(N^\circ \cap C_i) < N^\circ$ est un quasi groupe de Frobenius connexe avec complément définissable (lemme 4.2.1). On lui applique la proposition 4.2.3 pour obtenir que $N^\circ = A \rtimes (C_i \cap N^\circ)$. □

Si C est résoluble, pour obtenir un sous-groupe de Borel faiblement standard, il suffit de considérer un sous-groupe de Borel B contenant le groupe connexe résoluble N°.

Remarque 4.5.5. On rappelle qu'un sous-groupe de Borel faiblement standard B s'écrit sous la forme $B = B' \rtimes (B \cap C)$, avec $(B \cap C) \hookrightarrow K^\times$, pour un corps K. Par conséquent, si B contient un p-groupe unipotent, alors sous l'hypothèse de l'infinité des p-nombres premiers de Mersenne, le fait 4.1.7 nous permet de conclure que B est un sous-groupe de Borel standard.

Comment obtenir un sous-groupe de Borel fortement standard ? Dans [Frécon et Jaligot 2008], les auteurs considèrent plusieurs conjectures de généricité, notamment la *générosité des sous-groupes de Borel* (GB) et *la générosité des sous-groupes de Carter* (GC) (dans le deuxième cas, par [Jaligot 2006], cela implique la conjugaison des sous-groupes de Carter). On peut envisager ces hypothèses de générosité comme la reformulation dans le contexte rangé d'hypothèses topologiques de densité.

L'hypothèse (GB) est suffisante pour démontrer l'existence d'un sous-groupe de Borel fortement standard si le quasi complément de Frobenius est nilpotent. Néanmoins, dans les configurations de type CiBo, il existe des sous-groupes de Borel qui ne sont pas généreux ; voir [Cherlin et Jaligot 2004], par exemple le lemme 5.11.

Lemme 4.5.6. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable et nilpotent. Soit B un sous-groupe de Borel tel que $1 < B \cap C < B$. Si B est généreux, alors $C < B$.*

Démonstration. D'après le lemme 4.2.1, $(B \cap C) < B$ est un quasi groupe de Frobenius connexe avec complément définissable. Notamment, $B \cap C$ est généreux dans B, qui l'est dans G ; comme en outre B est connexe, par transitivité $B \cap C$ est généreux dans G. Notamment, $N_G(B \cap C)^\circ = (B \cap C)$. Par nilpotence de C et croissance des normalisateurs, $C < B$. □

4.6. *Théorèmes de classification.* Nous pouvons maintenant démontrer nos deux théorèmes de classification :

Théorème 1.0.6. *Soit $C < G$ un quasi groupe de Frobenius connexe rangé tel que C est définissable. Supposons que G est de degré pair et de type impair. Si les sous-groupes de Borel sont généreux, alors G est isomorphe à* $\mathrm{PGL}_2(K)$ *pour un corps K (rangé) algébriquement clos de caractéristique différente de* 2.

Théorème 1.0.7. *Soit $C < G$ un groupe de Frobenius connexe rangé de type impair avec C résoluble et G non résoluble. Si les sous-groupes de Borel sont généreux, alors C n'est pas nilpotent.*

On démontrera également les théorèmes précédents en remplaçant l'hypothèse de générosité des sous-groupes de Borel par celle de l'existence d'un sous-groupe de Borel fortement standard. Nous formulerons certains lemmes intermédiaires sous l'hypothèse de l'existence d'un sous-groupe de Borel faiblement standard (ou standard). Nous souhaitons dans la mesure du possible éviter d'utiliser toute la force de l'hypothèse de l'existence d'un sous-groupe de Borel fortement standard.

Pour les quasi groupes de Frobenius rangés connexes de degré pair, nous savons que le quasi complément de Frobenius est abélien. Néanmoins, rappelons qu'il pourrait s'agir d'un sous-groupe de Borel (une situation fortement non algébrique). On retrouve là l'un des contre-exemples (CiBo_2) minimaux potentiels à la conjecture de Cherlin–Zilber pour les groupes de petit rang de Prüfer ; voir [Deloro et Jaligot 2016]. Une résolution complète de la conjecture A_1 de [Deloro et Wiscons 2020] permettrait donc l'élimination de cette configuration pathologique.

L'identification de $\mathrm{PGL}_2(K)$ se fait via l'existence d'une (B, N)-paire scindée spéciale de rang 1. Dans tout groupe algébrique réductif, un sous-groupe de Borel B et $N_G(T)$ où T est un tore maximal contenu dans B forment une (B, N)-paire. Le rang de la (B, N)-paire correspond au nombre d'involutions qui engendrent le groupe $N/B \cap N$. Plus spécifiquement, dans $\mathrm{PGL}_2(K)$, une telle (B, N)-paire est scindée de rang 1 ; on a en effet $B = B_u \rtimes T$ où B_u est le radical unipotent et T le tore algébrique maximal. De plus, dans ce cas, un tore maximal T est un groupe TI presque autonormalisant tel que $N_G(T)/T$ est d'exposant 2.

Nous nous inspirons de la stratégie employée dans [Deloro et Jaligot 2016] ; voir en particulier la proposition 3. Nous renvoyons également à [Deloro 2007; Cherlin et Jaligot 2004].

Une (B, N)-paire scindée spéciale de rang 1 est un quadruplet de groupes B, N, $H = N \cap B$ et U un groupe nilpotent avec H normal dans N, qui satisfait les axiomes suivants :

(1) $G = \langle B, N\rangle$.

(2) $[N : H] = 2$.

(3) Pour tout $k \in N \setminus H$, on a $H = B \cap B^k$, $G = B \cup BkB$, et $B^k \neq B$.

(4) $B = U \rtimes H$.

Les deux théorèmes suivants permettent d'identifier des groupes algébriques via des (B, N)-paires dans le contexte rangé.

Fait 4.6.1 (adapté de [De Medts et Tent 2008, Theorem 2.1]). *Dans un univers rangé, soit (B, N, U, H) une (B, N)-paire définissable scindée spéciale de rang* 1 *telle que U est abélien divisible sans torsion. Alors $G \simeq \mathrm{PGL}_2(K)$.*

Fait 4.6.2 (adapté de [Wiscons 2011, Theorem 1.2]). *Dans un univers rangé, soit (B, N, U, H) une (B, N)-paire définissable scindée spéciale de rang* 1. *Supposons de plus que U est un groupe abélien contenant un élément d'ordre $p > 2$. Soient $L = \bigcap_G B^g$ et $M = \langle U^g : g \in G\rangle L$; si $H \neq L$ et si $(H \cap M)/L$ ne contient pas de p-groupe abélien élémentaire, alors $G/L \simeq \mathrm{PGL}_2(K)$.*

Sous-groupe de Borel standard.

Notation. Soit B un sous-groupe définissable et soit k une involution qui ne normalise pas B. On pose

$$T_B(k) = \{b \in B : b^k = b^{-1}\}.$$

Lemme 4.6.3. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable et résoluble. Supposons que G n'est pas résoluble et qu'il est de type impair. Soit B un sous-groupe de Borel standard et soit i une involution dans $B \cap C$ telle que $N_G(B) \cap i^G = B \cap i^G$. Alors $i^G \setminus N_G(B)$ et*

$$K_B = \{k \in i^G \setminus N_G(B) : \mathrm{RM}(T_B(k)) \geq \mathrm{RM}(B) - \mathrm{RM}(C_G(i))\}$$

sont génériques dans i^G.

Démonstration. Les involutions de G forment une classe de conjugaison i^G par le fait 4.2.13 et la remarque 4.2.14. Par hypothèse, $N_G(B) \cap i^G = B \cap i^G$. Supposons que $i^G \setminus N_G(B)$ n'est pas générique dans i^G, i.e., $i^G \cap N_G(B) = i^G \cap B$ est large dans i^G. Le groupe $N = \bigcap_G B^g = \bigcap_{1\leq i\leq n} B^{g_i}$ est un groupe définissable normal contenant certaines involutions, et donc toutes d'entre elles, car $i^G \setminus \bigcup_G B^g = i^G \setminus \bigcup_{1\leq i\leq n} B^{g_i}$ est non générique ; c'est en particulier un sous-groupe infini.

Si $H = C \cap N^\circ$ est trivial, alors par la proposition 4.2.3, $G = A \rtimes C$ pour un sous-groupe définissable abélien A et G est résoluble : contradiction. Le sous-groupe $H < N^\circ$ est donc un quasi groupe de Frobenius connexe avec complément définissable car $N^\circ \not\leq C$ (lemme 4.2.1). Mais pour tout $g \in G$, il existe $n \in N^\circ$

tel que $H^g = H^n$. Par conséquent, $G = N^\circ \cdot N_G(H)$. Mais $N_G(H) \le N_G(C)$. Or, $N_G(C)$ est résoluble ; le groupe G est donc résoluble : contradiction.

Supposons maintenant que $i^G \setminus N_G(B)$ est générique dans i^G. Dans la suite, nous reprenons la démonstration de la proposition 2 de [Deloro et Jaligot 2016] pour obtenir que

$$K_B = \{k \in i^G \setminus N_G(B) : \mathrm{RM}(T_B(k)) \ge \mathrm{RM}(B) - \mathrm{RM}(C_G(i))\}$$

est aussi générique dans i^G. On considère la fonction définissable $\phi : i^G \setminus N_G(B) \to G/B$ qui envoie k sur sa classe kB. Le domaine est de rang $\mathrm{RM}(i^G) = \mathrm{RM}(G) - \mathrm{RM}(C_G(i))$ et l'image de rang au plus $\mathrm{RM}(G) - \mathrm{RM}(B)$. La fibre $\phi^{-1}(\phi(k))$ pour k générique dans i^G est donc de rang au moins $\mathrm{RM}(B) - \mathrm{RM}(C_G(i))$. Mais si k et j appartiennent à la même fibre, on a $jB = kB$ et donc $kj \in T_B(k)$.

Par conséquent, pour k une involution générique, on a

$$\mathrm{RM}(T_B(k)) \ge \mathrm{RM}(\phi^{-1}\phi(k)) \ge \mathrm{RM}(B) - \mathrm{RM}(C_G(i)). \qquad \square$$

Lemme 4.6.4. *Soit $C < G$ un groupe de Frobenius rangé connexe de type impair avec C résoluble et G non résoluble. Supposons qu'il existe un sous-groupe de Borel standard B tel que B' est TI. Alors $T_B(k)$ est fini, pour $k \in K_B$; de même que $B \cap B^k$. De plus,* $\mathrm{RM}(B) \le \mathrm{RM}(C)$.

Démonstration. Soit $i \in C$ l'unique involution de C. Pour un sous-groupe H, on pose $I(H) = \{1 \ne h \in H : h^2 = 1\}$. On a $N_G(B) \cap i^G = B \cap i^G$. En effet, soit $j \in N_G(B)$, alors $C_B(j)$ n'est pas trivial, car B n'est pas nilpotent (théorème 4.3.5). Il suit que $1 < (B \cap C_j) < B$ est un groupe de Frobenius connexe. D'après le lemme 4.2.1, $(B \cap C_j)$ et $(B \cap C_i)$ sont conjugués dans B : le sous-groupe $B \cap C_j$ contient une involution qui est égale à j et donc $j \in B$. Enfin, $I(N_G(B \cap C)) \le I(N_G(C)) = I(C) = \{i\}$. D'après le lemme 4.6.3, $i^G \setminus N_G(B)$ et $K_B = \{k \in i^G \setminus N_G(B) : T_B(k) \ge \mathrm{RM}(B) - \mathrm{RM}(C_G(i))\}$ sont génériques dans i^G.

Soit $k \in K_B$; montrons que si $x \in B$ est inversé par k alors x est trivial. Supposons que $x \ne 1$. Supposons d'abord que $x \in B \cap C$. Alors k normalise $C_G(x)^\circ$ puis $B \cap C$, donc $k = i$: contradiction. De même pour $x \in (B \cap C)^b$ avec $b \in B$. D'après le théorème 4.3.5, $x \in B'$ et donc $x^{-1} \in B' \cap (B')^k$. Puisque B' est TI, on a $B' = (B')^k$. Comme $N_G(B')^\circ$ est résoluble (proposition 4.2.3), on obtient $B^k = B$: contradiction. Il suit que $B \cap B^k = \{1\}$: en effet, si x est dans l'intersection, alors $y = xx^{-k} \in B$ est inversé par k donc $y = 1$; ainsi $x = x^k$ et $x = 1$ pour les mêmes raisons.

Si $\mathrm{RM}(B) > \mathrm{RM}(C)$, alors $\mathrm{RM}(T_B(k)) \ge \mathrm{RM}(B) - \mathrm{RM}(C) > 0$, ce qui permet d'obtenir une contradiction. $\square$

Corollaire 4.6.5. *Soit $C < G$ un groupe strictement* 2*-transitif rangé de type impair avec C résoluble et G non résoluble. Soit $B = N_G(C_G(ij))$ pour deux involutions*

$i \neq j$, avec $i \in C$. Alors B est un sous-groupe de Borel, et $T_B(k)$ et l'intersection $B^k \cap B$ sont finis pour $k \in K_B(k)$.

Démonstration. On remarque que G est un groupe de Frobenius connexe. Soient $i \neq j$ deux involutions avec $i \in C$. Le groupe $N = N_G(C_G(ij)) = C_G(ij) \rtimes N_C(C_G(ij))$ est un groupe strictement 2-transitif par la proposition 11.51 de [Borovik et Nesin 1994]. De plus, $C_G(ij) = iI \cap jI$ est un sous-groupe TI [Borovik et Nesin 1994, Lemma 11.50]. Mais par le fait 7 et le théorème 10 de [Altınel et al. 2019], on a $N = K^+ \rtimes K^\times$, pour K un corps interprétable algébriquement clos. Montrons que N est un sous-groupe de Borel standard tel que N' est TI. Tout d'abord, d'après le théorème 4.3.5, $N' = C_G(ij)$ et $N' = \left(N \setminus \bigcup_N (C \cap N)^n\right) \cup \{1\}$. Soit B un sous-groupe de Borel contenant N. D'après le théorème 4.3.5, $B = B' \rtimes (B \cap C)$ et $B' = \left(B \setminus \bigcup_B (B \cap C)^b\right) \cup \{1\}$; les involutions i et j agissent par inversion sur le groupe abélien $B' \subseteq iI \cap jI = N' \subseteq B'$ et donc $B' = N'$. D'après la proposition 4.2.3, $N_G(B')^\circ$ est un quasi groupe de Frobenius connexe scindé et résoluble contenant B ; il lui est donc égal par maximalité de B. Par conséquent, on a $B = N$. Il suffit d'appliquer le lemme 4.6.4. □

Sous-groupe de Borel fortement standard.

Lemme 4.6.6. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable et résoluble. Supposons que G est de degré pair ou de degré* 1, *et de type impair. Supposons qu'il existe un sous-groupe de Borel fortement standard B. Alors $B^k \neq B$ pour tout $k \in N_G(C) \setminus C$. De plus, B est autonormalisant.*

Démonstration. Soit i l'involution de C ; d'après le théorème 4.3.5, on a $B = B' \rtimes C_i$. On rappelle que si G est de degré pair alors $N_G(C_i) \setminus C_i$ consiste d'involutions (fait 4.2.11) ; si G est de degré 1, alors $N_G(C_i) = C_i$. Soit $k \in N_G(C_i) \setminus C_i$ une involution (le cas de degré 1 est évident). Supposons par l'absurde que $B^k = B$. Le groupe $C_B(k)^\circ$ est infini, sinon B serait abélien et donc égal à C. D'après le lemme 4.2.1, $C_k < B$ et donc $k \in B$. Mais d'après le théorème 4.3.5, B est un groupe de Frobenius et donc $k \in C_i$: contradiction.

Montrons maintenant que B est autonormalisant. Soit $g \in N_G(B) \setminus B$; le groupe C_i^g est un quasi complément de Frobenius de B et par le lemme 4.2.1, il existe $b \in B$ tel que $C_i^{gb^{-1}} = C_i$. On a que $g \in B$ si et seulement si $gb^{-1} \in B$. On peut donc supposer que $g \in N_G(C_i)$. Si $g \in C_i$ alors on conclut. Sinon, g est une involution normalisant C_i et B ; le paragraphe précédent montre qu'alors $g = i \in C_i$. □

Lemme 4.6.7. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable et résoluble. Supposons que G n'est pas résoluble et qu'il est de type impair. On suppose qu'il existe un sous-groupe de Borel fortement standard B. Soit i une involution de $B \cap C$ telle que $i^G \setminus B$ et*

$$K_B = \{k \in i^G \setminus B : \mathrm{RM}(T_B(k)) \geq \mathrm{RM}(B) - \mathrm{RM}(C_G(i))\}$$

sont génériques dans i^G. *Alors pour toute involution* $k \in K_B$, *le groupe* $I_k = (B \cap B^k)^\circ$ *est un conjugué de* C *contenant une involution* j_k.

Démonstration. On a $B = B' \rtimes C$. Soit $k \in K_B$; on a tout d'abord $\mathrm{RM}(T_B(k)) \geq \mathrm{RM}(B) - \mathrm{RM}(C_G(i)) = \mathrm{RM}(B')$. Le sous-groupe I_k est k-invariant. Supposons que $C_{I_k}(k)^\circ \leq C_k$ est infini. Le groupe $B \cap C_k$ est un quasi complément de Frobenius de B donc conjugué à C et finalement $C_k < B$: contradiction.

Le sous-groupe $I_k \subseteq T_B(k) \subseteq B \cap B^k$ est donc abélien inversé par k : il suit que $\mathrm{RM}(I_k) = \mathrm{RM}(T_B(k)) \geq \mathrm{RM}(B')$. Si $I_k \subseteq (B \setminus B') \cup \{1\} = \bigcup_B C^b$ (théorème 4.3.5), alors il existe $1 \neq x \in I_k \cap C^b$, pour $b \in B$. Donc par commutativité, $I_k \leq C_G(x)^\circ \leq C^b$. Mais $\mathrm{RM}(C^b) = \mathrm{RM}(C) \leq \mathrm{RM}(B')$ (théorème 4.3.5), donc $C^b = I_k$. Sinon, $I_k \leq B'$. Puisque k agit par inversion et $\mathrm{RM}(I_k) \geq \mathrm{RM}(T_k(B)) \geq \mathrm{RM}(B')$, on a $B' = (B^k)'$. On considère $N_G(B')^\circ$ qui contient B et B^k. Mais alors $N_G(B')^\circ$ est un quasi groupe de Frobenius connexe contenant un groupe définissable connexe normal B' d'intersection triviale avec C et donc par la proposition 4.2.3, $N_G(B')^\circ = A \rtimes C$ où A est abélien. Le sous-groupe $N_G(B')^\circ$ est définissable connexe résoluble et contient B et B^k : contradiction, car k ne normalise pas B. □

Corollaire 4.6.8. *Soit* $C < G$ *un groupe de Frobenius connexe rangé de type impair avec* G *non résoluble et* C *résoluble. Alors* C *est un sous-groupe de Borel.*

Démonstration. Supposons que C est contenu strictement dans un sous-groupe de Borel B. Soit i l'unique involution de C. D'après le lemme 4.6.3, $i^G \setminus N_G(B)$ et $K_B = \{k \in i^G \setminus N_G(B) : \mathrm{RM}(T_B(k)) \geq \mathrm{RM}(B) - \mathrm{RM}(C_G(i))\}$ sont génériques dans i^G. Mais d'après le lemme 4.6.7, pour $k \in K_B$, le groupe $(B \cap B^k)^\circ = I_k$ est un conjugué de C. Mais on a alors $k \in N_G(I_k)$ et donc $k \in I_k \leq B$: contradiction, car k ne normalise pas B. □

Corollaire 4.6.9. *Soit* $C < G$ *un groupe de Frobenius connexe rangé de type impair avec* G *non résoluble et* C *résoluble. Si les sous-groupes de Borel sont généreux, alors* C *n'est pas nilpotent.*

Démonstration. D'après le théorème 4.5.4, il existe un sous-groupe de Borel faiblement standard qui est généreux par hypothèse. Si C est nilpotent, alors par le lemme 4.5.6, on a $C < B$: contradiction. □

Remarque 4.6.10. Clausen et Tent [2021] aboutissent également à un résultat de classification des groupes de Frobenius connexes de type impair avec complément nilpotent. Ils font l'hypothèse de l'absence de mauvais corps en caractéristique nulle.

Nous pouvons désormais compléter la suite des lemmes précédents pour identifier $\mathrm{PGL}_2(K)$ parmi les quasi groupes de Frobenius rangés connexes de degré pair.

Lemme 4.6.11. *Soit* $C < G$ *un quasi groupe de Frobenius rangé connexe tel que* C *est définissable. Supposons que* G *est de degré pair et de type impair. Supposons* *qu'il existe un sous-groupe de Borel fortement standard B. Soit i l'involution de $B \cap C$; supposons de plus que $i^G \setminus B$ et $K_B = \{k \in i^G \setminus B : \mathrm{RM}(T_B(k)) \geq \mathrm{RM}(B) - \mathrm{RM}(C_G(i))\}$ sont génériques dans i^G et que d'autre part $I_k = (B \cap B^k)^\circ$ est un conjugué de C contenant une involution j_k pour toute involution $k \in K_B$.*

Alors pour tout $g \in G \setminus B$, on a $G = B \cup BgB$ et donc G est un groupe 2-transitif (pour l'action sur G/B).

Démonstration. On rappelle que par le lemme 4.6.6, le groupe B est autonormalisant. Puisque le sous-groupe I_k est inversé par k, on a de plus $\mathrm{RM}(T_B(k)) = \mathrm{RM}(I_k) \geq \mathrm{RM}(B) - \mathrm{RM}(C) = \mathrm{RM}(B')$. Mais $I_k < B$ est un quasi groupe de Frobenius connexe avec complément définissable et par conséquent on peut interpréter un corps K tel que $\mathrm{RM}(B') \geq \mathrm{RM}(K^+) = \mathrm{RM}(K^\times) \geq \mathrm{RM}(I_k) \geq \mathrm{RM}(B')$ et donc $\mathrm{RM}(B') = \mathrm{RM}(I_k) = \mathrm{RM}(C)$. Cette égalité est vraie pour les conjugués de B et les conjugués de C qui y sont contenus. En particulier, B' est soit un groupe abélien divisible sans torsion, soit un p-groupe abélien élémentaire.

Considérons la fonction définissable de K_B vers i^B qui à chaque $k \in K_B$ associe l'unique involution j_k de I_k. De plus, si $j_k = j_\ell$ alors $B \cap B^\ell = B \cap B^k$ et donc ℓ normalise le conjugué $C_{j_k} = I_k$, i.e., $\ell \in C_G(j_k)$. Les fibres sont de rang $\mathrm{RM}(C_G(i))$. On a donc

$$\begin{aligned}\mathrm{RM}(G) - \mathrm{RM}(C_G(i)) &= \mathrm{RM}(i^G) = \mathrm{RM}(K_B) \\ &\leq \mathrm{RM}(i^B) + \mathrm{RM}(C_G(i)) = \mathrm{RM}(i^B) + \mathrm{RM}(C_B(i)) = \mathrm{RM}(B).\end{aligned}$$

Finalement, $\mathrm{RM}(G) \leq \mathrm{RM}(B) + \mathrm{RM}(C_i) = \mathrm{RM}(B) + \mathrm{RM}(B')$.

Soit $g \in G \setminus B$; montrons que $B' \cap B^g = \{1\}$. Supposons par l'absurde qu'il existe $1 \neq x \in B' \cap B^g$. On peut distinguer deux cas : $x \in (B')^g$ ou bien $x \in C_k \leq B^g$. Si $x \in (B')^g \cap B'$, alors x est une translation telle que $B', (B')^g \leq C_G(x)^\circ$. Le sous-groupe $C_G(x)^\circ$ est abélien et donc $B', (B')^g \leq N_G(B')^\circ$, mais $N_G(B')^\circ = B$ (proposition 4.2.3). On en déduit que $B' = B'^g$, puis $B = B^g$: contradiction. Si $x \in B' \cap C_k$, alors $C_k \leq B$ et $C_k = C^b$ pour $b \in B$. Puisque $B' = \left(B \setminus \bigcup_B C^b\right) \cup \{1\}$, on obtient une contradiction.

Par conséquent, les fibres de l'application de $B' \times B$ vers $B'gB$ qui à (f, b) associe fgb sont finies. On a donc $\mathrm{RM}(B'gB) = \mathrm{RM}(G)$. La double classe BgB est générique.

Par conséquent, pour tous $g, g' \in G \setminus B$, les doubles classes BgB et $Bg'B$ sont génériques dans G et donc $g \in Bg'B$ par connexité de G. On a donc $G = B \cup Bg'B$. Il suit que le groupe G est 2-transitif pour l'action de G sur G/B. □

Remarque 4.6.12. Le calcul de rang mis en œuvre dans la démonstration précédente utilise pleinement l'hypothèse de l'existence d'un sous-groupe de Borel *fortement* standard.

On peut désormais conclure la démonstration du théorème 1.0.6 :

Théorème 1.0.6. *Soit $C < G$ un quasi groupe de Frobenius rangé connexe tel que C est définissable. Supposons que G est de degré pair et de type impair. Si les sous-groupes de Borel sont généreux, alors $G \simeq \mathrm{PGL}_2(K)$, pour un corps K algébriquement clos de caractéristique différente de* 2.

Démonstration. D'après le fait 4.2.11, le groupe C est abélien et contient une unique involution i. De plus, le groupe G n'est pas résoluble (corollaire 4.3.3). Il existe un sous-groupe de Borel B qui contient strictement C_i, i.e., un sous-groupe de Borel fortement standard. En effet, d'après le théorème 4.5.4, il existe un sous-groupe de Borel B faiblement standard et puisque B est généreux par hypothèse, le lemme 4.5.6 montre que B est en fait un sous-groupe de Borel fortement standard.

Nous allons montrer que $(B, N_G(C_i))$ induit une (B, N)-paire (définissable) scindée spéciale de rang 1. D'après le lemme 4.6.6, pour tout $k \in N_G(C_i) \setminus C_i$, $B^k \neq B$. De plus, B est autonormalisant.

On rappelle qu'on pose $T_B(k) = \{b \in B : b^{-1} = b^k\}$ pour une involution k qui ne normalise pas B. D'après le lemme 4.6.3, l'ensemble $K_B = \{k \in N_G(B) \setminus B : \mathrm{RM}(T_B(k)) \geq \mathrm{RM}(B) - \mathrm{RM}(C_G(i))\}$ est générique dans i^G.

On a $B \cap N_G(C_i) = C_i$ car $C_i < B$ est un groupe de Frobenius connexe. On a donc $[N_G(C_i) : (N_G(C_i) \cap B)] = 2$ (fait 4.2.11). De plus, pour $k \in N_G(C_i) \setminus C_i$, on a $C_i \leq B \cap B^k$ et $(B \cap B^k) = C_i$, car $(B \cap B^k)^\circ$ est abélien inversé par k et $N_B(C_i) = C_i$. Par ailleurs, $B = B' \rtimes C_i$ où B' est abélien. On a donc vérifié l'axiome 4, l'axiome 2 et une partie de l'axiome 3.

Soit $k \in K_B$, alors d'après le lemme 4.6.7, $I_k = (B \cap B^k)^\circ$ est un conjugué de C_i contenant une involution j_k.

Finalement par le lemme 4.6.11, on obtient

$$G = B \cup BkB \quad \text{pour tout } k \in N_G(C_i) \setminus C_i \quad \text{et} \quad G = \langle B, N \rangle.$$

On a donc montré que $(B, N_G(C_i), C_i, B')$ forme une (B, N)-paire scindée spéciale de rang 1.

Or, $\bigcap_{g \in G} B^g$ est trivial. En effet, on a $B \cap B^k = C_i$ pour $k \in N_G(C_i) \setminus C_i$ et donc $\bigcap_{g \in G} B^g \leq C_i$. Donc si $\bigcap_{g \in G} B^g \neq \{1\}$, on trouve $G = N_G\big(\bigcap_{g \in G} B^g\big)^\circ \leq N_G(C_i)^\circ = C_i$: contradiction.

Si B' est un p-groupe abélien élémentaire, alors en remarquant que $U_p(C_i) = \{1\}$ (on peut interpréter un corps K tel que $C_i \leq K^\times$) on applique le fait 4.6.2 et on a $G/\bigcap_{g \in G} B^g \simeq \mathrm{PGL}_2(K)$ et donc $G \simeq \mathrm{PGL}_2(K)$. Sinon, B' est abélien divisible sans torsion et on applique le fait 4.6.1. Dans les deux cas, ceci permet de montrer que $G \simeq \mathrm{PGL}_2(K)$. □

En guise de conclusion, notons que tout l'enjeu est de montrer que le sous-groupe de Borel faiblement standard obtenu en section 4.5 est un sous-groupe de Borel fortement standard. Comme en témoigne l'étude des configurations de type CiBo, il s'agit d'un problème difficile. Néanmoins, nous espérons que les résultats présentés dans cet article seront susceptibles de rendre possible une approche inductive de ce problème, au moins en caractéristique positive (où l'existence de mauvais corps est improbable).

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Definable valuations on ordered fields

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We study the definability of convex valuations on ordered fields, with a particular focus on the distinguished subclass of henselian valuations. In the setting of ordered fields, one can consider definability both in the language of rings $\mathcal{L}_{\mathrm{r}}$ and in the richer language of ordered rings $\mathcal{L}_{\mathrm{or}}$. We analyse and compare definability in both languages and show the following contrary results: while there are *convex* valuations that are definable in the language $\mathcal{L}_{\mathrm{or}}$ but not in the language $\mathcal{L}_{\mathrm{r}}$, any $\mathcal{L}_{\mathrm{or}}$-definable *henselian* valuation is already $\mathcal{L}_{\mathrm{r}}$-definable. To prove the latter, we show that the value group and the *ordered* residue field of an ordered henselian valued field are stably embedded (as an ordered abelian group and an ordered field, respectively). Moreover, we show that in almost real closed fields *any* $\mathcal{L}_{\mathrm{or}}$-definable valuation is henselian.

1. Introduction

One of the main objectives in the model-theoretic study of fields is the analysis of first-order definable[1] sets and substructures. Given a field, it is a natural question to ask whether a given valuation ring is a definable subset in some expansion of the language $\mathcal{L}_{\mathrm{r}} = \{+, -, \cdot, 0, 1\}$ of rings. A key reason to study definability of valuation rings is to transfer questions of decidability and existential decidability (i.e., the question whether Hilbert's tenth problem has a positive solution) between different rings and fields. However, there is also a more recent motivation stemming from classifying fields within Shelah's classification hierarchy: whereas stable (or,

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[1]Throughout this work, "definable" always means "definable *with* parameters".

more generally, simple) fields do not admit any nontrivial $\mathcal{L}_{\mathrm{r}}$-definable valuations, a conjecture going back to Shelah predicts that infinite NIP fields which are neither real closed nor separably closed admit a nontrivial $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation. In a recent series of spectacular papers, this was shown to hold in the "finite-dimensional" (i.e., dp-finite) case by Johnson [2020]. For a survey on definability of henselian valuations, mostly in the language of rings, see [Fehm and Jahnke 2017].

In this work, we primarily study valuations on *ordered* fields. This allows us to also consider their definability in the language of *ordered* rings $\mathcal{L}_{\mathrm{or}} = \mathcal{L}_{\mathrm{r}} \cup \{<\}$. We focus on convex valuations, i.e., valuations whose valuation ring is convex with respect to the ordering, as these naturally induce an ordering on the residue field (see [Engler and Prestel 2005, Proposition 2.2.4]). Note that due to [Engler and Prestel 2005, Lemma 4.3.6], every henselian valuation on an ordered field is already convex.

By considering the expanded language $\mathcal{L}_{\mathrm{or}}$ rather than $\mathcal{L}_{\mathrm{r}}$, one may expect further definability results. Indeed, we present examples of ordered fields with *convex* valuations that are $\mathcal{L}_{\mathrm{or}}$-definable but not $\mathcal{L}_{\mathrm{r}}$-definable (see Examples 3.5 and 3.6). Rather surprisingly, for *henselian* valuations the language $\mathcal{L}_{\mathrm{or}}$ does not produce any further definability results, that is, every $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation is already $\mathcal{L}_{\mathrm{r}}$-definable (see Theorem 6.4). In the particular case of almost real closed fields, $\mathcal{L}_{\mathrm{or}}$-definability even suffices to ensure both henselianity (thus convexity) and $\mathcal{L}_{\mathrm{r}}$-definability (see Theorem 5.2).

The structure of this paper is as follows. After introducing preliminary notions and results in Section 2, we first turn to the definability of convex valuations in Section 3. We establish conditions on the value group and the residue field ensuring the definability (*with* and *without* parameters) of a given convex valuation (see Theorem 3.1 and Corollary 3.2). Subsequently we compare these results to other known definability conditions in the literature (see Remark 3.3) and construct our main examples — Examples 3.5 and 3.6 — to show that there are convex valuations that are $\varnothing$-$\mathcal{L}_{\mathrm{or}}$-definable but not $\mathcal{L}_{\mathrm{r}}$-definable. Lastly, we answer [Krapp et al. 2021, Question 7.1] positively by presenting in Example 3.7 an ordered valued field that is dense in its real closure but still admits a nontrivial $\varnothing$-$\mathcal{L}_{\mathrm{or}}$-definable convex valuation. In Section 4 we turn to ordered *henselian* valued fields and establish in Theorem 4.2 that their value group (as an ordered abelian group) and their residue field (as an ordered field) are always stably embedded as well as orthogonal. As a result, we obtain that within ordered fields, every $\mathcal{L}_{\mathrm{or}}$-definable coarsening of an $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation is already $\mathcal{L}_{\mathrm{r}}$-definable (see Corollary 4.4) and that any $\mathcal{L}_{\mathrm{or}}$-definable (not necessarily convex) valuation is comparable to any henselian valuation (see Proposition 4.5). The special class of almost real closed fields, which are ordered fields admitting a henselian valuation with real closed residue field, is studied in Section 5. We show in Theorem 5.2 that within almost real closed fields any $\mathcal{L}_{\mathrm{or}}$-definable valuation (which a priori does not have to be convex) is already henselian and $\mathcal{L}_{\mathrm{r}}$-definable, thereby giving a negative answer to [Krapp et al. 2021, Question 7.3]. Building on the results of the previous sections, we finally prove in Section 6 the main theorem of this paper stating as follows:[2]

Theorem A (main theorem). *Let $(K, <)$ be an ordered field and let v be a henselian valuation on K. If v is $\mathcal{L}_{\mathrm{or}}$-definable, then it is already $\mathcal{L}_{\mathrm{r}}$-definable.*

2. Preliminaries

We denote by $\mathbb{N}$ the set of natural numbers *without* 0 and by ω the set of natural numbers *with* 0.

We mostly follow the valuation-theoretic notation of [Engler and Prestel 2005]. For a valuation v on a field K, we write $\mathcal{O}_v$ for its valuation ring, $\mathcal{M}_v$ for the maximal ideal of $\mathcal{O}_v$, $Kv = \mathcal{O}_v/\mathcal{M}_v$ for the residue field and vK for its value group (written additively). For an element $x \in \mathcal{O}_v$, its residue $x + \mathcal{M}_v \in Kv$ is denoted $\bar{x}$, where the valuation v in question will always be clear from context.

Given an ordering $<$ on K (always compatible with the field structure), a valuation v is called *convex* (with respect to $<$) if $\mathcal{O}_v$ is a convex set in the usual sense. See [Engler and Prestel 2005, Section 2.2.2] for a number of equivalent conditions. In particular, v is convex if and only if the ordering $<$ induces an ordering on the residue field Kv in the natural way. This residue ordering is then also denoted by $<$, which should not lead to confusion.

For a given field K and ordered abelian group G, we write $K((G))$ for the *Hahn field* consisting of those formal sums $\sum_{g \in G} a_g t^g$ with coefficients $a_g \in K$, t a formal variable, whose support is well-ordered. See for instance [van den Dries 2014, Section 3.1] for details. We generally endow $K((G))$ with its natural valuation with value group G, assigning to each element of $K((G))^\times$ the order of the lowest nonzero coefficient. An ordering $<$ on K can naturally be extended to $K((G))$ by stipulating that an element of $K((G))^\times$ is positive if and only if its nonzero coefficient of lowest order is positive. We denote this ordering on $K((G))$ again by $<$.

For background on the model theory of valued fields, see [van den Dries 2014], or [Marker 2002] for model theory more generally. We consider fields as structures in the language of rings $\mathcal{L}_{\mathrm{r}} = \{+, -, \cdot, 0, 1\}$ and ordered abelian groups as structures in the language $\mathcal{L}_{\mathrm{og}} = \{+, -, 0, <\}$ in the natural way. Given an ordering $<$ on a field K, we consider the ordered field $(K, <)$ as a structure in the language $\mathcal{L}_{\mathrm{or}} = \mathcal{L}_{\mathrm{r}} \cup \{<\}$. Given a valuation v on a field K, we usually work in the one-sorted language $\mathcal{L}_{\mathrm{vf}} = \mathcal{L}_{\mathrm{r}} \cup \{\mathcal{O}\}$, where the unary predicate $\mathcal{O}$ is to be interpreted as the

[2] This result will be restated as Theorem 6.4.

valuation ring $\mathcal{O}_v \subseteq K$. In Section 4, we also work in a three-sorted language. For a field K with an ordering $<$ and a valuation v, we use the language $\mathcal{L}_{\mathrm{ovf}} = \mathcal{L}_{\mathrm{vf}} \cup \{<\}$ for $(K, <, v)$.

If a is an element of an ordered abelian group (or an ordered field), we denote its absolute value $\max\{a, -a\}$ by $|a|$.

A set is *$\mathcal{L}$-definable* if it is definable in the first-order language $\mathcal{L}$. If we wish to specify that the parameters can be chosen to come from a specific set A, we write *A-$\mathcal{L}$-definable*.

3. Convex valuations

We start by giving sufficient conditions on the value group or the residue field of a convex valuation v such that v is $\mathcal{L}_{\mathrm{or}}$-definable. By this, we strengthen all of the three cases given in [Krapp et al. 2021, Theorem 5.3]. Subsequently, we present several cases in which the given valuation is already $\mathcal{L}_{\mathrm{or}}$-definable without parameters and discuss how these cases generalise other known definability results in the literature.

Theorem 3.1. *Let $(K, <)$ be an ordered field and let v be a convex valuation on K. Suppose that at least one of the following holds*:

(i) *vK is discretely ordered*, i.e., *admits a least positive element.*

(ii) *vK is not closed in its divisible hull.*

(iii) *Kv is not closed in its real closure.*

Then v is $\mathcal{L}_{\mathrm{or}}$-definable. Moreover, *in the cases* (i) *and* (ii), *v is definable by a formula using only one parameter.*

Proof. We may assume that v is nontrivial.

(i) Since vK is discretely ordered, we can choose $b \in K^\times$ such that $v(b)$ is the minimal positive element of vK. Note that for every $x \in \mathcal{M}_v$, we have $v(x^2/b) = 2v(x) - v(b) > 0$. Since every element $y \in \mathcal{M}_v$ satisfies $|y| < 1$, we deduce that $\mathcal{M}_v = \{x \in K \mid |x^2/b| < 1\}$. Hence, $\mathcal{M}_v$ is $\{b\}$-$\mathcal{L}_{\mathrm{or}}$-definable, and $\mathcal{O}_v$ can be defined in terms of $\mathcal{M}_v$.

(ii) Since vK is not closed in its divisible hull, we can take $\gamma \in vK$ and $n > 1$ such that $\gamma \notin n \cdot vK$ but every open interval in vK containing γ contains an element of $n \cdot vK$. Let $b \in K$ with $b > 0$ and $v(b) = \gamma$, and set

$$S_b := \{x \in K \mid x \geq 0 \text{ and } x^n/b < 1\} = \{x \in K \mid x \geq 0 \text{ and } nv(x) > \gamma\} \qquad (3\text{-}1)$$

(where the equality uses that we cannot have $v(x^n/b) = 0$ since $\gamma \notin n \cdot vK$). It now suffices to prove that

$$\mathcal{O}_v = \{y \in K \mid y^4 S_b \subseteq S_b\}, \qquad (3\text{-}2)$$

since the set on the right-hand side is $\{b\}$-$\mathcal{L}_{\mathrm{or}}$-definable. The inclusion $\subseteq$ is clear since the condition $nv(x) > \gamma$ in (3-1) is stable under multiplying x by an element of $\mathcal{O}_v$.

For the inclusion $\supseteq$, suppose that $y \in K \setminus \mathcal{O}_v$, so $v(y) < 0$. By the choice of γ, we can take $z \in K$ with $z > 0$ and $\gamma + v(y) < nv(z) < \gamma - v(y)$. Now $z/y^2 \in S_b$ since $nv(z/y^2) = nv(z) - 2nv(y) > nv(z) - v(y) > \gamma$, but $y^4(z/y^2) \notin S_b$ since $nv(y^4(z/y^2)) = nv(z) + 2nv(y) < nv(z) + v(y) < \gamma$. This proves that S_b is not stable under multiplication by y^4, completing the proof of (3-2).

(iii) Let $f \in Kv[X]$ be the minimal polynomial of an element $x_0 \in R \setminus Kv$, where R denotes the real closure of $(Kv, <)$, such that x_0 can be arbitrarily approximated by elements of Kv. Then there are $a, b \in Kv$ with $a < x_0 < b$ such that the following hold:

(1) The polynomial f has exactly one zero in $\{x \in R \mid a \leq x \leq b\}$. In particular, f changes sign precisely once in this interval.

(2) For any $\varepsilon \in Kv$ with $0 < \varepsilon < b - a$, there exists $x \in Kv$ with $a < x < x + \varepsilon < b$ such that $f(x)f(x+\varepsilon) < 0$.

Passing to $-f$ if necessary, we may assume that $f(a) < 0 < f(b)$. Let $F \in K[X]$ be a lift of f, let $a_0, b_0 \in K$ be lifts of a and b, and consider the $\mathcal{L}_{\mathrm{or}}$-definable set S given by

$$\{x \in K \mid a_0 \leq x \leq b_0 \text{ and } F(x) < 0\} = \{x \in K \mid a_0 \leq x \leq b_0 \text{ and } f(\bar{x}) < 0\},$$

where the equality uses that f has no zero in Kv.

It now suffices to prove that for any $y \in K$ we have

$$y + S \subseteq S \iff y \in \mathcal{M}_v \text{ and } y \geq 0,$$

since then $\mathcal{M}_v$ and therefore $\mathcal{O}_v$ are $\mathcal{L}_{\mathrm{or}}$-definable. For the implication $\Leftarrow$, let $y \in \mathcal{M}_v$ be nonnegative and $x \in S$. Then we have $x + y \geq x \geq a_0$ and $f(\overline{x+y}) = f(\bar{x}) < 0$. In particular $\overline{x+y} < b$ and thus $x + y < b_0$. Hence $y + x \in S$, as desired.

For the implication $\Rightarrow$, let $y \in K$ with $y + S \subseteq S$. Since $a_0 \in S$, we must have $a_0 + y \in S$. Thus, $a_0 \leq a_0 + y \leq b_0$, implying that $0 \leq y \leq b_0 - a_0$ and $y \in \mathcal{O}_v$. Note that $\bar{y} \neq b - a$, as otherwise $f(\bar{y} + a) = f(b) > 0$, contradicting the fact that $y + a_0 \in S$. Hence, $\bar{y} < b - a$.

In order to show $y \in \mathcal{M}_v$, suppose for a contradiction that $v(y) = 0$, so $0 < \bar{y} < b - a$. By choice of f, we can find $z \in \mathcal{O}_v$ with $a < \bar{z} < b$ and $f(\bar{z}) < 0 < f(\overline{z+y})$. Then we have $z \in S$ but $z + y \notin S$, in contradiction to our assumption $y + S \subseteq S$. □

In the following corollary, we point out several distinguished cases in which we obtain $\mathcal{L}_{\mathrm{or}}$-definability without parameters.

Corollary 3.2. *Let $(K, <)$ be an ordered field and let v be a convex valuation on K. Suppose that at least one of the following holds*:

(i) *vK is p-regular but not p-divisible for some prime $p \in \mathbb{N}$.*[3]

(ii) *Kv is dense in its real closure but not real closed.*

Then v is $\emptyset$-$\mathcal{L}_{\mathrm{or}}$-definable.

Proof. In both cases, at least one of the three conditions in Theorem 3.1 is satisfied: if vK is p-regular but not p-divisible, then it is either discrete or not closed in its divisible hull (see [Krapp et al. 2022, Proposition 3.3]). Thus, there exists an $\mathcal{L}_{\mathrm{or}}$-formula $\psi(x, \underline{z})$ and a parameter tuple $\underline{b} \in K$ such that $\psi(x, \underline{b})$ defines $\mathcal{O}_v$.

(i) For any nontrivial convex subgroup $C \leq vK$ we have that vK/C is p-divisible (see [Hong 2014, page 14]). Thus, any strict coarsening of v has a p-divisible value group. As vK is not p-divisible, $\mathcal{O}_v$ is defined by the $\mathcal{L}_{\mathrm{or}}$-formula $\varphi(y)$ expressing the following:

$$\exists \underline{z}\ (\text{“}\psi(x, \underline{z}) \text{ defines a nontrivial convex valuation ring whose value group contains an element that is not } p\text{-divisible”} \wedge\ \psi(y, \underline{z})).$$

(ii) For any strict refinement w of v we have that Kw is real closed. Indeed, since Kv is dense in its real closure and the induced valuation $\overline{w}$ on Kv is nontrivial and convex, we have that $Kw = (Kv)\overline{w}$ is real closed (see [Krapp et al. 2021, Corollary 4.9]). Let θ be an $\mathcal{L}_{\mathrm{or}}$-sentence that is true in the theory of real closed fields but does not hold in Kv.[4] Then $\mathcal{O}_v$ is defined by the $\mathcal{L}_{\mathrm{or}}$-formula $\varphi(y)$ expressing the following:

$$\forall \underline{z}\ (\text{“}\psi(x, \underline{z}) \text{ defines a nontrivial convex valuation ring whose residue field does not satisfy } \theta\text{”} \rightarrow \psi(y, \underline{z})). \qquad \square$$

Remark 3.3. (i) The cases (i) and (ii) of Theorem 3.1 are optimal in the sense that, in general, one cannot obtain parameter-free definability. More precisely, in [Krapp et al. 2022, Examples 4.9 and 4.10] two ordered valued fields $(L_1, <, v_1)$ and $(L_2, <, v_2)$ are presented such that the following hold:

- v_1 and v_2 are henselian and thus convex;
- neither v_1 nor v_2 is $\emptyset$-$\mathcal{L}_{\mathrm{or}}$-definable;
- $v_1 L_1$ is discrete, and $v_2 L_2$ is not closed in its divisible hull.

[3] Equivalently, vK contains a rank 1 convex subgroup H that is not p-divisible but vK/H is p-divisible. See [Hong 2013, Section 2.2] for further characterisations of p-regular ordered abelian groups.

[4] For instance, θ may express that there exists a polynomial of a certain degree that does not have a zero.

(ii) The results for $\mathcal{L}_{\mathrm{r}}$-definability of henselian (rather than convex) valuations corresponding to Theorem 3.1(i) and (ii) as well as Corollary 3.2(i) are proven in [Hong 2014, Corollary 2], [Krapp et al. 2022, Theorem A] and [Hong 2013, Lemmas 2.3.6 and 2.3.7], respectively.

(iii) Corollary 3.2 applies in particular if vK is of rank 1 (i.e., v is the coarsest nontrivial convex valuation on K) but nondivisible, or if Kv is archimedean (i.e., v is the finest convex valuation) but not real closed.

We now apply the $\mathcal{L}_{\mathrm{or}}$-definability results above in order to obtain convex nonhenselian valuations that are definable in the language $\mathcal{L}_{\mathrm{or}}$ but not in the language $\mathcal{L}_{\mathrm{r}}$.

Lemma 3.4. *Let $K = \mathbb{Q}(s_i \mid i \in \omega)$, where $\{s_i \mid i \in \omega\}$ is algebraically independent over $\mathbb{Q}$. Suppose that v is any valuation on K with $v(s_i) \geq 0$ for any $i \in \mathbb{N}$ and $v(s_0) < 0$. Then v is not $\mathcal{L}_{\mathrm{r}}$-definable in K.*

Proof. First note that K is the $\mathcal{L}_{\mathrm{r}}$-definable closure of $S := \{s_i \mid i \in \omega\}$ in K. Hence, any $\mathcal{L}_{\mathrm{r}}$-definable subset of K is S-$\mathcal{L}_{\mathrm{r}}$-definable.

Assume, for a contradiction, that some $\mathcal{L}_{\mathrm{r}}$-formula $\varphi(x, \underline{s})$ defines $\mathcal{O}_v$, where $\underline{s} = (s_0, s_1, \ldots, s_n)$ for some $n \in \mathbb{N}$. Since also the set

$$\{s_0, s_1, \ldots, s_n, s_{n+1} + s_0, s_{n+2}, \ldots\}$$

is algebraically independent over $\mathbb{Q}$, we can set α to be the uniquely determined $\mathcal{L}_{\mathrm{r}}$-automorphism on K with

$$\alpha(s_i) := \begin{cases} s_i & \text{for } i \in \omega \setminus \{n+1\}, \\ s_{n+1} + s_0 & \text{for } i = n+1. \end{cases}$$

Now since $v(s_{n+1}) \geq 0$, we have

$$K \models \varphi(s_{n+1}, \underline{s}).$$

As $\alpha(s_{n+1}) = s_{n+1} + s_0$ and $\alpha(\underline{s}) = \underline{s}$, we obtain

$$K \models \varphi(s_{n+1} + s_0, \underline{s}),$$

i.e., $s_{n+1} + s_0 \in \mathcal{O}_v$. However, $v(s_{n+1} + s_0) = v(s_0) < 0$, a contradiction. $\square$

Example 3.5. We construct an ordered valued field $(K, <, v)$ such that K is a subfield of the Laurent series field $\mathbb{R}((\mathbb{Z}))$, vK is discretely ordered, Kv is archimedean and v is $\varnothing$-$\mathcal{L}_{\mathrm{or}}$-definable but not $\mathcal{L}_{\mathrm{r}}$-definable.

Let $k = \mathbb{Q}(s_1, s_2, \ldots) \subseteq \mathbb{R}$ for some set $\{s_i \mid i \in \mathbb{N}\} \subseteq \mathbb{R}$ that is algebraically independent over $\mathbb{Q}$. Consider the field $K = k(t)$, which we endow with the valuation v and ordering $<$ given as the restriction of the valuation and ordering on the Hahn field $k((t)) = k((\mathbb{Z}))$. Then $vK = \mathbb{Z}$ and $Kv = k$, which is archimedean. Corollary 3.2 shows that v is $\varnothing$-$\mathcal{L}_{\mathrm{or}}$-definable. Setting $s_0 = t^{-1}$, we obtain $K = \mathbb{Q}(s_0, s_1, \ldots)$ and

$v(s_0) = -1 < 0$ as well as $v(s_i) = 0$ for any $i \in \mathbb{N}$. Hence, Lemma 3.4 implies that v is not $\mathcal{L}_r$-definable. □

Example 3.6. We construct an ordered valued field $(K, <, v)$ such that K is a subfield of the Puiseux series field $\bigcup_{n\in\mathbb{N}} \mathbb{R}((t^{1/n})) \subseteq \mathbb{R}((\mathbb{Q}))$, vK is densely ordered, Kv is archimedean and v is $\varnothing$-$\mathcal{L}_{or}$-definable but not $\mathcal{L}_r$-definable.

Let $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{R}$ be an algebraically independent set over $\mathbb{Q}$. We set

$$s_0 = t^{-1} \in \mathbb{R}((\mathbb{Q})) \quad \text{and} \quad s_i = r_i t^{1/i} \in \mathbb{R}((\mathbb{Q}))$$

for any $i \in \mathbb{N}$. Let $K = \mathbb{Q}(s_0, s_1, \dots) \subseteq \mathbb{R}((\mathbb{Q}))$, and endow K with the ordering $<$ and valuation v given as the restriction of the Hahn field ordering and valuation on $\mathbb{R}((\mathbb{Q}))$. The archimedean residue field of K is not real closed, as $Kv \subseteq \mathbb{Q}(r_1, r_2, \dots)$ and thus, for instance, $\sqrt{2} \notin Kv$. Hence, Corollary 3.2(ii) shows that v is $\varnothing$-$\mathcal{L}_{or}$-definable. Since $v(s_i) = \frac{1}{i}$ for any $i \in \mathbb{N}$, we have $vK = \mathbb{Q}$. Finally, we can apply Lemma 3.4 to show that v is not $\mathcal{L}_r$-definable, as $v(s_0) = -1 < 0$ and $v(s_i) = \frac{1}{i} > 0$ for any $i \in \mathbb{N}$. □

To complete this section, we relate the property of an ordered field to admit a nontrivial $\mathcal{L}_{or}$-definable convex valuation to the property of being dense in the real closure. It is known that an ordered field is either dense in its real closure or admits a nontrivial $\mathcal{L}_{or}$-definable convex valuation (see [Jahnke et al. 2017, Proposition 6.5; Krapp et al. 2021, Fact 5.1]). However, the question whether these two cases are nonexclusive (see [Krapp et al. 2021, Question 7.1]) has so far been open. In the following, we answer this question positively by presenting an ordered field that is dense in its real closure and whose natural valuation is $\varnothing$-$\mathcal{L}_{or}$-definable.

Example 3.7. Let

$$K = \mathbb{R}(t^{1/n} \mid n \in \mathbb{N}) \subseteq \mathbb{R}((\mathbb{Q})),$$

and endow K with the ordering $<$ and valuation v given by restricting the ordering and the valuation of the Hahn field $\mathbb{R}((\mathbb{Q}))$. Since $Kv = \mathbb{R}$ is real closed and $vK = \mathbb{Q}$ is divisible and of rank 1, [Viswanathan 1977, Proposition 9] implies that K is dense in its real closure.

We first claim that the subset $\mathbb{R} \subseteq K$ is defined by the parameter-free $\mathcal{L}_r$-formula $\varphi(x)$ given by

$$\exists y \ 1 + x^4 = y^4.$$

Clearly $\mathbb{R} \subseteq \varphi(K)$, since for any $a \in \mathbb{R}$ we have $\sqrt[4]{1+a^4} \in \mathbb{R}$. For the other inclusion, take $a, y \in K$ with $1+a^4 = y^4$. There exists $N \in \mathbb{N}$ such that $a, y \in \mathbb{R}(t^{1/N})$. Letting $s = t^{1/N}$, we have $\mathbb{R}(s) \models \varphi(a)$, and therefore $a \in \mathbb{R}$ by [Maltsev 1960, Lemma 2]. This shows $\mathbb{R} = \varphi(K)$, as desired.

Therefore $\mathcal{O}_v$, the convex hull of $\mathbb{R}$ in K, is defined by the parameter-free $\mathcal{L}_{or}$-formula

$$\exists x_1, x_2\, (\varphi(x_1) \wedge \varphi(x_2) \wedge x_1 \le z \le x_2).$$

□

Note that Example 3.7 presents an ordered valued field with real closed residue field and divisible value group. Hence, none of the cases in Corollary 3.2 can be applied, but we still obtain $\varnothing$-$\mathcal{L}_{\mathrm{or}}$-definability of the valuation.

4. Stable embeddedness

In this section, we establish that the ordered value group and the ordered residue field of an ordered henselian valued field are stably embedded and orthogonal (see Theorem 4.2). Stable embeddedness and orthogonality are best known to hold in the unordered situation of henselian valued fields of equicharacteristic 0 (see, e.g., [van den Dries 2014, Section 5]), and various other well-behaved settings (see also [Aschenbrenner et al. 2017, Section 8.3; Jahnke and Simon 2020, pages 171–172]). As the key technical tool in the ordered context, we use Farré's embedding lemma [Farré 1993, Theorem 3.4].

We consider the three-sorted language $\mathcal{L}'_{\mathrm{ovf}}$ given by

$$\mathcal{L}'_{\mathrm{ovf}} = (\mathcal{L}_{\mathrm{or}}, \mathcal{L}_{\mathrm{or}}, \mathcal{L}_{\mathrm{og}}; \bar{\cdot}, v).$$

The three sorts are denoted by **f** (field sort), **r** (residue field sort) and **v** (value group sort). The two unary function symbols $\bar{\cdot}$ and v have sorts $\bar{\cdot} : \mathbf{f} \to \mathbf{r}$ and $v : \mathbf{f} \to \mathbf{v}$.

Let $(K, <, v)$ be an ordered valued field with convex valuation v. Then $(K, <, v)$ induces an $\mathcal{L}'_{\mathrm{ovf}}$-structure

$$\mathcal{K} = ((K, <), (Kv, <), vK; v, \bar{\cdot}),$$

where the domains of the valuation and the residue map are extended to K by setting $v(0) = 0$ and $\bar{a} = 0$ for any $a \in K \setminus \mathcal{O}_v$. When considering definability in $\mathcal{K}$, we allow parameters from all sorts as usual. An $\mathcal{L}_{\mathrm{or}}$-formula $\varphi(\underline{y})$ with parameters from Kv may be considered as an $\mathcal{L}'_{\mathrm{ovf}}$-formula with parameters from $\mathcal{K}$, where the variables $\underline{y}$ become **r**-variables. Similarly, an $\mathcal{L}_{\mathrm{og}}$-formula $\varphi(\underline{z})$ with parameters from vK may be considered as an $\mathcal{L}'_{\mathrm{ovf}}$-formula with parameters from $\mathcal{K}$, where the variables $\underline{z}$ become **v**-variables.

In the following, we prove a weak version of relative quantifier elimination; we only consider formulas whose variables are varying over residue field and value group.

Lemma 4.1. *Let $(K, <, v)$ be an ordered henselian valued field and let T be the diagram of the $\mathcal{L}'_{\mathrm{ovf}}$-structure $\mathcal{K}$ as above, i.e., the complete theory of $\mathcal{K}$ in the language $\mathcal{L}'_{\mathrm{ovf}}$ expanded by constants for all elements of $\mathcal{K}$. Further, let $\underline{y}$ and $\underline{z}$ be tuples of distinct* **r**- *and* **v**-*variables, respectively. Then any $\mathcal{L}'_{\mathrm{ovf}}$-formula $\varphi(\underline{y}, \underline{z})$ with parameters from $\mathcal{K}$ is T-equivalent to an $\mathcal{L}'_{\mathrm{ovf}}$-formula of the form*

$$(\psi_1(\underline{y}) \wedge \theta_1(\underline{z})) \vee \cdots \vee (\psi_N(\underline{y}) \wedge \theta_N(\underline{z})) \tag{4-1}$$

for some $N \in \mathbb{N}$, where all ψ_i are $\mathcal{L}_{\mathrm{or}}$-formulas with parameters from Kv and all θ_i are $\mathcal{L}_{\mathrm{og}}$-formulas with parameters from vK.

Proof. Let Θ be the set of all $\mathcal{L}'_{\mathrm{ovf}}$-formulas of the form (4-1), i.e., of all finite disjunctions of conjunctions of an $\mathcal{L}_{\mathrm{or}}$-formula and an $\mathcal{L}_{\mathrm{og}}$-formula with parameters from Kv and vK, respectively. Modulo logical equivalence, Θ contains $\top$ as well as $\perp$ and is closed under finite disjunctions, finite conjunctions and negation. Hence by [Aschenbrenner et al. 2017, Corollary B.9.3], we only have to verify the following:

> *Let p and q be any two complete T-realisable $(\underline{y}, \underline{z})$-types with $p \cap \Theta = q \cap \Theta$. Then $p = q$.*

Let p and q be as described above and let $\mathcal{M}$ be a sufficiently saturated elementary extension of $\mathcal{K}$ in which p and q are realised. Denote by $(M, <, v)$ the ordered henselian valued field inducing $\mathcal{M}$, and let $\underline{r}, \underline{r}^* \in Mv$ and $\underline{g}, \underline{g}^* \in vM$ with $\mathcal{M} \models p(\underline{r}, \underline{g}) \wedge q(\underline{r}^*, \underline{g}^*)$. By construction of the set Θ, we have

$$\mathrm{tp}^{vM}(\underline{g}/vK) = \mathrm{tp}^{vM}(\underline{g}^*/vK) \quad \text{and} \quad \mathrm{tp}^{(Mv,<)}(\underline{r}/Kv) = \mathrm{tp}^{(Mv,<)}(\underline{r}^*/Kv). \tag{4-2}$$

Now let $\mathcal{K} \preceq \mathcal{M}_0 \preceq \mathcal{M}$, with $\mathcal{M}_0$ smaller than the saturation of $\mathcal{M}$, such that $\underline{r} \in M_0 v$ and $\underline{g} \in vM_0$, where $(M_0, <, v)$ denotes the ordered henselian valued field inducing $\mathcal{M}_0$. Due to (4-2), we can fix an $\mathcal{L}_{\mathrm{or}}$-elementary embedding $\sigma : (M_0 v, <) \to (Mv, <)$ over Kv with $\sigma(\underline{r}) = \underline{r}^*$ and an $\mathcal{L}_{\mathrm{og}}$-elementary embedding $\rho : vM_0 \to vM$ over vK with $\rho(\underline{g}) = \underline{g}^*$ (see [Marker 2002, Proposition 4.1.5]).

The quotient vM_0/vK is torsion free, as $vK \preceq vM_0$. We can thus apply [Farré 1993, Theorem 3.4] (with all appearing levels equal to 1) in order to obtain an embedding $\iota : (M_0, <, v) \to (M, <, v)$ over K inducing both σ and ρ. Moreover, since both σ and ρ are elementary embeddings, [Farré 1993, Corollary 4.2(ii)] implies that $(\iota(M_0), <, v) \preceq (M, <, v)$. Let $\mathcal{M}'_0$ be the $\mathcal{L}'_{\mathrm{ovf}}$-structure induced by $(\iota(M_0), <, v)$ and denote by h the isomorphism $(\iota, \sigma, \rho) : \mathcal{M}_0 \to \mathcal{M}'_0$ over $\mathcal{K}$. For any $\varphi(\underline{y}, \underline{z}) \in p$ we have $\mathcal{M}_0 \models \varphi(\underline{r}, \underline{g})$. By applying h, we obtain $\mathcal{M}'_0 \models \varphi(\sigma(\underline{r}), \rho(\underline{g}))$ and hence $\mathcal{M} \models \varphi(\underline{r}^*, \underline{g}^*)$. This establishes $p \subseteq q$. The other inclusion follows likewise. $\square$

Theorem 4.2. *Let $(K, <, v)$ be an ordered henselian valued field inducing the $\mathcal{L}'_{\mathrm{ovf}}$-structure $\mathcal{K}$. Then for any $m, n \in \omega$ the following hold*:

(i) *Any subset of $(Kv)^m \times (vK)^n$ definable in $\mathcal{K}$ is a finite union of rectangles of the form $Y \times Z$, where $Y \subseteq (Kv)^m$ is $\mathcal{L}_{\mathrm{or}}$-definable in $(Kv, <)$ and $Z \subseteq (vK)^n$ is $\mathcal{L}_{\mathrm{og}}$-definable in vK.*

(ii) *For any set $B \subseteq \mathcal{O}_v^m$ that is $\mathcal{L}_{\mathrm{ovf}}$-definable in $(K, <, v)$, the set*

$$\overline{B} := \{(\overline{b_1}, \dots, \overline{b_m}) \mid (b_1, \dots, b_m) \in B\} \subseteq (Kv)^m$$

is $\mathcal{L}_{\mathrm{or}}$-definable in $(Kv, <)$.

(iii) *For any set $C \subseteq (K^\times)^n$ that is $\mathcal{L}_{\mathrm{ovf}}$-definable in $(K, <, v)$, the set*

$$v(C) := \{(v(c_1), \ldots, v(c_n)) \mid (c_1, \ldots, c_n) \in C\} \subseteq (vK)^n$$

is $\mathcal{L}_{\mathrm{og}}$-definable in vK.

Proof. By Lemma 4.1, any subset of $(Kv)^m \times (vK)^n$ definable in $\mathcal{K}$ can be defined by a formula of the form (4-1). This immediately implies (i). In order to obtain (ii) and (iii), it remains to notice that $\bar{B} \times v(C) \subseteq (Kv)^m \times (vK)^n$ is definable in $\mathcal{K}$. □

Corollary 4.3. *Let $(K, <, v)$ be an ordered henselian valued field and let w be an $\mathcal{L}_{\mathrm{ovf}}$-definable valuation on K. Then the following hold*:

(i) *If w is a refinement of v, then the valuation $\overline{w}$ induced by w on Kv is $\mathcal{L}_{\mathrm{or}}$-definable in $(Kv, <)$.*

(ii) *If w is a coarsening of v, then $v(\mathcal{O}_w^\times)$ is $\mathcal{L}_{\mathrm{og}}$-definable in vK.*

Proof. Both $\mathcal{O}_w^\times$ and $\mathcal{O}_w$ are $\mathcal{L}_{\mathrm{ovf}}$-definable in K. It remains to apply Theorem 4.2(ii) to $B = \mathcal{O}_w \subseteq \mathcal{O}_v$ in order to obtain (i) and Theorem 4.2(iii) to $C = \mathcal{O}_w^\times \subseteq K^\times$ in order to obtain (ii). □

Corollary 4.4. *Let $(K, <)$ be an ordered field, let v be an $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation on K and let w be an $\mathcal{L}_{\mathrm{or}}$-definable coarsening of v. Then w is already $\mathcal{L}_{\mathrm{r}}$-definable.*

Proof. Corollary 4.3(ii) shows that $H = v(\mathcal{O}_w^\times)$ is $\mathcal{L}_{\mathrm{og}}$-definable in vK. Since $wK = vK/H$, for any $x \in K$ we have $x \in \mathcal{O}_w$ if and only if $v(x) \geq 0 \vee v(x) \in H$. As v is $\mathcal{L}_{\mathrm{r}}$-definable in K, the latter can be expressed as an $\mathcal{L}_{\mathrm{r}}$-formula with parameters from K. □

For later use, we also deduce the following.

Proposition 4.5. *Let $(K, <, v)$ be an ordered henselian valued field. Then any $\mathcal{L}_{\mathrm{or}}$-definable valuation w on K is comparable to v.*

Proof. We may suppose that neither v nor w are trivial. We first claim that the valuation ring $\mathcal{O}_w$ contains a set $U \neq \varnothing$ which is open in the topology induced by v. This follows from a suitable form of relative quantifier elimination for henselian valued fields of residue characteristic zero: In the terminology of [Cluckers et al. 2022], the $\mathcal{L}_{\mathrm{vf}}$-theory of the valued field (K, v) is ω-h-minimal [Cluckers et al. 2022, Corollary 6.2.6(1.)]. Let P be the unary predicate on $\mathrm{RV} = K^\times/(1 + \mathcal{M}_v)$ given by $P(a(1 + \mathcal{M}_v))$ if and only if $a > 0$ for any $a \in K^\times$. Since elements of $1 + \mathcal{M}_v$ are squares and hence automatically positive, the positive cone of $(K, <)$ consists of all $a \in K^\times$ satisfying $P(a(1 + \mathcal{M}_v))$. As by [Cluckers et al. 2022, Theorem 4.1.19] ω-h-minimality is preserved under expansions by additional predicates on RV, we obtain that the $\mathcal{L}_{\mathrm{ovf}}$-theory of $(K, <, v)$ is ω-h-minimal. In models of ω-h-minimal theories, any infinite definable set contains a nonempty v-open ball [Cluckers et al. 2022, Lemma 2.5.2], proving our claim about $\mathcal{O}_w$.

It follows that w cannot be independent from v, since otherwise weak approximation [Engler and Prestel 2005, Theorem 2.4.1] would imply that the set $U \cap (K \setminus \mathcal{O}_w)$ is nonempty as the intersection of a v-open and a w-open set.

Let us now suppose for a contradiction that w and v are incomparable. Let v_0 be the finest common coarsening of v and w, and $\bar{v}$, $\overline{w}$ the induced valuations on the residue field Kv_0, which are nontrivial and independent. Writing $<$ for the induced ordering on Kv_0, we now have an ordered henselian valued field $(Kv_0, <, \bar{v})$ with a valuation $\overline{w}$ independent from $\bar{v}$. By Corollary 4.3(i), $\overline{w}$ is $\mathcal{L}_{\mathrm{or}}$-definable in $(Kv_0, <)$. Since $\overline{w}$ is independent from the henselian valuation $\bar{v}$ on Kv_0, this contradicts the first part of the proof. □

5. Almost real closed fields

Following the terminology of [Delon and Farré 1996], we call a field K *almost real closed* if it admits a henselian valuation v such that Kv is real closed. Almost real closed fields arise in many valuation-theoretic contexts, and they have been studied extensively (under varying names) both algebraically and model-theoretically (see, e.g., [Brown 1988; Becker et al. 1999; Delon and Farré 1996]). Due to the Baer–Krull representation theorem (see [Engler and Prestel 2005, pages 37–38]), any almost real closed field admits at least one ordering. In this section, we consider $\mathcal{L}_{\mathrm{or}}$- and $\mathcal{L}_{\mathrm{r}}$-definability of valuations (which are a priori not necessarily convex) in almost real closed fields. We establish in Theorem 5.2 that *any* $\mathcal{L}_{\mathrm{or}}$-definable valuation is already $\mathcal{L}_{\mathrm{r}}$-definable and henselian. Thereby we give a negative answer to [Krapp et al. 2021, Question 7.3].

Let K be an almost real closed field. Then for any prime $p \in \mathbb{N}$ there exists a coarsest henselian valuation on K, denoted by v_p, with the property that

$$Kv_p = (Kv_p)^p \cup [-(Kv_p)^p]$$

(see [Delon and Farré 1996, page 1126]).

Lemma 5.1. *Let $p \in \mathbb{N}$ be prime and let (K, v) be a henselian valued field with real closed residue field. Then $v(\mathcal{O}_{v_p}^\times)$ is the maximal p-divisible convex subgroup of vK.*

Proof. By [Delon and Farré 1996, Proposition 2.5(iv)], $v_pK = vK/v(\mathcal{O}_{v_p}^\times)$ has no nontrivial p-divisible convex subgroup, so $v(\mathcal{O}_{v_p}^\times)$ contains the maximal p-divisible convex subgroup of vK. On the other hand, v induces a valuation on the residue field Kv_p with value group $v(\mathcal{O}_{v_p}^\times)$, from which it is easy to see that $v(\mathcal{O}_{v_p}^\times)$ must itself be p-divisible by the defining property of v_p (or see [Delon and Farré 1996, Lemma 2.4(iii)]). □

With the results of the last section at our disposal, we can now imitate the proof of [Delon and Farré 1996, Theorem 4.4] to obtain the following.

Theorem 5.2. *Let K be an almost real closed field and let $<$ be any ordering on K. Then any $\mathcal{L}_{\mathrm{or}}$-definable valuation on $(K, <)$ is henselian and $\mathcal{L}_{\mathrm{r}}$-definable.*

Proof. Let us denote by v_K the canonical henselian valuation on K. Since K is almost real closed, v_K coincides with the natural, i.e., finest convex valuation v_{nat} on $(K, <)$ for any ordering $<$ on K, as v_{nat} is henselian [Delon and Farré 1996, Proposition 2.1(iv)].

Let v be an $\mathcal{L}_{\mathrm{or}}$-definable valuation on K. By Proposition 4.5, v and v_K are comparable. If v is a coarsening of v_K, then it is also henselian. Otherwise, v_K is a strict coarsening of v. Thus, by Corollary 4.3(i) the nontrivial valuation that v induces on Kv_K is $\mathcal{L}_{\mathrm{or}}$-definable in $(Kv_K, <)$, contradicting that Kv_K is real closed. Hence, v is henselian.[5]

In order to show that v is $\mathcal{L}_{\mathrm{r}}$-definable, by [Delon and Farré 1996, Theorem 4.4] it suffices to verify that $G_v := v_K(\mathcal{O}_v^\times)$ is $\mathcal{L}_{\mathrm{og}}$-definable in $v_K K$ and that $\mathcal{O}_{v_p} \subseteq \mathcal{O}_v$ for some prime $p \in \mathbb{N}$. The first condition follows from Corollary 4.3(ii). For the other condition, we distinguish between two cases.

Case 1: $v \neq v_K$. Then $G_v \neq \{0\}$ and by [Delon and Farré 1996, Corollary 4.3] we have $G_p \leq G_v$ for some prime $p \in \mathbb{N}$, where G_p denotes the maximal p-divisible convex subgroup of $v_K K$. Now $G_p = v_K(\mathcal{O}_{v_p}^\times)$ by Lemma 5.1. Hence, $\mathcal{O}_{v_p} \subseteq \mathcal{O}_v$, as required. This establishes that v is $\mathcal{L}_{\mathrm{r}}$-definable in K.

Case 2: $v = v_K$. Consider the set of formulas

$$p(x) = \{x > n \wedge v(x) = 0 \mid n \in \mathbb{N}\}.$$

This set is finitely satisfiable in $(K, <, v)$, i.e., a type. Hence, for some elementary extension $(L, <, v^*)$ of $(K, <, v)$, there is some $x \in L$ with $v^*(x) = 0$ and $x > n$ for all $n \in \mathbb{N}$. In particular, v^* is a strict coarsening of the natural valuation v_L on $(L, <)$. By Case 1, v^* is $\mathcal{L}_{\mathrm{r}}$-definable in L. Thus, there exists an $\mathcal{L}_{\mathrm{r}}$-formula $\varphi(x, \underline{y})$ such that

$$(L, <, v^*) \models \exists \underline{y} \forall x \, (v^*(x) \geq 0 \leftrightarrow \varphi(x, \underline{y})).$$

By elementary equivalence, there exists $\underline{b} \in K$ such that $\varphi(x, \underline{b})$ defines v in K. □

6. Henselian valuations

We now consider definability in general ordered henselian valued fields. Throughout this section, we freely use Farré's Ax–Kochen–Ershov principles [Farré 1993,

[5] An alternative argument is the following: If K is almost real closed, then one can show that $(K, <)$ has NIP. In this case, any $\mathcal{L}_{\mathrm{or}}$-definable valuation on K is henselian by [Halevi et al. 2020, Corollary 5.8].

Corollary 4.2] (with all levels equal to 1 in the notation there), stating that two ordered henselian valued fields are elementarily equivalent in $\mathcal{L}_{\mathrm{ovf}}$ if and only if the ordered residue fields and the value groups are so, and similarly for elementary extensions.

Our first step is to show that a henselian valuation that is "slippery" in a precise sense involving residue field and value group cannot be $\mathcal{L}_{\mathrm{or}}$-definable.

Lemma 6.1. *Let $(K, <, v)$ be an ordered henselian valued field satisfying*

$$(Kv, <) \equiv (L((\mathbb{Q})), <) \quad \textit{and} \quad vK \equiv \Gamma \oplus \mathbb{Q}$$

for some ordered field L and some ordered abelian group Γ. Then v is not $\mathcal{L}_{\mathrm{or}}$-definable.

Proof. Assume, for a contradiction, that v were $\mathcal{L}_{\mathrm{or}}$-definable. Fix an $\mathcal{L}_{\mathrm{or}}$-formula $\varphi(x, \underline{y})$ such that for some $\underline{b} \in K$ the valuation ring $\mathcal{O}_v$ is defined by $\varphi(x, \underline{b})$. Since some instance of $\varphi(x, \underline{y})$ also defines the valuation w in any

$$(M, <, w) \equiv (K, <, v),$$

we may assume that $(K, <, v)$ is in fact equal to

$$(L((\mathbb{Q}))\underbrace{((\mathbb{Q}))((\Gamma))}_{w}, <, w),$$

where w denotes the power series valuation with value group $\Gamma \oplus \mathbb{Q}$ ordered lexicographically. Let v_Γ denote the power series valuation on K with value group Γ and residue field $L((\mathbb{Q}))((\mathbb{Q}))$.

Applying Corollary 4.3(i) to the ordered residue field of $(K, <, v_\Gamma)$, the $\mathcal{L}_{\mathrm{or}}$-definability of v implies that $\bar{v}$ (i.e., the valuation induced by v on the residue field of its coarsening v_Γ) is also $\mathcal{L}_{\mathrm{or}}$-definable on $(L((\mathbb{Q}))((\mathbb{Q})), <)$.

Applying Corollary 4.3(ii) to the value group of $(L((\mathbb{Q}))((\mathbb{Q})), <, \bar{v})$, the convex subgroup $\mathbb{Q}$ corresponding to $\bar{v}$ is already $\mathcal{L}_{\mathrm{og}}$-definable in $\mathbb{Q} \oplus \mathbb{Q}$. This is a contradiction, as $\mathbb{Q} \oplus \mathbb{Q}$ is divisible, and divisible ordered abelian groups admit no nontrivial proper $\mathcal{L}_{\mathrm{og}}$-definable subgroups. □

We now prove a lemma used to define coarsenings of a valuation that is essentially already shown in [Jahnke and Koenigsmann 2017]. It states that although in an ordered abelian group G, the smallest convex subgroup containing a given element $\gamma \in G$ need not be definable (e.g., the convex subgroup generated by (0, 1) in the lexicographic sum $\mathbb{Q} \oplus \mathbb{Z}$ is not definable), it is definable up to p-divisible "noise".

Lemma 6.2. *Let $p \in \mathbb{N}$ be prime. There exists an $\mathcal{L}_{\mathrm{og}}$-formula $\varphi(x, y)$ such that the following holds: Let G be an ordered abelian group and $\gamma \in G^{>0}$, and let $\langle\gamma\rangle$ denote the smallest convex subgroup of G that contains γ. Then the set $\Delta_\gamma \subseteq G$*

defined by $\varphi(x, \gamma)$ in G is the maximal convex subgroup of G containing γ such that $\Delta_\gamma/\langle\gamma\rangle$ is p-divisible.

Proof. We set $\varphi(x, \gamma)$ to express

$$[0, p|x|] \subseteq [0, p\gamma] + pG.$$

By [Jahnke and Koenigsmann 2017, Lemma 4.1], Δ_γ is a convex subgroup of G with $\gamma \in \Delta_\gamma$ such that no nontrivial convex subgroup of G/Δ_γ is p-divisible. In particular, for every convex subgroup Δ of G properly containing Δ_γ, the group $\Delta/\langle\gamma\rangle$ is not p-divisible, since it has $\Delta/\Delta_\gamma \leq G/\Delta_\gamma$ as a quotient.

On the other hand, every positive element $\delta \in \Delta_\gamma$ can by definition be written as the sum of an element of $[0, p\gamma] \subseteq \langle\gamma\rangle$ and an element of pG, which implies that $\Delta_\gamma/\langle\gamma\rangle$ is p-divisible. □

We extract the following consequence of the definability results of [Jahnke and Koenigsmann 2015]. See the introduction of that paper for the notion of p-henselianity used in the proof.

Proposition 6.3. *Let (K, v) be a henselian valued field such that the residue field Kv is neither separably closed nor real closed. Then there exists an $\mathcal{L}_{\mathrm{r}}$-definable (not necessarily henselian) refinement w of v.*

Proof. Note that Kv either has a Galois extension of degree divisible by some prime $p \neq 2$ or it only has Galois extensions of 2-power degree.

In the latter case, since Kv is neither separably closed nor real closed, it has Galois extensions of degree 2 but is not Euclidean. Since v is henselian and thus, in particular, 2-henselian, we can now apply [Jahnke and Koenigsmann 2015, Corollary 3.3] to obtain that it admits an $\mathcal{L}_{\mathrm{r}}$-definable refinement. Indeed, let v_K^2 denote the canonical 2-henselian valuation (see [Jahnke and Koenigsmann 2015, page 743]). Then by [Jahnke and Koenigsmann 2015, Corollary 3.3], v_K^2 is $\mathcal{L}_{\mathrm{r}}$-definable if Kv_K^2 is non-Euclidean, otherwise the coarsest 2-henselian valuation with Euclidean residue field v_K^{2*} is $\mathcal{L}_{\mathrm{r}}$-definable. Either is a refinement of v.

Thus, we now assume that there is a prime $p \neq 2$ such that Kv has a finite Galois extension L of degree divisible by p. Then there is a finite separable extension M/Kv such that L/M is a finite Galois extension of degree p^n for some $n > 0$ (e.g., take M to be the fixed field of the p-Sylow subgroup of $\mathrm{Gal}(L/Kv)$ inside L). Let F_0/K be a finite separable extension such that the (by henselianity unique) prolongation of v to F_0 has residue field M (see [Engler and Prestel 2005, Theorem 5.2.7(2)] for the existence of F_0). Consider $F = F_0$ if the characteristic of K is p, and $F = F_0(\zeta_p)$ otherwise, where ζ_p is a primitive p-th root of unity. Then F/K is a finite separable extension, and the residue field of the unique prolongation u of v to F is a finite extension of M. In particular, Fu admits Galois extensions of p-power degree, e.g., the compositum of L and Fu. Therefore u is a henselian (and thus in particular p-henselian) valuation with $Fu \neq Fu(p)$, and hence coarsens the canonical p-henselian valuation v_F^p of F, which is $\varnothing$-$\mathcal{L}_{\mathrm{r}}$-definable in F by the main theorem of [Jahnke and Koenigsmann 2015].

Now F is interpretable in K as the splitting field of a separable polynomial (see [Marker 2002, page 31]). Hence $w = v_F^p|_K$ is $\mathcal{L}_{\mathrm{r}}$-definable in K. Lastly, $v = u|_K$ is a coarsening of w, as u is a coarsening of v_F^p in F. $\square$

We can now state our main theorem about definability of henselian valuations on ordered fields. In the proof, we need the following notion: for $n \in \mathbb{N}$, we say that a valuation v on a field K is *$n_{\leq}$-henselian* if Hensel's lemma holds in (K, v) for all polynomials of degree at most n. Note that for a fixed n, the property of $n_{\leq}$-henselianity is elementary in the language $\mathcal{L}_{\mathrm{vf}}$.

Theorem 6.4. *Let $(K, <, v)$ be an ordered henselian valued field. If v is $\mathcal{L}_{\mathrm{or}}$-definable, then it is $\mathcal{L}_{\mathrm{r}}$-definable.*

Proof. If v is trivial, then the proof is clear; thus we assume that v is nontrivial from now on. If K is almost real closed, then the result follows from Theorem 5.2, and hence we also assume that K is not almost real closed. Let v_K denote the canonical henselian valuation on K, i.e., the finest henselian valuation on K. In particular, v_K is a (not necessarily proper) refinement of v. The residue field Kv_K carries an ordering induced by the ordering on K, but is not real closed since K is not almost real closed by assumption. By Proposition 6.3, we may thus fix an $\mathcal{L}_{\mathrm{r}}$-definable refinement w of v_K (and hence of v).

Now, by Lemma 6.1, we can make the following case distinction.

Case 1: *$vK \not\equiv \Gamma \oplus \mathbb{Q}$ for any ordered abelian group Γ.*

Let $\Delta_v \leq wK$ be the convex subgroup such that $wK/\Delta_v = vK$. We first show that there is a prime p such that vK contains no nontrivial p-divisible convex subgroup. Assume for a contradiction that vK contains a nontrivial p-divisible convex subgroup for every prime p. Then any sufficiently saturated elementary extension G^* of vK contains a nontrivial convex divisible subgroup Q. Now, [Schmitt 1982, Lemma 1.11] implies

$$vK \equiv G^* \equiv G^*/Q \oplus Q \equiv G^*/Q \oplus \mathbb{Q},$$

contradicting that we are in Case 1.

Hence, we can fix some prime p such that vK does not contain any nontrivial p-divisible convex subgroup. By Lemma 6.2, there exists an $\mathcal{L}_{\mathrm{og}}$-formula $\varphi(x, y)$ such that for any positive $\gamma \in wK$, the subgroup Δ_γ of wK defined by $\varphi(x, \gamma)$ is the maximal convex subgroup of wK containing $\langle\gamma\rangle$ and such that $\Delta_\gamma/\langle\gamma\rangle$ is p-divisible. In case we choose $\gamma \in \Delta_v$, we have $\Delta_\gamma \leq \Delta_v$: otherwise

$$\langle\gamma\rangle \leq \Delta_v \lneq \Delta_\gamma$$

implies that $\Delta_\gamma/\Delta_v \leq vK$ is a nontrivial convex subgroup which is p-divisible since it is a quotient of the p-divisible group $\Delta_\gamma/\langle\gamma\rangle$.

For every $\gamma \in wK$, let u_γ be the $\mathcal{L}_{\mathrm{r}}$-definable coarsening of w on K with value group wK/Δ_γ. Since Δ_γ is uniformly $\mathcal{L}_{\mathrm{og}}$-definable in wK, also u_γ is uniformly $\mathcal{L}_{\mathrm{r}}$-definable in K, i.e., there exists an $\mathcal{L}_{\mathrm{r}}$-formula $\psi(x, \underline{y}, z)$ and a parameter tuple $\underline{b} \in K$ such that for every $a \in K^\times$ the formula $\psi(x, \underline{b}, a)$ defines $u_{w(a)}$.

If u_γ is already henselian for some $\gamma \in \Delta_v$, then v is an $\mathcal{L}_{\mathrm{or}}$-definable coarsening of an $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation and hence v is $\mathcal{L}_{\mathrm{r}}$-definable by Corollary 4.4. Thus, we assume that for every $\gamma \in \Delta_v$ the valuation u_γ is not henselian.

First suppose that there is some $n \in \mathbb{N}$ such that for every $\gamma \in \Delta_v$ we have that u_γ is not $n_{\leq}$-henselian. Let B be the $\mathcal{L}_{\mathrm{r}}$-definable subset of K consisting of all $a \in K^\times$ such that $u_{w(a)}$ is not $n_{\leq}$-henselian. We claim that $w(B) = \Delta_v$ holds. Let $a \in K^\times$ and set $\gamma = w(a)$. First suppose that $\gamma \in \Delta_v$. Then $u_\gamma = u_{w(a)}$ is not $n_{\leq}$-henselian. Thus, $a \in B$ and $\gamma \in w(B)$. Conversely, suppose that $\gamma \notin \Delta_v$. Then $\Delta_v \lneq \Delta_\gamma$ and thus u_γ is a strict coarsening of v. Since v is henselian, u_γ is $n_{\leq}$-henselian. Hence, $a \notin B$ and $\gamma \notin w(B)$, as required. Thus, in this case v is $\mathcal{L}_{\mathrm{r}}$-definable as $\mathcal{O}_v$ consists exactly of all $x \in K$ with $w(x) \geq 0 \vee w(x) \in w(B)$, which is an $\mathcal{L}_{\mathrm{r}}$-definable condition as w is $\mathcal{L}_{\mathrm{r}}$-definable.

Now suppose that for every $n \in \mathbb{N}$ there exists $\gamma_n \in \Delta_v$ such that u_{γ_n} is $n_{\leq}$-henselian. Then for every $n \in \mathbb{N}$, there is some $a_n \in K$ (with $w(a_n) = \gamma_n$) such that $\psi(x, \underline{b}, a_n)$ defines an $n_{\leq}$-henselian refinement of v in $(K, <, v)$. Hence, in some sufficiently saturated elementary extension $(K^*, <, v^*)$ of $(K, <, v)$, there exists $a \in K^*$ such that $\psi(x, \underline{b}, a)$ defines a henselian refinement u^* of v^*. Since v^* is $\mathcal{L}_{\mathrm{or}}$-definable by the same formula in K^* as v in K, it is an $\mathcal{L}_{\mathrm{or}}$-definable coarsening of the $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation u^*. Hence, v^* is $\mathcal{L}_{\mathrm{r}}$-definable in K^* by Corollary 4.4 and thus also v is $\mathcal{L}_{\mathrm{r}}$-definable in K.

Case 2: $(Kv, <) \not\equiv (L((\mathbb{Q})), <)$ for any ordered field $(L, <)$.

First suppose that $v \neq v_K$. Then v_K is strictly finer than v. Assume that $\overline{v_K}(Kv)$ is divisible, where $\overline{v_K}$ denotes the valuation induced by v_K on Kv. Then $\overline{v_K}(Kv)$ is elementarily equivalent to $\mathbb{Q}$ as an ordered abelian group, and thus we obtain

$$(Kv, <) \equiv (Kv_K((\overline{v_K}(Kv))), <) \equiv (Kv_K((\mathbb{Q})), <),$$

in contradiction to the assumption of Case 2.

Therefore we can assume that $\overline{v_K}(Kv) = \Delta$ is nondivisible. We show that there exists some $\mathcal{L}_{\mathrm{r}}$-definable henselian refinement u_γ of v.

We may assume that Δ does not have a rank 1 quotient: otherwise we could consider a sufficiently saturated elementary extension $(K^*, <^*, v^*, w^*)$ of $(K, <, v, w)$ in which — by the definability of the refinement w of v_K — $\overline{w^*}(K^*v^*)$ (and hence also $\overline{v_{K^*}}(K^*v^*)$) has no rank 1 quotient. Just as in Case 1, an $\mathcal{L}_{\mathrm{r}}$-definition of v^*

would give rise to an $\mathcal{L}_{\mathrm{r}}$-definition of v. We claim that there is a prime p such that Δ has no nontrivial p-divisible quotient. If not, then some saturated elementary extension Δ^* of Δ has a divisible nontrivial quotient Δ^*/Γ, where Γ is a convex proper subgroup. Then, as before, we have

$$\begin{aligned}(Kv, <) &\equiv (Kv_K((\overline{v_K}(Kv))), <) = (Kv_K((\Delta)), <)\\ &\equiv (Kv_K((\Delta^*)), <) \equiv (Kv_K((\Gamma))((\Delta^*/\Gamma)), <) \equiv (Kv_K((\Gamma))((\mathbb{Q})), <),\end{aligned}$$

contradicting that we are in Case 2. Hence, there is a prime p such that Δ is p-antiregular, i.e., Δ has no nontrivial p-divisible quotient and no rank 1 quotient (see [Jahnke and Koenigsmann 2017, page 670]).

Recall that w is an $\mathcal{L}_{\mathrm{r}}$-definable refinement of v_K. Then, there are convex subgroups

$$\Delta_{v_K} \lneq \Delta_v \leq wK$$

with $v_K K = wK/\Delta_{v_K}$ and $vK = wK/\Delta_v$.

Let $\gamma \in \Delta_v \setminus \Delta_{v_K}$ be positive, and let $\langle\gamma\rangle \leq wK$ denote the smallest convex subgroup containing γ. Since the convex subgroups of wK are ordered by inclusion, we have

$$\Delta_{v_K} \lneq \langle\gamma\rangle \leq \Delta_v.$$

Note that $\langle\gamma\rangle$ need not be $\mathcal{L}_{\mathrm{og}}$-definable in wK. However, by Lemma 6.2, the maximal convex subgroup $\Delta_\gamma \leq wK$ that contains $\langle\gamma\rangle$ and such that $\Delta_\gamma/\langle\gamma\rangle$ is p-divisible is $\mathcal{L}_{\mathrm{og}}$-definable in wK.

We claim that $\Delta_\gamma \leq \Delta_v$, i.e., that Δ_γ corresponds to an $\mathcal{L}_{\mathrm{r}}$-definable refinement of v. Assume for a contradiction that we have $\Delta_v \lneq \Delta_\gamma$. Since Δ_v contains $\langle\gamma\rangle$, the choice of Δ_γ implies that $\Delta_v/\langle\gamma\rangle$ is p-divisible. If $\langle\gamma\rangle \neq \Delta_v$, then $\Delta_v/\langle\gamma\rangle$ is a nontrivial p-divisible quotient of $\Delta = \Delta_v/\Delta_{v_K}$, a contradiction. Otherwise, $\langle\gamma\rangle = \Delta_v$ and the quotient of $\langle\gamma\rangle$ by its maximal convex subgroup not containing γ is of rank 1, also a contradiction. Thus, we have found an $\mathcal{L}_{\mathrm{r}}$-definable refinement of v. Since we have

$$\Delta_{v_K} \lneq \Delta_\gamma,$$

this refinement is a coarsening of v_K and thus henselian.

Now suppose that $v = v_K$. If Kv is not t-henselian, then v is $\mathcal{L}_{\mathrm{r}}$-definable by [Fehm and Jahnke 2015, Proposition 5.5]. Hence, suppose that Kv is t-henselian. Then for a sufficiently saturated elementary extension $(K^*, <^*, v^*) \succeq (K, <, v)$ the residue field $K^*v^* \succeq Kv$ is itself henselian. Since v is $\mathcal{L}_{\mathrm{or}}$-definable in K, also v^* is $\mathcal{L}_{\mathrm{or}}$-definable in K^*. However, since K^*v^* is henselian, v^* is not the canonical henselian valuation of K^*, and therefore by the arguments above v^* is already $\mathcal{L}_{\mathrm{r}}$-definable in K^*. Hence, also v is $\mathcal{L}_{\mathrm{r}}$-definable in K. □

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A note on geometric theories of fields

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Let T be a complete theory of fields, possibly with extra structure. Suppose that model-theoretic algebraic closure agrees with field-theoretic algebraic closure, or more generally that model-theoretic algebraic closure has the exchange property. Then T has uniform finiteness, or equivalently, it eliminates the quantifier $\exists^\infty$. It follows that very slim fields in the sense of Junker and Koenigsmann are the same thing as geometric fields in the sense of Hrushovski and Pillay. Modulo some fine print, these two concepts are also equivalent to algebraically bounded fields in the sense of van den Dries.

From the proof, one gets a one-cardinal theorem for geometric theories of fields: any infinite definable set has the same cardinality as the field. We investigate whether this extends to interpretable sets. We show that positive dimensional interpretable sets must have the same cardinality as the field, but zero-dimensional interpretable sets can have smaller cardinality. As an application, we show that any geometric theory of fields has an uncountable model with only countably many finite algebraic extensions.

1. Introduction

Throughout the paper, T denotes a complete theory. We use $\operatorname{acl}(-)$ to denote the model-theoretic algebraic closure. When T expands the theory of fields, we use $(-)^{\mathrm{alg}}$ to denote the field-theoretic algebraic closure. Following [Hrushovski and Pillay 1994; Gagelman 2005], we say that T is *geometric* if (1)–(2) hold and *pregeometric* if (1) holds:

(1) $\operatorname{acl}(-)$ satisfies the exchange property.

(2) T eliminates $\exists^\infty$, or equivalently, T has uniform finiteness.

A (*pre*)*geometric structure* is a structure M whose complete theory is (pre)geometric.

Using a simple argument, we show that pregeometric fields are geometric (Theorem 2.5). This seems to not be well-known. For example, it is implicitly unknown in [Hrushovski and Pillay 1994, Remark 2.12], and a special case of

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this implication is asked as an open problem in [Junker and Koenigsmann 2010, Question 1].

As a consequence of Theorem 2.5, several concepts in the literature are equivalent, namely the *very slim* fields of Junker and Koenigsmann [2010, Definition 1.1], the *geometric fields* of Hrushovski and Pillay [1994, Remark 2.10], and (modulo some fine print) the *algebraically bounded fields* of van den Dries [1989].

Our method also shows that in a geometric field, any infinite definable set has the same cardinality as the field (Proposition 3.1), which may be of independent interest. It is natural to ask whether this extends to interpretable sets. In Proposition 3.6, we show that if X is an interpretable set of positive dimension, then $|X| = |K|$, but there are models where all the zero-dimensional interpretable sets satisfy $|X| < |K|$. As an application, there is an uncountable model K with only countably many finite algebraic extensions (Corollary 4.3), which may be of interest to field theorists. In the special case of ω-free perfect PAC fields, this recovers examples such as [Bary-Soroker and Paran 2013, Example 2.2].

2. Uniform finiteness from the exchange property

If M is a geometric structure, there is a well-established dimension theory on M defined as follows. If $A \subseteq M$ and a is a tuple, we define $\dim(a/A)$ to be the length of the maximal acl_A-independent subtuple of a. This is well-defined by the exchange property. If X is an A-definable subset of M^n, we define

$$\dim(X) = \max\{\dim(x/A) : x \in X(\mathbb{U})\}$$

for some monster model $\mathbb{U} \succeq M$.

Fact 2.1. *Let M be a pregeometric structure, and X, Y be definable sets.*

(1) $\dim(X)$ *is well-defined, independent of the choice of A.*

(2) $\dim(X) > 0 \iff |X| = \infty$.

(3) $\dim(X \times Y) = \dim(X) + \dim(Y)$.

(4) $\dim(M^n) = n$ *unless M is finite.*

(5) *If $f : X \to Y$ is a definable injection or surjection, then* $\dim(X) \leq \dim(Y)$ *or* $\dim(X) \geq \dim(Y)$, *respectively.*

Gagelman observed that the dimension theory can also be extended to M^{eq}. If $a \in M^{\mathrm{eq}}$, then $\dim(a/A)$ is defined to be $\dim(b/Aa) - \dim(b/A)$ for any tuple b with $a \in \mathrm{acl}^{\mathrm{eq}}(Ab)$. If X is an interpretable set, then $\dim(X)$ is defined as for definable sets. By [Gagelman 2005, Lemma 3.3], these definitions are well-defined, and all of Fact 2.1 holds *except for* (4) and the $\Leftarrow$ direction of (2) [Gagelman 2005, Proposition 3.4 and p. 321].

Lemma 2.2. *Let K be a pregeometric field. Then K is perfect.*

Proof. Otherwise, K is an infinite field of characteristic p. Take $a \in K \setminus K^p$. Then the map

$$f(x, y) \mapsto x^p + ay^p$$

is a definable injection $f : K \times K \to K$. But $\dim(K^2) = 2 > \dim(K) = 1$, as K is infinite. □

Lemma 2.3. *Let K be a field and X be a subset. Then one of the following holds*:

(1) *The set $S = \{(y - y')/(x' - x) : x, y, x', y' \in X, x \neq x'\}$ equals K.*

(2) *There is an injection $f : X^2 \to K$ of the form $f(x, y) = ax + y$.*

Proof. If $a \in K \setminus S$, then the function $f(x, y) = ax + y$ is injective on X^2. □

Lemma 2.4. *Let K be an infinite pregeometric field*, *and let $X \subseteq K$ be definable. Then X is infinite if and only if $K = \{(y - y')/(x' - x) : x, y, x', y' \in X,\ x \neq x'\}$.*

Proof. If X is infinite, then $\dim(X) > 0$, so $\dim(X^2) \geq 2 > \dim(K) = 1$ and there are no definable injections $f : X^2 \to K$. Therefore, case (2) of Lemma 2.3 cannot hold, so (1) holds. Conversely, if X is finite, then case (1) of Lemma 2.3 cannot hold, as the set S would be finite. □

Theorem 2.5. *Let T be a complete theory of fields*, *possibly with extra structure. If T is pregeometric*, *then T is geometric.*

Proof. It suffices to eliminate $\exists^\infty$. If models of T are finite, then $\exists^\infty$ is trivially eliminated. If the models of T are infinite, then Lemma 2.4 gives a first-order criterion for telling whether a definable set $X \subseteq K$ is infinite. □

The proof of Theorem 2.5 is the same argument used in [Johnson 2018, Observation 3.1].

Remark 2.6. Let $(K, +, \cdot, \dots)$ be a field, possibly with extra structure. If $\operatorname{acl}(-)$ agrees with field-theoretic algebraic closure, then $\operatorname{acl}(-)$ satisfies exchange.

Definition 2.7 [Junker and Koenigsmann 2010, Definition 1.1]. A field K is *very slim* if $\operatorname{acl}(-)$ agrees with field-theoretic algebraic closure in any elementary extension of K.

Definition 2.8 [Hrushovski and Pillay 1994, Remark 2.10]. A *strongly geometric field* is a perfect field $(K, +, \cdot)$ with a geometric theory that is very slim.

Hrushovski and Pillay call these *geometric fields*, but we prefer the term *strongly geometric* to distinguish strongly geometric fields from the more general case of fields that are geometric (as structures). There are geometric fields that are not strongly geometric (Remark 2.16).

Corollary 2.9. *A field K is very slim if and only if it is strongly geometric.*

Proof. If K is very slim, then K is pregeometric, hence geometric by Theorem 2.5 and perfect by Lemma 2.2 (or [Junker and Koenigsmann 2010, Proposition 4.1]). □

Definition 2.10 [van den Dries 1989]. Let $(K, +, \cdot, \ldots)$ be an expansion of a field, and F be a subfield. Then K is *algebraically bounded over F* if for any formula $\varphi(\bar{x}, y)$, there are finitely many polynomials $P_1, \ldots, P_m \in F[\bar{x}, y]$ such that for any $\bar{a}$, *if* $\varphi(\bar{a}, K)$ is finite, then $\varphi(\bar{a}, K)$ is contained in the zero set of $P_i(\bar{a}, y)$ for some i such that $P_i(\bar{a}, y)$ does not vanish. Following the convention in [van den Dries 1989; Junker and Koenigsmann 2010] we say that K is *algebraically bounded* if it is algebraically bounded over K.

Lemma 2.11. *Suppose K is algebraically bounded over some subfield F.*

(1) *If A is a subset of K, then* $\mathrm{acl}(AF)$ *is the field-theoretic relative algebraic closure* $(AF)^{\mathrm{alg}} \cap K$.

(2) *K has uniform finiteness.*

(3) *If K^* is an elementary extension of K, then K^* is algebraically bounded over F.*

Proof. (1) Clearly $(AF)^{\mathrm{alg}} \cap K \subseteq \mathrm{acl}(AF)$. Conversely if $b \in \mathrm{acl}(AF)$, then b is in a finite set $\varphi(\bar{a}, K)$ for some tuple $\bar{a}$ from AF. By algebraic boundedness, $\varphi(\bar{a}, K)$ is contained in a finite zero set of some polynomial $P(\bar{a}, y)$, where $P(\bar{x}, y) \in F[\bar{x}, y]$. Therefore b is field-theoretically algebraic over AF.

(2) For a fixed formula $\varphi(\bar{x}, y)$, let $P_1, \ldots, P_m \in F[\bar{x}, y]$ be polynomials as in Definition 2.10. Then the cardinality of finite sets of the form $\varphi(\bar{a}, K)$ is bounded by the maximum of the degrees of the P_i's.

(3) By (2), the theory of K has uniform finiteness, and so $\exists^\infty$ is uniformly eliminated across elementary extension of K. It follows that for fixed $\varphi, P_1, \ldots, P_m$, the following condition is preserved in elementary extensions:

> For any $\bar{a}$, if $\varphi(\bar{a}, y)$ defines a finite set, then there is $i \in \{1, \ldots, m\}$ such that $P_i(\bar{a}, y)$ has finitely many zeros and $\varphi(\bar{a}, y) \to P_i(\bar{a}, y) = 0$.

Thus, algebraic boundedness transfers from K to any elementary extension K^*. □

Lemma 2.12. *Let $K = (K, +, \cdot, \ldots)$ be a field or an expansion of a field. Let F be a subfield. The following are equivalent*:

(1) *In any elementary extension $K^* \succeq K$, field-theoretic algebraic closure over F agrees with model-theoretic algebraic closure over F: if $A \subseteq K^*$ and $b \in \mathrm{acl}(AF)$, then $b \in (AF)^{\mathrm{alg}}$.*

(2) *Condition* (1) *holds and K has uniform finiteness, or equivalently, $\exists^\infty$ is eliminated in elementary extensions of K.*

(3) *K is algebraically bounded over F.*

Proof. (1)$\Longrightarrow$(2). Field-theoretic algebraic closure satisfies the exchange property. Therefore $\operatorname{acl}(-)$ satisfies exchange (after naming the elements of F as parameters). By Theorem 2.5, elementary extensions of K eliminate $\exists^\infty$.

(2)$\Longrightarrow$(3). Suppose (2) holds but K fails to be algebraically bounded over F, witnessed by some formula $\varphi(\bar{x}, y)$. Using elimination of $\exists^\infty$, we may assume that there is $n \in \mathbb{N}$ such that $|\varphi(\bar{a}, K)| \leq n$ for every $\bar{a}$. For any finite set of polynomials $P_1, \ldots, P_m \in F[\bar{x}, y]$, there is $\bar{a} \in K$ such that $\varphi(\bar{a}, K)$ is finite, but is not contained in the zero set of $P_i(\bar{a}, y)$ unless $P_i(\bar{a}, y) \equiv 0$. By compactness, there is an elementary extension $K^* \succeq K$ and a tuple $\bar{a} \in K^*$ such that $\varphi(\bar{a}, K^*)$ is finite, but is not contained in the zero set of $P(\bar{a}, y)$ for *any* $P \in F[\bar{x}, y]$ except those with $P(\bar{a}, y) \equiv 0$. Then $\varphi(\bar{a}, K^*)$ contains a point not in $F(\bar{a})^{\mathrm{alg}}$, contradicting (2).

(3)$\Longrightarrow$(1). Lemma 2.11 shows that if K^* is an elementary extension of K, then

- K^* is algebraically bounded over F, and
- field-theoretic and model-theoretic algebraic agree over F.

Therefore (1) holds. □

Specializing to the case where K is a pure field and F is the prime field, we get the following:

Theorem 2.13. *Let K be a pure field. Then K is algebraically bounded over the prime field if and only if K is very slim.*

We have thus answered [Junker and Koenigsmann 2010, Question 1] positively.

Remark 2.14. If $(K, +, \cdot, \ldots)$ is an *expansion* of a field, and $(K, +, \cdot, \ldots)$ is algebraically bounded over the prime field, then the reduct $(K, +, \cdot)$ is also algebraically bounded over the prime field, and so the underlying field $(K, +, \cdot)$ is very slim.

Remark 2.15. Reducts of geometric structures are geometric structures. This is folklore, but we include a proof for completeness. Let N be a geometric structure and M be a reduct. Without loss of generality, N and M are highly saturated. Uniform finiteness transfers from N to M in a trivial way: if M fails uniform finiteness, the same definable family fails uniform finiteness in N. Suppose $\operatorname{acl}^M(-)$ does not satisfy exchange. Then there are some $a, b \in M$ and $C \subseteq M$ such that $a \notin \operatorname{acl}^M(C)$, $b \notin \operatorname{acl}^M(Ca)$, but $a \in \operatorname{acl}^M(Cb)$. Let $p(x)$ and $q(x, y)$ be $\operatorname{tp}^M(a/C)$ and $\operatorname{tp}^M(a, b/C)$. The number of realizations of $p(x)$ is large, so we may find $a' \models p$ with $a' \notin \operatorname{acl}^N(C)$. Similarly, the number of realizations of $q(a', y)$ is large, so we may find a realization $b' \notin \operatorname{acl}^N(Ca')$. Then $a'b' \equiv_C ab$ in M, so $a' \in \operatorname{acl}^M(Cb') \subseteq \operatorname{acl}^N(Cb')$. Then a' and b' contradict the exchange property in N.

Therefore, any reduct of an algebraically bounded field is geometric, though not necessarily strongly geometric.

Remark 2.16. In future work, we will give an example of a pure field $(K, +, \cdot)$ of characteristic 0 with a subfield K_0 such that

(1) Field-theoretic algebraic closure and model-theoretic algebraic closure agree over K_0, and this remains true in elementary extensions.
(2) $\mathrm{acl}(\varnothing)$ contains elements of K_0 that are transcendental over $\mathbb{Q}$. In particular, field-theoretic algebraic closure and model-theoretic algebraic closure do not agree over $\mathbb{Q}$.

It follows that this field K is algebraically bounded over K_0, but not over $\mathbb{Q}$. In particular, K is algebraically bounded (over K) but not very slim and not a strongly geometric field.[1] Additionally, the field K is geometric (by Remark 2.15) but not strongly geometric.

Failure of $\mathrm{acl}(\varnothing)$ to be algebraic over the prime field is the *only* way an algebraically bounded field can fail to be very slim:

Proposition 2.17. *If $K = (K, +, \cdot, \dots)$ is algebraically bounded, then K is algebraically bounded over the subfield $F = \mathrm{dcl}(\varnothing)$.*

Proof. We use the criterion of Lemma 2.12(1) to show that K is algebraically bounded over F. Embed K into a monster model $\mathbb{K}$. Suppose $b, \bar{c} \in \mathbb{K}$ and $b \in \mathrm{acl}(F\bar{c})$. We must show $b \in F(\bar{c})^{\mathrm{alg}}$. By Remark 2.15, $\mathrm{Th}(K)$ is geometric, because it is geometric after naming parameters from F. Replacing $\bar{c}$ by a basis of $\bar{c}$ in the acl-pregeometry, we may assume that the tuple $\bar{c}$ is field-theoretically algebraically independent over F. Now suppose for the sake of contradiction that $b \notin F(\bar{c})^{\mathrm{alg}}$. Then $(\bar{c}, b)$ is also field-theoretically algebraically independent over F.

As $b \in \mathrm{acl}(F\bar{c}) = \mathrm{acl}(\bar{c})$, there is a 0-definable set $D \subseteq \mathbb{K}^n$ with $(b, \bar{c}) \in D$ and $D_{\bar{c}'}$ finite for each $\bar{c}'$. By algebraic boundedness over K, there are finitely many nonzero polynomials $P_1, \dots, P_i \in K[x, \bar{y}]$ such that D is contained in the union of the zero-sets of the P_i.

Let $M = \mathbb{K}^{\mathrm{alg}}$ and let V be the Zariski closure of D in M^{n+1}. The polynomials P_i show that $V \subsetneq M^{n+1}$, and so $\dim(V) < n + 1$. By elimination of imaginaries in ACF, there is a finite tuple e in M which codes V. Recall that $\mathbb{K}$ is perfect by Lemma 2.2. If $\sigma \in \mathrm{Aut}(M/\mathbb{K}) = \mathrm{Gal}(\mathbb{K})$, then $\sigma(D) = D$, $\sigma(V) = V$, and $\sigma(e) = e$. As the tuple e is fixed by $\mathrm{Gal}(M/\mathbb{K})$, it must be in the perfect field $\mathbb{K}$.

If σ is any automorphism of $\mathbb{K}$, then σ can be extended to an automorphism σ' of M. The fact that D is 0-definable implies that $\sigma'(D) = \sigma(D) = D$, which then implies $\sigma'(V) = V$ and $\sigma(e) = \sigma'(e) = e$. Thus e is $\mathrm{Aut}(\mathbb{K})$-invariant, which implies that e is in $F = \mathrm{dcl}(\varnothing)$. Therefore, in the structure M, the e-definable set V is in fact F-definable. However, the tuple $(b, \bar{c}) \in D \subseteq V$ is algebraically

[1]This contradicts the claim in [Junker and Koenigsmann 2010, p. 482] that algebraically bounded fields are very slim.

independent over F, so this implies $\dim(V) = n+1$, contradicting the earlier fact that $\dim(V) < n+1$. $\square$

Remark 2.18. Algebraically bounded fields are closely related to fields of size at most S in the sense of [Junker and Koenigsmann 2010]. For fields of size at most S, [Junker and Koenigsmann 2010, Proposition 3.4] and Lemma 2.12 show that they are algebraically bounded over $\mathrm{dcl}(\varnothing)$. On the other hand, by Lemma 2.12 and Proposition 2.17, any algebraically bounded field with $\mathrm{tr.deg}(\mathrm{dcl}(\varnothing)) \in \mathbb{N}$ is of size at most S.

Question 2.19. Is there a pure field K that is geometric, but not algebraically bounded?

3. Cardinalities

Fix a complete geometric theory T expanding the theory of fields, not necessarily algebraically bounded.

Proposition 3.1. *If $K \models T$ and $X \subseteq K^n$ is an infinite definable set, then $|X| = |K|$.*

Proof. Clearly $|X| \leq |K|$. We must show $|X| \geq |K|$. Replacing X with a projection onto one of the coordinate axes, we may assume $X \subseteq K^1$. By Lemma 2.4,

$$K = \{(y - y')/(x' - x) : x, y, x', y' \in X,\ x \neq x'\}.$$

Therefore $|K| \leq |X|^4 = |X|$. $\square$

Proposition 3.1 does not generalize to interpretable sets, as exhibited by the example of the value group $\mathbb{Z}$ in the geometric field $\mathbb{Q}_p$. In Proposition 3.6 below, we will see that the obstruction is precisely the zero-dimensional interpretable sets. Before proving this, we need a few general lemmas on geometric structures.

Lemma 3.2. *Suppose M is a geometric structure and X is an interpretable set in M with* $\dim(X) = d > 0$.

(1) *There is an interpretable set Y in M and finite-to-one interpretable functions $f : Y \to X$ and $g : Y \to M^d$ such that the image $g(Y)$ has dimension d.*

(2) *There is an infinite definable set D with $|X| \geq |D|$.*

Proof. Note that (1) implies (2), by taking $D = g(Y)$. We prove (1). Embed M into a monster model $\mathbb{M} \succeq M$. Take $e \in X$ with $\dim(e/M) = d$. Then $e \in \mathbb{M}^{\mathrm{eq}}$. Every imaginary is definable from a real tuple, so there is a real tuple $\bar{a} \in \mathbb{M}^m$ with $e \in \mathrm{acl}^{\mathrm{eq}}(M\bar{a})$. Replacing $\bar{a}$ with a subtuple, we may assume that $\bar{a}$ is independent over M. By [Gagelman 2005, Lemma 3.1], $\mathrm{acl}(-)$ continues to satisfy exchange after naming the parameter e. Therefore, we can meaningfully talk about real tuples being independent over eM. Write $\bar{a}$ as $\bar{b}\bar{c}$, where $\bar{b}$ is a maximal subtuple that is independent over eM. Then $\bar{c} \in \operatorname{acl}(\bar{b}eM)$. At the same time, $e \in \operatorname{acl}(\bar{b}\bar{c}M)$, so e is interalgebraic with $\bar{c}$ over $\bar{b}M$.

Meanwhile, $\dim(\bar{b}/eM) = \dim(\bar{b}/M) = |\bar{b}|$, because $\bar{b}$ is an independent tuple over eM. Then $\bar{b}$ and e are independent from each other over M, implying $\dim(e/\bar{b}M) = \dim(e/M) = d$ by symmetry. As e and $\bar{c}$ are interalgebraic over $\bar{b}M$, we have $\dim(\bar{c}/\bar{b}M) = d$. But $\bar{b}\bar{c}$ is an independent tuple over M, so $\dim(\bar{c}/\bar{b}M)$ is the length of $\bar{c}$. Thus $\bar{c} \in M^d$.

As e and $\bar{c}$ are interalgebraic over $\bar{b}M$, there is a $\bar{b}M$-interpretable set $Y_0 \subseteq X \times \mathbb{M}^d$ such that $(e, \bar{c}) \in Y_0$, and the projections $f_0 : Y_0 \to X$ and $g_0 : Y_0 \to \mathbb{M}^d$ have finite fibers. By saturation, there is a uniform bound N on the fiber size. The image $g_0(Y_0)$ is $\bar{b}M$-definable and contains the point $\bar{c}$ with $\dim(\bar{c}/\bar{b}M) = d$. Therefore $g_0(Y_0)$ has dimension d.

Now we have the desired configuration (Y_0, f_0, g_0), but defined over the parameter $\bar{b}$ outside M. Because $M \preceq \mathbb{M}$ and dimension is definable in families [Gagelman 2005, Fact 2.4], we can replace the parameter $\bar{b}$ with something in M, getting an M-definable configuration (Y, f, g) in which the fibers of f and g are still bounded in size by N. $\square$

Definition 3.3. Let M be a structure. A *definable notion of largeness*[2] on M is a partition of the M-definable sets into two classes — large and small — such that the following axioms hold:

(1) Any finite set is small.

(2) Any definable subset of a small set is small.

(3) If Y is small and $\{X_a\}_{a \in Y}$ is a definable family of small sets, then the union $\bigcup_{a \in Y} X_a$ is small.

(4) Smallness is definable in families: if $\{X_a\}_{a \in Y}$ is a definable family, then the set $\{a \in Y : X_a \text{ is small}\}$ is definable.

If $N \succeq M$, then any definable notion of largeness on M extends in a canonical way to a definable notion of largeness on N by extending the definition according to (4) above.

Fact 3.4. *Let M be a countable structure in a countable language. Fix a definable notion of largeness on M. Then there is an elementary extension $N \succeq M$ such that if X is definable in N, then X is uncountable if and only if X is large.*

Fact 3.4 is essentially Keisler's completeness theorem for $\mathcal{L}(Q)$ [Keisler 1970, Section 2], but the translation between these settings is sufficiently confusing that we give the details.

[2]This is a purely model-theoretic notion and should not be confused with the notion of large fields.

Proof of Fact 3.4. Let T be the elementary diagram of M. Let $\mathcal{L}$ be the language of T, and let $\mathcal{L}(Q)$ be the language obtained by adding a new quantifier (Qx). Let $\psi \mapsto \psi^*$ be the map from $\mathcal{L}(Q)$-formulas to $\mathcal{L}$-formulas interpreting $(Qx)\varphi(x, \bar{y})$ as "the set of x such that $\varphi(x, \bar{y})$ holds is large." More precisely,

- $\varphi^* = \varphi$ if φ doesn't involve the quantifier Q.
- $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$, and similarly for the other logical operators including $\exists$ and $\forall$.
- $((Qx)\varphi(x, \bar{y}))^*$ is the formula $\psi(\bar{y})$ such that in models $N \models T$,

$$N \models \psi(\bar{b}) \iff (\varphi^*(N, \bar{b}) \text{ is large}).$$

Let T' be the set of $\mathcal{L}(Q)$-sentences φ such that $T \vdash \varphi^*$. It is straightforward to verify that T' is closed under the rules of inference on pages 6–7 of [Keisler 1970]. For example, the "axioms of $\mathcal{L}(Q)$" [Keisler 1970, p. 6] correspond to the axioms in Definition 3.3. By the completeness theorem for $\mathcal{L}(Q)$ [Keisler 1970, Section 2], there is an $\mathcal{L}$-structure N which satisfies the sentences T', when (Qx) is interpreted as

"there are uncountably many x such that…".

Ignoring the sentences involving Q, we see that $N \models T$, and so $N \succeq M$. Finally, suppose $X = \varphi(N, \bar{b})$ is definable in N. Let $\psi(\bar{y})$ be the $\mathcal{L}$-formula such that $\psi(\bar{b})$ holds iff $\varphi(N, \bar{b})$ is large. Then T' contains the sentence

$$(\forall \bar{y})[\psi(\bar{y}) \leftrightarrow (Qx)\varphi(x, \bar{y})],$$

because its image under $(-)^*$ is the tautology

$$(\forall \bar{y})[\psi(\bar{y}) \leftrightarrow \psi(\bar{y})].$$

Therefore,

$$\varphi(N, \bar{b}) \text{ is uncountable} \iff N \models \psi(\bar{b}) \iff \varphi(N, \bar{b}) \text{ is large}. \qquad \square$$

Lemma 3.5. *Let T be a complete geometric theory in a countable language. Then there is a model $N \models T$ such that for any interpretable set X,*

$$\dim(X) > 0 \iff |X| > \aleph_0.$$

Proof. Take a model $M \models T$. There is a definable notion of largeness on M^{eq} in which X is large iff $\dim(X) > 0$. The requirements of Definition 3.3 hold by properties of dimension in M^{eq} [Johnson 2022, Propositions 2.8, 2.9, and 2.12]. Applying Fact 3.4, we get an elementary extension $N^{\mathrm{eq}} \succeq M^{\mathrm{eq}}$ such that if X is definable in N^{eq}, then X is uncountable iff $\dim(X) > 0$. $\square$

Proposition 3.6. *Let T be a complete, geometric theory of infinite fields, possibly with extra structure.*

(1) *If $K \models T$ and X is an interpretable set of positive dimension, then $|X| = |K|$.*
(2) *If the language is countable, there is a model $K \models T$ of cardinality $\aleph_1$ such that every zero-dimensional interpretable set is countable.*

Proof. (1) Suppose X has positive dimension. Lemma 3.2 shows that $|X| \geq |D|$ for some infinite definable set D. Then $|D| \geq |X| \geq |K|$ by Proposition 3.1. Finally, $|K| \geq |X|$ is clear.

(2) Lemma 3.5 gives an uncountable model K in which every zero-dimensional interpretable set is countable. By downward Löwenheim–Skolem, we can replace K with an elementary substructure of cardinality $\aleph_1$. □

4. Finite extensions

Let T be a complete, geometric theory of fields, possibly with extra structure, not necessarily algebraically bounded.

Proposition 4.1. *If $K \models T$ and $n \in \mathbb{N}_{>0}$, then the (interpretable) set of degree n finite extensions has dimension zero.*

Proof. By Lemma 2.2, K is perfect. Hence any finite extension of K is a simple extension. Let X be the set of irreducible monic polynomials of degree n. We can regard X as a definable subset of K^n by identifying a polynomial

$$P(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0$$

with the n-tuple $(c_0, c_1, \ldots, c_{n-1})$. Let Y be the interpretable set of degree n finite extensions. Let $f : X \to Y$ be the map sending $P(x)$ to the extension $K[x]/(P(x))$.

Note that $\dim(X) \leq \dim(K^n) = n$. By the definition of dimension for interpretable sets, it suffices to show that each fiber of f has dimension at least n. Fix some $b \in Y$ corresponding to a degree n extension L/K. We claim $f^{-1}(b)$ has dimension n. By identifying L with K^n, we can regard L as a definable set with $\dim(L) = n$. Because K is perfect, there are only finitely many fields between K and L. Let U be the union of the intermediate fields. Then U has lower dimension than n, so $\dim(L \setminus U) = n$. The elements of $L \setminus U$ are generators of L. The fiber $f^{-1}(b)$ is the set of minimal polynomials of elements of $L \setminus U$. Let $\rho : (L \setminus U) \to f^{-1}(b)$ be the map sending $a \in L \setminus U$ to its minimal polynomial. Then ρ is finite-to-one, so $\dim(f^{-1}(b)) \geq \dim(L \setminus U) = n$. □

We recover the following corollary, which is presumably well-known (for example, it follows from [Gagelman 2005, Corollary 3.6] and [Pillay and Poizat 1995, Théorème]).

Corollary 4.2. *If T eliminates imaginaries, then models of T have bounded Galois group — there are only finitely many extensions of degree n for each n.*

Proof. Zero-dimensional definable sets are finite. If elimination of imaginaries holds, then zero-dimensional interpretable sets are finite. □

Combining Propositions 3.6 and 4.1, we have the following.

Corollary 4.3. *If T is a geometric theory of infinite fields in a countable language, then there is an uncountable model $K \models T$ with countably many finite extensions of degree n for each n.*

For example, since perfect PAC fields are geometric [Chatzidakis and Hrushovski 2004], Corollary 4.3 recovers results such as [Bary-Soroker and Paran 2013, Example 2.2]. The fact that all very slim fields satisfy this property might have some field-theoretic consequences.

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