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We study the definability of convex valuations on ordered fields, with a particular focus on the distinguished subclass of henselian valuations. In the setting of ordered fields, one can consider definability both in the language of rings \mathcal{L}_r and in the richer language of ordered rings \mathcal{L}_{or} . We analyse and compare definability in both languages and show the following contrary results: while there are *convex* valuations that are definable in the language \mathcal{L}_{or} but not in the language \mathcal{L}_r , any \mathcal{L}_{or} -definable *henselian* valuation is already \mathcal{L}_r -definable. To prove the latter, we show that the value group and the *ordered* residue field of an ordered henselian valued field are stably embedded (as an ordered abelian group and an ordered field, respectively). Moreover, we show that in almost real closed fields *any* \mathcal{L}_{or} -definable valuation is henselian.

1. Introduction

One of the main objectives in the model-theoretic study of fields is the analysis of first-order definable¹ sets and substructures. Given a field, it is a natural question to ask whether a given valuation ring is a definable subset in some expansion of the language $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$ of rings. A key reason to study definability of valuation rings is to transfer questions of decidability and existential decidability (i.e., the question whether Hilbert’s tenth problem has a positive solution) between different rings and fields. However, there is also a more recent motivation stemming from classifying fields within Shelah’s classification hierarchy: whereas stable (or,

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¹Throughout this work, “definable” always means “definable *with* parameters”.

more generally, simple) fields do not admit any nontrivial \mathcal{L}_r -definable valuations, a conjecture going back to Shelah predicts that infinite NIP fields which are neither real closed nor separably closed admit a nontrivial \mathcal{L}_r -definable henselian valuation. In a recent series of spectacular papers, this was shown to hold in the “finite-dimensional” (i.e., dp-finite) case by Johnson [2020]. For a survey on definability of henselian valuations, mostly in the language of rings, see [Fehm and Jahnke 2017].

In this work, we primarily study valuations on *ordered* fields. This allows us to also consider their definability in the language of *ordered* rings $\mathcal{L}_{\text{or}} = \mathcal{L}_r \cup \{<\}$. We focus on convex valuations, i.e., valuations whose valuation ring is convex with respect to the ordering, as these naturally induce an ordering on the residue field (see [Engler and Prestel 2005, Proposition 2.2.4]). Note that due to [Engler and Prestel 2005, Lemma 4.3.6], every henselian valuation on an ordered field is already convex.

By considering the expanded language \mathcal{L}_{or} rather than \mathcal{L}_r , one may expect further definability results. Indeed, we present examples of ordered fields with *convex* valuations that are \mathcal{L}_{or} -definable but not \mathcal{L}_r -definable (see Examples 3.5 and 3.6). Rather surprisingly, for *henselian* valuations the language \mathcal{L}_{or} does not produce any further definability results, that is, every \mathcal{L}_{or} -definable henselian valuation is already \mathcal{L}_r -definable (see Theorem 6.4). In the particular case of almost real closed fields, \mathcal{L}_{or} -definability even suffices to ensure both henselianity (thus convexity) and \mathcal{L}_r -definability (see Theorem 5.2).

The structure of this paper is as follows. After introducing preliminary notions and results in Section 2, we first turn to the definability of convex valuations in Section 3. We establish conditions on the value group and the residue field ensuring the definability (*with* and *without* parameters) of a given convex valuation (see Theorem 3.1 and Corollary 3.2). Subsequently we compare these results to other known definability conditions in the literature (see Remark 3.3) and construct our main examples — Examples 3.5 and 3.6 — to show that there are convex valuations that are \emptyset - \mathcal{L}_{or} -definable but not \mathcal{L}_r -definable. Lastly, we answer [Krapp et al. 2021, Question 7.1] positively by presenting in Example 3.7 an ordered valued field that is dense in its real closure but still admits a nontrivial \emptyset - \mathcal{L}_{or} -definable convex valuation. In Section 4 we turn to ordered *henselian* valued fields and establish in Theorem 4.2 that their value group (as an ordered abelian group) and their residue field (as an ordered field) are always stably embedded as well as orthogonal. As a result, we obtain that within ordered fields, every \mathcal{L}_{or} -definable coarsening of an \mathcal{L}_r -definable henselian valuation is already \mathcal{L}_r -definable (see Corollary 4.4) and that any \mathcal{L}_{or} -definable (not necessarily convex) valuation is comparable to any henselian valuation (see Proposition 4.5). The special class of almost real closed fields, which are ordered fields admitting a henselian valuation with real closed residue field, is

studied in [Section 5](#). We show in [Theorem 5.2](#) that within almost real closed fields any \mathcal{L}_{or} -definable valuation (which a priori does not have to be convex) is already henselian and \mathcal{L}_r -definable, thereby giving a negative answer to [[Krapp et al. 2021](#), Question 7.3]. Building on the results of the previous sections, we finally prove in [Section 6](#) the main theorem of this paper stating as follows:²

Theorem A (main theorem). *Let $(K, <)$ be an ordered field and let v be a henselian valuation on K . If v is \mathcal{L}_{or} -definable, then it is already \mathcal{L}_r -definable.*

2. Preliminaries

We denote by \mathbb{N} the set of natural numbers *without* 0 and by ω the set of natural numbers *with* 0.

We mostly follow the valuation-theoretic notation of [[Engler and Prestel 2005](#)]. For a valuation v on a field K , we write \mathcal{O}_v for its valuation ring, \mathcal{M}_v for the maximal ideal of \mathcal{O}_v , $Kv = \mathcal{O}_v/\mathcal{M}_v$ for the residue field and vK for its value group (written additively). For an element $x \in \mathcal{O}_v$, its residue $x + \mathcal{M}_v \in Kv$ is denoted \bar{x} , where the valuation v in question will always be clear from context.

Given an ordering $<$ on K (always compatible with the field structure), a valuation v is called *convex* (with respect to $<$) if \mathcal{O}_v is a convex set in the usual sense. See [[Engler and Prestel 2005](#), Section 2.2.2] for a number of equivalent conditions. In particular, v is convex if and only if the ordering $<$ induces an ordering on the residue field Kv in the natural way. This residue ordering is then also denoted by $<$, which should not lead to confusion.

For a given field K and ordered abelian group G , we write $K((G))$ for the *Hahn field* consisting of those formal sums $\sum_{g \in G} a_g t^g$ with coefficients $a_g \in K$, t a formal variable, whose support is well-ordered. See for instance [[van den Dries 2014](#), Section 3.1] for details. We generally endow $K((G))$ with its natural valuation with value group G , assigning to each element of $K((G))^\times$ the order of the lowest nonzero coefficient. An ordering $<$ on K can naturally be extended to $K((G))$ by stipulating that an element of $K((G))^\times$ is positive if and only if its nonzero coefficient of lowest order is positive. We denote this ordering on $K((G))$ again by $<$.

For background on the model theory of valued fields, see [[van den Dries 2014](#)], or [[Marker 2002](#)] for model theory more generally. We consider fields as structures in the language of rings $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$ and ordered abelian groups as structures in the language $\mathcal{L}_{\text{og}} = \{+, -, 0, <\}$ in the natural way. Given an ordering $<$ on a field K , we consider the ordered field $(K, <)$ as a structure in the language $\mathcal{L}_{\text{or}} = \mathcal{L}_r \cup \{<\}$. Given a valuation v on a field K , we usually work in the one-sorted language $\mathcal{L}_{\text{vf}} = \mathcal{L}_r \cup \{\mathcal{O}\}$, where the unary predicate \mathcal{O} is to be interpreted as the

²This result will be restated as [Theorem 6.4](#).

valuation ring $\mathcal{O}_v \subseteq K$. In [Section 4](#), we also work in a three-sorted language. For a field K with an ordering $<$ and a valuation v , we use the language $\mathcal{L}_{\text{ovf}} = \mathcal{L}_{\text{vf}} \cup \{<\}$ for $(K, <, v)$.

If a is an element of an ordered abelian group (or an ordered field), we denote its absolute value $\max\{a, -a\}$ by $|a|$.

A set is \mathcal{L} -*definable* if it is definable in the first-order language \mathcal{L} . If we wish to specify that the parameters can be chosen to come from a specific set A , we write A - \mathcal{L} -*definable*.

3. Convex valuations

We start by giving sufficient conditions on the value group or the residue field of a convex valuation v such that v is \mathcal{L}_{or} -definable. By this, we strengthen all of the three cases given in [\[Krapp et al. 2021, Theorem 5.3\]](#). Subsequently, we present several cases in which the given valuation is already \mathcal{L}_{or} -definable without parameters and discuss how these cases generalise other known definability results in the literature.

Theorem 3.1. *Let $(K, <)$ be an ordered field and let v be a convex valuation on K . Suppose that at least one of the following holds:*

- (i) vK is discretely ordered, i.e., admits a least positive element.
- (ii) vK is not closed in its divisible hull.
- (iii) Kv is not closed in its real closure.

Then v is \mathcal{L}_{or} -definable. Moreover, in the cases (i) and (ii), v is definable by a formula using only one parameter.

Proof. We may assume that v is nontrivial.

(i) Since vK is discretely ordered, we can choose $b \in K^\times$ such that $v(b)$ is the minimal positive element of vK . Note that for every $x \in \mathcal{M}_v$, we have $v(x^2/b) = 2v(x) - v(b) > 0$. Since every element $y \in \mathcal{M}_v$ satisfies $|y| < 1$, we deduce that $\mathcal{M}_v = \{x \in K \mid |x^2/b| < 1\}$. Hence, \mathcal{M}_v is $\{b\}$ - \mathcal{L}_{or} -definable, and \mathcal{O}_v can be defined in terms of \mathcal{M}_v .

(ii) Since vK is not closed in its divisible hull, we can take $\gamma \in vK$ and $n > 1$ such that $\gamma \notin n \cdot vK$ but every open interval in vK containing γ contains an element of $n \cdot vK$. Let $b \in K$ with $b > 0$ and $v(b) = \gamma$, and set

$$S_b := \{x \in K \mid x \geq 0 \text{ and } x^n/b < 1\} = \{x \in K \mid x \geq 0 \text{ and } nv(x) > \gamma\} \quad (3-1)$$

(where the equality uses that we cannot have $v(x^n/b) = 0$ since $\gamma \notin n \cdot vK$). It now suffices to prove that

$$\mathcal{O}_v = \{y \in K \mid y^4 S_b \subseteq S_b\}, \quad (3-2)$$

since the set on the right-hand side is $\{b\}$ - \mathcal{L}_{or} -definable. The inclusion \subseteq is clear since the condition $nv(x) > \gamma$ in (3-1) is stable under multiplying x by an element of \mathcal{O}_v .

For the inclusion \supseteq , suppose that $y \in K \setminus \mathcal{O}_v$, so $v(y) < 0$. By the choice of γ , we can take $z \in K$ with $z > 0$ and $\gamma + v(y) < nv(z) < \gamma - v(y)$. Now $z/y^2 \in S_b$ since $nv(z/y^2) = nv(z) - 2nv(y) > nv(z) - v(y) > \gamma$, but $y^4(z/y^2) \notin S_b$ since $nv(y^4(z/y^2)) = nv(z) + 2nv(y) < nv(z) + v(y) < \gamma$. This proves that S_b is not stable under multiplication by y^4 , completing the proof of (3-2).

(iii) Let $f \in Kv[X]$ be the minimal polynomial of an element $x_0 \in R \setminus Kv$, where R denotes the real closure of $(Kv, <)$, such that x_0 can be arbitrarily approximated by elements of Kv . Then there are $a, b \in Kv$ with $a < x_0 < b$ such that the following hold:

- (1) The polynomial f has exactly one zero in $\{x \in R \mid a \leq x \leq b\}$. In particular, f changes sign precisely once in this interval.
- (2) For any $\varepsilon \in Kv$ with $0 < \varepsilon < b - a$, there exists $x \in Kv$ with $a < x < x + \varepsilon < b$ such that $f(x)f(x + \varepsilon) < 0$.

Passing to $-f$ if necessary, we may assume that $f(a) < 0 < f(b)$. Let $F \in K[X]$ be a lift of f , let $a_0, b_0 \in K$ be lifts of a and b , and consider the \mathcal{L}_{or} -definable set S given by

$$\{x \in K \mid a_0 \leq x \leq b_0 \text{ and } F(x) < 0\} = \{x \in K \mid a_0 \leq x \leq b_0 \text{ and } f(\bar{x}) < 0\},$$

where the equality uses that f has no zero in Kv .

It now suffices to prove that for any $y \in K$ we have

$$y + S \subseteq S \iff y \in \mathcal{M}_v \text{ and } y \geq 0,$$

since then \mathcal{M}_v and therefore \mathcal{O}_v are \mathcal{L}_{or} -definable. For the implication \Leftarrow , let $y \in \mathcal{M}_v$ be nonnegative and $x \in S$. Then we have $x + y \geq x \geq a_0$ and $f(\overline{x+y}) = f(\bar{x}) < 0$. In particular $\overline{x+y} < b$ and thus $x + y < b_0$. Hence $y + x \in S$, as desired.

For the implication \Rightarrow , let $y \in K$ with $y + S \subseteq S$. Since $a_0 \in S$, we must have $a_0 + y \in S$. Thus, $a_0 \leq a_0 + y \leq b_0$, implying that $0 \leq y \leq b_0 - a_0$ and $y \in \mathcal{O}_v$. Note that $\bar{y} \neq b - a$, as otherwise $f(\bar{y} + a) = f(b) > 0$, contradicting the fact that $y + a_0 \in S$. Hence, $\bar{y} < b - a$.

In order to show $y \in \mathcal{M}_v$, suppose for a contradiction that $v(y) = 0$, so $0 < \bar{y} < b - a$. By choice of f , we can find $z \in \mathcal{O}_v$ with $a < \bar{z} < b$ and $f(\bar{z}) < 0 < f(\overline{z+y})$. Then we have $z \in S$ but $z + y \notin S$, in contradiction to our assumption $y + S \subseteq S$. \square

In the following corollary, we point out several distinguished cases in which we obtain \mathcal{L}_{or} -definability without parameters.

Corollary 3.2. *Let $(K, <)$ be an ordered field and let v be a convex valuation on K . Suppose that at least one of the following holds:*

- (i) vK is p -regular but not p -divisible for some prime $p \in \mathbb{N}$.³
- (ii) Kv is dense in its real closure but not real closed.

Then v is \emptyset - \mathcal{L}_{or} -definable.

Proof. In both cases, at least one of the three conditions in [Theorem 3.1](#) is satisfied: if vK is p -regular but not p -divisible, then it is either discrete or not closed in its divisible hull (see [\[Krapp et al. 2022, Proposition 3.3\]](#)). Thus, there exists an \mathcal{L}_{or} -formula $\psi(x, \underline{z})$ and a parameter tuple $\underline{b} \in K$ such that $\psi(x, \underline{b})$ defines \mathcal{O}_v .

(i) For any nontrivial convex subgroup $C \leq vK$ we have that vK/C is p -divisible (see [\[Hong 2014, page 14\]](#)). Thus, any strict coarsening of v has a p -divisible value group. As vK is not p -divisible, \mathcal{O}_v is defined by the \mathcal{L}_{or} -formula $\varphi(y)$ expressing the following:

$$\exists \underline{z} \left(\text{“}\psi(x, \underline{z}) \text{ defines a nontrivial convex valuation ring whose value group contains an element that is not } p\text{-divisible”} \wedge \psi(y, \underline{z}) \right).$$

(ii) For any strict refinement w of v we have that Kw is real closed. Indeed, since Kv is dense in its real closure and the induced valuation \bar{w} on Kv is nontrivial and convex, we have that $Kw = (Kv)\bar{w}$ is real closed (see [\[Krapp et al. 2021, Corollary 4.9\]](#)). Let θ be an \mathcal{L}_{or} -sentence that is true in the theory of real closed fields but does not hold in Kv .⁴ Then \mathcal{O}_v is defined by the \mathcal{L}_{or} -formula $\varphi(y)$ expressing the following:

$$\forall \underline{z} \left(\text{“}\psi(x, \underline{z}) \text{ defines a nontrivial convex valuation ring whose residue field does not satisfy } \theta \text{”} \rightarrow \psi(y, \underline{z}) \right). \quad \square$$

Remark 3.3. (i) The cases (i) and (ii) of [Theorem 3.1](#) are optimal in the sense that, in general, one cannot obtain parameter-free definability. More precisely, in [\[Krapp et al. 2022, Examples 4.9 and 4.10\]](#) two ordered valued fields $(L_1, <, v_1)$ and $(L_2, <, v_2)$ are presented such that the following hold:

- v_1 and v_2 are henselian and thus convex;
- neither v_1 nor v_2 is \emptyset - \mathcal{L}_{or} -definable;
- $v_1 L_1$ is discrete, and $v_2 L_2$ is not closed in its divisible hull.

³Equivalently, vK contains a rank 1 convex subgroup H that is not p -divisible but vK/H is p -divisible. See [\[Hong 2013, Section 2.2\]](#) for further characterisations of p -regular ordered abelian groups.

⁴For instance, θ may express that there exists a polynomial of a certain degree that does not have a zero.

(ii) The results for \mathcal{L}_r -definability of henselian (rather than convex) valuations corresponding to [Theorem 3.1\(i\)](#) and [\(ii\)](#) as well as [Corollary 3.2\(i\)](#) are proven in [\[Hong 2014, Corollary 2\]](#), [\[Krapp et al. 2022, Theorem A\]](#) and [\[Hong 2013, Lemmas 2.3.6 and 2.3.7\]](#), respectively.

(iii) [Corollary 3.2](#) applies in particular if vK is of rank 1 (i.e., v is the coarsest nontrivial convex valuation on K) but nondivisible, or if Kv is archimedean (i.e., v is the finest convex valuation) but not real closed.

We now apply the \mathcal{L}_{or} -definability results above in order to obtain convex non-henselian valuations that are definable in the language \mathcal{L}_{or} but not in the language \mathcal{L}_r .

Lemma 3.4. *Let $K = \mathbb{Q}(s_i \mid i \in \omega)$, where $\{s_i \mid i \in \omega\}$ is algebraically independent over \mathbb{Q} . Suppose that v is any valuation on K with $v(s_i) \geq 0$ for any $i \in \mathbb{N}$ and $v(s_0) < 0$. Then v is not \mathcal{L}_r -definable in K .*

Proof. First note that K is the \mathcal{L}_r -definable closure of $S := \{s_i \mid i \in \omega\}$ in K . Hence, any \mathcal{L}_r -definable subset of K is S - \mathcal{L}_r -definable.

Assume, for a contradiction, that some \mathcal{L}_r -formula $\varphi(x, \underline{s})$ defines \mathcal{O}_v , where $\underline{s} = (s_0, s_1, \dots, s_n)$ for some $n \in \mathbb{N}$. Since also the set

$$\{s_0, s_1, \dots, s_n, s_{n+1} + s_0, s_{n+2}, \dots\}$$

is algebraically independent over \mathbb{Q} , we can set α to be the uniquely determined \mathcal{L}_r -automorphism on K with

$$\alpha(s_i) := \begin{cases} s_i & \text{for } i \in \omega \setminus \{n+1\}, \\ s_{n+1} + s_0 & \text{for } i = n+1. \end{cases}$$

Now since $v(s_{n+1}) \geq 0$, we have

$$K \models \varphi(s_{n+1}, \underline{s}).$$

As $\alpha(s_{n+1}) = s_{n+1} + s_0$ and $\alpha(\underline{s}) = \underline{s}$, we obtain

$$K \models \varphi(s_{n+1} + s_0, \underline{s}),$$

i.e., $s_{n+1} + s_0 \in \mathcal{O}_v$. However, $v(s_{n+1} + s_0) = v(s_0) < 0$, a contradiction. \square

Example 3.5. We construct an ordered valued field $(K, <, v)$ such that K is a subfield of the Laurent series field $\mathbb{R}((\mathbb{Z}))$, vK is discretely ordered, Kv is archimedean and v is \emptyset - \mathcal{L}_{or} -definable but not \mathcal{L}_r -definable.

Let $k = \mathbb{Q}(s_1, s_2, \dots) \subseteq \mathbb{R}$ for some set $\{s_i \mid i \in \mathbb{N}\} \subseteq \mathbb{R}$ that is algebraically independent over \mathbb{Q} . Consider the field $K = k(t)$, which we endow with the valuation v and ordering $<$ given as the restriction of the valuation and ordering on the Hahn field $k((t)) = k((\mathbb{Z}))$. Then $vK = \mathbb{Z}$ and $Kv = k$, which is archimedean. [Corollary 3.2](#) shows that v is \emptyset - \mathcal{L}_{or} -definable. Setting $s_0 = t^{-1}$, we obtain $K = \mathbb{Q}(s_0, s_1, \dots)$ and

$v(s_0) = -1 < 0$ as well as $v(s_i) = 0$ for any $i \in \mathbb{N}$. Hence, [Lemma 3.4](#) implies that v is not \mathcal{L}_r -definable. \square

Example 3.6. We construct an ordered valued field $(K, <, v)$ such that K is a subfield of the Puiseux series field $\bigcup_{n \in \mathbb{N}} \mathbb{R}((t^{1/n})) \subseteq \mathbb{R}((\mathbb{Q}))$, vK is densely ordered, Kv is archimedean and v is \emptyset - \mathcal{L}_{or} -definable but not \mathcal{L}_r -definable.

Let $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{R}$ be an algebraically independent set over \mathbb{Q} . We set

$$s_0 = t^{-1} \in \mathbb{R}((\mathbb{Q})) \quad \text{and} \quad s_i = r_i t^{1/i} \in \mathbb{R}((\mathbb{Q}))$$

for any $i \in \mathbb{N}$. Let $K = \mathbb{Q}(s_0, s_1, \dots) \subseteq \mathbb{R}((\mathbb{Q}))$, and endow K with the ordering $<$ and valuation v given as the restriction of the Hahn field ordering and valuation on $\mathbb{R}((\mathbb{Q}))$. The archimedean residue field of K is not real closed, as $Kv \subseteq \mathbb{Q}(r_1, r_2, \dots)$ and thus, for instance, $\sqrt{2} \notin Kv$. Hence, [Corollary 3.2\(ii\)](#) shows that v is \emptyset - \mathcal{L}_{or} -definable. Since $v(s_i) = \frac{1}{i}$ for any $i \in \mathbb{N}$, we have $vK = \mathbb{Q}$. Finally, we can apply [Lemma 3.4](#) to show that v is not \mathcal{L}_r -definable, as $v(s_0) = -1 < 0$ and $v(s_i) = \frac{1}{i} > 0$ for any $i \in \mathbb{N}$. \square

To complete this section, we relate the property of an ordered field to admit a nontrivial \mathcal{L}_{or} -definable convex valuation to the property of being dense in the real closure. It is known that an ordered field is either dense in its real closure or admits a nontrivial \mathcal{L}_{or} -definable convex valuation (see [[Jahnke et al. 2017](#), Proposition 6.5; [Krapp et al. 2021](#), Fact 5.1]). However, the question whether these two cases are nonexclusive (see [[Krapp et al. 2021](#), Question 7.1]) has so far been open. In the following, we answer this question positively by presenting an ordered field that is dense in its real closure and whose natural valuation is \emptyset - \mathcal{L}_{or} -definable.

Example 3.7. Let

$$K = \mathbb{R}(t^{1/n} \mid n \in \mathbb{N}) \subseteq \mathbb{R}((\mathbb{Q})),$$

and endow K with the ordering $<$ and valuation v given by restricting the ordering and the valuation of the Hahn field $\mathbb{R}((\mathbb{Q}))$. Since $Kv = \mathbb{R}$ is real closed and $vK = \mathbb{Q}$ is divisible and of rank 1, [[Viswanathan 1977](#), Proposition 9] implies that K is dense in its real closure.

We first claim that the subset $\mathbb{R} \subseteq K$ is defined by the parameter-free \mathcal{L}_r -formula $\varphi(x)$ given by

$$\exists y \, 1 + x^4 = y^4.$$

Clearly $\mathbb{R} \subseteq \varphi(K)$, since for any $a \in \mathbb{R}$ we have $\sqrt[4]{1+a^4} \in \mathbb{R}$. For the other inclusion, take $a, y \in K$ with $1+a^4 = y^4$. There exists $N \in \mathbb{N}$ such that $a, y \in \mathbb{R}(t^{1/N})$. Letting $s = t^{1/N}$, we have $\mathbb{R}(s) \models \varphi(a)$, and therefore $a \in \mathbb{R}$ by [[Maltsev 1960](#), Lemma 2]. This shows $\mathbb{R} = \varphi(K)$, as desired.

Therefore \mathcal{O}_v , the convex hull of \mathbb{R} in K , is defined by the parameter-free \mathcal{L}_{or} -formula

$$\exists x_1, x_2 \, (\varphi(x_1) \wedge \varphi(x_2) \wedge x_1 \leq z \leq x_2).$$

\square

Note that [Example 3.7](#) presents an ordered valued field with real closed residue field and divisible value group. Hence, none of the cases in [Corollary 3.2](#) can be applied, but we still obtain \emptyset - \mathcal{L}_{or} -definability of the valuation.

4. Stable embeddedness

In this section, we establish that the ordered value group and the ordered residue field of an ordered henselian valued field are stably embedded and orthogonal (see [Theorem 4.2](#)). Stable embeddedness and orthogonality are best known to hold in the unordered situation of henselian valued fields of equicharacteristic 0 (see, e.g., [\[van den Dries 2014, Section 5\]](#)), and various other well-behaved settings (see also [\[Aschenbrenner et al. 2017, Section 8.3; Jahnke and Simon 2020, pages 171–172\]](#)). As the key technical tool in the ordered context, we use Farré’s embedding lemma [\[Farré 1993, Theorem 3.4\]](#).

We consider the three-sorted language $\mathcal{L}'_{\text{ovf}}$ given by

$$\mathcal{L}'_{\text{ovf}} = (\mathcal{L}_{\text{or}}, \mathcal{L}_{\text{or}}, \mathcal{L}_{\text{og}}; \bar{\cdot}, v).$$

The three sorts are denoted by \mathbf{f} (field sort), \mathbf{r} (residue field sort) and \mathbf{v} (value group sort). The two unary function symbols $\bar{\cdot}$ and v have sorts $\bar{\cdot} : \mathbf{f} \rightarrow \mathbf{r}$ and $v : \mathbf{f} \rightarrow \mathbf{v}$.

Let $(K, <, v)$ be an ordered valued field with convex valuation v . Then $(K, <, v)$ induces an $\mathcal{L}'_{\text{ovf}}$ -structure

$$\mathcal{K} = ((K, <), (Kv, <), vK; v, \bar{\cdot}),$$

where the domains of the valuation and the residue map are extended to K by setting $v(0) = 0$ and $\bar{a} = 0$ for any $a \in K \setminus \mathcal{O}_v$. When considering definability in \mathcal{K} , we allow parameters from all sorts as usual. An \mathcal{L}_{or} -formula $\varphi(\underline{y})$ with parameters from Kv may be considered as an $\mathcal{L}'_{\text{ovf}}$ -formula with parameters from \mathcal{K} , where the variables \underline{y} become \mathbf{r} -variables. Similarly, an \mathcal{L}_{og} -formula $\varphi(\underline{z})$ with parameters from vK may be considered as an $\mathcal{L}'_{\text{ovf}}$ -formula with parameters from \mathcal{K} , where the variables \underline{z} become \mathbf{v} -variables.

In the following, we prove a weak version of relative quantifier elimination; we only consider formulas whose variables are varying over residue field and value group.

Lemma 4.1. *Let $(K, <, v)$ be an ordered henselian valued field and let T be the diagram of the $\mathcal{L}'_{\text{ovf}}$ -structure \mathcal{K} as above, i.e., the complete theory of \mathcal{K} in the language $\mathcal{L}'_{\text{ovf}}$ expanded by constants for all elements of \mathcal{K} . Further, let \underline{y} and \underline{z} be tuples of distinct \mathbf{r} - and \mathbf{v} -variables, respectively. Then any $\mathcal{L}'_{\text{ovf}}$ -formula $\varphi(\underline{y}, \underline{z})$ with parameters from \mathcal{K} is T -equivalent to an $\mathcal{L}'_{\text{ovf}}$ -formula of the form*

$$(\psi_1(\underline{y}) \wedge \theta_1(\underline{z})) \vee \cdots \vee (\psi_N(\underline{y}) \wedge \theta_N(\underline{z})) \quad (4-1)$$

for some $N \in \mathbb{N}$, where all ψ_i are \mathcal{L}_{or} -formulas with parameters from Kv and all θ_i are \mathcal{L}_{og} -formulas with parameters from vK .

Proof. Let Θ be the set of all $\mathcal{L}'_{\text{ovf}}$ -formulas of the form (4-1), i.e., of all finite disjunctions of conjunctions of an \mathcal{L}_{or} -formula and an \mathcal{L}_{og} -formula with parameters from Kv and vK , respectively. Modulo logical equivalence, Θ contains \top as well as \perp and is closed under finite disjunctions, finite conjunctions and negation. Hence by [Aschenbrenner et al. 2017, Corollary B.9.3], we only have to verify the following:

Let p and q be any two complete T -realisable $(\underline{y}, \underline{z})$ -types with $p \cap \Theta = q \cap \Theta$. Then $p = q$.

Let p and q be as described above and let \mathcal{M} be a sufficiently saturated elementary extension of \mathcal{K} in which p and q are realised. Denote by $(M, <, v)$ the ordered henselian valued field inducing \mathcal{M} , and let $\underline{r}, \underline{r}^* \in Mv$ and $\underline{g}, \underline{g}^* \in vM$ with $\mathcal{M} \models p(\underline{r}, \underline{g}) \wedge q(\underline{r}^*, \underline{g}^*)$. By construction of the set Θ , we have

$$\text{tp}^{vM}(\underline{g}/vK) = \text{tp}^{vM}(\underline{g}^*/vK) \quad \text{and} \quad \text{tp}^{(Mv, <)}(\underline{r}/Kv) = \text{tp}^{(Mv, <)}(\underline{r}^*/Kv). \quad (4-2)$$

Now let $\mathcal{K} \leq \mathcal{M}_0 \leq \mathcal{M}$, with \mathcal{M}_0 smaller than the saturation of \mathcal{M} , such that $\underline{r} \in M_0v$ and $\underline{g} \in vM_0$, where $(M_0, <, v)$ denotes the ordered henselian valued field inducing \mathcal{M}_0 . Due to (4-2), we can fix an \mathcal{L}_{or} -elementary embedding $\sigma : (M_0v, <) \rightarrow (Mv, <)$ over Kv with $\sigma(\underline{r}) = \underline{r}^*$ and an \mathcal{L}_{og} -elementary embedding $\rho : vM_0 \rightarrow vM$ over vK with $\rho(\underline{g}) = \underline{g}^*$ (see [Marker 2002, Proposition 4.1.5]).

The quotient vM_0/vK is torsion free, as $vK \leq vM_0$. We can thus apply [Farré 1993, Theorem 3.4] (with all appearing levels equal to 1) in order to obtain an embedding $\iota : (M_0, <, v) \rightarrow (M, <, v)$ over K inducing both σ and ρ . Moreover, since both σ and ρ are elementary embeddings, [Farré 1993, Corollary 4.2(ii)] implies that $(\iota(M_0), <, v) \leq (M, <, v)$. Let \mathcal{M}'_0 be the $\mathcal{L}'_{\text{ovf}}$ -structure induced by $(\iota(M_0), <, v)$ and denote by h the isomorphism $(\iota, \sigma, \rho) : \mathcal{M}_0 \rightarrow \mathcal{M}'_0$ over \mathcal{K} . For any $\varphi(\underline{y}, \underline{z}) \in p$ we have $\mathcal{M}_0 \models \varphi(\underline{r}, \underline{g})$. By applying h , we obtain $\mathcal{M}'_0 \models \varphi(\sigma(\underline{r}), \rho(\underline{g}))$ and hence $\mathcal{M} \models \varphi(\underline{r}^*, \underline{g}^*)$. This establishes $p \subseteq q$. The other inclusion follows likewise. \square

Theorem 4.2. *Let $(K, <, v)$ be an ordered henselian valued field inducing the $\mathcal{L}'_{\text{ovf}}$ -structure \mathcal{K} . Then for any $m, n \in \omega$ the following hold:*

- (i) *Any subset of $(Kv)^m \times (vK)^n$ definable in \mathcal{K} is a finite union of rectangles of the form $Y \times Z$, where $Y \subseteq (Kv)^m$ is \mathcal{L}_{or} -definable in $(Kv, <)$ and $Z \subseteq (vK)^n$ is \mathcal{L}_{og} -definable in vK .*
- (ii) *For any set $B \subseteq \mathcal{O}_v^m$ that is \mathcal{L}_{ovf} -definable in $(K, <, v)$, the set*

$$\bar{B} := \{(\bar{b}_1, \dots, \bar{b}_m) \mid (b_1, \dots, b_m) \in B\} \subseteq (Kv)^m$$

is \mathcal{L}_{or} -definable in $(Kv, <)$.

(iii) For any set $C \subseteq (K^\times)^n$ that is \mathcal{L}_{ovf} -definable in $(K, <, v)$, the set

$$v(C) := \{(v(c_1), \dots, v(c_n)) \mid (c_1, \dots, c_n) \in C\} \subseteq (vK)^n$$

is \mathcal{L}_{og} -definable in vK .

Proof. By Lemma 4.1, any subset of $(Kv)^m \times (vK)^n$ definable in \mathcal{K} can be defined by a formula of the form (4-1). This immediately implies (i). In order to obtain (ii) and (iii), it remains to notice that $\bar{B} \times v(C) \subseteq (Kv)^m \times (vK)^n$ is definable in \mathcal{K} . \square

Corollary 4.3. *Let $(K, <, v)$ be an ordered henselian valued field and let w be an \mathcal{L}_{ovf} -definable valuation on K . Then the following hold:*

- (i) *If w is a refinement of v , then the valuation \bar{w} induced by w on Kv is \mathcal{L}_{or} -definable in $(Kv, <)$.*
- (ii) *If w is a coarsening of v , then $v(\mathcal{O}_w^\times)$ is \mathcal{L}_{og} -definable in vK .*

Proof. Both \mathcal{O}_w^\times and \mathcal{O}_w are \mathcal{L}_{ovf} -definable in K . It remains to apply Theorem 4.2(ii) to $B = \mathcal{O}_w \subseteq \mathcal{O}_v$ in order to obtain (i) and Theorem 4.2(iii) to $C = \mathcal{O}_w^\times \subseteq K^\times$ in order to obtain (ii). \square

Corollary 4.4. *Let $(K, <)$ be an ordered field, let v be an \mathcal{L}_r -definable henselian valuation on K and let w be an \mathcal{L}_{or} -definable coarsening of v . Then w is already \mathcal{L}_r -definable.*

Proof. Corollary 4.3(ii) shows that $H = v(\mathcal{O}_w^\times)$ is \mathcal{L}_{og} -definable in vK . Since $wK = vK/H$, for any $x \in K$ we have $x \in \mathcal{O}_w$ if and only if $v(x) \geq 0 \vee v(x) \in H$. As v is \mathcal{L}_r -definable in K , the latter can be expressed as an \mathcal{L}_r -formula with parameters from K . \square

For later use, we also deduce the following.

Proposition 4.5. *Let $(K, <, v)$ be an ordered henselian valued field. Then any \mathcal{L}_{or} -definable valuation w on K is comparable to v .*

Proof. We may suppose that neither v nor w are trivial. We first claim that the valuation ring \mathcal{O}_w contains a set $U \neq \emptyset$ which is open in the topology induced by v . This follows from a suitable form of relative quantifier elimination for henselian valued fields of residue characteristic zero: In the terminology of [Cluckers et al. 2022], the \mathcal{L}_{vf} -theory of the valued field (K, v) is ω -h-minimal [Cluckers et al. 2022, Corollary 6.2.6(1.)]. Let P be the unary predicate on $\text{RV} = K^\times/(1 + \mathcal{M}_v)$ given by $P(a(1 + \mathcal{M}_v))$ if and only if $a > 0$ for any $a \in K^\times$. Since elements of $1 + \mathcal{M}_v$ are squares and hence automatically positive, the positive cone of $(K, <)$ consists of all $a \in K^\times$ satisfying $P(a(1 + \mathcal{M}_v))$. As by [Cluckers et al. 2022, Theorem 4.1.19] ω -h-minimality is preserved under expansions by additional predicates on RV , we obtain that the \mathcal{L}_{ovf} -theory of $(K, <, v)$ is ω -h-minimal. In models of ω -h-minimal

theories, any infinite definable set contains a nonempty v -open ball [Cluckers et al. 2022, Lemma 2.5.2], proving our claim about \mathcal{O}_w .

It follows that w cannot be independent from v , since otherwise weak approximation [Engler and Prestel 2005, Theorem 2.4.1] would imply that the set $U \cap (K \setminus \mathcal{O}_w)$ is nonempty as the intersection of a v -open and a w -open set.

Let us now suppose for a contradiction that w and v are incomparable. Let v_0 be the finest common coarsening of v and w , and \bar{v} , \bar{w} the induced valuations on the residue field Kv_0 , which are nontrivial and independent. Writing $<$ for the induced ordering on Kv_0 , we now have an ordered henselian valued field $(Kv_0, <, \bar{v})$ with a valuation \bar{w} independent from \bar{v} . By Corollary 4.3(i), \bar{w} is \mathcal{L}_{or} -definable in $(Kv_0, <)$. Since \bar{w} is independent from the henselian valuation \bar{v} on Kv_0 , this contradicts the first part of the proof. \square

5. Almost real closed fields

Following the terminology of [Delon and Farré 1996], we call a field K *almost real closed* if it admits a henselian valuation v such that Kv is real closed. Almost real closed fields arise in many valuation-theoretic contexts, and they have been studied extensively (under varying names) both algebraically and model-theoretically (see, e.g., [Brown 1988; Becker et al. 1999; Delon and Farré 1996]). Due to the Baer–Krull representation theorem (see [Engler and Prestel 2005, pages 37–38]), any almost real closed field admits at least one ordering. In this section, we consider \mathcal{L}_{or} - and \mathcal{L}_r -definability of valuations (which are a priori not necessarily convex) in almost real closed fields. We establish in Theorem 5.2 that any \mathcal{L}_{or} -definable valuation is already \mathcal{L}_r -definable and henselian. Thereby we give a negative answer to [Krapp et al. 2021, Question 7.3].

Let K be an almost real closed field. Then for any prime $p \in \mathbb{N}$ there exists a coarsest henselian valuation on K , denoted by v_p , with the property that

$$Kv_p = (Kv_p)^p \cup [-(Kv_p)^p]$$

(see [Delon and Farré 1996, page 1126]).

Lemma 5.1. *Let $p \in \mathbb{N}$ be prime and let (K, v) be a henselian valued field with real closed residue field. Then $v(\mathcal{O}_{v_p}^\times)$ is the maximal p -divisible convex subgroup of vK .*

Proof. By [Delon and Farré 1996, Proposition 2.5(iv)], $v_p K = vK/v(\mathcal{O}_{v_p}^\times)$ has no nontrivial p -divisible convex subgroup, so $v(\mathcal{O}_{v_p}^\times)$ contains the maximal p -divisible convex subgroup of vK . On the other hand, v induces a valuation on the residue field Kv_p with value group $v(\mathcal{O}_{v_p}^\times)$, from which it is easy to see that $v(\mathcal{O}_{v_p}^\times)$ must itself be p -divisible by the defining property of v_p (or see [Delon and Farré 1996, Lemma 2.4(iii)]). \square

With the results of the last section at our disposal, we can now imitate the proof of [Delon and Farré 1996, Theorem 4.4] to obtain the following.

Theorem 5.2. *Let K be an almost real closed field and let $<$ be any ordering on K . Then any \mathcal{L}_{or} -definable valuation on $(K, <)$ is henselian and \mathcal{L}_r -definable.*

Proof. Let us denote by v_K the canonical henselian valuation on K . Since K is almost real closed, v_K coincides with the natural, i.e., finest convex valuation v_{nat} on $(K, <)$ for any ordering $<$ on K , as v_{nat} is henselian [Delon and Farré 1996, Proposition 2.1(iv)].

Let v be an \mathcal{L}_{or} -definable valuation on K . By Proposition 4.5, v and v_K are comparable. If v is a coarsening of v_K , then it is also henselian. Otherwise, v_K is a strict coarsening of v . Thus, by Corollary 4.3(i) the nontrivial valuation that v induces on Kv_K is \mathcal{L}_{or} -definable in $(Kv_K, <)$, contradicting that Kv_K is real closed. Hence, v is henselian.⁵

In order to show that v is \mathcal{L}_r -definable, by [Delon and Farré 1996, Theorem 4.4] it suffices to verify that $G_v := v_K(\mathcal{O}_v^\times)$ is \mathcal{L}_{og} -definable in $v_K K$ and that $\mathcal{O}_{v_p} \subseteq \mathcal{O}_v$ for some prime $p \in \mathbb{N}$. The first condition follows from Corollary 4.3(ii). For the other condition, we distinguish between two cases.

Case 1: $v \neq v_K$. Then $G_v \neq \{0\}$ and by [Delon and Farré 1996, Corollary 4.3] we have $G_p \leq G_v$ for some prime $p \in \mathbb{N}$, where G_p denotes the maximal p -divisible convex subgroup of $v_K K$. Now $G_p = v_K(\mathcal{O}_{v_p}^\times)$ by Lemma 5.1. Hence, $\mathcal{O}_{v_p} \subseteq \mathcal{O}_v$, as required. This establishes that v is \mathcal{L}_r -definable in K .

Case 2: $v = v_K$. Consider the set of formulas

$$p(x) = \{x > n \wedge v(x) = 0 \mid n \in \mathbb{N}\}.$$

This set is finitely satisfiable in $(K, <, v)$, i.e., a type. Hence, for some elementary extension $(L, <, v^*)$ of $(K, <, v)$, there is some $x \in L$ with $v^*(x) = 0$ and $x > n$ for all $n \in \mathbb{N}$. In particular, v^* is a strict coarsening of the natural valuation v_L on $(L, <)$. By Case 1, v^* is \mathcal{L}_r -definable in L . Thus, there exists an \mathcal{L}_r -formula $\varphi(x, \underline{y})$ such that

$$(L, <, v^*) \models \exists \underline{y} \forall x (v^*(x) \geq 0 \leftrightarrow \varphi(x, \underline{y})).$$

By elementary equivalence, there exists $\underline{b} \in K$ such that $\varphi(x, \underline{b})$ defines v in K . \square

6. Henselian valuations

We now consider definability in general ordered henselian valued fields. Throughout this section, we freely use Farré's Ax–Kochen–Ershov principles [Farré 1993,

⁵An alternative argument is the following: If K is almost real closed, then one can show that $(K, <)$ has NIP. In this case, any \mathcal{L}_{or} -definable valuation on K is henselian by [Halevi et al. 2020, Corollary 5.8].

Corollary 4.2] (with all levels equal to 1 in the notation there), stating that two ordered henselian valued fields are elementarily equivalent in \mathcal{L}_{ovf} if and only if the ordered residue fields and the value groups are so, and similarly for elementary extensions.

Our first step is to show that a henselian valuation that is “slippery” in a precise sense involving residue field and value group cannot be \mathcal{L}_{or} -definable.

Lemma 6.1. *Let $(K, <, v)$ be an ordered henselian valued field satisfying*

$$(Kv, <) \equiv (L((\mathbb{Q})), <) \quad \text{and} \quad vK \equiv \Gamma \oplus \mathbb{Q}$$

for some ordered field L and some ordered abelian group Γ . Then v is not \mathcal{L}_{or} -definable.

Proof. Assume, for a contradiction, that v were \mathcal{L}_{or} -definable. Fix an \mathcal{L}_{or} -formula $\varphi(x, \underline{y})$ such that for some $\underline{b} \in K$ the valuation ring \mathcal{O}_v is defined by $\varphi(x, \underline{b})$. Since some instance of $\varphi(x, \underline{y})$ also defines the valuation w in any

$$(M, <, w) \equiv (K, <, v),$$

we may assume that $(K, <, v)$ is in fact equal to

$$(L((\mathbb{Q})) \underbrace{((\mathbb{Q}))((\Gamma))}_w, <, w),$$

where w denotes the power series valuation with value group $\Gamma \oplus \mathbb{Q}$ ordered lexicographically. Let v_Γ denote the power series valuation on K with value group Γ and residue field $L((\mathbb{Q}))((\mathbb{Q}))$.

Applying Corollary 4.3(i) to the ordered residue field of $(K, <, v_\Gamma)$, the \mathcal{L}_{or} -definability of v implies that \bar{v} (i.e., the valuation induced by v on the residue field of its coarsening v_Γ) is also \mathcal{L}_{or} -definable on $(L((\mathbb{Q}))((\mathbb{Q})), <)$.

Applying Corollary 4.3(ii) to the value group of $(L((\mathbb{Q}))((\mathbb{Q})), <, \bar{v})$, the convex subgroup \mathbb{Q} corresponding to \bar{v} is already \mathcal{L}_{og} -definable in $\mathbb{Q} \oplus \mathbb{Q}$. This is a contradiction, as $\mathbb{Q} \oplus \mathbb{Q}$ is divisible, and divisible ordered abelian groups admit no nontrivial proper \mathcal{L}_{og} -definable subgroups. \square

We now prove a lemma used to define coarsenings of a valuation that is essentially already shown in [Jahnke and Koenigsmann 2017]. It states that although in an ordered abelian group G , the smallest convex subgroup containing a given element $\gamma \in G$ need not be definable (e.g., the convex subgroup generated by $(0, 1)$ in the lexicographic sum $\mathbb{Q} \oplus \mathbb{Z}$ is not definable), it is definable up to p -divisible “noise”.

Lemma 6.2. *Let $p \in \mathbb{N}$ be prime. There exists an \mathcal{L}_{og} -formula $\varphi(x, y)$ such that the following holds: Let G be an ordered abelian group and $\gamma \in G^{>0}$, and let $\langle \gamma \rangle$ denote the smallest convex subgroup of G that contains γ . Then the set $\Delta_\gamma \subseteq G$*

defined by $\varphi(x, \gamma)$ in G is the maximal convex subgroup of G containing γ such that $\Delta_\gamma / \langle \gamma \rangle$ is p -divisible.

Proof. We set $\varphi(x, \gamma)$ to express

$$[0, p|x|] \subseteq [0, p\gamma] + pG.$$

By [Jahnke and Koenigsmann 2017, Lemma 4.1], Δ_γ is a convex subgroup of G with $\gamma \in \Delta_\gamma$ such that no nontrivial convex subgroup of G/Δ_γ is p -divisible. In particular, for every convex subgroup Δ of G properly containing Δ_γ , the group $\Delta/\langle \gamma \rangle$ is not p -divisible, since it has $\Delta/\Delta_\gamma \leq G/\Delta_\gamma$ as a quotient.

On the other hand, every positive element $\delta \in \Delta_\gamma$ can by definition be written as the sum of an element of $[0, p\gamma] \subseteq \langle \gamma \rangle$ and an element of pG , which implies that $\Delta_\gamma / \langle \gamma \rangle$ is p -divisible. \square

We extract the following consequence of the definability results of [Jahnke and Koenigsmann 2015]. See the introduction of that paper for the notion of p -henselianity used in the proof.

Proposition 6.3. *Let (K, v) be a henselian valued field such that the residue field Kv is neither separably closed nor real closed. Then there exists an \mathcal{L}_r -definable (not necessarily henselian) refinement w of v .*

Proof. Note that Kv either has a Galois extension of degree divisible by some prime $p \neq 2$ or it only has Galois extensions of 2-power degree.

In the latter case, since Kv is neither separably closed nor real closed, it has Galois extensions of degree 2 but is not Euclidean. Since v is henselian and thus, in particular, 2-henselian, we can now apply [Jahnke and Koenigsmann 2015, Corollary 3.3] to obtain that it admits an \mathcal{L}_r -definable refinement. Indeed, let v_K^2 denote the canonical 2-henselian valuation (see [Jahnke and Koenigsmann 2015, page 743]). Then by [Jahnke and Koenigsmann 2015, Corollary 3.3], v_K^2 is \mathcal{L}_r -definable if Kv_K^2 is non-Euclidean, otherwise the coarsest 2-henselian valuation with Euclidean residue field v_K^{2*} is \mathcal{L}_r -definable. Either is a refinement of v .

Thus, we now assume that there is a prime $p \neq 2$ such that Kv has a finite Galois extension L of degree divisible by p . Then there is a finite separable extension M/Kv such that L/M is a finite Galois extension of degree p^n for some $n > 0$ (e.g., take M to be the fixed field of the p -Sylow subgroup of $\text{Gal}(L/Kv)$ inside L). Let F_0/K be a finite separable extension such that the (by henselianity unique) prolongation of v to F_0 has residue field M (see [Engler and Prestel 2005, Theorem 5.2.7(2)] for the existence of F_0). Consider $F = F_0$ if the characteristic of K is p , and $F = F_0(\zeta_p)$ otherwise, where ζ_p is a primitive p -th root of unity. Then F/K is a finite separable extension, and the residue field of the unique prolongation u of v to F is a finite extension of M . In particular, Fu admits Galois extensions of p -power degree, e.g., the compositum of L and Fu . Therefore u is a henselian (and

thus in particular p -henselian) valuation with $Fu \neq Fu(p)$, and hence coarsens the canonical p -henselian valuation v_F^p of F , which is \emptyset - \mathcal{L}_r -definable in F by the main theorem of [Jahnke and Koenigsmann 2015].

Now F is interpretable in K as the splitting field of a separable polynomial (see [Marker 2002, page 31]). Hence $w = v_F^p|_K$ is \mathcal{L}_r -definable in K . Lastly, $v = u|_K$ is a coarsening of w , as u is a coarsening of v_F^p in F . \square

We can now state our main theorem about definability of henselian valuations on ordered fields. In the proof, we need the following notion: for $n \in \mathbb{N}$, we say that a valuation v on a field K is n_{\leq} -henselian if Hensel's lemma holds in (K, v) for all polynomials of degree at most n . Note that for a fixed n , the property of n_{\leq} -henselianity is elementary in the language \mathcal{L}_{vf} .

Theorem 6.4. *Let $(K, <, v)$ be an ordered henselian valued field. If v is \mathcal{L}_{or} -definable, then it is \mathcal{L}_r -definable.*

Proof. If v is trivial, then the proof is clear; thus we assume that v is nontrivial from now on. If K is almost real closed, then the result follows from Theorem 5.2, and hence we also assume that K is not almost real closed. Let v_K denote the canonical henselian valuation on K , i.e., the finest henselian valuation on K . In particular, v_K is a (not necessarily proper) refinement of v . The residue field Kv_K carries an ordering induced by the ordering on K , but is not real closed since K is not almost real closed by assumption. By Proposition 6.3, we may thus fix an \mathcal{L}_r -definable refinement w of v_K (and hence of v).

Now, by Lemma 6.1, we can make the following case distinction.

Case 1: $vK \not\equiv \Gamma \oplus \mathbb{Q}$ for any ordered abelian group Γ .

Let $\Delta_v \leq wK$ be the convex subgroup such that $wK/\Delta_v = vK$. We first show that there is a prime p such that vK contains no nontrivial p -divisible convex subgroup. Assume for a contradiction that vK contains a nontrivial p -divisible convex subgroup for every prime p . Then any sufficiently saturated elementary extension G^* of vK contains a nontrivial convex divisible subgroup Q . Now, [Schmitt 1982, Lemma 1.11] implies

$$vK \equiv G^* \equiv G^*/Q \oplus Q \equiv G^*/Q \oplus \mathbb{Q},$$

contradicting that we are in Case 1.

Hence, we can fix some prime p such that vK does not contain any nontrivial p -divisible convex subgroup. By Lemma 6.2, there exists an \mathcal{L}_{og} -formula $\varphi(x, y)$ such that for any positive $\gamma \in wK$, the subgroup Δ_γ of wK defined by $\varphi(x, \gamma)$ is the maximal convex subgroup of wK containing $\langle \gamma \rangle$ and such that $\Delta_\gamma/\langle \gamma \rangle$ is p -divisible. In case we choose $\gamma \in \Delta_v$, we have $\Delta_\gamma \leq \Delta_v$; otherwise

$$\langle \gamma \rangle \leq \Delta_v \leq \Delta_\gamma$$

implies that $\Delta_\gamma/\Delta_v \leq vK$ is a nontrivial convex subgroup which is p -divisible since it is a quotient of the p -divisible group $\Delta_\gamma/\langle\gamma\rangle$.

For every $\gamma \in wK$, let u_γ be the \mathcal{L}_r -definable coarsening of w on K with value group wK/Δ_γ . Since Δ_γ is uniformly \mathcal{L}_{og} -definable in wK , also u_γ is uniformly \mathcal{L}_r -definable in K , i.e., there exists an \mathcal{L}_r -formula $\psi(x, \underline{y}, z)$ and a parameter tuple $\underline{b} \in K$ such that for every $a \in K^\times$ the formula $\psi(x, \underline{b}, a)$ defines $u_{w(a)}$.

If u_γ is already henselian for some $\gamma \in \Delta_v$, then v is an \mathcal{L}_{or} -definable coarsening of an \mathcal{L}_r -definable henselian valuation and hence v is \mathcal{L}_r -definable by [Corollary 4.4](#). Thus, we assume that for every $\gamma \in \Delta_v$ the valuation u_γ is not henselian.

First suppose that there is some $n \in \mathbb{N}$ such that for every $\gamma \in \Delta_v$ we have that u_γ is not n_\leq -henselian. Let B be the \mathcal{L}_r -definable subset of K consisting of all $a \in K^\times$ such that $u_{w(a)}$ is not n_\leq -henselian. We claim that $w(B) = \Delta_v$ holds. Let $a \in K^\times$ and set $\gamma = w(a)$. First suppose that $\gamma \in \Delta_v$. Then $u_\gamma = u_{w(a)}$ is not n_\leq -henselian. Thus, $a \in B$ and $\gamma \in w(B)$. Conversely, suppose that $\gamma \notin \Delta_v$. Then $\Delta_v \not\leq \Delta_\gamma$ and thus u_γ is a strict coarsening of v . Since v is henselian, u_γ is n_\leq -henselian. Hence, $a \notin B$ and $\gamma \notin w(B)$, as required. Thus, in this case v is \mathcal{L}_r -definable as \mathcal{O}_v consists exactly of all $x \in K$ with $w(x) \geq 0 \vee w(x) \in w(B)$, which is an \mathcal{L}_r -definable condition as w is \mathcal{L}_r -definable.

Now suppose that for every $n \in \mathbb{N}$ there exists $\gamma_n \in \Delta_v$ such that u_{γ_n} is n_\leq -henselian. Then for every $n \in \mathbb{N}$, there is some $a_n \in K$ (with $w(a_n) = \gamma_n$) such that $\psi(x, \underline{b}, a_n)$ defines an n_\leq -henselian refinement of v in $(K, <, v)$. Hence, in some sufficiently saturated elementary extension $(K^*, <, v^*)$ of $(K, <, v)$, there exists $a \in K^*$ such that $\psi(x, \underline{b}, a)$ defines a henselian refinement u^* of v^* . Since v^* is \mathcal{L}_{or} -definable by the same formula in K^* as v in K , it is an \mathcal{L}_{or} -definable coarsening of the \mathcal{L}_r -definable henselian valuation u^* . Hence, v^* is \mathcal{L}_r -definable in K^* by [Corollary 4.4](#) and thus also v is \mathcal{L}_r -definable in K .

Case 2: $(Kv, <) \not\equiv (L((\mathbb{Q})), <)$ for any ordered field $(L, <)$.

First suppose that $v \neq v_K$. Then v_K is strictly finer than v . Assume that $\overline{v_K}(Kv)$ is divisible, where $\overline{v_K}$ denotes the valuation induced by v_K on Kv . Then $\overline{v_K}(Kv)$ is elementarily equivalent to \mathbb{Q} as an ordered abelian group, and thus we obtain

$$(Kv, <) \equiv (Kv_K((\overline{v_K}(Kv))), <) \equiv (Kv_K((\mathbb{Q})), <),$$

in contradiction to the assumption of Case 2.

Therefore we can assume that $\overline{v_K}(Kv) = \Delta$ is nondivisible. We show that there exists some \mathcal{L}_r -definable henselian refinement u_γ of v .

We may assume that Δ does not have a rank 1 quotient: otherwise we could consider a sufficiently saturated elementary extension $(K^*, <^*, v^*, w^*)$ of $(K, <, v, w)$ in which — by the definability of the refinement w of v_K — $\overline{w^*}(K^*v^*)$ (and hence also $\overline{v_K^*}(K^*v^*)$) has no rank 1 quotient. Just as in Case 1, an \mathcal{L}_r -definition of v^*

would give rise to an \mathcal{L}_t -definition of v . We claim that there is a prime p such that Δ has no nontrivial p -divisible quotient. If not, then some saturated elementary extension Δ^* of Δ has a divisible nontrivial quotient Δ^*/Γ , where Γ is a convex proper subgroup. Then, as before, we have

$$\begin{aligned} (Kv, <) &\equiv (Kv_K((\overline{v_K}(Kv))), <) = (Kv_K((\Delta)), <) \\ &\equiv (Kv_K((\Delta^*)), <) \equiv (Kv_K((\Gamma))((\Delta^*/\Gamma)), <) \equiv (Kv_K((\Gamma))((\mathbb{Q})), <), \end{aligned}$$

contradicting that we are in Case 2. Hence, there is a prime p such that Δ is p -antiregular, i.e., Δ has no nontrivial p -divisible quotient and no rank 1 quotient (see [Jahnke and Koenigsmann 2017, page 670]).

Recall that w is an \mathcal{L}_t -definable refinement of v_K . Then, there are convex subgroups

$$\Delta_{v_K} \leq \Delta_v \leq wK$$

with $v_K K = wK/\Delta_{v_K}$ and $vK = wK/\Delta_v$.

Let $\gamma \in \Delta_v \setminus \Delta_{v_K}$ be positive, and let $\langle \gamma \rangle \leq wK$ denote the smallest convex subgroup containing γ . Since the convex subgroups of wK are ordered by inclusion, we have

$$\Delta_{v_K} \leq \langle \gamma \rangle \leq \Delta_v.$$

Note that $\langle \gamma \rangle$ need not be \mathcal{L}_{og} -definable in wK . However, by Lemma 6.2, the maximal convex subgroup $\Delta_\gamma \leq wK$ that contains $\langle \gamma \rangle$ and such that $\Delta_\gamma/\langle \gamma \rangle$ is p -divisible is \mathcal{L}_{og} -definable in wK .

We claim that $\Delta_\gamma \leq \Delta_v$, i.e., that Δ_γ corresponds to an \mathcal{L}_t -definable refinement of v . Assume for a contradiction that we have $\Delta_v \leq \Delta_\gamma$. Since Δ_v contains $\langle \gamma \rangle$, the choice of Δ_γ implies that $\Delta_v/\langle \gamma \rangle$ is p -divisible. If $\langle \gamma \rangle \neq \Delta_v$, then $\Delta_v/\langle \gamma \rangle$ is a nontrivial p -divisible quotient of $\Delta = \Delta_v/\Delta_{v_K}$, a contradiction. Otherwise, $\langle \gamma \rangle = \Delta_v$ and the quotient of $\langle \gamma \rangle$ by its maximal convex subgroup not containing γ is of rank 1, also a contradiction. Thus, we have found an \mathcal{L}_t -definable refinement of v . Since we have

$$\Delta_{v_K} \leq \Delta_\gamma,$$

this refinement is a coarsening of v_K and thus henselian.

Now suppose that $v = v_K$. If Kv is not t -henselian, then v is \mathcal{L}_t -definable by [Fehm and Jahnke 2015, Proposition 5.5]. Hence, suppose that Kv is t -henselian. Then for a sufficiently saturated elementary extension $(K^*, <^*, v^*) \succeq (K, <, v)$ the residue field $K^*v^* \succeq Kv$ is itself henselian. Since v is \mathcal{L}_{or} -definable in K , also v^* is \mathcal{L}_{or} -definable in K^* . However, since K^*v^* is henselian, v^* is not the canonical henselian valuation of K^* , and therefore by the arguments above v^* is already \mathcal{L}_t -definable in K^* . Hence, also v is \mathcal{L}_t -definable in K . \square

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